ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

ESTER GIARRUSSO
DIANA NUNZIANTE

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Annales de la faculté des sciences de Toulouse 5e série, tome 7, nº 1 (1985), p. 57-73
<http://www.numdam.org/item?id=AFST_1985_5_7_1_57_0>

Annales Faculté des Sciences Toulouse
Vol VII, 1985, P. 57 à 73

COMPARISON THEOREMS FOR A CLASS OF FIRST ORDER HAMILTON-JACOBI EQUATIONS [3]

Ester Giarrusso [2] **Diana Nunziante** [1]

(1) Dipartimento di Matematica e Applicazioni «R. Caccioppoli», Università di Napoli, Via Mezzocannone 8, 80134-Napoli - Italy.

(2) Temp. University of Maryland, Dept. of Math., College Park, Maryland 20742 - U.S.A.

(3) Work supported by Ministero Pubblica Istruzione of Italy (40 % - 1983).

Résumé : Nous étudions une certaine classe d'équations de Hamilton-Jacobi du premier ordre. Tout d'abord, en utilisant des techniques de symétrisation, nous comparons une solution du problème considéré avec la solution à symétrie sphérique décroissante d'un problème symétrisé. Enfin nous démontrons un théorème d'existence des solutions de viscosité.

Summary : We study a certain class of first order Hamilton-Jacobi equations. First, by means of symmetrization technique, we compare a solution of the considered problem with the decreasing spherically symmetric solution of a symmetrized problem. Next we prove an existence theorem of viscosity solution.

1. - INTRODUCTION AND RESULTS

From the same point of view as in [9], making use of symmetrization techniques, we study the Dirichlet problem :

$$(1.1)\qquad \begin{cases} |Du| - \lambda u = f(x) & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $x \equiv (x_1,\dots,x_n)$ is a point of R^n, $Du = \text{grad } u$, and $f(x)$ is a measurable real-valued function defined in Ω.

We assume that :

a) Ω is an open subset of R^n with finite measure M ;

b) $f(x) \in L^p(\Omega)$, $p \geq 1$;

c) λ is a positive real number ;

d) there exists a generalized solution u of (1.1), that is there exists a function $u \in W_0^{1,p}(\Omega)$, $p \geq 1$, which satisfies the equation $|Du| - \lambda u = f(x)$ a.e. in Ω.

Existence theorems for (1.1) can be found in [11], so that the last assumption makes sense.

Herein our main goal is to compare a solution $u(x)$ of (1.1) with the unique decreasing spherically symmetric solution of a problem :

- which is of the same type as (1.1) ;
- given in a ball $\Omega^\star \subset R^n$ having measure M ;
- for which the right hand-side has the same distribution function as f(x).

In order to state our results more precisely, let us recall that one denotes by

$$\mu(t) = \text{meas} \left\{ x \in \Omega : |u(x)| > t \right\}$$

the *distribution function* of a measurable real-valued function u defined in Ω, that

$$u^*(s) = \inf \left\{ t \geq 0 : \mu(t) < s \right\}$$

is the *decreasing rearrangement* of u, and that

$$u^\star(x) = u^*(C_n |x|^n),$$

where C_n is the measure of the n-dimensional unit-ball of R^n, is the *spherically symmetric decreasing rearrangement* of u.

Also we consider the *increasing rearrangement* of u :

$$u_*(s) = u^*(\text{meas}\,\Omega - s)$$

and the *spherically symmetric increasing rearrangement* of u :

$$u_\star(x) = u_*(C_n\,|x|^n).$$

The function u and its rearrangements have the same distribution function and, as well known, the following inequality holds (see [10], [13]) :

$$\int_\Omega |u\,v|\,dx \leqslant \int_0^M u^*(s)\,v^*(s)ds = \int_{\Omega^*} u^\star(x)\,v^\star(x)dx \tag{1.2}$$

Here and below $\Omega^\star$ is the ball of R^n centered at the origin with the same measure M as Ω. Denoted by u(x) a generalized solution of (1.1), using auxiliary lemmas of section 2, in section 3 we prove the following results :

