ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

M. MILLA MIRANDA

L.A. MEDEIROS

Hidden regularity for semilinear hyperbolic partial differential equations

Annales de la faculté des sciences de Toulouse 5e série, tome 9, nº 1 (1988), p. 103-120

<http://www.numdam.org/item?id=AFST_1988_5_9_1_103_0>

Annales Faculté des Sciences de Toulouse Vol. IX, n°1, 1988

Hidden Regularity for Semilinear Hyperbolic Partial Differential Equations

M. MILLA MIRANDA [1]and L.A. MEDEIROS [2]

RÉSUMÉ.— "Hidden regularity" est un concept introduit par J.L. LIONS dans [6] pour l'équation d'onde nonlinéaire $u_{tt} - \Delta u + |u|^\rho u = 0$. Dans ce travail, les auteurs obtiennent le même type de régularité pour l'équation $u_{tt} - \Delta u + F(u) = 0$ où $F: \mathbf{R} \mapsto \mathbf{R}$ satisfait les conditions de W. A. STRAUSS [9], c'est-à-dire, F est continue et $sF(s) \geq 0$. Dans §3 les auteurs développent certaines notions sur la trace de la dérivée normale.

ABSTRACT.— " Hidden Regularity" is a concept introduced by J.L. LIONS in [6], for the nonlinear wave equation $u_{tt} - \Delta u + |u|^\rho u = 0$. In the present work, the authors prove the same type of regularity for the equation $u_{tt} - \Delta u + F(u) = 0$ under the hypothesis of W. A. Strauss [9], i.e., $F: \mathbf{R} \mapsto \mathbf{R}$ is continuous and $sF(s) \geq 0$. In §3 The authors develop certain notions about the trace of the normal derivative.

1. Introduction

Let Ω be a bounded open set of $\mathbf{R}^n$ with smooth boundary Γ. By Q we represent the cylinder $\Omega \times]0, T[$, T an arbitrary positive real number. Let $F: \mathbf{R} \mapsto \mathbf{R}$ be a function satisfaying:

$$F \text{ is continuous and } sF(s) \geq \quad \text{for all } s \text{ in } \mathbf{R} \tag{1.1}$$

In the cylinder Q we consider the similinear hyperbolic equation:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + F(u) = 0 \text{ in } Q \tag{1.2}$$

(1) Partially supported by CNPq – Brasil
(2) Instituto di Matemática –UFRJ Caixa Postal 68 530 CEP 21944, Rio de Janeiro, RJ - - Brasil

with initial data

$$u(x,0) = u_0(x), \quad \frac{\partial}{\partial t}u(x,0) = u_1(x), \quad x \in \Omega. \tag{1.3}$$

Let us represent by G the function

$$G(s) = \int_0^s F(r)\,dr. \tag{1.4}$$

It was proved by STRAUSS [9], that if F satisfies (1.1) and

$$u_0 \in H_0^1(\Omega), \quad G(u_0) \in L^1(\Omega), \quad u_1 \in L^2(\Omega), \tag{1.5}$$

then the equation (1.2), with initial conditions (1.3), has one solution u such that

$$u \in L^\infty(0,T : H_0^1(\Omega)), \quad u' \in L^\infty(0,T;\, H^2(\Omega)), \tag{1.6}$$

and we have the energy inequality

$$E(t) \le E(0) = E_0. \tag{1.7}$$

The energy $E(t)$ is given by:

$$E(t) = \frac{1}{2}\int_\Omega [|u'(t)|^2 + |\nabla u(t)|^2]\,dx + \int_\Omega G(u(t))\,dx.$$

In this paper we prove the following.

THEOREM 1.1.— *If we assume (1.1), (1.5), then the equation (1.2) has one solution u satisfying (1.6) the initial conditions (1.3), and*

$$\frac{\partial u}{\partial \nu} \in L^2(\Sigma) \tag{1.8}$$

such that:

$$\|\frac{\partial u}{\partial \nu}\|_{L^2(\Sigma)} \le CE_0, \dagger \tag{1.9}$$

† By ν we re present the normal to Γ, directed towards the exterior of Ω and by $\Sigma = \Gamma \times]0,T[$ the lateral boundary of the cylindar Q. By $\|v\|_{L^2(\Sigma)}$ we represent the norm of v in $L^2(\Sigma)$, that is,

$$\|v\|_{L^2(\Sigma)} = \left(\int_0^T \int_\Gamma |v|^2\,d\Gamma\,dt\right)^{1/2}$$

the constant C depending only on T and Ω.

Remark 1.1.— From the properties (1.6) of the solutions u of (1.2) given by Theorem 1.1 we shall prove that:

$$\frac{\partial u}{\partial \nu} \in L^1\left(0,T;W^{(1/p)-2,p'}(\Gamma)\right) + H^{-1}\left(0,T;H^{(1/2)}(\Gamma)\right). \tag{1.10}$$

For the proof, look Proposition 3.4, §3 of this paper, where $p = 2$ if $n = 1,2,3$, $p < \dfrac{n}{2}$ if $n \geq 4$, and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Observe that (1.8) does not follows from (1.10). This phenomenom was denominated "*Hidden Regularity*" by LIONS, cf. [6], for the case $F(s) = |s|^\rho s$. We also can find results of hidden regularity motivated by problems of optimal control in LIONS [5], for the linear case of (1.2), i.e., $F = F(x,t)$.

In §2 we give the proof of Theorem 1.1. In §3 we study the trace of the normal derivative $\dfrac{\partial u}{\partial \nu}$ for functions u that belong to the space

$$E = \{v;\ v \in L^{p'}(\Omega),\ \Delta v \in L^1(\Omega)\},$$

p' as in the Remark 1.1. We did not find in the literature this direct proof.

