Comparison theorems for a class of first order Hamilton-Jacobi equations


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COMPARISON THEOREMS FOR A CLASS OF FIRST ORDER HAMILTON-JACOBI EQUATIONS (3)

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Résumé : Nous étudions une certaine classe d'équations de Hamilton-Jacobi du premier ordre. Tout d'abord, en utilisant des techniques de symétrisation, nous comparons une solution du problème considéré avec la solution à symétrie sphérique décroissante d'un problème symétrisé. Enfin nous démontrons un théorème d'existence des solutions de viscosité.

Summary : We study a certain class of first order Hamilton-Jacobi equations. First, by means of symmetrization technique, we compare a solution of the considered problem with the decreasing spherically symmetric solution of a symmetrized problem. Next we prove an existence theorem of viscosity solution.
1. - INTRODUCTION AND RESULTS

From the same point of view as in [9], making use of symmetrization techniques, we study the Dirichlet problem:

\[
\begin{cases}
|Du| - \lambda u = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

where \( x = (x_1, \ldots, x_n) \) is a point of \( \mathbb{R}^n \), \( Du = \text{grad } u \), and \( f(x) \) is a measurable real-valued function defined in \( \Omega \).

We assume that:

a) \( \Omega \) is an open subset of \( \mathbb{R}^n \) with finite measure \( M \);

b) \( f(x) \in L^p(\Omega), \quad p \geq 1 \); 

c) \( \lambda \) is a positive real number;

d) there exists a generalized solution \( u \) of (1.1), that is there exists a function \( u \in W^{1,p}_0(\Omega), \quad p \geq 1 \), which satisfies the equation \(|Du| - \lambda u = f(x)\) a.e. in \( \Omega \).

Existence theorems for (1.1) can be found in [11], so that the last assumption makes sense.

Herein our main goal is to compare a solution \( u(x) \) of (1.1) with the unique decreasing spherically symmetric solution of a problem:

- which is of the same type as (1.1);
- given in a ball \( \Omega^* \subset \mathbb{R}^n \) having measure \( M \);
- for which the right hand-side has the same distribution function as \( f(x) \).

In order to state our results more precisely, let us recall that one denotes by

\[ \mu(t) = \text{meas } \{ x \in \Omega : |u(x)| > t \} \]

the distribution function of a measurable real-valued function \( u \) defined in \( \Omega \), that

\[ u^*(s) = \inf \left\{ t \geq 0 : \mu(t) < s \right\} \]

is the decreasing rearrangement of \( u \), and that

\[ u^*(x) = u^*(C_n |x|^n), \]
where $C_n$ is the measure of the $n$-dimensional unit-ball of $\mathbb{R}^n$, is the spherically symmetric decreasing rearrangement of $u$.

Also we consider the increasing rearrangement of $u$ :

$$u^*_*(s) = u^*(\text{meas } \Omega - s) \tag{1.2}$$

and the spherically symmetric increasing rearrangement of $u$ :

$$u_k(x) = u_k(C_n \| x \| n).$$

The function $u$ and its rearrangements have the same distribution function and, as well known, the following inequality holds (see [10], [13]) :

$$\int_{\Omega} |u^*| \, dx \leq \int_0^M u^*(s) \, v^*(s) \, ds = \int_{\Omega^*} u^*(x) \, v^*(x) \, dx$$

Here and below $\Omega^*$ is the ball of $\mathbb{R}^n$ centered at the origin with the same measure $M$ as $\Omega$. Denoted by $u(x)$ a generalized solution of (1.1), using auxiliary lemmas of section 2, in section 3 we prove the following results :

