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A SYMMETRIZATION RESULT FOR ELLIPTIC EQUATIONS WITH LOWER-ORDER TERMS

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Résumé : Dans cet article sont établies des majorations a priori pour les solutions d'équations elliptiques avec termes d'ordre inférieur par utilisation de la méthode de rearrangement de fonctions. Cette méthode se base sur la comparaison avec la solution d'un problème particulier à symétrie radiale pour lequel l'effet des termes d'ordre inférieur est également pris en compte.

Summary : The technique of radially symmetric rearrangements of functions is used to obtain an a priori bound for the solutions of a class of linear second-order elliptic equations in divergence form with lower-order terms. It is based on comparison with the solution of a particular radially symmetric problem wherein the effect of the lower-order terms is also taken into account.

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1. - INTRODUCTION

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geqslant 2$, and let $u \in H_0^1(\Omega) = W_0^{1,2}(\Omega)$ be a weak solution of the equation

$$(1.1) \qquad Lu \equiv - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) u_{x_j}) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (b_i(x) u) + c(x) u = f(x),$$

where the coefficients and data satisfy

$$(1.2.a) \qquad a_{ij} \in L^\infty(\Omega), \ b_i(x) \in L^\infty(\Omega), \ c(x) \in L^{n/2}(\Omega),$$

(1.2.b) for a.e. $x \in \Omega$ and every $\xi = (\xi_i) \in \mathbb{R}^n$.

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geqslant |\xi|^2,$$

$$(1.2.c) \qquad c(x) \geqslant 0 \quad \text{a.e.} \quad \text{in } \Omega,$$

$$(1.2.d) \qquad \sum_{i=1}^{n} \frac{\partial}{\partial x_i} b_i(x) + c(x) \geqslant 0 \quad \text{in } \mathscr{D}'(\Omega),$$

and finally

$$(1.2.e) \qquad f \in L^{2n/(n+2)}(\Omega) \quad \text{if} \quad n > 2,$$

$$f \in L^p(\Omega) \text{ for every } p > 1 \text{ if } n = 2.$$

The aim of this work is to establish an a priori bound on u in terms of the above quantities by comparing it with the solution of a certain simple equation with radial symmetry. In doing this we use the technique of *Schwarz symmetrization* or radially symmetric rearrangement of functions.

Let $\Omega^\# = B_R(0)$ be the ball in $\mathbb{R}^n$ centered at 0 with same measure as Ω. Let

$$(1.3) \qquad B = \| (\sum (b_i(x))^2)^{1/2} \|_{L^\infty(\Omega)}$$

and let $k(x)$ be a measurable function : $\Omega \to [0,\infty)$ such that

$$(1.4.a) \qquad c(x) \geqslant k(x) \quad \text{a.e.} \quad \text{in } \Omega,$$

$$(1.4.b) \qquad \sum \frac{\partial}{\partial x_i} b_i(x) + c(x) \geqslant k(x) \quad \text{in } \mathscr{D}'(\Omega).$$

Given a function $u \in L^1(\Omega)$ we denote by $u^\#$ the *symmetric decreasing rearrangement* of u, i.e. the unique nonnegative function in $L^1(\Omega^\#)$ that is radially symmetric nonincreasing and has the same distribution function as $|u|$. If we change nonincreasing into nondecreasing in the above definition we obtain the *symmetric increasing rearrangement* of u, denoted by $u_\#$.

We now consider the problem

$$(1.5)\qquad \begin{cases} -\Delta v - B \sum_{i=1}^{n} \frac{\partial}{\partial y_i}\left(\frac{y_i}{|y|} v\right) + k_\#(y) v = f^\#(y) \quad \text{in } \Omega^\# \\ v \in H_0^1(\Omega^\#). \end{cases}$$

Due to be radial symmetry of the equation the unique solution v of (1.5) is radially symmetric. In case $k_\# = 0$ it is also nonincreasing, $v = v^\#$. Let $\Omega_o \subset \Omega$ be the set where k vanishes and $\Omega_o^\# = B_{R_o}(0)$; we have $0 \leqslant R_o \leqslant R$ and $k_\#(y) = 0$ if $0 \leqslant |y| \leqslant R_o$.