2. Proof of Theorem 1.1

We will represent by $(\,,\,)$, $|\cdot|$ and $((\,,\,))$, $\|\cdot\|$ the inner product and norm, respectively, in $L^2(\Omega)$ and $H_0^1(\Omega)$. By $H_0^1(\Omega)$ we represent the Sobolev space of order one whose functions have trace zero on the boundary of Ω and by $L^2(\Omega)$ the space of square integrable numerical functions on Ω. All functions considered in this paper are real valued. The existence proof follows the idea of STRAUSS [9].

Suppose u_0, u_1 given by (1.5). For each natural number j, let us consider the function $\beta_j\colon \mathbf{R} \mapsto \mathbf{R}$ defined by:

$$\beta_j(s) = s \text{ if } |s| \leq j; \quad \beta_j(s) = j \text{ if } s > j; \quad \beta_j(s) = -j \text{ if } s < -j.$$

It follows by KINDERLEHRER-STAMPACCHIA [2], that $\beta_j(u_0) = u_{0j}$ belongs to $H_0^1(\Omega)$ for all $j \in \mathbf{N}$.

Let F and G as in (1.1), (1.4) and represent by F_k the Strauss approximation of F, that is, F_k, $k \in \mathbf{N}$, is a continuous function defined by:

$$\begin{cases} F_k(s) = (-k)\left[G\left(s - \frac{1}{k}\right) - G(s)\right] & \text{if } -k \leq s \leq -\frac{1}{k} \\ F_k(s) = k\left[G\left(s + \frac{1}{k}\right) - G(s)\right] & \text{if } \frac{1}{k} \leq s \leq k \\ F_k(s) \text{ linear by parts} & \text{in } -\frac{1}{k} < s \leq \frac{1}{k} \text{ with } F_k(0) = 0 \\ F_k(s) \text{ appropriate constants} & \text{for } |s| > k. \end{cases} \tag{2.1}$$

It follows by STRAUSS [9], COOPER-MEIDEROS [1], that F_k is Lipschitz for each k, $sF_k(s) \geq 0$ and (F_k) converges to F uniformly on the compacts subsets of $\mathbf{R}$. Represent by $G_k(s) \int_0^s F_k(r)\,dr$ and we obtain $G_k(0) = F_k(0)$, for all $k \in \mathbf{N}$.

Since $u_{0j} \in H_0^1(\Omega)$, let $(\varphi_{\mu j})_{\mu \in \mathbf{N}}$ and $(\Psi_\mu)_{\mu \in \mathbf{N}}$ two sequences of elements of $\mathcal{D}(\Omega)$, space of C^∞ functions with compact support in Ω, such that

$$\varphi_{\mu j} \to u_{0j} \text{ in } H_0^1(\Omega), \quad \Psi_\mu \to u_1 \text{ in } L^2(\Omega), \quad \text{as } \mu \to \infty. \tag{2.2}$$

It follows by the above hypothesis, that there exists only one function $u_{\mu jk}$ which we represent by u such that:

$$\begin{cases} u \in L^\infty\left(0,T; H^2(\Omega) \cap H_0^1(\Omega)\right) \\ u' \in L^\infty\left(0,T; H_0^1(\Omega)\right) \\ u'' \in L^2\left(0,T; L^2(\Omega)\right) \end{cases} \tag{2.3}$$

and u is a weak solution of the problem

$$\begin{cases} u'' - \Delta u + F_k(u) = 0 \\ u(0) = \varphi_{\mu j}, \quad u'(0) = \Psi_\mu. \end{cases} \tag{2.4}$$

The energy identity is verified by the solution $u = u_{\mu jk}$, i.e.,

$$\frac{1}{2}|u'(t)|^2 + \frac{1}{2}\|u(t)\|^2 + \int_\Omega G_k(u(t))\,dx = \frac{1}{2}|\Psi_\mu|^2 + \frac{1}{2}\|\varphi_{\mu j}\|^2 + \int_\Omega G_k(\varphi_{\mu j})\,dx. \tag{2.5}$$

This result can be found in LIONS [4], STRAUSS [8].

The next step is to prove that $(u_{\mu jk})$ converges to a solution of the initial value problem (1.2), (1.3) and the conditions (1.8), (1.9) are verified. So, we divide the proof in two parts. First on the existence of solutions and second on the estimate (1.9) of the normal derivative.

1..— Existence of Solutions

In this step we bounded the second member of (2.5) by a constant independent of μ, j and k. We obtain:

$$\|u_0 j\| \leq \|u_0\| \tag{2.6}$$

and

$$\int_\Omega G_k(\varphi_{\mu j})\,dx \;\to\; \int_\Omega G_k(u_{0j})\,dx \quad \text{as } \mu \to \alpha. \tag{2.7}$$

We have:

$$G_k(u_{0j}(x)) \;\to\; G(u_{0j}(x)), \quad k \to \infty, \quad \text{uniformly a.e. in } \Omega,$$

hence there exists a subsequence (G_{k_j}) of (G_k), which is denoted by (G_j), such that:

$$\int_\Omega |G_j(u_{0j}) - G(u_{0j})|\,dx \to 0, \quad \text{as } j \to \infty. \tag{2.8}$$

We also have $G(u_{0j}) \to G(u_0)$ a.e. in Ω and $G(u_{0j}) \leq G(u_0)$. As $G(u_0) \in L^1(\Omega)$, we have then

$$\int_\Omega |G(u_{0j}) - G(u_0)|\,dx \to 0, \quad \text{as } j \to \infty. \tag{2.9}$$

