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Convergence of the coincidence set in the homogenization of the obstacle problem

CONVERGENCE OF THE COINCIDENCE SET
IN THE HOMOGENIZATION OF THE OBSTACLE PROBLEM

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Résumé : On étudie le comportement asymptotique, dans la métrique de Hausdorff, des ensembles de contact d'une suite d'inéquations variationnelles avec obstacles qui ont des solutions convergentes au sens de l'homogénéisation. On considère aussi la convergence en mesure des ensembles de contact.

Summary : In this paper we study the asymptotic behaviour, in the Hausdorff metric, of the coincidence sets of a sequence of variational inequalities with obstacles, which have convergent solutions in the homogenization sense. We also consider the convergence in measure of the coincidence sets.

INTRODUCTION

In section 1 we introduce the suitable notations for a sequence of variational inequalities with obstacles and we recall some related results on the homogenization in order to provide a sufficient framework to our study.

Section 2 deals with the main result of this paper on the Hausdorff convergence of the

coincidence sets in homogenization of variational inequalities with obstacles. This exploits the convergence properties of the solutions following an idea of Pironneau and Saguez [PS], and some regularity in the limit problem, as in a preceding work of one of the authors [R1]. We also include a result due to F. Murat on the convergence in measure of the coincidence sets when the obstacle is zero.

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1. - HOMOGENIZATION OF VARIATIONAL INEQUALITIES WITH OBSTACLE

Let Ω be a bounded domain of $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, and consider the sequence of bilinear forms and associated operators

$$(1.1) \qquad a^\epsilon(u,v) = \sum_{i,j} \int_\Omega a_{ij}^\epsilon \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx, \quad A^\epsilon u = -\sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij}^\epsilon \frac{\partial u}{\partial x_j} \right),$$

where the coefficients $a_{ij}^\epsilon(x)$ satisfy

$$(1.2) \qquad \exists \alpha > 0 \; : \; \sum_{i,j} a_{ij}^\epsilon \, \xi_i \, \xi_j \; \geqslant \; \alpha \, |\xi|^2, \quad \text{a.e. in } \Omega, \quad \forall \, \xi \in \mathbb{R}^n, \quad \forall \, \epsilon > 0,$$

$$(1.3) \qquad \exists \, M > 0 \; : \; |a_{ij}^\epsilon(x)| \leqslant M \quad \text{a.e. in } \Omega, \quad \forall \, i,j = 1,...,n, \quad \forall \, \epsilon > 0.$$

Here ϵ is a real parameter taking values in a sequence going to zero. We consider the following problem : find

$$(1.4) \qquad u^\epsilon \in \mathbb{K}(\psi^\epsilon) : a^\epsilon(u^\epsilon, v - u^\epsilon) \; \geqslant \; <f, v - u^\epsilon> \; , \quad \forall \, v \in \mathbb{K}(\psi^\epsilon)$$

where the convex set $\mathbb{K}(\psi)$ is defined by

$$(1.5) \qquad \mathbb{K}(\psi) = \left\{ v \in H_0^1(\Omega) : v \geqslant \psi \quad \text{a.e. in } \Omega \right\}.$$

It is well known that if the functions f and ψ^ϵ are given in appropriate spaces ($f \in H^{-1}(\Omega)$ and $\psi^\epsilon \in H^1(\Omega)$) and $\psi^\epsilon \leqslant 0$ on $\partial\Omega$ (then $\mathbb{K}(\psi^\epsilon) \neq \phi$), there exists an unique solution u^ϵ of (1.4). We are interested in the asymptotic behaviour of u^ϵ when $\epsilon \searrow 0$.

Under hypothesis (1.2) and (1.3) there exists a subsequence $A^{\epsilon'}$ and an operator A^0 (defined by a bilinear form $a^0(\cdot,\cdot)$ with coefficients $a_{i,j}^0(x)$ verifying (1.2) with the same α and (1.3) with possibly a different M) such that (see [DS] and [T]) :

(1.6) $(A^{\epsilon'})^{-1} f \xrightarrow{\epsilon} (A^o)^{-1} f$ in $H_o^1(\Omega)$-weak, $\forall f \in H^{-1}(\Omega)$.