THEOREM 1.1. *If* $n > 1$, *we have :*

$$\|u\|_\infty \leqslant \|w\|_\infty$$

where, if $M \leqslant C_n\left(\frac{n-1}{\lambda}\right)^n$, w(x) *is the unique decreasing spherically symmetric solution of the problem :*

$$\begin{cases} |Dw| - \lambda w = f^\star(x) & \text{in} \quad \Omega^\star \\ w = 0 & \text{on} \quad \partial\Omega^\star \end{cases} \tag{1.4}$$

while, if $M > C_n\left(\frac{n-1}{\lambda}\right)^n$, w(x) *is the unique decreasing spherically symmetric solution of the problem :*

$$\begin{cases} |Dw| - \lambda w = \hat{f}(C_n\,|x|^n) & \text{in} \quad \Omega^\star \\ w = 0 & \text{on} \quad \partial\Omega^\star \end{cases} \tag{1.5}$$

$\hat{f}(s)$ *being a function with the same distribution function as* f(x). *(see* REMARK 3.1 *for an explicit definition of* $\hat{f}$ *).*

THEOREM 1.2. *If* $n > 1$, *we have :*

$$\| u \|_1 \leqslant \| z \|_1 \tag{1.6}$$

where $z(x)$ *is the unique decreasing spherically symmetric solution of the problem :*

$$\begin{cases} |Dz| - \lambda z = \check{f}(C_n |x|^n) & \text{in } \Omega^\star \\ z = 0 & \text{on } \partial\Omega^\star \end{cases} \tag{1.7}$$

$\check{f}(s)$ *being a fixed function having the same distribution function as* $f(x)$.

THEOREM 1.3. *If* $n \geqslant 1$ *and* $\lambda > (n-1)\,[C_n / M]^{1/n}$, *we have :*

$$u^\star(x) \leqslant q(x) \qquad \text{in } \Omega^\star - \Omega_0^\star \tag{1.8}$$

where $\Omega_0^\star$ *is a ball of* R^n *centered at the origin and with radius* $\dfrac{n-1}{\lambda}$, *and* $q(x)$ *is the unique spherically symmetric decreasing solution of the problem :*

$$\begin{cases} |Dq| - \lambda q = f_\star(x) & \text{in } \Omega^\star \\ q = 0 & \text{on } \partial\Omega^\star \end{cases} \tag{1.9}$$

In particular, from Theorem 1.3 we derive the following :

COROLLARY 1.1. *If* $n = 1$, *then :*

$$u^\star(x) \leqslant q(x) \quad \text{in} \quad \left(-\frac{M}{2}, \frac{M}{2}\right)$$

$q(x)$ *being the unique solution, depending only on* $|x|$, *of the problem :*

$$\begin{cases} \left|\dfrac{dq}{ds}\right| - \lambda q = f_\star(x) & \text{in} \quad \left(-\dfrac{M}{2}, \dfrac{M}{2}\right) \\ q\left(-\dfrac{M}{2}\right) = q\left(\dfrac{M}{2}\right) = 0 \end{cases}$$

Now, let us recall that, more generally, for a first order Hamilton-Jacobi equation : $H(x,u(x), Du(x)) = 0$ (H being a continuous function in $\Omega \times R \times R^n$), besides the definition of generalized solutions, M.G. Crandall and P.L. Lions have introduced the notion of *viscosity solution.*

We refer to [5], [6], [11] for the exact definition and for the properties of viscosity solutions.

Only let us mention that a viscosity solution u of the equation H = 0 need to be continuous but not necessarily differentiable in anywhere ; however, if u is differentiable at some x_o, then $H(x_o, u(x_o), Du(x_o)) = 0$.

Furthermore some uniqueness and stability problems can be solved introducing this new notion of solution (see [5], [6], [11]).

In section 4, under more restrictive assumptions, we prove an existence theorem for viscosity solutions of (1.1) (see Th. 4.1 for the exact statement).

ACKNOWLEDGEMENT. We are thankful to Prof. P.L. Lions for some useful colloquia and for suggesting us the way to prove Th. 4.1.