We can now state our main result :

THEOREM 1. *For every* $r \in (0,R)$ *we have*

$$(1.6)\qquad \int_{|y|\leqslant r} u^\#(y)dy \leqslant \int_{|y|\leqslant r} v(y)dy \left(\leqslant \int_{|y|\leqslant r} v^\#(y)dy\right).$$

Moreover

$$(1.7)\qquad 0 \leqslant u^\#(y) \leqslant v(y) \quad \textit{for} \quad 0 \leqslant y \leqslant R_o.$$

A well-known result of G. Hardy, J. Littlewood and G. Polya, cf. [6], [7], allows to derive from the *comparison of integrals in balls* (1.6) the following interesting consequence.

COROLLARY 1. *For every convex nondecreasing function* $\Phi : [0,\infty) \to [0,\infty)$ *we have*

$$(1.8)\qquad \int \Phi(|u|)dy \leqslant \int \Phi(v)dy$$

In particular we obtain an estimate in every L^p *space,* $1 \leqslant p \leqslant \infty$:

$$(1.9)\qquad \|u\|_{L^p(\Omega)} \leqslant \|v\|_{L^p(\Omega^\#)}.$$

We are interested in choosing k as large as possible so that v be as small as possible. In particular if $k(x) > 0$ a.e. in Ω the pointwise estimate reduces to $u^\#(0) = \|u\|_\infty \leqslant v(0)$.

The technique of rearrangements of functions has been used to obtain a priori estimates for the solutions of different classes of elliptic and parabolic equations via comparison with a worst case consisting in a radially symmetric equation that can be explicitly solved or at least estimated in a convenient way. The first result in this line seems to be due to H. Weinberger [17] in the case of a second-order elliptic equation without lower-order terms, where a comparison of the L^∞-norms is obtained.

Subsequently G. Talenti [13] proved the following sharp result : let $u \in H_0^1(\Omega)$ be a weak solution of the Dirichlet problem for the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}u_{x_j}) + c(x)u = f(x) \quad \text{in } \Omega, \tag{1.10}$$

under the above assumptions on a_{ij},c,f, and let v be the solution of

$$-\Delta v = f^{\#}(y), \quad y \in \Omega^{\#}, \tag{1.11}$$

in $H_0^1(\Omega^{\#})$. Then the following pointwise comparison holds

$$u^{\#}(y) \leqslant v(y) \quad \text{a.e.} \quad \text{in } \Omega^{\#}. \tag{1.12}$$

It corresponds to (1.7) above for the choice $k = 0$. In this way a strong result is obtained but the influence of the lower-order term c(x)u is disregarded. Results that take into account this term in (1.10) have been obtained by several authors : G. Chiti [7] , P.L. Lions [10] and J.L. Vazquez [16] . In all of them a comparison like (1.6), and not (1.12), holds. In [7] the model equation is $-\Delta v + k(y)v = f^{\#}$ with a nonnegative function k related to c and different from ours, but a result like (1.6)-(1.7) is obtained ; c has to be bounded (for more details see § 3). In [10] a result is given for (1.10) with $c \in L^\infty(\Omega)$: one compares u with the solution v of $-\Delta v + c_{\#}(y)v = f^{\#}(y)$ (in our notation). Actually it even includes some c(x) with changing sign, cf. Theorem 2, [10] . Finally [16] considers the semilinear equation $-\Delta u + \beta(u) = f$ in $\Omega = \mathbb{R}^n$, where β is continuous, nondecreasing with $\beta(0) = 0$. The assumptions $a_{ij} = \delta_{ij}$, $\Omega = \mathbb{R}^n$ are made only for simplicity. If v is the solution of $-\Delta v + \beta(v) = f^{\#}$ in $\mathbb{R}^n$ then (1.6) holds with u, v replaced respectively by $\beta(u^{\#})$, $\beta(v)$.

