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OPTIMAL CONTROL OF UNSTABLE NON LINEAR EVOLUTION SYSTEMS

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Résumé : Dans le cas où U_{ad} a un intérieur non vide, l'auteur obtient un système d'optimalité, pour deux problèmes de contrôle optimal provenant d'un système d'évolution non linéaire où la variable de contrôle apparaît à la frontière.

Summary : We show that if we suppose the interior of U_{ad} non empty, then we obtain an optimality system for two problems of optimal control of unstable non linear evolution systems where the control variable appears on the boundary.

INTRODUCTION.

Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ. We study the problems of optimal control related to the partial differential equation

(1) $$\frac{\partial z}{\partial t} - \Delta z - z^3 = f, \quad \text{in } Q = \Omega \times \,]0,T[$$

where the control variable v is a function definite on $\Sigma = \Gamma \times \,]0,T[$.

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We will show that if the interior of U_{ad} (the set of admissible controls) is non empty, there exists an optimality system characterizing the optimal couple.

In Section 1 we given an abstract statement for problems of optimal control of singular systems, we show the existence of an optimal couple (u,y) and we make some remarks on the penalized problem.

In Section 2 we study the case where the state equation is given by (1), (2) and (3), where :

$$\frac{\partial z}{\partial \nu} = v, \quad \text{on } \Sigma \tag{2}$$

$$z(x,0) = y_0(x), \quad \text{in } \Omega. \tag{3}$$

In Section 3 we consider the problem of optimal control of the system governed by (1), (2'), (3), where :

$$z = v, \quad \text{on } \Sigma. \tag{2'}$$

The plan is as follows :

1. The abstract problem.
2. Unstable non linear evolution system : Case of the Neumann condition.
3. Unstable non linear evolution system : Case of the Dirichlet condition.

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1. THE ABSTRACT PROBLEM

1.1. Setting of the Problem.

Let U and H be two Hilbert spaces on IR and let Z be a reflexive Banach spaces on IR. We consider the control variable $v \in U$ and the state $z \in Z$ related by the state equation :

$$\mathcal{A} z = f + Bv \tag{1.1}$$

where f is given in H, $\mathcal{A}$ is an operator (non necessarly linear) from the domain $D(\mathcal{A}) \subset Z$ into

H and B is an operator from U into H.

In the usual theory (Lions [1]) we assume that the equation (1.1) has a unique solution for each v in U. At the present we ignore the existence or the uniqueness of the solutions of (1.1). For each control v we define the set :

(1.2) $$Z(v) = \mathcal{A}^{-1}\left\{f + Bv\right\} = \left\{z \in D(\mathcal{A})\ ;\ \mathcal{A}z = f + Bv\right\}.$$

Also, for each $M \subset U$ we consider the set given by :

(1.3) $$\hat{M} = \left\{v \in M\ ;\ \ Z(v) \neq \emptyset\right\}.$$

The cost function is given by :

(1.4) $$J(v,z) = \Phi(z) + (Nv \mid v)_U\ ,\ \ (v,z)\ \text{in}\ U \times Z$$

in which Φ is a positive real function defined on Z and $N : U \to U$ is a linear operator.

Let U_{ad} be a subset of U such that $\hat{U}_{ad}$ is non empty. The optimal control problem is :

(1.5) Find a couple (u,y) in $U_{ad} \times Z$ such that $y \in Z(u)$ and

$$J(u,y) = \inf\left\{J(v,z)\ ; v \in U_{ad}\ , z \in Z(v)\right\}.$$

THEOREM 1.1. *Let us suppose that the following hypothesis* (1.6) (1.7) (1.8) (1.9) (1.10) *are fulfilled :*

(1.6) *The graph of* $\mathcal{A}$ *is closed in the weak topology of* $U \times Z$.

(1.7) *The graph of* B *is a weakly closed, convex subset of* $U \times H$. *Also, if* K *is a bounded set of* U, *then* B(K) *is a bounded set of* H.

(1.8) Φ *is a convex, weakly lower semi-continuous function from* Z *into* $R_+ = [0, +\infty[$ *such that :*

$$\Phi(z) \to +\infty, \quad as \quad \| z \|_Z \to +\infty.$$

(1.9) $N \in \mathcal{L}(U)$ *is hermitian, positive definite.*

(1.10) U_{ad} *is a closed, convex subset of* U *such that* $\hat{U}_{ad} \neq \emptyset$.

Then there exists a couple (u,y) *satisfying* (1.5).

Proof. Let X_{ad} be the set defined by :

(1.11) $$X_{ad} = \{(v,z) ; v \in U_{ad} , z \in Z(v)\}$$

From (1.10) we deduce that X_{ad} is non empty and then inf $J(X_{ad})$ is finite.