2. - TWO LEMMAS

Henceforth let u(x) be a solution of (1.1) and $\mu(t)$ its distribution function, for each $s \in [0, \text{meas } \Omega]$ consider a measurable subset D(s) of Ω, such that :

$$\text{meas } D(s) = s \,;$$

$$s_1 < s_2 \Rightarrow D(s_1) \subset D(s_2) \,;$$

$$D(s) = \{ x \in \Omega : |u(x)| > t \} \quad \text{if } s = \mu(t).$$

Then by b) $\int_{D(s)} f(x)\, dx$ is an absolutely continuous function; and so there exists a function $\tilde{f}(t)$ such that :

$$\int_0^s \tilde{f}(t)\, dt = \int_{D(s)} |f(x)|\, dx. \tag{2.1}$$

Furthermore the following lemma holds (see [1] for the proof) :

LEMMA 2.1. *There exists a sequence* $\{ f_h(s) \}$ *of functions which have the same distribution function as* f(x) *and such that, if* $p > 1$:

$$f_h(s) \rightharpoonup \tilde{f}(s) \quad \text{in } L^p([0,M]),$$

while if $p = 1$:

$$\lim_h \int_0^M f_h(s)\, g(s) ds = \int_0^M \tilde{f}(s)\, g(s) ds$$

for each function $g(s)$ *belonging to the space* $BV([0,M])$ *of the functions of bounded variation.*

Now we give a sketch of the proof of the well known :

LEMMA 2.2. *Let* $\phi(s)$ *and* $f(s)$ *be two given measurable functions in* $[0,M]$. *Then there exists* $\hat{f}(s)$, *which has the same distribution function as* $f(s)$ *and which depends on* $\phi(s)$ *such that :*

$$\int_0^M f(s)\phi(s) ds \leqslant \int_0^M \hat{f}(s)\phi(s) ds = \int_0^M f^*(s)\phi^*(s) ds \tag{2.2}$$

(compare for example with [4] *or* [12]*)*.

- Denoted by $\nu_\phi(t)$ the distribution function of $\phi(s)$, for each $s \in [0,M]$ we can fix a measurable subset $E(s) \subseteq [0,M]$, such that :

$$\text{meas}\, E(s) = s\,;$$

$$s_1 < s_2 \Rightarrow E(s_1) \subset E(s_2)\,;$$

$$E(s) = \{\sigma : |\phi(\sigma)| > t\} \quad \text{if } s = \nu_\phi(t).$$

Then let

$$s(\sigma) = \inf \{\bar{s} \in [0,M] : \sigma \in E(\bar{s})\}\,,\ \sigma \in [0,M]\,; \tag{2.3}$$

the required function is :

$$\hat{f}(\sigma) = f^*(s(\sigma)). \tag{2.4}$$

Moreover, denoting by $\nu_f(t)$ the distribution function of $f(s)$, we have :

$$\int_0^M \hat{f}(\sigma)\phi(\sigma) d\sigma = \int_0^\infty dt \int_{\hat{f} > t} \phi(\sigma) d\sigma = \int_0^\infty dt \int_{E(\nu_f(t))} \phi(\sigma) d\sigma =$$

$$= \int_0^\infty dt \int_0^{\nu_f(t)} \phi^*(\sigma) d\sigma = \int_0^\infty dt \int_{f^* > t} \phi^*(\sigma) d\sigma = \int_0^M f^*(s)\phi^*(s) ds.$$

3. - PROOF OF THEOREMS 1.1, 1.2, 1.3

For the sake of clearness, first of all we prove two lemmas.

LEMMA 3.1. *We have :*

$$u^{\star}(x) \leqslant \frac{1}{nC_n^{1/n}} \int_{C_n |x|^n}^{M} (\lambda u^*(s) + \tilde{f}(s))s^{1/n-1}\,ds \tag{3.1}$$

a.e. in $\Omega^{\star}$.

Proof. By the isoperimetric inequality (see [7]) :

$$nC_n^{1/n}\,\mu(t)^{1-1/n} \leqslant P\{x : |u(x)| > t\},$$

where P is the perimeter in the sense of De Giorgi, and by the Fleming-Rishel formula ([8]) :

$$\int_{|u|>t} |Du|\,dx = \int_0^{\infty} P\{x : |u(x)| > \xi\}\,d\xi$$

we get :

$$nC_n^{1/n}\,\mu(t)^{1-1/n} \leqslant -\frac{d}{dt}\int_{|u|>t} |Du|\,dx. \tag{3.2}$$

On the other hand, since u is a solution of (1.1) :

$$\frac{1}{h}\int_{t<|u|\leqslant t+h} |Du|\,dx \leqslant \frac{1}{h}\int_{t<|u|\leqslant t+h} (\lambda\,|u| + |f|)dx, \qquad h>0,$$

hence for $h \to 0$:

$$-\frac{d}{dt}\int_{|u|>t} |Du|\,dx \leqslant -\frac{d}{dt}\int_{|u|>t} (\lambda\,|u| + |f|)dx. \tag{3.3}$$