**THEOREM 1.1.** If $n > 1$, we have :

$$\| u \|_\infty \leq \| w \|_\infty$$

where, if $M \leq C_n \left( \frac{n-1}{\lambda} \right)^n$, $w(x)$ is the unique decreasing spherically symmetric solution of the problem :

$$\begin{cases}
\| Dv \! - \! \lambda w = f^*(x) & \text{in } \Omega^* \\
w = 0 & \text{on } \partial \Omega^*
\end{cases} \tag{1.4}$$

while, if $M > C_n \left( \frac{n-1}{\lambda} \right)^n$, $w(x)$ is the unique decreasing spherically symmetric solution of the problem :

$$\begin{cases}
\| Dw \! - \! \lambda w = \hat{f}(C_n \| x \| n) & \text{in } \Omega^* \\
w = 0 & \text{on } \partial \Omega^*
\end{cases} \tag{1.5}$$

$\hat{f}(s)$ being a function with the same distribution function as $f(x)$. (see REMARK 3.1 for an explicit definition of $\hat{f}$).
THEOREM 1.2. If \( n > 1 \), we have:

\[
\|u\|_1 \leq \|z\|_1
\]

where \( z(x) \) is the unique decreasing spherically symmetric solution of the problem:

\[
\begin{cases}
|Dz| - \lambda z = \tilde{f}(C_n \, |x|^n) & \text{in } \Omega^* \\
z = 0 & \text{on } \partial\Omega^*
\end{cases}
\]

\( \tilde{f}(s) \) being a fixed function having the same distribution function as \( f(x) \).

THEOREM 1.3. If \( n \geq 1 \) and \( \lambda > (n-1) \left[ C_n / M \right]^{1/n} \), we have:

\[
u^*(x) \leq q(x) \quad \text{in } \Omega^* - \Omega_0^*
\]

where \( \Omega_0^* \) is a ball of \( \mathbb{R}^n \) centered at the origin and with radius \( \frac{n-1}{\lambda} \), and \( q(x) \) is the unique spherically symmetric decreasing solution of the problem:

\[
\begin{cases}
|Dq| - \lambda q = f_q(x) & \text{in } \Omega^* \\
q = 0 & \text{on } \partial\Omega^*
\end{cases}
\]

In particular, from Theorem 1.3 we derive the following:

COROLLARY 1.1. If \( n = 1 \), then:

\[
u^*(x) \leq q(x) \quad \text{in } \left( -\frac{M}{2}, \frac{M}{2} \right)
\]

where \( q(x) \) is the unique solution, depending only on \( |x| \), of the problem:

\[
\begin{cases}
\frac{d}{ds} q - \lambda q = f_q(x) & \text{in } \left( -\frac{M}{2}, \frac{M}{2} \right) \\
q \left( -\frac{M}{2} \right) = q \left( \frac{M}{2} \right) = 0
\end{cases}
\]

Now, let us recall that, more generally, for a first order Hamilton-Jacobi equation:

\( H(x,u(x), Du(x)) = 0 \) (\( H \) being a continuous function in \( \Omega \times \mathbb{R} \times \mathbb{R}^n \)), besides the definition of generalized solutions, M.G. Crandall and P.L. Lions have introduced the notion of viscosity solution.
We refer to [5], [6], [11] for the exact definition and for the properties of viscosity solutions.

Only let us mention that a viscosity solution $u$ of the equation $H = 0$ need to be continuous but not necessarily differentiable in anywhere; however, if $u$ is differentiable at some $x_0$, then $H(x_0, u(x_0), Du(x_0)) = 0$.

Furthermore some uniqueness and stability problems can be solved introducing this new notion of solution (see [5], [6], [11]).

In section 4, under more restrictive assumptions, we prove an existence theorem for viscosity solutions of (1.1) (see Th. 4.1 for the exact statement).

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2. - TWO LEMMAS

Henceforth let $u(x)$ be a solution of (1.1) and $\mu(t)$ its distribution function, for each $s \in [0, \text{meas } \Omega]$ consider a measurable subset $D(s)$ of $\Omega$, such that:

- $\text{meas } D(s) = s$
- $s_1 < s_2 \Rightarrow D(s_1) \subset D(s_2)$
- $D(s) = \left\{ x \in \Omega : |u(x)| > t \right\}$ if $s = \mu(t)$.