It is interesting to remark that the comparison of integrals in balls of the form (1.6) is the result typically obtained for many parabolic equations, cf. the results of [6] . It is shown in [16] how to reduce the parabolic result for the equation $u_t = \Delta\varphi(u)$, with $\varphi = \beta^{-1}$, to the above semilinear elliptic result.

The influence of the first-order term $\sum (b_i u)_{x_i}$ is studied by A. Alvino and G. Trombetti, [2] , [3] , under the conditions (1.2). They obtain pointwise comparison, (1.12), by setting $k = 0$, i.e. disregarding the zero-order term. This result has been extended by G. Talenti [14] to the case $b_i \in L^p(\Omega)$, $p > n$. We do not quite recover these results because of condition (1.2.d).

For other results for elliptic equations related to the above discussion see for instance [1], [4], [5], [8], [10].

Finally a word about our assumptions (1.2) : the parts a), b), c) and e) are standard in related works. We introduce d) to control the two lower-order terms ; in fact d) appears as a usual condition (together with a), b)) for the maximum principle to hold, cf. [9], § 8.1.

Theorem 1 is proved in Section 2. Several comments and applications are briefly treated in Section 3. In particular, in Theorem 2, we obtain a result like (1.9) for a dual class of equations.

2. - PROOF OF THEOREM 1

2.1. - We review for the reader's convenience some notations and results that we shall use. Let u be a measurable function defined in a bounded open set $\Omega \in IR^n$, $n \geqslant 1$, with measure meas $(\Omega) = |\Omega|$, then

i) the *distribution function* of u is the function $\mu : [0,\infty) \to IR^+$ defined by

$$\mu(t) = \text{meas}\left\{x \in \Omega : |u(x)| > t\right\}. \tag{2.1}$$

ii) the *decreasing rearrangement* of u, u* is defined by

$$u^*(s) = \inf\left\{t \geqslant 0 : \mu(t) < s\right\}, \quad 0 \leqslant s \leqslant |\Omega|. \tag{2.2}$$

iii) the *increasing rearrangement* of u, u_*, is defined by

$$u_*(s) = u^*(|\Omega| - s), \quad 0 \leqslant s \leqslant |\Omega| \tag{2.3}$$

iv) the *symmetric decreasing rearrangement* of u, $u^\#$ (resp. symmetric increasing rearrangement of u, $u_\#$) are given by the formulas

$$u^\#(x) = u^*(C_n |x|^n) \quad , \quad 0 \leqslant |x| \leqslant R, \tag{2.4.a}$$

$$u_\#(x) = u_*(C_n |x|^n) \quad , \quad 0 \leqslant |x| \leqslant R, \tag{2.4.b}$$

where $C_n = \pi^{n/2} / \Gamma(1 + \frac{n}{2})$ is the volume of the unit ball in IR^n and $|\Omega| = C_n R^n$.

A number of well-known formulas relate the functions u, u^*, u_*, $u^\#$, $u_\#$, cf. [6], [13] for instance. Let us only point out the following fact as a consequence of the Hardy-Littlewood inequality :

$$(2.5)\qquad \int_\Omega |uv|\,dx \leqslant \int_0^{|\Omega|} u^*(s)v^*(s)ds = \int_{\Omega^\#} u^\#(x)v^\#(x)dx,$$

valid if u, v $\in L^2(\Omega)$, we have not only

$$(2.6)\qquad \int_\Omega |uv|\,dx \leqslant \int_0^{|\Omega|} u_*(s)v_*(s)ds = \int_{\Omega^\#} u_\#(x)v_\#(x)dx,$$

but also

$$(2.7)\qquad \int_\Omega |uv|\,dx \geqslant \int_0^{|\Omega|} u_*(s)v^*(s)ds = \int_{\Omega^\#} u_\#(x)v^\#(x)dx.$$

Before proceeding with the proof let us remark that we can reduce ourselves to case $f \geqslant 0$, $u \geqslant 0$ a.e. without loss of generality by using the maximum principle. This assumption is therefore made in the sequel.