Condition (1.6) is usually taken as the definition of the so-called G-convergence of operators. This convergence is rather general and does not give much information on the limit operator A^o. But if A^{ϵ} has a special structure, as in a few but important cases, it is possible to obtain more precise results on the structure of the coefficients of the homogenized operator. For instance, if the coefficients are in the form $a_{ij}^{\epsilon}(x) = a_{ij}(\frac{x}{\epsilon})$, where the a_{ij} are periodic functions, we have a family of operators with rapidly oscillating coefficients, and it is well known that A^o is an operator with constant coefficients which can be computed explicitly (see [BLP] , for instance). An easier exemple is the case of layers, that is, when the coefficients only depend on one coordinate x_k : $a_{ij}^{\epsilon}(x) = a_{ij}^{\epsilon}(x_k)$; then we can also determine a_{ij}^o from a_{ij}^{ϵ}.

There are several results, with different hypothesis, on the homogenization of variational inequalities with obstacles (see Boccardo and Cappuzzo Dolcetta [BC] Murat [M1] , Boccardo and Marcellini [BMa] , Attouch [A] and for more references De Giorgi [DG]).

We just recall without proof a recent result by Boccardo and Murat [BMu] (see also Murat [M1]) to provide a general framework for our study in section 2.

PROPOSITION 1 [M1] Let $f \in H^{-1}(\Omega)$ and consider a sequence of operators A^{ϵ} G-convergent to A^o. If one assumes that

(1.7) $\exists \zeta \in H_o^1(\Omega) : \zeta \geqslant \psi^{\epsilon}, \quad \forall \epsilon \geqslant 0 \text{ and } \psi^{\epsilon} \rightarrow \psi^o \text{ in } L^{\infty}(\Omega)\text{-strong,}$

then the solutions u^{ϵ} *of (1.4) are such that*

(1.8) $u^{\epsilon} \xrightarrow{\epsilon} u^o$ *in* $H_o^1(\Omega)$-weak,

(1.9) $A^{\epsilon} u^{\epsilon} \xrightarrow{\epsilon} A^o u^o$ *in* $H^{-1}(\Omega)$-weak,

where u^o *is the solution of the homogenized variational inequality :*

(1.10) $u^o \in \mathbb{K}(\psi^o) : a^o(u^o, v - u^o) \geqslant <f, v - u^o> , \quad \forall v \in \mathbb{K}(\psi^o).$

Assumption (1.7) is an exemple of a sufficient condition to the convergence of the convex sets $\mathbb{K}(\psi^{\epsilon})$ to $\mathbb{K}(\psi^o)$ in the Mosco sense, which is a general condition to get (1.8) and (1.9) for homogenization of unilateral variational inequalities (see [A] and [BMu]).

Since in the next section we need the uniform convergence of the solution u^{ϵ}, we give here a sufficient condition to this convergence. We note that this result has been proved by others authors with different hypothesis (see Th. 5 of [Bi] , for instance).

PROPOSITION 2. *In conditions of Proposition 1, if in adition one assumes that* $f \in W^{-1,p}(\Omega)$, *for* $p > n$, *and that*

(1.11) ψ^ϵ *are uniformly bounded in* $C^{0,\lambda}(\overline{\Omega})$, *for some* $0 < \lambda < 1$,

then one also has

(1.12) $u^\epsilon(x) \xrightarrow[\epsilon]{} u^0(x)$ *uniformy for* $x \in \overline{\Omega}$.