Then by b) $\int_{D(s)} f(x) \, dx$ is an absolutely continuous function; and so there exists a function $\tilde{f}(t)$ such that:

(2.1) $\int_0^s \tilde{f}(t) \, dt = \int_{D(s)} |f(x)| \, dx.$

Furthermore the following lemma holds (see [1] for the proof):

**Lemma 2.1.** There exists a sequence $\left\{ f_h(s) \right\}$ of functions which have the same distribution function as $f(x)$ and such that, if $p > 1$:

$$f_h(s) \to \tilde{f}(s) \quad \text{in } L^p([0,M]),$$
while if \( p = 1 \):

\[
\lim_{h \to 0} \int_0^M f_h(s) g(s) ds = \int_0^M \hat{f}(s) g(s) ds
\]

for each function \( g(s) \) belonging to the space \( BV([0,M]) \) of the functions of bounded variation.

Now we give a sketch of the proof of the well known:

**LEMMA 2.2.** Let \( \phi(s) \) and \( f(s) \) be two given measurable functions in \([0,M]\). Then there exists \( \hat{f}(s) \), which has the same distribution function as \( f(s) \) and which depends on \( \phi(s) \) such that:

\[
\int_0^M f(s) \phi(s) ds \leq \int_0^M \hat{f}(s) \phi(s) ds = \int_0^M f^*(s) \phi^*(s) ds
\]

(compare for example with [4] or [12]).

- Denoted by \( \nu_\phi(t) \) the distribution function of \( \phi(s) \), for each \( s \in [0,M] \) we can fix a measurable subset \( E(s) \subseteq [0,M] \), such that:

\[
\text{meas } E(s) = s; \quad s_1 < s_2 \Rightarrow E(s_1) \subseteq E(s_2);
\]

\[
E(s) = \left\{ \sigma : \mid \phi(\sigma) \mid > t \right\} \text{ if } s = \nu_\phi(t).
\]

Then let

\[
s(\sigma) = \inf \left\{ s \in [0,M] : \sigma \in E(s) \right\} , \sigma \in [0,M];
\]

the required function is:

\[
\hat{f}(\sigma) = f^*(s(\sigma)).
\]

Moreover, denoting by \( \nu_f(t) \) the distribution function of \( f(s) \), we have:

\[
\int_0^M \hat{f}(\sigma) \phi(\sigma) d\sigma = \int_0^\infty \int_0^t \phi(\sigma) d\sigma = \int_0^\infty \int_0^{\nu_f(t)} \phi(\sigma) d\sigma = \int_0^\infty dt \int_0^{\nu_f(t)} \phi(\sigma) d\sigma = \int_0^M f^*(s) \phi^*(s) ds.
\]
3. - PROOF OF THEOREMS 1.1, 1.2, 1.3

For the sake of clearness, first of all we prove two lemmas.

LEMMA 3.1. We have:

\[ u^*(x) \leq \frac{1}{n C_n^{1/n}} \int_{C_n}^{M} (\lambda u^*(s) + \tilde{F}(s))^s \frac{1}{1/n - 1} ds \]

a.e. in \( \Omega^* \).

Proof. By the isoperimetric inequality (see [7]):

\[ n C_n^{1/n} \mu(t)^{1-1/n} \leq P \{ x : |u(x)| > t \} , \]

where \( P \) is the perimeter in the sense of De Giorgi, and by the Fleming-Rishel formula ([8]):

\[ \int_{|u| > t} |Du| dx = \int_0^\infty P \{ x : |u(x)| > \xi \} d\xi \]

we get:

\[ n C_n^{1/n} \mu(t)^{1-1/n} \leq -\frac{d}{dt} \int_{|u| > t} |Du| dx . \]