2.2. - The proof begins as in [13](see also [2]) : we consider the function $\Phi(t)$, $t \geqslant 0$, given by

$$(2.8)\qquad \Phi(t) = \int_{u>t} \left\{ \sum_{i,j} a_{ij}(x) u_{x_i} u_{x_j} - \sum_i b_i(x) u u_{x_i} \right\} dx.$$

Using the fact that $u \in H_0^1(\Omega)$ is a weak solution of (1.1) one obtains for a.e. $t > 0$:

$$(2.9)\qquad -\Phi'(t) = \int_{u>t} f(x)dx - \int_{u>t} c(x)udx.$$

Now let $Du = (u_{x_1},\dots,u_{x_n})$. The ellipticity condition (1.2.b) implies that

$$(2.10)\qquad \left(-\frac{d}{dt}\int_{u>t} |Du|\,dx\right)^2 \leqslant -\mu'(t)\left(-\frac{d}{dt}\int_{u>t} \sum a_{ij}(x) u_{x_i} u_{x_j}\,dx\right).$$

Arguing also as in [2] we have

$$(2.11)\qquad -\frac{d}{dt}\int_{u>t} \sum b_i(x) u u_{x_i}\,dx \leqslant -Bt \int_{u>t} |Du|\,dx$$

There three last inequalities give

$$(2.12)\qquad \frac{\left(-\frac{d}{dt}\int_{u>t} |Du|\,dx\right)^2}{(-\mu'(t))} - Bt\left(-\frac{d}{dt}\int_{u>t} |Du|\,dx\right) \leqslant \int_{u>t} f(x)dx - \int_{u>t} c(x)udx$$

We now need the following lemma, where the condition (1.2.d) comes in :

LEMMA 1. *Under the hypotheses* (1.2), (1.4) *we have for a.e.* $t > 0$

$$\int_{u>t} k(u)u\,dx \leqslant \int_{u>t} f(x)dx \tag{2.13}$$

Proof of the lemma. Because of (1.2.b) the quantity

$$\int_{u>t} \sum_{i,j} a_{ij}(x)u_{x_i}u_{x_j}\,dx$$

is nonincreasing in t. Therefore, it follows from (2.9) that

$$\frac{d}{dt}\int_{u>t}\left(\sum_i b_i u u_{x_i}\right)dx \leqslant \int_{u>t} f(x)dx - \int_{u>t} c(x)u\,dx. \tag{2.14}$$

On the other hand since $(h > 0)$

$$\left|\frac{1}{h}\int_{t<u\leqslant t+h}(\sum b_i u u_{x_i})dx - \frac{t}{h}\int_{t<u\leqslant t+h}(\sum b_i u_{x_i})dx\right| \leqslant \frac{B}{h}\int_{t<u\leqslant t+h} |Du|\,|u-t|\,dx \leqslant B\int_{t<u\leqslant t+h}|Du|\,dx\,,$$

and this expression tends to 0 with h, we obtain

$$\frac{d}{dt}\int_{u>t}(\sum_i b_i u u_{x_i})\,dx = t\,\frac{d}{dt}\int_{u>t}(\sum_i b_i u_{x_i})\,dx. \tag{2.15}$$

Now let us define the function $\varphi \in H_0^1(\Omega)$ as follows :

$$\varphi(x) = \begin{cases} 0 & \text{if} \quad u(x) \leqslant t, \\ u(x)-t & \text{if} \quad t < u(x) \leqslant t+h, \\ h & \text{if} \quad u(x) > t. \end{cases}$$