Proof. This is an immediate consequence of the De Giorgi - Nash - Moser estimate extended to variational inequalities with obstacles (see [MS] and [Bi2]). Indeed from theorem 2 of [Bi2] the solution u^ϵ are bounded in $C^{0,\nu}(\Omega)$, for some $0 < \nu < \lambda < 1$, independently of ϵ ; then (1.12) follows by the Ascoli - Arzelà theorem. ∎

2. - ON THE ASYMPTOTIC BEHAVIOUR OF THE FREE BOUNDARY

We can divide $\overline{\Omega}$ into two subsets,

(2.1) $E^\epsilon = \left\{ x \in \overline{\Omega} : u^\epsilon(x) = \psi^\epsilon(x) \right\}$, $(\epsilon \geqslant 0)$,

which are usually called the coincidence sets of the corresponding variational inequalities, and

$$Q^\epsilon = \left\{ x \in \overline{\Omega} : u^\epsilon(x) > \psi^\epsilon(x) \right\}.$$

We shall assume that the functions u^ϵ and ψ^ϵ are continuous and therefore E^ϵ and Q^ϵ are respectively compact and open sets of $\overline{\Omega}$. The associated free boundaries, defined by

(2.2) $\Gamma^\epsilon = \partial E^\epsilon \cap \Omega$, $(\epsilon \geqslant 0)$

may have, in general, a very complicated structure. With some regularity on Γ^0 we can study the asymptotic behaviour of the coincidence sets E^ϵ as $\epsilon \searrow 0$ and give some kind of convergence of the free boundaries.

Let us recall that the Hausdorff distance $\delta(F,G)$ between two sets F and G may be defined by

$$\delta(F,G) = \max \left[\sup_{x \in F} d(x,G) , \sup_{y \in G} d(y,F) \right] , \text{where } d(x,G) = \inf_{y \in G} |x-y|.$$

It is well known that the familly of all compact subsets of a compact set of $\mathbb{R}^n$, is a compact metric space for the Hausdorff distance (see [De] pg. 42, for instance), and it is not very difficult to show that $F^\epsilon \xrightarrow[\epsilon]{} F$ (Hausdorff) in $\overline{\Omega}$, if and only if, for every $\eta \in C^0(\overline{\Omega})$

(2.3)
$$\sup_{x \in F^\epsilon} |\eta(x)| \xrightarrow{\epsilon} \sup_{x \in F} |\eta(x)|.$$

LEMMA 1. *Consider a sequence of non-negative measures μ^ϵ verifying*

$$\mu^\epsilon \xrightarrow{\epsilon} \mu \quad in \quad \sigma(\mathcal{M}(\Omega), C_K^0(\overline{\Omega})),$$

and a sequence of compact sets of $\overline{\Omega}$ verifying

$$F^\epsilon \xrightarrow{\epsilon} F \quad (Hausdorff).$$

Then if supp $\mu^\epsilon \subset F^\epsilon$, one has supp $\mu \subset F$.

Proof. Let $\eta \in C^0(\overline{\Omega})$, $\eta \geqslant 0$ and $\eta = 0$ on F. We have, for each $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geqslant 0$

$$0 \leqslant < \mu^\epsilon, \eta\varphi > \leqslant \sup_{x \in F^\epsilon} |\eta(x)| < \mu^\epsilon, \varphi >.$$

In the limit, by the hypothesis and using (2.3), one finds $< \mu, \eta\varphi > = 0$. Therefore, since $\varphi \geqslant 0$ is arbitrary, one has

$$< \mu, \eta > = 0, \quad \text{for all} \quad \eta \geqslant 0 \quad \text{vanishing on F.}$$

This implies supp $\mu \subset F$, and the lemma is proved. ∎

Now we require that the following supplementary hypothesis for the limit problem (1.10) hold : the non-homogeneous term f, the obstacle ψ^0 and the coefficients of the limit operator A^0 are such that :

(2.4) $A^0\psi^0 - f \neq 0$ in the distributional sense in all open subsets of Ω ;   (*)

moreover we assume that the limit coincidence set verifies

(2.5) $\overline{int\ E^0} = E^0,$

which is a regularity hypothesis on the free boundary Γ^0.