On the other hand, since \( u \) is a solution of (1.1):

\[ \frac{1}{h} \int_{t < |u| < t+h} |Du| dx \leq \frac{1}{h} \int_{t < |u| < t+h} (\lambda |u| + |fl|) dx, \quad h > 0 , \]

hence for \( h \to 0 \):

\[ -\frac{d}{dt} \int_{|u| > t} |Du| dx \leq -\frac{d}{dt} \int_{|u| > t} (\lambda |u| + |fl|) dx . \]

Moreover, since

\[ \int_{|u| > t} |u| dx = \int_0^{\mu(t)} u^*(s) ds , \]

(2.1), (3.2) and (3.3) give:

\[ n C_n^{1/n} \mu(t)^{1-1/n} \leq [\lambda u^*(\mu(t)) + \tilde{F}(\mu(t))] \left( -\mu'(t) \right) \]

that is:
Now, integrating both sides of last inequality from 0 to t, we have:

\[ t \leq \frac{1}{nC_{1/n}^{1/n}} \int_{0}^{t} [\lambda u^*(\mu(\tau)) + \gamma(\mu(\tau))] \frac{-\mu'(\tau)}{\mu(\tau)^{1-1/n}} d\tau \]

which implies, by the definition of decreasing rearrangement:

\[ u^*(s) \leq \frac{1}{nC_{1/n}^{1/n}} \int_{s}^{M} [\lambda u^*(\tau) + \gamma(\tau)] \tau^{-1/n} d\tau, \]

which gives (3.1) replacing \( s \) by \( C_{1/n} |x|^n \).

**Lemma 3.2.** We have a.e.:

\[ u^*(x) \leq v(x) \]

where \( v(x) \) is the unique decreasing spherically symmetric solution of the problem:

\[
\begin{cases}
|Dv| - \lambda v(x) = \gamma(C_{1/n} |x|^n) & \text{in } \Omega^*, \\
v(x) = 0 & \text{on } \partial \Omega^*
\end{cases}
\]

**Proof.** Define a sequence \( \{v_k(x)\} \) of functions in \( \Omega \) in this way:

\[ v_0(x) = u^*(x) \]

and

\[ v_k(x) = \frac{1}{nC_{1/n}^{1/n}} \int_{C_{1/n} |x|^n}^{M} [\lambda v_{k-1}^*(s) + \gamma(s)] s^{-1+1/n} ds. \]

Of course \( v_k(x) \leq v_{k+1}(x) \) by (3.1). Moreover, as we prove now, \( \{v_k(x)\} \) converges in \( L^p(\Omega^*) \).

In fact first of all we derive from (3.7), changing the variable on the right-hand side, that:
Now for simplicity set \( p = \frac{1}{n} \) and

By the following Hardy inequality (\([10]\)):

\[
\int_{0}^{\infty} y^{r} \left( \int_{y}^{\infty} a(t) dt \right)^{p} dy \leq \left( \frac{p}{r+1} \right)^{p} \int_{0}^{\infty} y^{r} (y a(y))^{p} dy,
\]

where \( p > 1 \) and \( r > -1 \), we get from (3.8):

\[
\int_{0}^{\infty} \rho^{r} (\omega_{k+1}(\rho) - \omega_{k}(\rho))^{p} d\rho = \int_{0}^{\infty} \rho^{r} \left( \int_{0}^{\infty} \lambda(\omega_{k}(r) - \omega_{k-1}(r)) dr \right)^{p} d\rho
\]

\[
\leq \left( \frac{p}{r+1} \right)^{p} \int_{0}^{\infty} \rho^{r+p} \lambda^{p} (\omega_{k} - \omega_{k-1})^{p} d\rho \leq \left( \frac{\lambda p}{r+1} \left( \frac{M}{C_{n}} \right)^{1/n} \right)^{p} \int_{0}^{\infty} \rho^{r} (\omega_{k} - \omega_{k-1})^{p} d\rho.
\]