Then $\varphi \geqslant 0$ and using (1.4.b) we have

$$\frac{t}{h}\int_{t<u\leqslant t+h}(\sum b_i u_{x_i})dx = \frac{t}{h}\int_{\Omega}(\sum b_i \varphi_{x_i})\,dx =$$

$$= \frac{t}{h}\int_{\Omega}(\sum b_i \varphi_{x_i} - c(x)\varphi)\,dx + \frac{t}{h}\int_{\Omega} c(x)\varphi dx \leqslant \frac{t}{h}\int_{\Omega} -k(x)\varphi(x)dx + \frac{t}{h}\int_{\Omega} c(x)\varphi(x)dx.$$

As $h \to 0$ this expression tends for a.e. $t > 0$ to

$$\int_{u>t}(c(x)-k(x))t\,dx \leqslant \int_{u>t}(c(x)-k(x))u(x)dx.$$

Therefore

$$(2.16)\qquad -\frac{d}{dt}\int_{u>t}\left(\sum b_i u u_{x_i}\right)dx \leqslant \int_{u>t}(c(x)-k(x))u(x)dx.$$

From (2.14) the desidered result follows. ■

We return now to the proof of the Theorem. Consider the quadratic expression in Y :

$$(2.17)\qquad P(Y)=\frac{Y^2}{(-\mu'(t))}-BtY-\left(\int_{u>t}f(x)dx-\int_{u>t}k(x)u(x)dx\right).$$

It follows from the lemma that $P(0)\leqslant 0$. It also follows from (2.12) that $P(Y_1)\leqslant 0$ for $Y_1 = -\dfrac{d}{dt}\displaystyle\int_{u>t}|Du|\,dx$ since $c(x)\geqslant k(x)$ a.e. . It follows that $P(Y)\leqslant 0$ for every $Y\in[0,Y_1]$. On the other hand we have

$$(2.18)\qquad 0\leqslant nC_n^{1/n}(t)^{1\;1/n}\leqslant -\frac{d}{dt}\int_{u>t}|Du|\,dx$$

as a consequence of de Giorgi's isoperimetric inequality plus Fleming-Rishel's formula, cf. [13] . Hence, putting this value into (2.17) and using (2.7) results in

$$(2.19)\qquad \frac{(nC_n^{1/n}\mu(t)^{1-1/n})^2}{(-\mu'(t))}-Bt(nC_n^{1/n}\mu(t)^{1-1/n})+\int_0^{\mu(t)}k_*(s)u^*(s)ds\leqslant\int_0^{\mu(t)}f^*(s)ds.$$

Using the definition of u* (loosely speaking the inverse function of μ) we easily obtain from (2.19)

$$(2.20)\qquad -\frac{d}{ds}u^*(s)-\frac{B}{nC_n^{1/n}}s^{1/n-1}u^*(s)+\frac{s^{2/n-2}}{n^2C_n^{2/n}}\left[\int_0^s k_*(r)u^*(r)dr-\int_0^s f^*(r)dr\right]\leqslant 0.$$

This is the fundamental inequality satisfied by u*. Let us recall that since $u\in H_0^1(\Omega)$ then $u^\#\in H_0^1(\Omega^\#)$, cf. [6] , hence u* is absolutely continuous in $[\alpha, |\Omega|]$ $\forall\,\alpha>0$. On the other hand it is easy to see that the solution v to (1.5) satisfies for every $0<s<|\Omega|$

$$(2.21)\qquad -\frac{d}{ds}\hat{v}(s)-\frac{B}{nC_n^{1/n}}s^{(1/n)-1}\hat{v}(s)+\frac{s^{(2/n)-2}}{n^2C_n^{2/n}}\left[\int_0^s k(r)\hat{v}(r)dr-\int_0^s f^*(r)dr\right]=0,$$

where, obviously, $\hat{v}(C_n|y|^n)=v(y)$ for every $y\in\Omega^\#$. If we now define