Moreover, since

$$\int_{|u|>t} |u|\,dx = \int_0^{\mu(t)} u^*(s)ds,$$

(2.1), (3.2) and (3.3) give :

$$nC_n^{1/n}\,\mu(t)^{1-1/n} \leqslant [\lambda u^*(\mu(t)) + \tilde{f}(\mu(t))]\,(-\mu'(t))$$

that is :

$$1 \leqslant \frac{-\mu'(t)}{nC_n^{1/n}\,\mu(t)^{1-1/n}}\ [\lambda u^*(\mu(t)) + \tilde{f}(\mu(t))]\,.$$

Now, integrating both sides of last inequality from 0 to t, we have :

$$t \leqslant \frac{1}{nC_n^{1/n}} \int_0^t [\lambda u^*(\mu(\tau)) + \tilde{f}(\mu(\tau))]\ \frac{-\mu'(\tau)}{\mu(\tau)^{1-1/n}}\ d\tau$$

$$\leqslant \frac{1}{nC_n^{1/n}} \int_{\mu(\tau)}^M [\lambda u^*(\tau) + \tilde{f}(\tau)]\,\tau^{1/n-1}\ d\tau,$$

which implies, by the definition of decreasing rearrangement :

$$u^*(s) \leqslant \frac{1}{nC_n^{1/n}} \int_s^M [\lambda u^*(\tau) + \tilde{f}(\tau)]\,\tau^{1/n-1}\ d\tau,$$

which gives (3.1) replacing s by $C_n\,|x|^n$.

LEMMA 3.2. *We have a.e. :*

$$u^\star(x) \leqslant v(x) \tag{3.4}$$

where v(x) *is the unique decreasing spherically symmetric solution of the problem :*

$$\begin{cases} |Dv| - \lambda v(x) = \tilde{f}(C_n\,|x|^n) & \text{in} \quad \Omega^\star, \\ v(x) = 0 & \text{on} \quad \partial\Omega^\star \end{cases} \tag{3.5}$$

Proof. Define a sequence $\{v_k(x)\}$ of functions in Ω in this way :

$$v_0(x) = u^\star(x) \tag{3.6}$$

and

$$v_k(x) = \frac{1}{nC_n^{1/n}} \int_{C_n\,|x|^n}^M [\lambda v_{k-1}^*(s) + \tilde{f}(s)]\,s^{-1+1/n}\,ds. \tag{3.7}$$

Of course $v_k(x) \leqslant v_{k+1}(x)$ by (3.1). Moreover, as we prove now, $\{v_k(x)\}$ converges in $L^p(\Omega^\star)$.

In fact first of all we derive from (3.7), changing the variable on the right-hand side, that :

$$\text{(3.8)} \qquad v_k(x) = \int_{|x|}^{(M/C_n)^{1/n}} [\lambda v^*_{k-1}(C_n\tau^n) + f(C_n\tau^n)]\, d\tau.$$

Now for semplicity set $\rho = |x|$ and

$$\omega_k(\rho) = \begin{cases} v_k(x) & \text{if } \rho = |x| \text{ and } \rho \leqslant (M/C_n)^{1/n} \\ 0 & \text{if } \rho > (M/C_n)^{1/n}. \end{cases}$$

By the following Hardy inequality ([10]) :

$$\text{(3.9)} \qquad \int_0^\infty y^r \left(\int_y^\infty a(t)dt \right)^p dy \leqslant \left(\frac{p}{r+1} \right)^p \int_0^\infty y^r (ya(y))^p\, dy,$$

where $p \geqslant 1$ and $r > -1$, we get from (3.8) :

$$\int_0^\infty \rho^r (\omega_{k+1}(\rho) - \omega_k(\rho))^p\, d\rho = \int_0^\infty \rho^r \left(\int_\rho^\infty \lambda(\omega_k(\tau) - \omega_{k-1}(\tau))d\tau \right)^p d\rho$$

$$\leqslant \left(\frac{p}{r+1} \right)^p \int_0^\infty \rho^{r+p} \lambda^p (\omega_k - \omega_{k-1})^p\, d\rho \leqslant \left(\frac{\lambda p}{r+1} \left(\frac{M}{C_n} \right)^{1/n} \right)^p \int_0^\infty \rho^r (\omega_k - \omega_{k-1})^p\, d\rho.$$