Then repeating the above \( k \) times, finally we have:

\[
(3.10) \int_{0}^{\infty} \rho^{r} (\omega_{k+1}(\rho) - \omega_{k}(\rho))^{p} d\rho \leq \left( \frac{\lambda p}{r+1} \left( \frac{M}{C_{n}} \right)^{1/n} \right)^{kp} \int_{0}^{\infty} \rho^{r} (\omega_{1}(\rho) - \omega_{0}(\rho))^{p} d\rho.
\]

Hence, fixed \( \tau > r = \lambda p \left( \frac{M}{C_{n}} \right)^{1/n} - 1 \) and \( r \gg n-1 \), the sequence \( \{ \omega_{k}(\rho) \} \) converges in the space of the functions \( \phi(\rho) \) such that:

\[
\left( \int_{0}^{\infty} \rho^{r} |\phi(\rho)|^{p} d\rho \right)^{1/p} < +\infty.
\]

Then if \( n-1 > \tau \) we conclude that the sequence \( \{ v_{k}(x) \} \) converges in \( L^{P}(\Omega^*) \) since:

\[
nC_{n} \int_{0}^{\infty} \rho^{n-1} |\omega_{k}(\rho) - \omega_{n}(\rho)|^{p} d\rho = \int_{\Omega^*} |v_{k}(x) - v_{n}(x)|^{p} dx
\]

If on the contrary \( n-1 \leq \tau \), we can fix a positive number \( r > \tau \) and an integer \( m > 0 \).
such that $r - mp = n - 1$ \((1)\) and so, as above, by inequality (3.9):

$$
\int_0^\infty \rho^{r-p} |\omega_{k+1}(\rho) - \omega_{h+1}(\rho)|^p \, d\rho \leq \left( \frac{\lambda p}{r-p+1} \right)^p \int_0^\infty \rho^r |\omega_k(\rho) - \omega_h(\rho)|^p \, d\rho.
$$

Then the sequence $\{\omega_k(\rho)\}$ converges in the space of the functions $\phi(\rho)$ such that:

$$
\left( \int_0^\infty \rho^{r-p} |\phi(\rho)|^p \, d\rho \right)^{1/p} < \infty.
$$

In this way, after $m$ steps, we obtain the convergence of $\{\omega_k(\rho)\}$ in the space of the functions $\phi(\rho)$ for which:

$$
\left( \int_0^\infty \rho^{n-1} |\phi(\rho)|^p \, d\rho \right)^{1/p} < \infty
$$

that is the convergence of $\{v_k(x)\}$ in $L^p(\Omega^*)$.

Say $v(x) = \lim v_k(x)$ in $L^p(\Omega^*)$. Then of course also:

$$
v(x) = \lim v_k(x) \quad \text{a.e. in } \Omega^*.
$$

Hence we get from (3.6):

$$
(3.11) \quad u^*(x) \leq v(x).
$$

On the other hand, interchanging integration and limit process on the right-hand side of (3.7), for $v(x)$ we get:

$$
(3.12) \quad v(x) = \frac{1}{nC_n^{1/n}} \int_{C_n \times \mathbb{R}^n} \left[ \lambda v^*(s) + \tilde{T}(s) \right] s^{1/n-1} \, ds.
$$

Then $v(x)$ solves (3.5) and so it is the unique solution of (3.5), which is spherically symmetric decreasing and by (3.11) we get our claim.

**Proof of Theorem 1.1.** The decreasing spherically symmetric solution of (3.5) is:

$$
v(x) = e^{-\lambda |x|} \int_{1/|x|}^{(M/C_n)^{1/n}} \tilde{T}(t/C_n)^{1/n} \lambda \, dt
$$

or

$$
v(x) = \frac{1}{nC_n^{1/n}} \int_{C_n \times \mathbb{R}^n} \tilde{T}(t) e^{\lambda t/C_n} t^{1/n-1} \, dt,
$$

\((1)\) - In fact we can choose $m > \frac{r - (n-1)}{p}$ and $r = (n-1) + mp$. 


then for \( s = C_n \| x \|^n \):

\[
\nu^*(s) = \frac{e^{-\lambda(s/C_n)^{1/n}}}{nC_n^{1/n}} \int_s^M \Gamma(t) e^{\lambda(t/C_n)^{1/n} t^{1/n-1}} dt.
\]  

(3.13)

Set \( \alpha = C_n \left( \frac{n-1}{\lambda} \right)^n \).