THEOREM 1. *Assume that the solutions u^ϵ of the variational inequalities (1.4) with obstacles $\psi^\epsilon \xrightarrow{\epsilon} \psi^0$ in $C^0(\Omega)$, converge uniformly to u^0, solution of (1.10), and that $A^\epsilon u^\epsilon \xrightarrow{\epsilon} A^0 u^0$*

(*)   Assumption (2.4) means that for all open $\omega \subset \Omega$ there exists $\varphi \in \mathcal{D}(\omega)$ such that $< A^0\psi^0 - f, \varphi > \neq 0$. If $F \equiv A^0\psi^0 - f$ is a locally integrable function it means $F \neq 0$ almost everywhere in Ω.

in $H^{-1}(\Omega)$*-weak. Then under assumptions (2.4) and (2.5) the coincidence sets verify :*

$$E^{\epsilon} \xrightarrow[\epsilon]{} E^{o} \quad \text{in Hausdorff metric.}$$

We remark that Proposition 2 provides, together with (2.4) and (2.5), a framework for this theorem that we state more explicity as the following.

COROLLARY 1. *Assume that* A^{ϵ} G - *convergence to* A^{o}, $f \in W^{-1,p}(\Omega)$ *for some* p > n, *and the obstacles verify* $\psi^{\epsilon} \xrightarrow[\epsilon]{} \psi^{o}$ *in* $C^{o,\lambda}(\overline{\Omega})$ *for some* $0 < \lambda < 1$, *and* $\exists \, \varsigma \in H_0^1(\Omega) \quad \varsigma \geqslant \psi^{\epsilon}$,

$\forall \, \epsilon \geqslant 0$. *Then, if (2.4) and (2.5) hold, the coincidence set* E^{ϵ} *associated to the variational inequality (1.4), converges in the Hausdorff distance to the coincidence set* E^{o} *of the homogenized variational inequality (1.10).*

Proof of Theorem 1. Since $\left\{ E^{\epsilon} \right\}$ is a sequence of compact subsets of $\overline{\Omega}$, there exists a subsequence, still denoted by $\left\{ E^{\epsilon} \right\}$, and a compact set $E^* \subset \overline{\Omega}$ such that

$$E^{\epsilon} \xrightarrow[\epsilon]{} E^* \quad \text{(Hausdorff).}$$

We proceed by steps to prove that

$$(2.6) \qquad\qquad E^* = E^{o}.$$

1st *step.* $E^* \subset E^{o}$: since u^{o}, ψ^{o} are continuous and $u^{\epsilon} \xrightarrow[\epsilon]{} u^{o}$, $\psi^{\epsilon} \xrightarrow[\epsilon]{} \psi^{o}$ uniformly, using (2.3) we have

$$\sup_{x \in E^*} |u^{o}(x) - \psi^{o}(x)| = \lim_{\epsilon} \sup_{x \in E^{\epsilon}} |u^{o}(x) - \psi^{o}(x)|$$

$$\leqslant \lim_{\epsilon} \| u^{\epsilon} - u^{o} \|_{C^{o}(\overline{\Omega})} + \lim_{\epsilon} \| \psi^{\epsilon} - \psi^{o} \|_{C^{o}(\overline{\Omega})} = 0,$$

and the inclusion follows.

2nd *step.* $E^{o} = E^* \cup N$, where int $N = \phi$: assume to the contrary that there exists an open ball B such that $B \subset E^{o} \setminus E^*$; from $A^{\epsilon}u^{\epsilon} \xrightarrow[\epsilon]{} A^{o}u^{o}$, $\mu^{\epsilon} = A^{\epsilon}u^{\epsilon} - f$ is a convergent sequence to $\mu^{o} = A^{o}u^{o} - f$ of non negative measures verifying supp $\mu^{\epsilon} \subset E^{\epsilon}$; by Lemma 1 it follows that supp $(A^{o}u^{o} - f) \subset E^*$; but then in $B \subset E^{o}$ we have $u^{o} = \psi^{o}$ and that implies

$$(2.7) \qquad\qquad A^{o}\psi^{o} = f \quad \text{in} \quad B \quad (\text{since } B \cap E^* = \phi),$$

which is a contradiction with assumption (2.4).