Then repeating the above k times, finally we have :

$$\text{(3.10)} \int_0^\infty \rho^r (\omega_{k+1}(\rho) - \omega_k(\rho))^p\, d\rho \leqslant \left(\frac{\lambda p}{r+1} \left(\frac{M}{C_n} \right)^{1/n} \right)^{kp} \int_0^\infty \rho^r (\omega_1(\rho) - \omega_0(\rho))^p\, d\rho.$$

Hence, fixed $\bar{r} > r = \lambda p \left(\frac{M}{C_n} \right)^{1/n} - 1$ and $r \geqslant n-1$, the sequence $\{\omega_k(\rho)\}$ converges in the space of the functions $\phi(\rho)$ such that :

$$\left(\int_0^\infty \rho^r |\phi(\rho)|^p\, d\rho \right)^{1/p} < +\infty.$$

Then if $n-1 > \bar{r}$ we conclude that the sequence $\{v_k(x)\}$ converges in $L^p(\Omega^\star)$ since :

$$nC_n \int_0^\infty \rho^{n-1} |\omega_k(\rho) - \omega_h(\rho)|^p\, d\rho = \int_{\Omega^\star} |v_k(x) - v_h(x)|^p\, dx$$

If on the contrary $n-1 \leqslant \bar{r}$, we can fix a positive number $r > \bar{r}$ and an integer $m > 0$

such that $r - mp = n-1$ (1) and so, as above, by inequality (3.9) :

$$\int_0^\infty \rho^{r-p} |\omega_{k+1}(\rho) - \omega_{h+1}(\rho)|^p \, d\rho \leqslant \left(\frac{\lambda p}{r-p+1}\right)^p \int_0^\infty \rho^r |\omega_k(\rho) - \omega_h(\rho)|^p \, d\rho.$$

Then the sequence $\{\omega_k(\rho)\}$ converges in the space of the functions $\phi(\rho)$ such that :

$$\left(\int_0^\infty \rho^{r-p} |\phi(\rho)|^p \, d\rho\right)^{1/p} < +\infty.$$

In this way, after m steps, we obtain the convergence of $\{\omega_k(\rho)\}$ in the space of the functions $\phi(\rho)$ for which :

$$\left(\int_0^\infty \rho^{n-1} |\phi(\rho)|^p \, d\rho\right)^{1/p} < +\infty$$

that is the convergence of $\{v_k(x)\}$ in $L^p(\Omega^\star)$.

Say $v(x) = \lim v_k(x)$ in $L^p(\Omega^\star)$. Then of course also :

$$v(x) = \lim v_k(x) \quad \text{a.e. in } \Omega^\star.$$

Hence we get from (3.6) :

$$u^\star(x) \leqslant v(x). \tag{3.11}$$

On the other hand, interchanging integration and limit process on the right-hand side of (3.7), for v(x) we get :

$$v(x) = \frac{1}{nC_n^{1/n}} \int_{C_n |x|^n}^{M} [\lambda v^*(s) + \tilde{f}(s)] \, s^{1/n-1} \, ds. \tag{3.12}$$

Then v(x) solves (3.5) and so it is the unique solution of (3.5), which is spherically symmetric decreasing and by (3.11) we get our claim.

Proof of Theorem 1.1. The decreasing spherically symmetric solution of (3.5) is :

$$v(x) = e^{-\lambda |x|} \int_{|x|}^{(M/C_n)^{1/n}} \tilde{f}(C_n t^n) e^{\lambda t} \, dt$$

$$= \frac{e^{-\lambda |x|}}{nC_n^{1/n}} \int_{C_n |x|^n}^{M} \tilde{f}(t) \, e^{\lambda (t/C_n)^{1/n}} \, t^{1/n-1} \, dt,$$

(1) - In fact we can choose $m > \dfrac{\bar{r}-(n-1)}{p}$ and $r = (n-1) + mp$.

then for $s = C_n |x|^n$:

$$v^*(s) = \frac{e^{-\lambda(s/C_n)^{1/n}}}{nC_n^{1/n}} \int_s^M \tilde{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}\, dt. \tag{3.13}$$

Set $\alpha = C_n \left(\frac{n-1}{\lambda}\right)^n$.