The function \( \phi(t) = e^{\lambda(t/C_n)^{1/n} t^{1/n-1}} \) decreases in \((0, \alpha]\), then, if \( M \leq \alpha \), from Lemma 2.1 and inequality (1.2) we derive:

Now assume \( M > \alpha \). As above:

\[
\nu^*(s) \leq \lim_{h \to 0} \frac{1}{nC_n^{1/n}} \int_s^M f_h(t) e^{\lambda(t/C_n)^{1/n} t^{1/n-1}} dt
\]

\[
\leq \frac{1}{nC_n^{1/n}} \int_0^M f(t) e^{\lambda(t/C_n)^{1/n} t^{1/n-1}} dt
\]

\[
= \sup \frac{e^{-\lambda(s/C_n)^{1/n}}}{nC_n^{1/n}} \int_s^M f(t) e^{\lambda(t/C_n)^{1/n} t^{1/n-1}} dt
\]

\[
= \sup \nu^*(s)
\]

and from here, since \( w(x) = w^*(C_n \| x \|^n) \), by (3.4) it follows (1.3) in the case \( M \leq \alpha \).

Now assume \( M > \alpha \). As above:

\[
\nu^*(s) \leq \lim_{h \to 0} \frac{1}{nC_n^{1/n}} \int_s^M f(t) e^{\lambda(t/C_n)^{1/n} t^{1/n-1}} dt.
\]

Now, since the functions \( f_h(t) \) and \( f(t) \) are equidistributed, by (2.2), there exists a function \( \tilde{f}(t) \) having the same distribution function as \( f(t) \) such that:

\[
\nu^*(s) \leq \frac{1}{nC_n^{1/n}} \int_0^M \tilde{f}(t) e^{\lambda(t/C_n)^{1/n} t^{1/n-1}} dt.
\]

From this inequality and from (3.4) we derive the result, since the function

\[
w(x) = \frac{e^{-\lambda \| x \|^n}}{nC_n^{1/n}} \int_0^M \tilde{f}(t) e^{\lambda(t/C_n)^{1/n} t^{1/n-1}} dt
\]

is the unique decreasing spherically symmetric solution of the problem (1.5).

**Remark 3.1.** In this case we can define precisely the function \( \tilde{f} \) given by (2.4). In fact, if \( \nu_\phi \) is the distribution function of

\[
\phi(t) = e^{\lambda(t/C_n)^{1/n} t^{1/n-1}},
\]
we have

\[ \hat{f}(t) = f^*(\varphi(t)) , \quad t \in [0,M]. \]

Such a function verifies the following properties:

- each level set of \( \hat{f} \) is a level set of \( \phi \);
- \( \hat{f} \) is decreasing in \([0,\alpha]\) and increasing in \([\alpha,M]\);
- \( \hat{f} \) is equimeasurable with \( f^* \).

**Proof of Theorem 1.2.** By Lemma 2. and (3.13) we have:

\[
\| u \|_1 = \int_0^M u^*(s)ds \leq \int_0^M v^*(s)ds
\]

\[
= \frac{1}{nC_{n}^{1/n}} \int_0^M e^{-\lambda(s/C_n)^{1/n}} \int_s^M \hat{T}(t)e^{\lambda(t/C_n)^{1/n}t^{1/n-1}}dt ds
\]

\[
= \frac{1}{nC_{n}^{1/n}} \int_0^M \hat{T}(s)e^{\lambda(s/C_n)^{1/n}s^{1/n-1}}ds \int_0^s e^{-\lambda(t/C_n)^{1/n}t^{1/n-1}}dt.
\]