From (2.8) and (2.9) we obtain

$$\int_\Omega G_j(u_{0j})\,dx \to \int_\Omega G(u_0)\,dx \quad \text{as } j \to \infty. \tag{2.10}$$

Thus, from the convergences (2.2), (2.7), (2.10) and the property (2.6), it follows that for every $\varepsilon > 0$, the energy equality (2.5) can be estimated as follows:

$$\frac{1}{2}|u'_{\mu j}(t)|^2 + \frac{1}{2}\|u_{\mu j}(t)\|^2 + \int_\Omega G_j(u_{\mu j}(t))\,dx \leq E_0 + \varepsilon \tag{2.11}$$

for all $t \in [0,T]$ and $\mu \geq \mu_0$, $j \leq j_0$, where E_0 is defined by (1.7).

It follows from the estimate (2.11) that there exists subsequences $(u_{\mu j})$, (u_j) and a function u such that:

$$\begin{cases} u_{\mu j} \to u_j & \text{in } L^\infty(0,\,T;\,H_0^1(\Omega)) \text{ weak star as } \mu \to \infty \\ u'_{\mu j} \to u'_j & \text{in } L^\infty(0,\,T;\,L^2(\Omega)) \text{ weak star as } \mu \to \infty, \end{cases} \tag{2.12}$$

and

$$\begin{cases} u_j \to u & \text{in } L^\infty(0,\, T;\; H_0^1(\Omega)) \text{ weak star} \\ u_j' \to u' & \text{in } L^\infty(0,\, T;\; L^2(\Omega)) \text{ weak star .} \end{cases} \tag{2.13}$$

Taking limits in the approximated system (2.4) and using the convergences (2.12), we obtain:

$$\begin{cases} u'' - \Delta u_j + F_j(u_j) = 0 \\ u_j(0 = u_{0j}, \quad u_j'(0) = u_1. \end{cases} \tag{2.14}$$

Remark 2.1.— Strauss proved in [9], cf. Lions [4], Lemma 1.3, a convergence theorem for sequence of measurable functions, which permit us to pass to the limit in (2.14). This result says that if $F_j(u_j) \to F(u)$ a.e. in Q and

$$\int_0^T (F_j(u_j), u_j)\, dt < C, \forall j \tag{2.15}$$

then

$$f_j(u_j) \to F(u) \text{ strongly in } L^1(Q). \tag{2.16}$$

Let us apply the result of Remark 2.1 in order to obtain the limit of (2.14) as $j \to \infty$. It is sufficient to verify the conditions (2.15). In fact, from (2.14) we obtain

$$\int_0^T (F_j(u_j), u_j)\; dt = (u_1, u_{0j}) - \big(u_j'(T), u_j(T)\big) + \int_0^T |u_j'|^2\, dt - \int_0^T \|u_j\|^2\, dt,$$

which by the inequality (2.11) is bounded by a constant independent of j, thus conditions of Remark 2.1 are verified.

Therefore, it is permissible to pass to the limit in (2.14) and obtain a solution u of (1.2). To verify the initial conditions (1.3) we use the usual argument, as in Lions [4], Strauss [9].

2..— Estimates for the Normal Derivative

The method used in this section is one applied by Lions [6]. First of all we prove a Lemma. Note that we use in ntis section the summation convention, i.e., terms like $h_i \dfrac{\partial u}{\partial x_i}$ means summation in i from one to n.

We consider functions h_i such that

$$h_i \in C^2(\overline{\Omega}) \text{ and } h_i = \nu_i \text{ on } \Gamma, \qquad i = 1, 2, \ldots, n.$$

LEMMA 2.1.— *Let $w \in H^2(\Omega) \cap H_0^1(\Omega)$. Then*

$$\frac{\partial w}{\partial x_i} = \nu_i \frac{\partial w}{\partial \nu}, \qquad \nu = (\nu_1, \nu_2, \dots, \nu_n).$$

From this it follows that $|\nabla w|^2 = \left(\frac{\partial w}{\partial \nu}\right)^2$.

Proof.— In fact, let $\xi \in \mathcal{D}(\Gamma)$ and let $\overline{\xi} \in H^m(\Omega)$, with $m > \max(n/2, 2)$, such that trace $\gamma_0 \overline{\xi}$ on Γ is ξ. We know that $\overline{\xi}$ exists because $\mathcal{D}(\Gamma) \subset H^{m-1/2}(\Gamma)$. We have:

$$\int_\Omega \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}(w h_j \overline{\xi})\, dx = \int_\Gamma \nu_i \frac{\partial}{\partial x_j}(w h_j \overline{\xi})\, d\Gamma = \int_\Gamma \nu_i \frac{\partial w}{\partial \nu} \xi \, d\Gamma$$

for all $\xi \in \mathcal{D}(\Gamma)$. Note that Ω is regular. We also obtain

$$\int_\Omega \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i}(w h_j \overline{\xi}) = \int_\Gamma \nu_j \frac{\partial}{\partial x_i}(w h_j \overline{\xi})\, d\Gamma = \int_\Gamma \nu_j \frac{\partial w}{\partial x_i} \nu_j \xi = \int_\Gamma \frac{\partial w}{\partial x_i} \xi \, d\Gamma.$$

It follows that

$$\int_\Gamma \nu_i \frac{\partial w}{\partial \nu} \xi \, d\Gamma = \int_\Gamma \frac{\partial w}{\partial x_i} \xi \, d\Gamma \text{ for all } \xi \in \mathcal{D}(\Gamma)$$

which implies the proof of Lemma 2.1.