The function $\phi(t) = e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}$ decreases in $(0,\alpha]$, then, if $M \leqslant \alpha$, from Lemma 2.1 and inequality (1.2) we derive :

$$v^*(s) \leqslant \lim_h \frac{1}{nC_n^{1/n}} \int_s^M f_h(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}\, dt$$

$$\leqslant \frac{1}{nC_n^{1/n}} \int_0^M f^*(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}\, dt$$

$$= \sup \frac{e^{-\lambda(s/C_n)^{1/n}}}{nC_n^{1/n}} \int_s^M f^*(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}\, dt$$

$$= \sup w^*(s)$$

and from here, since $w(x) = w^\star(C_n |x|^n)$, by (3.4) it follows (1.3) in the case $M \leqslant \alpha$.

Now assume $M > \alpha$. As above :

$$v^*(s) \leqslant \lim_h \frac{1}{nC_n^{1/n}} \int_s^M f_h(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}\, dt.$$

Now, since the functions $f_h(t)$ and $f(x)$ are equidistribuited, by (2.2), there exists a function $\hat{f}(t)$ having the same distribution function as $f(x)$ such that :

$$v^*(s) \leqslant \frac{1}{nC_n^{1/n}} \int_0^M \hat{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}\, dt.$$

From this inequality and from (3.4) we derive the result, since the function

$$w(x) = \frac{e^{-\lambda |x|}}{nC_n^{1/n}} \int_{C_n |x|^n}^M \hat{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}\, dt$$

is the unique decreasing spherically symmetric solution of the problem (1.5).

Remark 3.1. In this case we can define precisely the function $\hat{f}$ given by (2.4). In fact, if ν_ϕ is the distribution function of

$$\phi(t) = e^{\lambda(t/C_n)^{1/n}} t^{1/n-1},$$

we have

$$\hat{f}(t) = f^*(\nu_\phi(\phi(t))) \quad , \quad t \in [0,M] .$$

Such a function verifies the following properties :

each level set of $\hat{f}$ is a level set of ϕ ;

$\hat{f}$ is decreasing in $[0,\alpha]$ and increasing in $[\alpha,M]$;

$\hat{f}$ is equimeasurable with f^*.

Proof of Theorem 1.2. By Lemma 2.1 and (3.13) we have :

$$\| u \|_1 = \int_0^M u^*(s)ds \leqslant \int_0^M v^*(s)ds$$

$$= \frac{1}{nC_n^{1/n}} \int_0^M e^{-\lambda(s/C_n)^{1/n}} ds \int_s^M \tilde{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt$$

$$= \frac{1}{nC_n^{1/n}} \int_0^M \tilde{f}(s) e^{\lambda(s/C_n)^{1/n}} s^{1/n-1} ds \int_0^s e^{-\lambda(t/C_n)^{1/n}} dt.$$

Then by Lemma 2.1 and Lemma 2.2, there exists a function $\check{f}(t)$, which has the same distribution function as $f(x)$, such that :

$$\| u \|_1 \leqslant \frac{1}{nC_n^{1/n}} \int_0^M \check{f}(s) e^{\lambda(s/C_n)^{1/n}} s^{1/n-1} ds \int_0^s e^{-\lambda(t/C_n)^{1/n}} dt$$

$$= \| z^*(s) \|_1,$$

where

$$z^*(s) = \frac{e^{-\lambda(s/C_n)^{1/n}}}{nC_n^{1/n}} \int_s^M \check{f}(s) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt$$

and from there we get the result, since $z(x) = z^*(C_n |x|^n)$.

Remark 3.2. We could exhibit the function $\check{f}$ in the same way we did for $\hat{f}$ in Remark 3.1.

Proof of Theorem 1.3. Set

$$\chi_s(t) = \begin{cases} 0, & 0 \leqslant t \leqslant s \\ 1, & s \leqslant t \leqslant M \end{cases}$$

Obviously :

$$\int_s^M \tilde{f}(t) e^{\lambda (t/C_n)^{1/n}} t^{1/n-1} dt = \int_0^M \tilde{f}(t)\chi_s(t) e^{\lambda (t/C_n)^{1/n}} t^{1/n-1}\, dt$$