Then by Lemma 2.1 and Lemma 2.2, there exists a function \( \tilde{f}(t) \), which has the same distribution function as \( f(x) \), such that:

\[
\| u \|_1 \leq \frac{1}{nC_{n}^{1/n}} \int_0^M \tilde{f}(s)e^{\lambda(s/C_n)^{1/n}s^{1/n-1}}ds \int_0^s e^{-\lambda(t/C_n)^{1/n}t^{1/n-1}}dt
\]

\[
= \| z^*(s) \|_1,
\]

where

\[
z^*(s) = e^{-\lambda(s/C_n)^{1/n}} \int_s^M \tilde{T}(s)e^{\lambda(s/C_n)^{1/n}s^{1/n-1}}dt.
\]

and from there we get the result, since \( z(x) = z^*(C_n | x | n) \).

**Remark 3.2.** We could exhibit the function \( \tilde{f} \) in the same way we did for \( \hat{f} \) in Remark 3.1.

**Proof of Theorem 1.3.** Set
Obviously:
\[
X_s(t) = \begin{cases} 
0, & 0 \leq t \leq s \\
1, & s \leq t \leq M 
\end{cases}
\]

\[
\int_s^M T(t) e^{\lambda(t/C_n)^{1/n} t_1/n-1} dt = \int_0^M T(t) X_s(t) e^{\lambda(t/C_n)^{1/n} t_1/n-1} dt
\]

On the other hand if \( s \geq C_n \left( \frac{n-1}{\lambda} \right)^n = \alpha \), then the function \( X_s(t) e^{\lambda(t/C_n)^{1/n} t_1/n-1} \) is increasing in \([0,M]\) and hence, by Lemma 2.1 and (1.2), it follows that:

\[
\int_0^M f(t) X_s(t) e^{\lambda(t/C_n)^{1/n} t_1/n-1} dt \leq \int_0^M f^*(t) X_s e^{\lambda(t/C_n)^{1/n} t_1/n-1} dt.
\]

Then by (3.4) and (3.13), for \( s \geq \alpha \) we have:

\[
u^*(s) \leq e^{-\lambda(s/C_n)^{1/n}} \int_s^M f^*(t) e^{\lambda(t/C_n)^{1/n} t_1/n-1} dt = q^*(s).
\]

**Remark 3.3.** As in section 3 of [9] it is possible to extend the previous results to the more general problem:

\[
\begin{cases} 
H(u,Du) = f(x,u) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

where \( H(p,q) \) and \( f(x,p) \) are given real valued functions, satisfying the hypotheses:

\( \tilde{a} \) = a) ;

\( \tilde{b} \) \( \exists K : R \to R_+ \) strictly increasing such that:

\[
K( |q| - \lambda p) \leq H(p,q), \quad \forall p \in R \quad \text{and} \quad \forall q \in R^n ;
\]

\( \tilde{c} \) \( K^{-1}(f(x,0)) \in L^p, \quad p \geq 1 ;
\]

\( \tilde{d} \) \( f(x,p) \leq f(x,0), \quad \forall (x,p) \in \Omega \times R ;
\]

\( \tilde{e} \) analogous to hypothesis d).

In fact we can compare the solution \( u(x) \) of such a problem with the unique spherically symmetric decreasing solution of a spherically symmetric problem in \( \Omega^* \) for which the right-hand side is a function depending only on \( |x| \), equidistributed with \( f(x,0) \).
4. - AN EXISTENCE THEOREM FOR VISCOSITY SOLUTIONS OF (1.1)

In all this paragraph we assume that:

i) \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \);
ii) \( f(x) \in C(\Omega), \ 0 \leq f(x) \leq L \);
iii) \( \lambda > 0 \).

Then, denoting by \( B_R \) a ball of \( \mathbb{R}^n \), centered at the origin, with radius \( R \) and containing \( \Omega \), the function:

\[
g(x) = \frac{L}{\lambda} + \frac{L}{e^{\lambda(R-|x|)}}, \quad |x| \leq R,
\]

is a viscosity (and generalized) solution of:

\[
(4.1) \quad |Dg| - \lambda g = L \quad \text{in} \ B,
\]

with boundary condition \( g = 0 \) on \( \partial B_R \).