Let (v_m,z_m) ($m \in \mathbb{N}$) be a minimizing sequence for the Problem (1.5). Then the sequence (v_m,z_m) ($m \in \mathbb{N}$) is bounded in U X Z and then we may extract a subsequence, again denoted by (v_m,z_m), such that, as $m \to \infty$:

(1.12) $$(v_m,z_m) \to (u,y), \quad \text{weakly in } U \times Z.$$

Since $\mathcal{A}\, z_m = f + Bv_m$, from (1.7) we obtain that the sequence z_m($m \in \mathbb{N}$) is bounded in H and we may assume, by extraction of a subsequence, that, as $m \to \infty$:

(1.13) $$z_m \to h, \text{ weakly in } H.$$

Hence :

(1.14) $$(v_m,Bv_m) = (v_m, \mathcal{A}\, z_m - f) \to (u,h-f), \text{ weakly in } U \times H$$

$$(z_m, \mathcal{A}\, z_m) \to (y,h), \quad \text{weakly in } Z \times H.$$

and, from (1.6) (1.7) (1.10), we obtain :

(1.15) $$(u,y) \in U_{ad} \times D(\mathcal{A}), \quad Bu = h-f, \quad \mathcal{A}\, y = h = f + bu$$

Then the couple (u,y) belongs to X_{ad} and by standard arguments, using (1.8) and (1.9) we show that (u,y) verifies (1.5).

Remark 1.1. There is no uniqueness of (1.5) in general.

1.2. The Penalized Problem.

For given $\epsilon > 0$ we define the penalized cost function by :

(1.16) $$J_\epsilon(v,z) = J(v,z) + \epsilon^{-1} \| \mathcal{A}\, z - f - Bv \|_H^2 , \quad v \in U, \quad z \in D(\mathcal{A}) \; (*)$$

(*) By introducing an extra term in ∂_ϵ , as in V. BARBU [10] (cf. also J.L. LIONS [5]), the results which follow are valid for *every* optimal couple $\{u,y\}$.

THEOREM 1.2. *Under the hypothesis of Theorem 1.1, there exists a couple* (u_ϵ, y_ϵ) *such that :*

(1.17) $$u_\epsilon \in U_{ad} \ , \ y_\epsilon \in D(\mathcal{A})$$

(1.18) $$J_\epsilon(u_\epsilon, y_\epsilon) = \inf \{ J_\epsilon(v,z) \ ; v \in U_{ad} \ , z \in D(\mathcal{A}) \}$$

Proof. Let (v_m, z_m) $(m \in \mathbb{N})$ be a minimizing sequence for the penalized problem (1.18). If we set $h_m = \mathcal{A} z_m - f - Bv_m$, the sequence (v_m, z_m, h_m) is bounded in U X Z X H and we may assume, by extraction of a subsequence, that, as $m \to \infty$:

(1.19) $$(v_m, z_m, h_m) \to (u_\epsilon, y_\epsilon, h_\epsilon), \text{ weakly in } U \times Z \times H$$

From (1.7) we have that Bv_m $(m \in \mathbb{N})$ is a bounded sequence in H. Hence, we may assume hat :

(1.20) $$Bv_m \to b_\epsilon \ , \quad \text{weakly in H,}$$

Then (1.6) (1.7) (1.19) (1.20) imply :

(1.21) $$u_\epsilon \in U'_{ad} \ , \ y_\epsilon \in D(\mathcal{A})$$

(1.22) $$Bu_\epsilon = b_\epsilon \ , \ \mathcal{A} \, y_\epsilon = h_\epsilon + f + b_\epsilon$$

From (1.8) (1.9) (1.19) (1.20) (1.21) (1.22) we obtain that (u_ϵ, y_ϵ) is a solution of the penalized problem (1.17).

1.3. Convergence of (u_ϵ, y_ϵ).

For each $\epsilon > 0$ let $p_\epsilon \in H$ be defined by

(1.23) $$p_\epsilon = -\epsilon^{-1} \{ \mathcal{A} \, y_\epsilon - f - Bu_\epsilon \}$$

THEOREM 1.3. *Under the hypothesis of Theorem 1.1, there exists a solution* (u,y) *of* (1.5) *and there exists a sequence* ϵ_m $(m \in \mathbb{N})$, *which converges to* 0, *such that, as* $m \to \infty$:

(1.24) $$J_{\epsilon_m}(u_{\epsilon_m}, y_{\epsilon_m}) \to J(u,y)$$

(1.25) $$u_{\epsilon_m} \to u, \text{ in H}$$

(1.26) $$\Phi(y_{\epsilon_m}) \to \Phi(y) \quad \textit{and} \quad y_{\epsilon_m} \to y, \quad \textit{weakly in } Z$$

(1.27) $$\sqrt{\epsilon_m}\, p_{\epsilon_m} \to 0, \quad \textit{in } H.$$

Proof. Since $X_{ad} \subset U_{ad} \times D(\mathcal{A})$ we have :