On the other hand if $s \geqslant C_n \left(\frac{n-1}{\lambda}\right)^n = \alpha$, then the function $\chi_s(t) e^{\lambda (t/C_n)^{1/n}} t^{1/n-1}$ is increasing in [0,M] and hence, by Lemma 2.1 and (1.2), it follows that :

$$\int_0^M \tilde{f}(t)\chi_s(t) e^{\lambda (t/C_n)^{1/n}} t^{1/n-1}\, dt \leqslant \int_0^M f_*(t)\chi_s\, e^{\lambda (t/C_n)^{1/n}} t^{1/n-1}\, dt.$$

Then by (3.4) and (3.13), for $s \geqslant \alpha$ we have :

$$u^*(s) \leqslant \frac{e^{-\lambda (s/C_n)^{1/n}}}{n C_n^{1/n}} \int_s^M f_*(t) e^{\lambda (t/C_n)^{1/n}} t^{1/n-1}\, dt = q^*(s).$$

Remark 3.3. As in section 3 of [9] it is possible to extend the previous results to the more general problem :

$$\begin{cases} H(u,Du) = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where H(p,q) and f(x,p) are given real valued functions, satisfying the hipotheses :

$\bar{a}) = a)$;

$\bar{b})$ $\exists K : R \to R_+$ strictly increasing such that :

$$K(|q| - \lambda p) \leqslant H(p,q), \quad \forall\, p \in R \text{ and } \forall\, q \in R^n ;$$

$\bar{c})$ $K^{-1}(f(x,0)) \in L^p$, $p \geqslant 1$;

$\bar{d})$ $f(x,p) \leqslant f(x,0)$, $\forall\, (x,p) \in \Omega \times R$;

$\bar{e})$ analogous to hipothesis d).

In fact we can compare the solution u(x) of such a problem with the unique spherically symmetric decreasing solution of a spherically symmetric problem in Ω^* for which the right-hand side is a function depending only on $|x|$, equidistribuited with f(x,0).

4. - AN EXISTENCE THEOREM FOR VISCOSITY SOLUTIONS OF (1.1)

In all this paragraph we assume that :

i) Ω is a bounded open subset of R^n ;

ii) $f(x) \in C(\bar{\Omega})$, $0 \leq f(x) \leq L$;

iii) $\lambda > 0$.

Then, denoting by B_R a ball of R^n, centered at the origin, with radius R and containing Ω, the function :

$$g(x) = -\frac{L}{\lambda} + \frac{L}{\lambda} e^{\lambda(R-|x|)}, \quad |x| \leq R,$$

is a viscosity (and generalized) solution of :

(4.1) $$|Dg| - \lambda g = L \quad \text{in} \quad B \quad ,$$

with boundary condition $g = 0$ on ∂B_R .

Also g(x) is a viscosity (and generalized) supersolution of the equation :

(4.2) $$|Du| - \lambda u = f(x) \quad \text{in} \quad \Omega.$$

Of course the function $u_o \equiv 0$ is a viscosity (and generalized) subsolution of (4.2).

For (4.2) the following theorem holds :

THEOREM 4.1. *In the interval* $[0,g]$ *there exists a minimal and a maximal viscosity solution of* (4.2), *which are in* $C(\bar{\Omega})$ *and which are zero on* $\partial\Omega$.

Proof. Using an iterative process, by Prop. 7.3 and by Prop. 3.4 of [11] , we can construct a sequence of functions belonging to $W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$, such that :

- u_n satisfies the equation

(4.n) $$|Du| + u = f(x) + (\lambda+1)u_{n-1} \quad \text{in} \quad \Omega$$

almost everywhere, and $u_n = 0$ on $\partial\Omega$;

- u_n is a viscosity solution of (4.n) ;
- $0 \leq u_o \leq u_{n-1} \leq u_n$.

Furthermore, applying Th. 1.11 of [11], we have also :

- $u_n \leqslant g(x)$.

Then the sequence $\{u_n\}$ is equibounded in $W^{1,\infty}(\Omega)$ and so $u_n \to \bar{u} \in C(\bar{\Omega})$ uniformly. By stability theorem 1.2 of [6] $\bar{u}$ is a viscosity solution of the equation $|Du| + u = (\lambda+1)\bar{u}$ in Ω, that is $\bar{u}$ is a viscosity solution of (4.2). Of course $0 \leqslant \bar{u}(x) \leqslant g(x)$, and $\bar{u} = 0$ on $\partial\Omega$.