3rd step. $E^o \subset E^*$: from assumption (2.5) and the 2nd step we have

$$E^o = \overline{int\ E^o} = \overline{int\ E^*} \subset E^*.$$

Then (2.6) is proved and the proof of theorem 1 is achieved.   ■

Remark 1. If we denote by $S^\epsilon = supp\ (A^\epsilon u^\epsilon - f)$ for $\epsilon \geqslant 0$, we may assume (at least for a subsequence) that $S^\epsilon \xrightarrow[\epsilon]{} S^*$ (Hausdorff) as $\epsilon \searrow 0$, where S^* is a compact subset of $\overline{\Omega}$. As a consequence of Lemma 1, one has $S^o \subset S^*$. Since $S^\epsilon \subset E^\epsilon$ implies $S^* \subset E^*$, step 1, which is independent of assumptions (2.4) and (2.5), implies the following chain

$$S^o \subset S^* \subset E^* \subset E^o.$$

Then conclusion of Theorem 1 still holds if we replace (2.4) and (2.5) by the single hypothesis

(2.8) $supp\ (A^o u^o - f) \supset E^o,$

which is also an assumption on the limit problem, different but near to (2.4).

For instance, (2.8) holds if $E^o \neq \phi$ is regular and $A^o \psi^o - f$ is a strictly positive continuous function in Ω.   ■

Remark 2. The assumption (2.4) seems essential since it is necessary to assure that the region where $u^o = \psi^o$ (the coincidence set) and the region where u^o verifies the equation $A^o u^o = f$ have intersection with void interior.

On the contrary, hypothesis (2.5) stands only to assure that the global difference $N = E^o \setminus E^*$ is in fact the void set. From our proof (see step 3) one can also obtain a local result. Assume (2.4) and replace (2.5) by the following local hypothesis : if F is a compact subset of Ω, such that

$$E^o \cap F = \overline{int\ (E^o \cap F)},$$

then one has $E^\epsilon \cap F \xrightarrow[\epsilon]{} E^o \cap F$ (Hausdorff).   ■

Remark 3. Step 2 of our proof could also be proved by replacing Lemma 1 by the technic of Pironneau and Saguez [PS] based on the fact that for each $v \in H_0^1\ (\Omega \setminus E^*)$, there exists a sequence of functions $v^\epsilon \in H_0^1\ (\Omega \setminus E^\epsilon)$, such that $\tilde{v}^\epsilon \xrightarrow[\epsilon]{} \tilde{v}$ in $H_0^1\ (\Omega)$ - strong ($\tilde{\ }$ denotes the extension by zero). So from

$$< A^o u^o - f, \tilde{v} > = \lim_\epsilon\ < A^\epsilon u^\epsilon - f, \tilde{v}^\epsilon > = 0, \quad \forall v \in H_0^1\ (\Omega \setminus E^*),$$

we also deduce (2.7) in $B \subset E^o \setminus E^*$.   ■

Remark 4. The asymptotic behaviour of the coincidence sets in the Hausdorff metric is not suffi-cient to assure the convergence of the free boundaries, except in the simple case of one dimension if we assume for instance that E^ϵ is an interval.

With the further assumption that the inclusion $E^0 \subset E^\epsilon$ holds for all ϵ, or else assu-ming that the free boundaries Γ^ϵ are lipschitz, uniformly with respect to ϵ, one can also prove the Hausdorff convergence of Γ^ϵ to Γ^0 (see [R1]). However it is not likely that these conditions be fullfiled in homogenization theory.   ■