$$(2.22)\qquad y(s)=\int_0^s(\hat{v}(r)-u^*(r))k_*(r)dr$$

the above formulas imply that y satisfies

$$
(2.23)\qquad \begin{cases} \left(\dfrac{y'}{k_*}\right)' + \dfrac{B}{nC_n^{1/n}}\, s^{1/n-1}\, \dfrac{y'}{k_*} - \dfrac{s^{2/n-2}}{n^2C_n^{2/n}}\, y \leqslant 0 \quad \text{for} \quad |\Omega_o| < s < |\Omega| \\[2ex] y(|\Omega_o|) = 0, \quad y'(|\Omega|) = 0, \end{cases}
$$

therefore, by the maximum principle we conclude that $y(s) \geqslant 0$ for every $s \in [|\Omega_o|, |\Omega|]$ and hence for $0 \leqslant s \leqslant |\Omega|$. This implies, cf. [6], that

$$
\int_{|\Omega_o|}^{s} u^*(s)ds \leqslant \int_{|\Omega_o|}^{s} \hat{v}(s)ds \;, \qquad |\Omega_o| \leqslant s \leqslant \Omega \,,
$$

from which $\hat{v}(|\Omega_o|) \geqslant u^*(|\Omega_o|)$ follows by continuity.

In case $|\Omega_o| > 0$ it follows from (2.20), (2.21) that the function $w(s) = v^*(s) - u^*(s)$ satisfies

$$
(2.24)\qquad \begin{cases} \dfrac{d}{ds} w(s) + \dfrac{B}{nC_n^{1/n}}\, s^{(1/n)-1}\, w(s) \leqslant 0, \qquad 0 < s < |\Omega_o|, \\[2ex] w(|\Omega_o|) \geqslant 0. \end{cases}
$$

From which it follows that $w(s) \geqslant 0$, i.e. $v^*(s) \geqslant u^*(s)$ in $[0, |\Omega_o|]$ or $u^{\#}(y) \leqslant v(y)$ for $|y| \leqslant R_o$.

Remark. Actually in the proof of the Theorem we obtain the estimate $y(s) \geqslant 0$, that can be reformulated as

$$
(2.25)\qquad \int_{|y| \leqslant r} k_{\#}(y) u^{\#}(y)dy \leqslant \int_{|y| \leqslant r} k_{\#}(y) v(y)dy \,, \quad 0 \leqslant r \leqslant R.
$$

(2.25) and (1.7) together imply (1.6).

3. - COMMENTS AND APPLICATIONS

3.1. The function k_* used in the theorem can be replaced by any nonnegative function $K : [0, |\Omega|] \to \mathbb{R}$ such that

$$(3.1)\qquad \int_0^s k_*(r)u^*(r)dr \geqslant \int_0^s K(r)u^*(r)ds$$

for every $0 \leqslant s \leqslant |\Omega|$ (substitute K for k_* from (2.20) onwards). (3.1) follows from the assumption

$$(3.2)\qquad \int_0^s k_*(r)dr \geqslant \int_0^s K(r)dr \ , \quad 0 < s < |\Omega|,$$

by integration by parts. Then Theorem 1 holds, v being the solution of (1.5) with $k_{\#}(y)$ replaced by $K(C_n |y|^n)$.

Of course we are interested in taking K as large as possible. In case $b_i = 0$ and

$$(3.3)\qquad 0 \leqslant c(x) \leqslant \gamma_1 \ , \quad \int_\Omega c(x)dx = \gamma_2$$

the function $K(s) = 0$ for $0 \leqslant s \leqslant s_o$, $K(s) = \gamma_1$ for $s_o \leqslant s$ where $|\Omega| - s_o = \gamma_2 / \gamma_1$ satisfies (3.2) is nondecreasing and has the same L^∞-norm as c. We obtain thus Chiti's result, [7].

3.2. Our result is not optimal in the following sense : typically the comparison involves a generic solution u of a class of problems and the solution v of a certain *worst case* in this class (see for instance [13], [1]). In our case the problem (1.5) does not belong to the class of problems (1.1), (1.2) because $b_i(x) = -Bx_i |x|^{-1}$ and $c(x) = k(x)$ do not satisfy (1.2.d). Therefore we are comparing with a *relaxed* problem.