Let $u_{\mu j} = u$ in the class (2.3) which is the solution of (2.4). We use $F_j = F$, $G_j = G$ to simplify the notation. Multiply both terms of (2.4) by $h_i \frac{\partial u}{\partial x_i}$ and integrate on Q, which is permissible. We obtain:

$$\int_Q u'' h_i \frac{\partial u}{\partial x_i}\, dx\, dt - \int_Q \Delta u \cdot h_i \frac{\partial u}{\partial x_i}\, dx\, dt + \int_Q (u) h_i \frac{\partial u}{\partial x_i}\, dx\, dt = 0. \quad (2.15)$$

We obtain:

$$\int_Q u'' h_i \frac{\partial u}{\partial x_i}\, dx\, dt = N - \frac{1}{2} \int_Q h_i \frac{\partial}{\partial x_i}(u')^2\, dx\, dt,$$

where

$$N = \left| \int_\Omega u' h_i \frac{\partial u}{\partial x_i}\, dx \right|_{t=0}^{t=T};$$

$$\int_Q h_i \frac{\partial}{\partial x_i}(u')^2\, dx\, dt = \int_\Sigma h_i (u')^2 \nu_i \, d\Sigma - \int_Q (u')^2 \frac{\partial h_i}{\partial x_i}\, dx\, dt,$$

consequently, observing that $u'(t) \in H_0^1(\Omega)$ in $]0,T[$, we have:

$$\int_Q u'' h_i \frac{\partial u}{\partial x_i}\, dx\, dt = N + \frac{1}{2}\int_Q (u')^2 \frac{\partial h_i}{\partial x_i}\, dx\, dt. \tag{2.16}$$

We also obtain:

$$-\int_Q \Delta u h_i \frac{\partial u}{\partial x_i}\, dx\, dt = \int_Q \frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\left(h_i \frac{\partial u}{\partial x_i}\right) dx\, dt - \int_\Sigma \left(\frac{\partial u}{\partial \nu}\right)^2 d\Sigma \tag{2.17}$$

and

$$\begin{aligned}
&\int_Q \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}\left(h_i \frac{\partial u}{\partial x_i}\right) dx\, dt = \\
&\quad = \frac{1}{2}\int_Q h_i \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right)^2 dx\, dt + \int_Q \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial h_i}{\partial x_j}\, dx\, dt = \\
&\quad = \frac{1}{2}\int_\Sigma h_i \left(\frac{\partial u}{\partial x_j}\right)^2 \nu_i\, d\Sigma - \frac{1}{2}\int_Q \left(\frac{\partial u}{\partial x_j}\right)^2 \frac{\partial h_i}{\partial x_i}\, dx\, dt + \\
&\qquad\qquad + \int_Q \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial h_i}{\partial x_j}\, dx\, dt.
\end{aligned}$$

Applying Lemma 2.1 to the first integral in second member of the above inequality, we get:

$$\begin{aligned}
&-\int_Q \Delta u h_i \frac{\partial u}{\partial x_i}\, dx\, dt = \\
&= -\frac{1}{2}\int_\Sigma \left(\frac{\partial u}{\partial \nu}\right) d\Sigma - \frac{1}{2}\int_Q |\nabla u|^2 \frac{\partial h_i}{\partial x_i}\, dx\, dt + \int_Q \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial h_i}{\partial x_j}\, dx\, dt. \quad (2.18\ )
\end{aligned}$$

Nothing that G is of class C^1 with bounded derivative and $G(0) = 0$, i.e., $G(u) \in L^2(0,\ T;\ H_0^1(\Omega))$, we obtain

$$\int_Q F(u) h_i \frac{\partial u}{\partial x_i}\, dx\, dt = -\int_Q G(u)\frac{\partial h_i}{\partial x_i}\, dx\, dt. \tag{2.19}$$

Substituting (2.16), (2.18), (2.19) in (2.15) we obtain:

$$\begin{aligned}
\frac{1}{2}\int_\Sigma \left(\frac{\partial u}{\partial \nu}\right)^2 d\Sigma = N + \int_Q \frac{\partial h_i}{\partial x_i}\left[\frac{1}{2}u'^2 - \frac{1}{2}|\nabla u|^2 - G(u)\right] dx\, dt + \\
+ \int_Q \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial h_i}{\partial x_j}\, dx\, dt.
\end{aligned}$$

Using the inequality (2.11) in the second member of the last equality, we get:

$$\int_{\Sigma}\left(\frac{\partial}{\partial\nu}u_{\mu j}\right)^2 d\Sigma \le C(E_0+\varepsilon),$$

for all $\mu \ge \mu_0$, $j \ge j_0$, $\varepsilon > 0$, where $C > 0$ does not depend of μ, j, E_0, ε. Thus, we obtain a subsequence, still represented by $(u_{\mu j})$, such that

$$\frac{\partial}{\partial\nu}u_{\mu j} \to \chi \quad \text{weakly in } L^2(0,T;L^2(\Gamma)) \text{ as } \mu, j \to \infty, \tag{2.20}$$

and as ε is arbitrary, we have:

$$\|\chi\|_{L^2(\Sigma)} \le CE_0. \tag{2.21}$$

In the next section we will prove that $\frac{\partial}{\partial\nu}u_j \to \frac{\partial u}{\partial\nu}$ weak star in

$$\left(L^\infty\left(0,T;W^{2-\frac{1}{2},p}(\Gamma)\right)\right)\bigcap H_0^1\left(0,T;H^{-\frac{1}{2}}(\Gamma)\right)',$$

for $p \ge 2$ and $p > \frac{n}{2}$. We note that by W' we represent the topological dual of W. It follows from (2.20) that $\left(\frac{\partial}{\partial\nu}u_j\right)$ converges to χ in the above space and consequently $\chi = \frac{\partial u}{\partial\nu}$. This fact and (2.21) complete the proof of Theorem 1.1. QED