Remark 5. A classical situation in which assumptions (2.4) and (2.5) are fullfilled is the following case in dimension n = 2, considered by Lewy and Stampacchia [LS] : let $\Omega \subset \mathbb{R}^2$ be a convex smooth domain, $f \equiv 0$, A^0 an operator with constant coefficients and ψ^0 a strictly concave obsta-cle verifying :

$$(2.9) \qquad \psi^0 \in C^2(\overline{\Omega}), \quad \psi^0 < 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \max_{\Omega} \psi^0 > 0 \ ;$$

then the coincidence set E^0 is a simply connected domain equal to the closure of ist interior. Moreover E^0 is bounded by an analytic Jordan curve if the obstacle ψ^0 is also analytic.   ■

Remark 6. In higher dimensions and without assuming convexity hypothesis on Ω and on $- \psi^0$, the results of Caffarelli [C] on the regularity of the free boundaries provide, at least locally, also sufficients conditions to assumption (2.5) : suppose that A^0 has constant coefficients (which is true for periodic homogenization, for instance), $f \in C^{0,\lambda}(\Omega)$, ψ^0 verifies (2.9) and for some $\nu > 0$, one has

$$A^0\psi^0 - f \geq \nu > 0 \qquad \text{in } \Omega \ ; \quad \text{(which implies (2.4))} \ ;$$

then, far from eventually some singular points, the free boundary Γ^0 is a C^1-surface.   ■

Assuming different hypothesis on the data, Murat [M2] has proved another type of convergence of the coincidence sets for the homogenization of unilateral variationale inequalities.

Precisely, suppose that $\psi^\epsilon = \psi^0 \equiv 0$, $f \in L^2(\Omega)$ and let A^ϵ be a sequence of operatores G - convergent to A^0. Note that in this case Proposition 1 holds. Assume that we can write for the solutions u^ϵ and u^0 of the variational inequalities (1.4) and (1.10) :

$$(2.10) \qquad A^\epsilon u^\epsilon = f(1 - \chi^\epsilon) \qquad \text{a.e. in } \Omega, \text{ for each } \epsilon \geq 0,$$

where χ^ϵ is the characteristic function of E^ϵ. This equality is verified if we assume, for instance, that $a^\epsilon_{ij} \in C^1(\Omega)$ (since then $u^\epsilon \in H^2(\Omega)$ see [LS] , for example), without hypothesis of unifor-mity in ϵ, of course. This is the case in periodic homogenization $a^\epsilon_{ij}(x) = a_{ij}(\frac{x}{\epsilon})$ with $a_{ij} \in C^1(\overline{Y})$ and Y - periodic (see [BLP])

Then letting $\epsilon \searrow 0$ in (2.10) and observing that $\chi^\epsilon \xrightarrow[\epsilon]{} q$ in $L^\infty(\Omega)$-weak * (at least for a subsequence), one finds $fq = f\chi^0$, by the classical homogenization theory for equations with second term converging strongly in $H^{-1}(\Omega)$ or by Proposition 1.

Now assuming $f \neq 0$ a.e. in Ω it follows $q = \chi^0$. But since if characteristic functions converge weakly to a characteristic function they also converge strongly, one deduces that $\chi^\epsilon \xrightarrow[\epsilon]{} \chi^0$ in $L^p(\Omega)$-strong ($\forall\, p < \infty$), and we have proved the

THEOREM 2 (Murat [M2]). *Consider a sequence of operators* A^ϵ G - *convergent to* A^0, *obstacles* $\psi^\epsilon \equiv 0$ *and* $f \in L^2(\Omega)$. *Assume that* $f \neq 0$ *a.e. in* Ω *and that the solutions* u^ϵ *and* u^0 *of the variational inequalities (1.4) and (1.10) are such that (2.10) holds. Then the coincidence sets* E^ϵ *converge asymptoticly in measure to* E^0, *i.e., their characteristic functions converge strongly in* $L^1(\Omega)$. ∎

We observe that in general there is no equivalence between convergence in measure and convergence in the Hausdorff sense (see [B]).

Remark 7. These results on the convergence of the coincidence set for the homogenization of the obstacle problem have applications to the dam problem (see Codegone and Rodrigues [CR]). They may be extended to the parabolic case of the homogenization of the one phase Stefan problem (see Rodrigues [R2]). ∎

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