3.3. As in [2] we may apply the above result to obtain a comparison result for a dual problem. In fact, let $w \in H_o^1(\Omega)$ be a weak solution of the problem

$$(3.4)\qquad \begin{cases} \hat{L}w = -\sum_{i,j=1}^{n} (a_{ij}(x)w_{x_i})_{x_j} \ \sum_{i=1}^{n} b_i w_{x_i} + c(x)w = g(x) & \text{in } \Omega \\ w = 0 \ \text{on } \partial\Omega \end{cases}$$

where the domain, coefficients and data satisfy the conditions of § 1 and let $z \in H_o^1(\Omega^{\#})$ be the (radially symmetric) weak solution of

$$(3.5)\qquad \begin{cases} -\Delta z + B \sum_{i=1}^{n} \frac{x_i}{|x|} z_{x_i} + k_{\#}(x)z = g^{\#}, \\ z = 0. \end{cases}$$

Then the following result holds :

THEOREM 2. *For every* $p \in [1,\infty]$ *we have*

$$\| w \|_{L^p(\Omega)} \leqslant \| z \|_{L^p(\Omega)} . \tag{3.6}$$

Proof. This is based on Theorem 1 and the duality between the operators L and $\hat{L}$, on the one hand, and between the first-member operators of equations (1.5) and (3.5), that we shall denote by Λ and $\hat{\Lambda}$, on the other hand.

Let $\hat{L}w = g$ as in (3.4) and let $f \in L^\infty$ and $g \in C_0^\infty(\Omega)$. If u, v, z are the solutions of (1.1), (1.5), (3.5) we have :

$$\int_\Omega w \,.\, f = \int_\Omega w \,.\, Lu = \int_\Omega \hat{L}w \,.\, u = \int_\Omega g \,.\, u = \int_0^{|\Omega|} g^* u^* =$$

$$= \int_0^{|\Omega|} - dg^*(s) \left(\int_0^s u^*(t)dt \right) \leqslant \int_0^{|\Omega|} - dg^*(s) \left(\int_0^s v(t)dt \right) = \int_0^{|\Omega|} g^* v =$$

$$= \int_{\Omega^\#} \hat{\Lambda} z \,.\, v = \int_{\Omega^\#} z \,.\, \Lambda v = \int_{\Omega^\#} z \,.\, f^\# \leqslant \int z^\# f^\# .$$

By density argument we have

$$\| w \|_p \leqslant \| z^\# \|_p = \| z \|_p \qquad \text{for} \quad p \in [1,\infty[. \tag{3.7}$$

In the limit we obtain (3.6) for $p = \infty$ ■

3.4. The above results apply to other related classes of elliptic equations. Let us point out here the case of *degenerate* elliptic equations.

Thus, if we take Theorem 1, the result (1.6) holds true even if we replace (1.2.b) by

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqslant \nu(x) \, |\xi|^2 \tag{3.8}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, where ν is a nonnegative measurable function in Ω such that

$$(1/\nu)(x) \in L^p(\Omega) \ \text{ for some } \ p \in (1,\infty). \tag{3.9}$$

(Existence and regularity of solutions of (1.1) under the above assumptions is studied for instance in [11], [14]).

Then we have to replace the term $-\Delta v$ in (1.5) by

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(\underline{\nu}(C_n\,|y|^n)\,\frac{\partial v(y)}{\partial y_i}\right), \tag{3.10}$$

where $\underline{\nu} : \Omega^{\#} \to [0,\infty)$ depends not only on ν but also on u. For the construction of $\underline{\nu}$ cf. [3], where the problem with c = 0 is treated. For the proof we repeat the argument in Section 2 with the necessary changes due to (3.8) that can be taken from [3].

3.5. Theorem 1 should be a good starting point for the study of second-order parabolic equations with lower-orden terms along the ideas of [16].

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