3. Trace of Normal Derivative

We summarize this section as follows. First of all we prove that $-\Delta u = -F(u)-u''$ can be written in the form $-\Delta u = \Delta y + \Delta z'$; then we show that $u = -y - z'$; we prove that the traces of $\frac{\partial y}{\partial\nu}, \frac{\partial z'}{\partial\nu}$ are defined, consequently the trace of $\frac{\partial u}{\partial\nu}$ is defined, and to complete the argument we obtain the convergence of $\frac{\partial}{\partial\nu}u_j = -\frac{\partial}{\partial\nu}y_j - \frac{\partial}{\partial\nu}z_j'$ to $\frac{\partial u}{\partial\nu} = -\frac{\partial y}{\partial\nu} - \frac{\partial z'}{\partial\nu}$ in an appropriate space which contains $L^2(\Sigma)$, equipped with the weak topology. The main difficult in this procedure is because the nonlinear term $F(u)$ belongs to $L^1\left(0,\, T;\, L^1(\Omega)\right)$.

In order to have a better notation we represent by $\gamma_0 w$, $\gamma_1 w$ the traces, respectively, of the function w and of its normal derivative, as is done usually.

By the symbol (f, g) we still represent the integral on Ω of fg and by $\langle f, g\rangle$ the duality pairing between W and its topological dual W'. In all this section the numbers p, p' satisfy the conditions:

$$p = 2 \text{ if } n = 1, 2, 3; \quad p > \frac{n}{2} \text{ if } n \geq 4; \quad \text{and } \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.1}$$

It follows that $W^{2,p}(\Omega)$ is continuously embedded in $C(\overline{\Omega})$.

The begin we prove the existence of solutions for the problem:

$$\begin{cases} -\Delta y = f & \text{in } \Omega, \text{ with } f \in L^1(\Omega) \\ y = 0 & \text{on } \Gamma. \end{cases}$$

This will be proved by transposition method, cf. Lions [3] and Lions-Magenes [7].

Proposition 3.1.— *If $f \in L^1(\Omega)$, there exist only one function $y \in L^{p'}(\Omega)$ such that*

$$\int_\Omega y(-\Delta w)\, dx = \int_\Omega fw\, dx, \quad \textit{for each } w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{3.2}$$

The application $Tf = y$ from $L^1(\Omega)$ in $L^{p'}(\Omega)$ is linear and continuous and $-\Delta y = f$.

Proof.— Let $h \in L^p(\Omega)$ and w be the solution of the problem:

$$\begin{cases} -\Delta w = h & \\ w = 0 & \text{on } \Gamma. \end{cases} \tag{3.3}$$

Then, by the regularity of the solutions of elliptic equations, it follows that $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Let S be the application

$$Sh = w \text{ from } L^p(\Omega) \text{ in } C(\overline{\Omega}),$$

where w is the solution of (3.3). Then S is linear and continuous. Let S^* be the transpose of S, that is:

$$S^*\colon (C(\overline{\Omega}))' \to L^{p'}(\Omega).$$

The function $y = S^*f$ satisfies the conditions (3.2). In fact, $\langle S^*f, h\rangle = \langle f, Sh\rangle$, that is:

$$\int_\Omega y(-\Delta w)\, dx = \int_\Omega fw\, dx.$$

To prove the uniqueness, let y_1, y_2 in $L^{p'}(\Omega)$ satisfying (3.2). Then

$$\int_\Omega (y_1 - y_2)(-\Delta w)\,dx = 0, \quad \text{for all } w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Let $h \in L^p(\Omega)$ and w solution of problem (3.3). Then by the last equality we have:

$$\int_\Omega (y_1 - y_2) h\,dx = \qquad \text{for all } h \in L^p(\Omega),$$

which implies $y_1 = y_2$. Therefore, the uniqueness is proved.

Since $T = S^*$ on $L^1(W)$ and S^* is linear and continuous, it follows that T has the same properties. QED

Let us represent by E the Banach space

$$E = \{v \in L^{p'}(\Omega);\ \Delta v \in L^1(\Omega)\}$$

with the norm:

$$\|v\|_E = \|v\|_{L^{p'}(\Omega)} + \|\Delta v\|_{L^1(\Omega)}.$$

The next step in our argument, is to prove that $\gamma_1 v$ is defined for all $v \in E$. We follows the usual method, as in LIONS [3]. By $\mathcal{D}(\overline{\Omega})$ we represent the restrictions of the test functions φ of $\mathcal{D}(\mathbf{R}^n)$ to the bounded open set Ω.

LEMMA 3.1.— *$\mathcal{D}(\overline{\Omega})$ is dense in E.*

Proof.— Let $M \in E'$ be such that $M\varphi = 0$ for all $\varphi \in \mathcal{D}(\overline{\Omega})$.

We must prove that $M = 0$.