(1.28) $$J_\epsilon(u_\epsilon, y_\epsilon) \leqslant \inf J(X_{ad})$$

from which we have that, as $\epsilon \to 0_+$, $(u_\epsilon, y_\epsilon, \sqrt{\epsilon}\; p_\epsilon)$ is in a bounded set of $U \times Z \times H$ and from (1.7) we obtain that Bu_ϵ is in a bounded set of H. Hence, we may extract a sequence, again denoted by $(u_\epsilon, y_\epsilon, \sqrt{\epsilon}\; p_\epsilon)$, such that, as $\epsilon \to 0_+$:

(1.29) $$(u_\epsilon, y_\epsilon) \to (u,y) \;, \quad \text{weakly in } U \times Z$$

(1.30) $$\epsilon\, p_\epsilon \to 0 \;, \quad \text{in } H$$

(1.31) $$Bv_\epsilon \to b_\epsilon \;, \quad \text{weakly in } H.$$

From the relation $\mathcal{A}\, y_\epsilon = f + bu_\epsilon - \epsilon\, p_\epsilon$ and by the same arguments given in the proof of the Theorem 1.2 we obtain that $(u,y) \in X_{ad}$. Hence :

$$\inf J(X_{ad}) \leqslant J(u,y) \leqslant \underline{\lim}\, J(u_\epsilon, y_\epsilon) \leqslant \overline{\lim}\, J_\epsilon(u_\epsilon, y_\epsilon) \leqslant \inf J(X_{ad})$$

from which we obtain that $J(u,y) = \inf J(X_{ad})$, i.e. : (u,y) is an optimal couple.

We have again the properties (1.24) (1.27) and

(1.32) $$J(u_\epsilon, y_\epsilon) \to J(u,y) \;, \quad \text{as } \epsilon \to 0_+ .$$

If we set :

$$a_\epsilon = \Phi(y_\epsilon) \;, \; b_\epsilon = \| N^{1/2} u_\epsilon \|_U^2 \;, \quad a = \Phi(y) \;, \quad b = \| N^{1/2} u \|_U^2$$

from (1.8) (1.9) (1.29) and (1.32) we obtain :

$$a = \underline{\lim}\, a_\epsilon \;, \quad b = \underline{\lim}\, b_\epsilon \;, \quad a_\epsilon + b_\epsilon \to a + b$$

from which we obtain that $a_\epsilon \to a$, $b_\epsilon \to b$, as $\epsilon \to 0_+$. Hence :

(1.33) $$\Phi(y_\epsilon) \to \Phi(y) \text{ and } \| N^{1/2} u \|_U \to \| N^{1/2} u \|_U .$$

We deduce from (1.29) (1.33) the strong convergence (1.25).

Remark 1.2. If we assume that J is Gateaux-differentiable, and $B(\mathcal{A})$ is a convex subset of Z, the couple (u_ϵ, y_ϵ) verifies :

$$J'_\epsilon(u_\epsilon, y_\epsilon) \,.\, (v-u_\epsilon, z-y_\epsilon) \geqslant 0, \quad v \in U_{ad}\,,\; z \in D(\mathcal{A}) \tag{1.34}$$

Remark 1.3. If we assume that p_ϵ is bounded in H, by passing to the limit in (1.34) we can obtain a set of relations to characterize one optimal couple (u,y). In Sections 2 and 3 with the additional (strong) condition «Int $U_{ad} \neq \emptyset$» we prove that, as $\epsilon \to 0_+$, p_ϵ remains in a bounded subset of H. For the case where $\mathcal{A}$ and B are linear operators, we refer to Rivera [8], others examples are given in Lions [3], [4], [5] and Murat [7].

2. - UNSTABLE NON LINEAR EVOLUTION SYSTEM : CASE OF THE NEUMAN CONDITION

2.1. Setting of the Problem.

Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ and let T be a positive number. We shall use the following notation :

$$Q = \Omega \times \,]0,T[\quad ; \qquad \Sigma = \Gamma \times \,]0,T[\,.$$

Let us assume that the control variable v and the state z satisfy the state equation given by :

$$(2.1) \qquad \begin{cases} z' - \Delta z - z^3 = f, \ \text{ in } Q & (' = \dfrac{\partial}{\partial t} \\ \dfrac{\partial z}{\partial \nu} = \psi + v, \ \text{ on } \Sigma & \\ z(x,0) = y_0(x), \ \text{ in } \Omega & \end{cases}$$

with v and z satisfying the contraints conditions :

$$v \in L^2(\Sigma)\,, \quad z \in L^6(Q). \tag{2.2}$$

In (2.1) (f,ψ,y_0) is given in $L^2(Q) \times L^2(\Sigma) \times H^1(\Omega)$.