Also \( g(x) \) is a viscosity (and generalized) supersolution of the equation:

\[
(4.2) \quad |Du| - \lambda u = f(x) \quad \text{in} \ \Omega.
\]

Of course the function \( u_0 \equiv 0 \) is a viscosity (and generalized) subsolution of (4.2).

For (4.2) the following theorem holds:

THEOREM 4.1. In the interval \([0,g]\) there exists a minimal and a maximal viscosity solution of (4.2), which are in \( C(\overline{\Omega}) \) and which are zero on \( \partial \Omega \).

Proof. Using an iterative process, by Prop. 7.3 and by Prop. 3.4 of [11], we can construct a sequence of functions belonging to \( W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \), such that:

- \( u_n \) satisfies the equation

\[
(4.n) \quad |Du| + u = f(x) + (\lambda+1)u_{n-1} \quad \text{in} \ \Omega
\]

almost everywhere, and \( u_n = 0 \) on \( \partial \Omega \);

- \( u_n \) is a viscosity solution of (4.n);

- \( 0 \leq u_0 \leq u_{n-1} \leq u_n \).
Furthermore, applying Th. 1.11 of [11], we have also:

- \( u_n \leq g(x) \).

Then the sequence \( \{ u_n \} \) is equibounded in \( W^{1,\infty}(\Omega) \) and so \( u_n \to \overline{u} \in C(\overline{\Omega}) \) uniformly. By stability theorem 1.2 of [6] \( \overline{u} \) is a viscosity solution of the equation \( |D u| + u = (\lambda + 1) \overline{u} \) in \( \Omega \), that is \( \overline{u} \) is a viscosity solution of (4.2). Of course \( 0 \leq \overline{u}(x) \leq g(x) \), and \( \overline{u} = 0 \) on \( \partial \Omega \).

Now we show that \( \overline{u} \) is minimal in the interval \([0, g(x)]\) in the sense that, if \( u \) is a viscosity solution of (4.2) such that:

- \( u(x) \in C(\overline{\Omega}) \), \( u = 0 \) on \( \partial \Omega \),
- \( 0 \leq u(x) \leq g(x) \),

then:

\[
\overline{u}(x) \leq u(x).
\]

In fact, by Th. 1.11 of [11], we get:

\[
u_n(x) \leq u(x),
\]

which implies (4.3) immediately.

Now set \( u^*_0(x) = g(x) \). As above, by Prop. 7.3 and Prop. 3.4 of [11], by an iterative process we can construct a sequence \( \{ u^*_n \} \) of functions belonging to \( W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \), such that:

- \( u^*_n \) satisfies the equation:

\[
(4.n)\quad |D u| + u = f(x) + (\lambda + 1)u^*_{n-1} \quad \text{in } \Omega
\]
a.e. and \( u^*_n = 0 \) on \( \partial \Omega \).

- \( u^*_n \) is a viscosity solution of \( (4.n)^* \);
- \( 0 \leq u^*_n(x) \).

Furthermore, by Th. 1.11 and by Prop. 3.4 of [10], we have:

- \( u^*_n \leq u^*_{n-1} \leq u^*_o = g(x) \).
Then, analogously as before, we can conclude that $u_n^\prime$ converges uniformly to a function $\tilde{u}^\prime$, which is a viscosity solution of (4.2), verifying the boundary condition $\tilde{u}^\prime = 0$ on $\partial\Omega$ and for which

$$u^\prime(x) \leq \tilde{u}^\prime(x),$$

for each viscosity solution $u^\prime(x)$ of (4.2) such that:

- $u^\prime(x) \in C(\bar{\Omega})$, $u^\prime(x) = 0$ on $\partial\Omega$,
- $0 \leq u^\prime(x) \leq g(x)$. 
REFERENCES


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