We can consider E as a closed subspace of $L^{p'}(\Omega) \times L^1(\Omega) = \mathcal{W}$. Let $\widetilde{M}$ be the continuous linear extension of M to $\mathcal{W}$. Then there exists $f \in L^p(\Omega)$ and $h \in L^\infty(\Omega)$ such that

$$\widetilde{M}([\xi, \eta]) = (f, \xi) + (h, \eta) \quad \text{for all } [\xi, \eta] \in \mathcal{W}.$$

In particular,

$$Mv = (f, v) + h, \Delta v) \qquad \text{for all } v \in E. \tag{3.4}$$

Let $\tilde{f}$ and $\tilde{h}$ be the extension of f and h to $\mathbf{R}^n$, zero outside Ω. Consider $\emptyset \in \mathcal{D}(\mathbf{R}^n)$ and let $\varphi = \emptyset$ on $\overline{\Omega}$. By (3.4) we have:

$$\int_{\mathbf{R}^n} [\tilde{f} + \Delta\tilde{h}]\emptyset\,dx = \int_{\mathbf{R}^n} \tilde{f}\emptyset\,dx + \int_{\mathbf{R}^n} \tilde{h}\Delta\emptyset\,dx = \int_\Omega f\varphi\,dx + \int_\Omega h\Delta_\varphi\,dx = 0$$

that is,

$$\widetilde{f} + \Delta \widetilde{h} = 0. \tag{3.5}$$

We have $\widetilde{f} \in L^p(\mathbf{R}^n)$, $\widetilde{h} \in L^p(\mathbf{R}^n)$ and by (3.5) $-\Delta\widetilde{h} = \widetilde{f}$. As $p \geq 2$ and Ω is bounded it follows that

$$\widetilde{f} \in L^2(\mathbf{R}^n), \quad \widetilde{h} \in L^2(\mathbf{R}^n) \quad \text{and} \quad -\Delta\widetilde{h} = \widetilde{f}.$$

By Fourier transform we obtain $\widetilde{h} \in H^2(\mathbf{R}^n)$. As Ω is regular, we deduce from there that

$$h \in H_0^2(\Omega). \tag{3.6}$$

Consider an open ball B of $\mathbf{R}^n$ which contains Ω. Let w be the solution of the problem:

$$\begin{cases} -\Delta w = \widetilde{f} & \text{in } B \\ w = 0 & \text{on the boundary of } B. \end{cases} \tag{3.7}$$

As $\widetilde{f} \in L^p(B)$, $\widetilde{f}$ restrict to B, it follows by the regularity theorem, that

$$w \in W^{2,p}(B) \cap W_0^{1,p}(B). \tag{3.8}$$

By (3.6) and (3.8) it follows that

$$\widetilde{h}, w \in H^2(B) \cap H_0^1(B)$$

and by (3.5), (3.7) these both functions are solutions of

$$\begin{cases} -\Delta z = \widetilde{f} & \text{in } B \\ z = 0 & \text{on the boundary of } B. \end{cases}$$

By the uniqueness result, we have $\widetilde{h} = w$. It then follows that $\widetilde{h}$ belongs to $W^{2,p}_{\text{loc}}(\mathbf{R}^n)$, hence

$$h \in W_0^{2,p}(\Omega). \tag{3.9}$$

Let $v \in E$. By (3.9) there exists a sequence (φ_ν) of elements of $\mathcal{D}(\Omega)$ such that (φ_ν) converges to h in $W_0^{2,p}(\Omega)$. By the limits in $(\varphi_\nu, \Delta v) = (\Delta\varphi_\nu, v)$ it follows that

$$(h, \Delta v) = (\Delta h, v). \tag{3.10}$$

Substituting (3.10) in (3.4) and observing that $f + \Delta h = 0$ in Ω we prove that $M = 0$. QED

PROPOSITION 3.2.— *There exists an application $\gamma v = [\gamma_0 v, \gamma_1 v]$ from E to $W^{\frac{1}{2}-1,p'}(\Gamma) \times W^{\frac{1}{2}-2,p'}(\Gamma)$, linear and continuous such that :*

$$\gamma\varphi = \left[\varphi\Big|\Gamma, \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma}\right], \quad \textit{for all } \varphi \textit{ in } \mathcal{D}(\overline{\Omega}).$$

Proof.— We shall use the notation:

$$X = W^{2-\frac{1}{p},p}(\Gamma), \quad Y = W^{1-\frac{1}{p},p}(\Gamma), \quad Z = X \times Y.$$

By the trace theorem, LIONS [3], for each $[\xi,\eta] \in Z$, there exists a function $w \in W^{2,p}(\Omega)$ such that $\gamma_0 w = \xi$ and $\gamma_1 w = \eta$. The condition (3.1) implies by Sobolev theorem that $W^{2,p}(\Omega)$ is continuously embedded in $C(\overline{\Omega})$. For each $v \in E$ we define the functional T_v on Z by

$$T_v([\xi,\eta]) = (v, \Delta w) - (\Delta v, w). \tag{3.11}$$

It is easy to show that T_v is well defined. We have:

$$|T_v[\xi,\eta]| \leq C\,[\|v\|_{L^{p'}\Omega} + \|\Delta v\|_{L^1\Omega}]\,[\|\xi\|_X + \|\eta\|_Y] = C\|v\|_E\|[\xi,\eta]\|_Z.$$

Thus,

$$T_v \in Z' \quad \text{and} \quad \|T_v\|_{Z'} \leq C\|v\|_E. \tag{3.12}$$

Let $\varphi \in \mathcal{D}(\overline{\Omega})$. By the definition (3.11) and Green formula, we obtain:

$$T_\varphi([\xi,\eta]) = \langle\gamma_0\varphi, \eta\rangle - \langle\gamma_1\varphi, \xi\rangle = \langle[-\gamma_1\varphi, \gamma_0\varphi], [\xi,\eta]\rangle.$$

From (3.12) and (3.13) it follows that we have established an application σ given by:

$$\sigma(\varphi) = [-\gamma_1\varphi, \gamma_0\varphi] \quad \text{from } \mathcal{D}(\overline{\Omega}) \text{ to } X' \times Y',$$

linear and continuous, where $\mathcal{D}(\overline{\Omega})$ is equipped with the topology induced by that one of E.