The cost function is given by :

(2.3) $$J(v,z) = \frac{1}{6} \| z - z_d \|^6_{L^6(Q)} + \frac{1}{2}(Nv \mid v)_\Sigma \ , \quad (v,z) \text{ as in (2.2)}$$

where z_d belongs to $L^6(Q)$, $N \in \mathcal{L}(L^2(\Sigma))$ is an hermitian, definite positive operator on $L^2(\Sigma)$ and where $(\ \mid\ )_\Sigma$ denotes the inner product in $L^2(\Sigma)$ and $|\ \ |_\Sigma$ the norm.

Let U_{ad} be a subset of $L^2(\Sigma)$ such that :

(2.4) U_{ad} is a closed, convex subset of $L^2(\Sigma)$ and there exists v in U_{ad} for which the Problem (2.1) admits solution $z \in L^6(Q)$.

The problem of optimal control is :

(2.5) Find (u,y) in $U_{ad} \times L^6(Q)$ verifying (2.1) and

$$J(u,y) = \inf \{ J(v,z) \ ; v \in U_{ad} \ , \ z \text{ verifies (2.1) (2.2)} \}.$$

2.2. Abstract formulation for the Problem (2.5).

In order to set the optimal control problem (2.5) in the abstract form that was given in the Section 1, we consider :

(2.6) $$U = L^2(\Sigma), \ Z = L^6(Q), \ H = L^2(Q) \times L^2(\Sigma) \times H^1(\Omega)$$

$$D(\mathcal{A}) = \left\{ z \in L^6(Q) \ ; z' - \Delta z \in L^2(Q), \ \frac{\partial z}{\partial \nu} \in L^2(\Sigma), \ z(0) \in H^1(\Omega) \right\},$$

$$z = (z' - \Delta z - z^3, \ \frac{\partial z}{\partial \nu}, \ z(0)), \quad \text{for } z \text{ in } D(\mathcal{A}),$$

(2.8) $$Bv = (0,v,0), \quad v \in U$$

(2.9) $$f_o = (f, \psi, y_o)$$

(2.10) $$N_o = \frac{1}{2} N$$

(2.11) $$\Phi(z) = \frac{1}{6} \| z - z_d \|^6_{L^6(Q)} \ , \quad z \in Z = L^6(Q).$$

We verify easily that the Problems (1.5) and (2.5) are equivalent and the hypothesis (1.7) (1.8) (1.9) (1.10) are fulfilled. We have :

PROPOSITION 2.1. *The graph of the operator* $\mathcal{A}$ *given by* (2.7) *is weakly closed in* Z X H.

Proof. Let z_m ($m \in \mathbb{N}$) be a sequence in $D(\mathcal{A})$ such that, as $m \to \infty$:

$$z_m \to z\ ,\ \text{weakly in } L^6(Q)$$

(2.12)
$$\frac{\partial}{\partial \nu} z_m \to \gamma\ ,\ \text{weakly in } L^2(\Sigma)$$

$$z_m(0) \to z_o\ ,\ \text{weakly in } H^1(\Omega)$$

$$z'_m - \Delta z_m - z_m^3 \to \chi\ ,\ \text{weakly in } L^2(Q).$$

Then the sequence z_m ($m \in \mathbb{N}$) is bounded in $L^2(0,T;H^{3/2}(\Omega))$ (Lions-Magenes [6]) and we may extract a subsequence, again denoted by z_m, such that, as $m \to \infty$:

(2.13)
$$z_m \to z,\ \text{weakly in } L^2(0,T;H^{3/2}(\Omega)).$$

Since the embedding $H^{1/2}(\Omega) \subset L^2(\Omega)$ is compact, we may assume that z_m converges to z strongly in $L^2(Q)$ and therefore

$$z_m^3(x,t) \to z^3(x,t)\ ,\ \text{a.e. in } Q.$$

But, from (2.12) z_m^3 ($m \in \mathbb{N}$) is a bounded sequence in $L^2(Q)$, hence we may assume that, as $m \to \infty$:

(2.14)
$$z_m^3 \to z^3\ ,\ \text{weakly in } L^2(Q).$$

From (2.12) (2.14) we obtain :

(2.15)
$$z' - \Delta z - z^3 = \chi\ ,\ \text{in } \mathcal{D}'(Q).$$

Since $\Delta \in \mathcal{L}(H^{3/2}(\Omega), H^{-1/2}(\Omega))$, we deduce from (2.12) (2.13) that :

(2.16)
$$z'_m \to z',\ \text{weakly in } L^2(0,T;H^{-1/2}(\Omega)).$$

From (2.12) (2.13) (2.16) we obtain :

(2.17)
$$\frac{\partial z}{\partial \nu} = \gamma,\ \text{on } \Sigma\ ;\ z(0) = z_o :$$

Hence, $z \in D(\mathcal{A})$ and $\mathcal{A} z = (\chi,\gamma,z_o)$ and Proposition 2.1 is proved.