Now we show that $\bar{u}$ is minimal in the interval $[0,g(x)]$ in the sense that, if u is a viscosity solution of (4.2) such that :

- $\underline{u}(x) \in C(\bar{\Omega})$, $\underline{u} = 0$ on $\partial\Omega$,
- $0 \leqslant \underline{u}(x) \leqslant g(x)$,

then :

(4.3) $$\bar{u}(x) \leqslant \underline{u}(x).$$

In fact, by Th. 1.11 of [11], we get :

$$u_n(x) \leqslant \underline{u}(x),$$

which implies (4.3) immediately.

Now set $u'_o(x) = g(x)$. As above, by Prop. 7.3 and Prop. 3.4 of [11], by an iterative process we can construct a sequence $\{u'_n\}$ of functions belonging to $W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$, such that :

- u'_n satisfies the equation :

(4.n)' $$|Du| + u = f(x) + (\lambda+1)u'_{n-1} \quad \text{in } \Omega$$

a.e. and $u'_n = 0$ on $\partial\Omega$.

- u'_n is a viscosity solution of (4.n)' ;
- $0 \leqslant u'_n(x)$.

Furthermore, by Th. 1.11 and by Prop. 3.4 of [10], we have :

- $u'_n \leqslant u'_{n-1} \leqslant u'_o = g(x)$.

Then, anagously as before, we can conclude that u'_n converges uniformly to a function $\bar{u}'$, which is a viscosity solution of (4.2), verifying the boundary condition $\bar{u}' = 0$ on $\partial\Omega$ and for which

$$\underline{u}'(x) \leqslant \bar{u}'(x),$$

for each viscosity solution $\underline{u}'(x)$ of (4.2) such that :

- $\underline{u}'(x) \in C(\bar{\Omega})$, $\underline{u}'(x) = 0$ on $\partial\Omega$,

- $0 \leqslant \underline{u}'(x) \leqslant g(x)$.

REFERENCES

[1] A. ALVINO - G. TROMBETTI. *«Sulle migliori costanti di maggio razione per una classe di equazioni ellittiche degeneri»*. Ricerche Mat. 27 (1978), n. 2, 413-428.

[2] A. ALVINO - G. TROMBETTI. *«Equazioni ellittiche con termini di ordine inferiore e riordinamenti»*. Atti Accad. Naz. Lincei (8) 66 (1979), n. 3, 194-200.

[3] G. CHITI. *«Norme di Orlicz delle soluzioni di una classe di equazioni ellittiche»*. Boll. U.M.I. A(5) 16 (1979), 178-185.

[4] K.M. CHONG - N.M. RICE. *«Equimeasurable rearrangements of functions»*. Queens Papers in Pure and Applied Math., n^o 28, 1971.

[5] M.G. CRANDALL - L.C. EVANS - P.L. LIONS. *«Some properties of viscosity solutions of Hamilton-Jacobi equations»*. Trans. A.M.S. 282 (1984), n^{o}2, 487-502.

[6] M.G. CRANDALL - P.L. LIONS. *«Viscosity solutions of Hamilton-Jacobi equations»*. Trans. Amer. Math. Soc. 277 (1983), 1-42.

[7] E. DE GIORGI. *«Su una teoria generale della misura $(r-1)$-dimensionale in una spazio ad r dimensioni»*. Ann. Mat. Pura Appl. (4) 36 (1954), 191-213.

[8] F. FLEMING - R. RISHEL. *«An integral formula for total gradient variation»*. Arch. Math. 11 (1960), 218-222.

[9] E. GIARRUSSO - D. NUNZIANTE. *«Symmetrization in a class of first-order Hamilton-Jacobi equations»*. Nonlinear Analysis T.M.A. 8 (1984), n^o 4, 289-299.

[10] G.H. HARDY - J.E. LITTLEWOOD - G. POLYA. *«Inequalities»*. Cambridge University Press (1964).

[11] P.L. LIONS. *«Generalized solutions of Hamilton-Jacobi equations»*. Pitman, London (1982).

[12] G.G. LORENTZ. *«Some new functional spaces»*. Ann. of Math. (2) 51 (1950), 37-55.

[13] G. TALENTI. *«Elliptic equations and rearrangements»*. Ann. Scuola Norm. Sup. Pisa (4) 3 (1976), n. 4, 697-718.

(Manuscrit reçu le 4 juin 1984)