Let τ be the application

$$\tau([-\gamma_1\varphi, \gamma_0\varphi]) = [\gamma_0\varphi, \gamma_1\varphi] \quad \text{from } Z' \text{ to } Z'.$$

Since $\mathcal{D}(\overline{\Omega})$ is dense in E, it follows that the extension of $\gamma = \tau \cdot \sigma$ to E satisfies the conditions of Propsição 3.2. QED

Let u be the solution obtained in Theorem 1.1. Then

$$-\Delta u = -F(u) - u'',$$

with $F(u) \in L^1\left(0,T;L^1(\Omega)\right)$ and $u' \in L^2\left(0,T;L^2(\Omega)\right)$. By the Propositions 3.1, 3.2 and regularity Theorem for elliptic equations, it follows that there exists unique functions

$$y \in L^1\left(0,T;L^{p'}(\Omega)\right) \text{ and } z \in L^2\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right) \tag{3.14}$$

such that

$$-\Delta y = F(u) \text{ and } -\Delta z = u'.$$

Consequently

$$-\Delta u = \Delta y + (\Delta z)'. \tag{3.15}$$

Remark 3.1.— Let $v \in L^{p_1}(0,T;L^{q_1}(\Omega))$, $\Delta v \in L^{p_2}(0,T;L^{q_2}(\Omega))$, $1 \le p_i,\ q_i \le \infty$, $i=1,2$. Then

$$\Delta \int_0^T v(t)\theta(t)\,dt = \int_0^T (\Delta v)(t)\theta(t)\,dt \quad \text{for all } \theta \in \mathcal{D}(0,T).$$

This is a consequence or the fact that $\langle \Delta v, \theta\varphi\rangle = \langle v, \theta\Delta\varphi\rangle$ for all $\varphi \in \mathcal{D}(\Omega)$.

PROPOSITION 3.3.— *We have*

$$u = -y - z'$$

where y,z are defined by (3.15).

Proof.— We observe that $u \in L^\infty\left(0,T;H_0^1(\Omega)\right)$. Let $\theta \in \mathcal{D}(0,T)$. By Remark 3.1 and nothing that $-\Delta$ is continuous from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, we obtains from (3.15):

$$-\Delta \int_0^T u\theta\,dt = \Delta \int_0^T y\theta\,dt - \Delta \int_0^T z\theta'\,dt,$$

that is

$$-\Delta\left[\int_0^T u\theta\,dt - \int_0^T z\theta'\,dt\right] = -\Delta \int_0^T (-y)\theta\,dt. \tag{3.16}$$

Let us consider:

$$U = \int_0^T u\theta\, dt - \int_0^T z\theta'\, dt, \quad V = \int_0^T (-y)\theta\, dt \quad \text{and } f = (-\Delta)V.$$

We want to show that U and V are solutions of the problem

$$\begin{cases} v \in L^{p'}(\Omega) \\ \displaystyle\int_\Omega v(-\Delta w)\, dx = \int_\Omega fw\, dx, \quad \text{for all } w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \end{cases} \tag{3.17}$$

Therefore, by the uniqueness given by Proposition 3.1, we get $U = V$ and we have the proof of the Proposition 3.3.

We have $U \in L^{p'}(\Omega)$ because $p \geq 2$. Let $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Then

$$\int_\Omega U(-\Delta w)\, dx = \int_0^T \theta \left[\int_\Omega u(-\Delta w)\, dx\right] dt - \int_0^T \theta' \left[\int_\Omega z(-\Delta w)\, dx\right] dt.$$

As $-\Delta u \in W^{-1,p'}(\Omega)$, because $p \geq 2$, it follows that $\langle -\Delta u, w\rangle = (u, -\Delta w)$. Also, $\langle -\Delta z, w\rangle = (z, -\Delta w)$. Therefore,

$$\int_\Omega U(-\Delta w)\, dx = \left\langle \int_0^T (-\Delta u)\theta\, dt, w \right\rangle - \left\langle \int_0^T (-\Delta z)\theta', w \right\rangle.$$

From (3.15) and Remark 3.1, it follows:

$$\int_\Omega U(-\Delta x)\, dx = \int_\Omega fw\, dx.$$

It is clear that V is a solution of (3.17). Then U, V are solutions of (3.17) which proves Proposition 3.3. QED

To know in what space $\gamma_1 u$ is localized we need the following lemma.

LEMMA 3.2.— *We have* $\gamma_1 z' = (\gamma_1 z)'$.

Proof.— The set

$$\{\alpha\varphi\,;\, \alpha \in \mathcal{D}(0,T)\,,\, \varphi \in \mathcal{D}(\overline{\Omega})\}$$

is total in $L^2\left(0,T; H^2(\Omega)\right)$. Then there exists a sequence (z_μ) such that

$$z_\mu = \sum_{j=1}^{\mu} \alpha_{j\mu}\varphi_{j\mu} \to z \text{ in } L^2\left(0,T; H^2(\Omega)\right). \tag{3.18}$$

We have then

$$\int_0^T z_\mu \theta' \, dt \to \int_0^T z\theta' \, dt \quad \text{in } H^2(\Omega), \quad \theta \in \mathcal{D}(0,T)$$

whence

$$\gamma_1 \int_0^T z_\mu \theta' \, dt \to \gamma_1 \int_0^T z\theta' \, dt \quad \text{in } H^{1/2}(\Gamma). \tag{3.19}$$