By Proposition 2.1 and the previous remarks, we are in the conditions to apply Theorems 1.1, 1.3 and we obtain the followings results :

THEOREM 2.1. *Let us suppose that the state equation and the cost function are given by* (2.1) *and* (2.3) *respectively. If* U_{ad} *verifies condition* (2.4), *there exists a solution of the optimal control problem* (2.5).

THEOREM 2.2. *For each* $\epsilon > 0$ *there exists* (u_ϵ, y_ϵ) *in* $U_{ad} \times D(\mathcal{a})$ *such that, if we consider :*

(2.18) $$p_\epsilon = -\epsilon^{-1}\{ y'_\epsilon - \Delta y_\epsilon - y_\epsilon^3 - f \}$$

(2.19) $$\gamma_\epsilon = -\epsilon^{-1}\{ \frac{\partial y_\epsilon}{\partial \nu} - \psi - u_\epsilon \}$$

(2.20) $$y_{\epsilon_o} = \epsilon^{-1}\{ y_\epsilon(0) - y_o \}$$

we have the following relations :

(2.21) $$(p_\epsilon \mid z' - \Delta z - 3y_\epsilon^2 z)Q = \int_{\mathcal{a}} (y_\epsilon - z_d)^5 z + (y_{\epsilon_o} \mid z(0))_{H^1(\Omega)} - (\gamma_\epsilon \mid \frac{\partial z}{\partial \nu})_\Sigma$$

for z *in* $D(\mathcal{a})$.

(2.22) $$(\gamma_\epsilon + N u_\epsilon \mid v - u_\epsilon)_\Sigma \geqslant 0, \quad \text{for v in } U_{ad}$$

We have also :

(2.23) *As* $\epsilon \to 0$, (u_ϵ, y_ϵ) *remains in a bounded subset of* $L^2(\Sigma) \times L^6(Q)$

(2.24) *There exists a sequence, again denoted by* (u_ϵ, y_ϵ), *and there exists a solution* (u,y) *of the Próblem* (2.5) *such that :*

$$(u_\epsilon, y_\epsilon) \to (u,y), \ \textit{in } L^2(\Sigma) \times L^6(Q), \quad \textit{as } \epsilon \to 0_+ .$$

Proof. We consider the penalized cost function given by

$$J_\epsilon(v,z) = J(v,z) + (2\epsilon)^{-1} \| \ z - f_o - Bv \|_H^2 .$$

By Theorem 1.2 we obtain a couple (u_ϵ, y_ϵ) in $U_{ad} \times D(\mathcal{a})$ such that :

$$J_\epsilon(u_\epsilon, y_\epsilon) = \inf \{ J_\epsilon(v,z) \ ; v \in U_{ad} \ , \ z \in D(\mathcal{a}) \}$$

(u_ϵ, y_ϵ) verifies (2.23) (2.24).

Since $U_{ad} \times D(\mathcal{A})$ is a convex subset of $L^2(\Sigma) \times L^6(Q)$, the couple (u_ϵ, y_ϵ) is characterized by :

$$J'(u_\epsilon, y_\epsilon) \cdot (v - u_\epsilon, z - y_\epsilon) \geqslant 0, \quad (v,z) \in U_{ad} \times D(\mathcal{A})$$

from which we obtain (2.21) and (2.22).

2.3. Estimates for p_ϵ, $\epsilon > 0$.

In order to obtain estimates for p_ϵ, $\epsilon > 0$, we shall assume that :

(2.25) $$\Omega \subset R^3.$$

For $\rho \geqslant 1$ given, we define the space $W^{2,1;\rho}(Q)$ as the space of functions Φ in $L^\rho(Q)$ such that the partial derivatives $\frac{\partial \Phi}{\partial t}$, $\frac{\partial \Phi}{\partial \sigma_i}$, $\frac{\partial^2 \Phi}{\partial x_i x_j}$ (i,j = 1,2,3) belong to $L^\rho(Q)$.

With the norm defined by

$$\| \Phi \|_{W^{2,1;\rho}(Q)} = \sum_{\substack{|a| \leqslant 2 \\ a \in IN^3}} \| D^a \Phi \|_{L^\rho(Q)} + \left\| \frac{\partial \Phi}{\partial t} \right\|_{L^\rho(Q)},$$

$W^{2,1;\rho}(Q)$ is a Banach space and we have the following property :

PROPOSITION 2.2. *Let us assume that* (2.25) *holds and* $\rho < 5/2$. *If we consider the real number* $\rho^* = 5\rho/(5-2\rho)$, *we have the following embeddings :*

$W^{2,1;\rho}(Q) \subset L^{\rho^*}(Q)$, *with continuous embedding.*

$W^{2,1;\rho}(Q) \subset L^p(Q)$ *with compact embedding, for* $1 \leqslant p < \rho^*$.

Proof. See Becov, Ilin & Nikolski [11] and Lions [5].

COROLLARY 2.1. *The embedding of* $W^{2,1;6/5}(Q)$ *in* $L^2(Q)$ *is compact.*

We need also the following results.

PROPOSITION 2.3. *Let* Φ_m $(m \in IN)$ *be a bounded sequence in* $L^2(Q)$ *such that* $\Phi_m(0) = 0$, $\Phi_m = 0$ *(on* Σ*) and* $\Phi'_m - \Delta \Phi_m$ *is a bounded sequence in* $L^{6/5}(Q)$. *Then the sequence* Φ_m

($m \in \mathbb{N}$) *is bounded in* $W^{2,1;6/5}(Q)$.

This result is classical.

PROPOSITION 2.4. *As* $\epsilon \to 0_+$, p_ϵ *belongs to a bounded subset of* $L^2(Q)$.

If Proposition 2.4 was wrong, then :

(2.26) $$a_\epsilon = |p_\epsilon|_Q^{-1} \to 0, \quad \text{as } \epsilon \to 0_+ .$$

If we set :

(2.27) $$q_\epsilon = a_\epsilon \, p_\epsilon$$

from (2.21) we have :

(2.28) $$(q_\epsilon \mid z' - \Delta z - 3y^2 z) = a_\epsilon \int_Q (y_\epsilon - z_d)^5 z, \quad \text{for } z \text{ in } D_o$$

where :

(2.29) $$D_o = \{ z \in D(\mathcal{A}) \; ; \quad z \mid \Sigma = 0, \quad z(0) = 0 \}$$

and we have that q_ϵ is a solution of :

$$-q'_\epsilon - \Delta q - 3y^2 q_\epsilon = a_\epsilon \, (y_\epsilon - z_d)^5, \text{ on } Q$$

$$q_\epsilon \mid \Sigma = 0 \;, \quad q_\epsilon(T) = 0.$$

From (2.23) (2.26) (2.27) we have that (q_ϵ, y_ϵ) is bounded in $L^2(Q) \times L^6(Q)$, therefore $g_\epsilon = a_\epsilon(y_\epsilon - z_d)^5 + 3y_\epsilon^2 \, q_\epsilon$ is bounded in $L^{6/5}(Q)$. If we define $\Phi_\epsilon(t) = q_\epsilon(T-t)$ and $F_\epsilon(t) = g_\epsilon(T-t)$, from (2.30) we obtain :

$$\Phi'_\epsilon - \Delta \Phi_\epsilon = F_\epsilon \text{ is bounded in } L^{6/5}(Q)$$

$$\Phi_\epsilon \mid \Sigma = 0 \;, \quad \Phi_\epsilon(0) = 0$$

and Proposition 2.3 gives that Φ_ϵ is bounded in $W^{2,1;6/5}(Q)$. It follows that q_ϵ is bounded in the same space and by Corollary 2.1 we may suppose that :

(2.31) $$q_\epsilon \to q \;, \quad \text{in } L^2(Q).$$

From (2.24) (2.26) (2.28) (2.31) we obtain :

(2.32) $$|q|_Q = 1$$

(2.33) $$(q \mid z' - 3y^2 z - \Delta z)_Q = 0\,, \text{ for } z \text{ in } D_o$$

and (2.33) gives :

$$-q' - \Delta q - 3y^2 q = 0\,, \text{ in } Q$$

$$q \mid \Sigma = 0\,,\ q(T) = 0$$

from which it follows that

(2.34) $$q = 0\,, \text{ in } Q.$$

Since (2.32) and (2.34) give a contradiction, we have that Proposition 2.4 holds.

COROLLARY 2.2. *As* $\epsilon \to 0_+$,

(2.35) p_ϵ *remins in a bounded subset of* $W^{2,1;6/5}(Q)$

(2.36) $p_\epsilon(0)$ *remains in a bounded subset of* $W^{1,6/5}(\Omega)$.

Proof. From (2.21) we have that p_ϵ is solution of

$$-p'_\epsilon - \Delta p_\epsilon - 3y_\epsilon^2 p_\epsilon = (y_\epsilon - z_d)^5\,, \text{ in } Q$$

$$p_\epsilon \mid \Sigma = 0\,,\ p_\epsilon(T) = 0.$$

Since (p_ϵ, y_ϵ) is bounded in $L^2(Q) \times L^6(Q)$, we obtain that $3y_\epsilon^2 p_\epsilon + (y_\epsilon - z_d)^5$ is bounded in $L^{6/5}(Q)$ and Proposition 2.3 gives the estimate (2.35).

If we set $X = W^{2,6/5}(\Omega)$, $Y = W^{1,6/5}(\Omega)$, $Z = L^{6/5}(\Omega)$ we obtain :

$$W^{2,1;6/5}(Q) = \{\Phi \in L^{6/5}(0,T;X)\ ;\Phi' \in L^{6/5}(0,T;Z)\}.$$

Hence (2.35) implies (2.36) (Lions [2]).