Also, by (3.18) we have:

$$\gamma_1 z_\mu \to \gamma_1 z \quad \text{in } L^2\left(0,T; H^{1/2}(\Gamma)\right)$$

therefore,

$$\int_0^T (\gamma_1 z_\mu)\theta' \, dt \to \int_0^T (\gamma_1 z)\theta' \, dt \quad \text{in } H^{1/2}(\Gamma). \tag{3.20}$$

From (3.19), (3.20), since

$$\gamma_1 \int_0^T z_\mu \theta' \, dt = \int_0^T (\gamma_1 z_\mu)\theta' \quad \text{for all } \theta' \in \mathcal{D}(0,T).$$

It follows the proof of Lemma 3.2. QED

Remark 3.2.— As $u \in L^2\left(0,T; H_0^1(\Omega)\right)$, by the same argument used in the proof of Lemma 3.2, we obtain:

$$\gamma_0 u^{(k)} = (\gamma_0 u)^{(k)} = 0, \quad u^{(k)} = d^k u / dt^k,$$

for all natural number k.

As a consequence of (3.14), Proposition 3.2, 3.3 and Lemma 3.2, it follows that

PROPOSITION 3.4.— *We have*

$$\gamma_1 u \in L^1\left(0,T; W^{\frac{1}{p}-2,p'}(\Gamma)\right) + H^{-1}\left(0,T; H^{1/2}(\Gamma)\right).$$

Let u_j be the approximate solution introduced in the proof of Theorem 1.1, that is, $-\Delta u_j = -F_j(u_j) - u_j''$. By the Proposition 3.3 we can write

$$u_j = -y_j - z_j' \quad \textit{where} \ -\Delta y_j = F_j(u_j) \quad \textit{and} \ -\Delta z_j = u_j'. \tag{3.21}$$

We have that $\gamma_1 u_j$ belongs to the space given in Proposition 3.4.

In this conditions we obtain the following result:

PROPOSITION 3.5.— *We have*

$$\gamma_1 u_j \to \gamma_1 u \quad in \quad \left(L^\infty\left(0,T;W^{2-\frac{1}{p},p}(\Gamma)\right)\bigcap H_0^1\left(0,T;H^{\frac{1}{2}}(\Gamma)\right)\right)' \tag{3.22}$$

weak star.

Proof.— We note that $F_j(u_j) \to F(u)$ in $L^1\left(0,T;L^1(\Omega)\right)$; consequently from

$$\|y_j - y\|_E = \|y_j - y\|_{L^{p'}(\Omega)} + \|\Delta y_j - \Delta y\|_{L^1(\Omega)} \leq c\|F_j(u_j) - F(u)\|_{L^1(\Omega)},$$

it follows that

$$y_j \to y \text{ in } L^1(0,T;E),$$

which, by Proposition 3.2 implies:

$$\gamma_1 y_j \to \gamma_1 y \text{ in } L^1\left(0,T;W^{\frac{1}{p}-2,p'}(\Gamma)\right). \tag{3.23}$$

Also nothing that $u_j' \to u'$ in $L^2\left(0,T;L^2(\Omega)\right)$ weak and by

$$\|\gamma_1 z_j - \gamma_1 z\|_{H^{1/2}(\Gamma)} \leq c\|z_j - z\|_{H^2(\Omega)} \leq c\|u_j' - u'\|_{L^2(\Omega)},$$

we obtain that

$$\gamma_1 z_j \to \gamma_1 z \text{ in } L^2\left(0,T;H^{1/2}(\Gamma)\right) \text{ weak.} \tag{3.24}$$

Let $\xi \in W$ where W' is the space in (3.22). One has by Lemma 3.2

$$\langle \gamma_1 u_j, \xi\rangle = -\langle \gamma_1 y_j, \xi\rangle + \langle \gamma_1 z_j, \xi'\rangle.$$

Then, by (3.23), (3.24) we obtain

$$\langle \gamma_1 u_j, \xi\rangle \to \langle \gamma_1 u, \xi\rangle$$

which proves Proposition 3.5 and consequently the proof of Theorem 1.1 is complete. QED

Références

[1] COOPER (J.) and MEIDEROS (L.A.). — *The Cauchy problem for nonlinear wave equations in domains with moving boundary*, Annali Scuola Norm. Sup. Pisa XXVI 1972, p. 829-838.

[2] KINDERLEHRER (D.) and STAMPACCHIA (G.) . — *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York, 1980.

[3] LIONS (J.L.) . — *Problèmes aux Limites dans les Equations aux Dérivées Partielles*, Les Presses de l'Université de Montreal, Montreal, 1965.

[4] LIONS (J.L.) . — *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[5] LIONS (J.L.) . — *Contrôle des Systèmes Distribués singuliers*, Gauthiers-Villars, Paris, 1983.

[6] LIONS (J.L.) . — *Hidden regularity in some nonlinear hyperbolic equations*, Matemática Aplicada e Computcional, Vol. 6, N° 1 1987, p. 7-15.

[7] LIONS (J.L.) and MAGENES (E.) . — *Problèmes aux Limites Non Homogènes et Applications*, Vol. I, Dunod, Paris, 1968.

[8] STRAUSS (W.A.) . — *The Energy Method in Nonlinear Partial Differential Equations*, Notas de Matemática 47, IMPA, Rio de Janeiro, RJ, 1969.

[9] STRAUSS (W.A.) . — *On weak solutions of semilinear hyperbolic equations*, An. Acad. Bras. Ciências 42 (4) 1970, p. 645-651.

(Manuscrit reçu le 25 mars 1987)