2.4. Estimates for $p_\epsilon \mid \Sigma, \epsilon > 0$.

LEMMA 2.1. *The set* $M = \left\{ \frac{\partial z}{\partial \nu} \; ; z \in D(\mathcal{A}) \, , \; z(0) = 0 \right\}$ *is dense in* $L^2(\Sigma)$.

Proof. By the Trace Theorem (Lions-Magenes [6]) we verify easily that $M_o = \{ \psi \otimes \theta \; ; \psi \in H^{1/2}(\Omega) \, , \; \theta \; C_o([0,T]) \} \subset M$, from which we obtain that the Lemma 2.1 holds, because M_o is dense in $L^2(\Sigma)$.

PROPOSITION 2.6. *We assume that* U_{ad} *has non empty interior. Then, as* $\epsilon \to 0_+$, p_ϵ *remains in a bounded subset of* $L^2(\Sigma)$.

Proof. First we note that (2.21) and (2.37) imply :

(2.38) $$p_\epsilon \mid \Sigma = \gamma_\epsilon$$

(2.39) $$(y_{\epsilon_o} \mid \varphi)_{H^1(\Omega)} = - \int_\Omega p_\epsilon(0) \varphi \, , \quad \text{for } \varphi \text{ in } \mathscr{D}(\Omega).$$

Since $\Omega \subset \mathbb{R}^3$, by Sobolev's embedding Theorem (Sobolev [9]) we have that $H^1(\Omega) \subset L^6(\Omega)$ with continuous embedding. Hence, (2.35) and (2.39) imply :

(2.40) $$y_{\epsilon_o} \text{ is in a bounded subset of } H^1(\Omega).$$

From Lemma 2.1 and the hypothesis made on U_{ad} , we may find a real number $r > 0$ and φ_o such that :

(2.41) $$\varphi_o \in D(\mathcal{A}) \, , \; \varphi_o(0) = 0 \, , \; v_o = \frac{\partial \varphi_o}{\partial \nu} - \psi \in U_{ad}$$

(2.42) $$D_r(v_o) = \{ v \in L^2(\Sigma) \; ; \; | v - v_o |_\Sigma \leqslant r \} \subset U_{ad}.$$

From (2.25) and (2.42) we obtain :

(2.43) $$(\gamma_\epsilon + N u_\epsilon \mid w)_\Sigma \geqslant (\gamma_\epsilon \mid u_\epsilon - v_o)_\Sigma - (N u_\epsilon \mid v_o + w)_\Sigma \, , \text{ if } | w |_\Sigma \leqslant r$$

If we substitute z by $y_\epsilon - \varphi_o$ in (2.21) we obtain :

(2.44) $$(\gamma_\epsilon \mid u_\epsilon - v_o)_\Sigma = \epsilon \left\{ | p_\epsilon |_Q^2 + | \gamma_\epsilon |_\Sigma^2 + \| y_{\epsilon_o} \|^2_{H^1(\Omega)} \right\} + K_\epsilon$$

where :

(2.45) $$K_\epsilon = (y_{\epsilon_o} \mid y_o)_{H^1(\Omega)} + (y_\epsilon - z_d)^5 (y_\epsilon - \varphi_o)$$

$$+ (p_\epsilon \mid \varphi_o' - \Delta\varphi_o - f - 2y_\epsilon^3 - 3y_\epsilon^2 \varphi_o)_Q.$$

We deduce from (2.23) (2.40) and Proposition 2.5 that :

$$c_o = \sup \left\{ \mid K_\epsilon - (Nu_\epsilon \mid v_o + w)_\Sigma \mid ; \epsilon > 0, \mid w \mid_\Sigma \leqslant r \right\}$$

is finite. Therefore (2.43) (2.44) imply :

(2.46) $$(\gamma_\epsilon + Nu_\epsilon \mid w) \geqslant -c_o \quad , \quad \mid w \mid_\Sigma \leqslant r, \ \ \epsilon > 0$$

from which we obtain :

(2.47) $$\mid \gamma_\epsilon + Nu_\epsilon \mid \leqslant c_o \, r^{-1} \ , \ \ \epsilon > 0.$$

From (2.23) (2.38) and (2.47) we obtain that Proposition 2.6 holds.

2.5. The optimality system.

The estimates that we found in Proposition 2.5 and 2.6 are sufficient to pass to the limit in (2.21) (2.22) and we obtain the following result :

THEOREM 2.3. *We assume that* $\Omega \subset \mathbb{R}^3$ *and* U_{ad} *has non empty interior. Then there exists* (u,y,p) *such that :*

(2.48) $$u \in U_{ad} \ , \ \ y \in L^6(Q) \cap L^2(0,T; H^{3/2}(\Omega))$$

$$p \in W^{2,1;6/5}(Q) \ , \ \ p \mid \Sigma \in L^2(\Sigma)$$

(2.49) $$\left.\begin{array}{l} y' - \Delta y - y^3 = f \\ -p' - \Delta p - 3\,y^2 p = (y - z_d)^5 \end{array}\right. \ , \text{ in } Q$$

(2.50) $$\frac{\partial y}{\partial \nu} = \psi + u \quad , \quad \frac{\partial p}{\partial \nu} = 0, \quad \text{on } \Sigma$$

(2.51) $$y(x,0)=y_o(x)\ ,\quad p(x,T)=0,\quad \text{in } \Omega$$

(2.52) $$(p+Nu\mid v-u)_\Sigma \geqslant 0\ ,\quad v \text{ in } U_{ad}$$

(2.53) (u,y) is solution of the optimal control problem (2.5).

Remark 2.1. In the case $\Omega \subset \mathrm{IR}^2$ the mapping $\Phi \to \Phi \mid \Sigma$ is continuous from $W^{2,1;6/5}(Q)$ into $L^{9/4}(\Sigma) \subset L^2(\Sigma)$ and in this case we obtain directely from Proposition 2.5, that $p_\epsilon \mid \Sigma$ is bounded in $L^2(\Sigma)$. Hence : in the case $\Omega \subset \mathrm{IR}^2$ we obtain the optimality system (2.48) (2.49) (2.50) (2.52) (2.53) without the hypothesis that the interior of U_{ad} is non empty.

3. - UNSTABLE NON LINEAR EVOLUTION SYSTEM : CASE OF THE DIRICHLET CONDITION

Let us assume that the control variable v and the state z are related by the following state equation :

$$z' - \Delta z - z^3 = f\ ,\quad \text{in } Q \qquad (' = \frac{\partial}{\partial t}$$

(3.1) $$z = \psi + v\ ,\quad \text{on } \Sigma$$

$$z(x,0) = y_o(x)\ ,\quad \text{in } \Omega$$

(3.2) $$v \in L^2(\Sigma)\ ,\quad z \in L^6(Q)$$

where (f,ψ,y_o) is given in $L^2(Q) \times L^2(\Sigma) \times L^2(\Omega)$.

The cost function is defined by :

(3.3) $$J(v,z) = \frac{1}{6}\,\| z - z_d \|^6_{L^6(Q)} + \frac{1}{2}\,(Nv\mid v)_\Sigma\ ,\quad v \in L^2(\Sigma),\ \ z \in L^6(Q)$$

where z_d is given in $L^6(Q)$ and $N \in \mathcal{L}(L^2(\Sigma))$ is an hermitian, positive definite operator on $L^2(\Sigma)$.

Let U_{ad} be a subset of $L^2(\Sigma)$ such that :

(3.4) U_{ad} is a closed convex subset of $L^2(\Sigma)$ and there exists v in U_{ad} for which (3.1) (3.2) has solution.

The problem of optimal control is :

(3.5) Find (u,y) in $U_{ad} \times L^6(Q)$ verifying (3.1) and

$$J(u,y) = \inf\{J(v,z) ; v \in U_{ad}, \ z \text{ verifies (3.1) (3.2)}\}.$$

Remark 3.1. If v and z verify (3.1) then $z' + z - \Delta z = f + z + z^3$ belongs to $L^2(Q)$, from which we obtain that $z \in L^2(0,T;H^{1/2}(\Omega))$.

By analogous arguments as those used in Section 2, se obtain the following results :

THEOREM 3.1. *We assume that the state equation and that the cost function are given by* (3.1) *and* (3.3) *respectively and we assume that* (3.4) *holds. Then there exists a solution* (u,y) *of the Problem* (3.5).

THEOREM 3.2. *We assume that* $\Omega \subset \mathbb{R}^3$ *and that the interior of* U_{ad} *is non empty. Then there exists a solution* (u,y) *of the Problem* (3.5) *and there exists* p *in* $L^2(Q)$ *such that :*

(3.6) $$u \in U_{ad}, \ y \in L^6(Q) \cap L^2(0,T;H^{1/2}(\Omega))$$

(3.7) $$p \in W^{2,1;6/5}(Q), \quad \frac{\partial p}{\partial \nu} \in L^2(\Sigma)$$

(3.8) $$\left.\begin{array}{l} y' - \Delta y - y^3 = f \\ -p' - \Delta p - 3y^2 p = (y - z_d)^5 \end{array}\right\} , \text{ in } Q$$

(3.9) $$y \mid \Sigma = \psi + u \ , \quad p \mid \Sigma = 0$$

(3.10) $$y(x,0) = y_0(x) \ , \ p(x,T) = 0, \quad \text{in } \Omega$$

(3.11) $$\left(-\frac{\partial p}{\partial \nu} + Nu \mid v-u\right)_\Sigma \geqslant 0, \quad v \text{ in } U_{ad}.$$

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