ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

LABROS DOKAS
JOHN STABAKIS

Comments on the completeness of order complements and on the Prüfer numbers

Annales de la faculté des sciences de Toulouse 5e série, tome 10, nº 1 (1989), p. 65-73
<http://www.numdam.org/item?id=AFST_1989_5_10_1_65_0>

Annales Faculté des Sciences de Toulouse Vol. X, n°1, 1989

Comments on the completeness of order complements and on the Prüfer numbers

LABROS DOKAS[1] AND JOHN STABAKIS[1]

ABSTRACT. — Given a poset we define two chain – extensions of it; the first is the system of all its chains without end points, the order is a system of directed sets for which a linear subset is cofinal therein. The latter is complete, in a meaning, while the two completions applied to the set of non negative numbers ordered by division, give isomorphie complements. Each complement consists the underlying set for a system which is closed under the operations of the product, g.c.d. and l.c.m. of any number of elements (Prüfer numbers).

Subj. Class. AMS 1981, 06A10, 06A23.

1. Introduction

The present note is concerned with two order-complements introduced by the authors in [1] and [5]. The aim of this reference is to give a notion of "completeness" of order-complements and, as an application, to make comments on a complement of the set $\mathbf{N}_0$ of the non-negative integers, ordered by divisibility.

The main theorems will include the following results : given the order structure $(E, \leq)$, symbolize by E^* and E_{K_τ} the f^* and the KRASNER complement respectively. Then

a) $(E^*)^* = E^*$.

(1) University of Patras, Departement of Mathematics, 26110, Patras, Greece.

b) In the complement E^*, every cut of a maximal chain is not a gap.

c) The complement $(\mathbf{N}_0)_{K_r}$, as well as the $(\mathbf{N}_0)^*$, is the set of Prüfer numbers.

Propositions a) and b) state that the completion of a complement, as well as the restriction of the whole space to a chain subset, does not give new elements; the structure is complete. This completeness is non valid in other cases, in say the Mac Neille's complement. But it is remarkable that another version of the completion procedure, for example the consideration of the lower classes A of the cuts (A, B) instead of the cuts themselves, leads to another complement, which is connected with the former by a surjective isotone function, but has not such a completeness. This is the difference between the two complements of our case.

Statement c) says that the corresponding complement of $\mathbf{N}_0$ is the underlying set of an algebraic system which generalizes the well-known *Prüfer group* p^∞).

From another point of view the mentioned structures are not lattices in general, but they could be characterized, both of them, as chain-extensions ([2], §1); by this we mean that they are defined by subsets wich are cofinal to totally ordered sets.

2. KRASNER complement and f^*–complement of an order structure

We briefly restate the two complements. We always refer to an order structure $(E, \leq)$.

2.1. Consider a subset L of E ordered by a total order α and an element $a \in L$. The subset

$$L_a = \{x \in L : a\alpha x\}$$

is called *final section of L with origin a*. The structure (L, α) is called *monotone*, if α is the restriction on L of $<$ or $>$ (in wich cases (L, α) is called *increasing* or *decreasing* resp.). Such a structure (L, α) cannot be increasing and at the same time decreasing unless if it is a singleton. If there exists a final section L_a of (L, α) such that (L_a, α) is monotone, then (L, α) is called *asymptotically monotone (as. monotone)* and *a the origin of the monotony*. If for an $a \in L, (L_a, \alpha)$ admits a maximum or minimum, (L, α) is called *as. constant*.

Two as. monotone structures (L, α_1) and (M, α_2) are called *equivalent* ($L \equiv M$ in symbols) if for each origin of monotony a of L and b of M, we can, for any $x \in L_a$, find a $y \in M_b$ and a $z \in L_a$ such that $x\alpha_1 z$ and $y \in [x, z]$, where $[x, z]$ is the segment of E. If $L \equiv M$ and L has a last element e, M also has e as last element, otherwise L and M are increasing and decreasing simultaneously.

Evidently the relation is an equivalence.

If now $M(E)$ is the set of monotone structures of E, define an order $<$ in $M(E) : (L, \alpha_1) < (M, \alpha_2)$ if there exist origins of monotony a and b of L and M respectively such that for any $x \in L_a$ and any $y \in M_b$, $x < y$. The order $<$ is an extension of $\leq$, so in the sequel we use the same symbol for both of them.

The set $M(E)/\equiv$ ordered by the above relation is called KRASNER *complement of* E (symbolize by E_{K_r}). In the complement, an element $e \in E$ is identified with a class $\widetilde{a} \in M(E)$, whose elements are subsets of E that have e as last element.

2.2 The f^*–complement is an imitation of Mac Neille's complement, whose classes are subsets cofinal to a totally ordered set.

DEFINITION 1. — *([5], Def.1)*

A couple (A, B) of subsets of an ordered set E is called f^*–cut, if it fulfils the next properties :

(1) The subsets A, B are directed (right and left respectively).

(2) If $x \in A$ and $y \in B$, $x < y$.

(3) There does not exist any element between x and y.

(4) There exist chains L_1, L_2 subsets of A and B respectively, L_1 being cofinal to A and L_2 coinitial to B (that is, cofinal for the opposite direction).

(5) If $x \in A$ and $x' < x$, then $x' \in A$ as well as if $y \in B$ and $y < y'$, $y' \in B$.

The subsets A and B are called *the lower* and the *upper class* resp. of (A, B). If there exists the maximum of A and the minimum of B, the f^*–cut is called f^*–jump, if neither of them exists, it is called f^*–gap.

Symbolizing by $L^*(E)$ the set of f^*–gaps, we call f^**–complement of* E the set $E^* = EUL^*(E)$ ordered by the following order $\leq$ (extension of the given $\leq$).

If x, y belong to E and (A_1, B_1), (A_2, B_2) are non-equal elements of $L^*(E)$, then :

$$x \leq (A_1, B_1) \Leftrightarrow x \in A_1, (A_1, B_1) \leq x \Leftrightarrow x \in B_1,$$
$$\text{and } (A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_2 \cap B_1 \neq \emptyset.$$

Remark.— If $\wedge^*(E)$ is the set of the classes of the f^*–cuts of E without end points, we define an order relation on $E \cup \wedge^*(E)$ in such a way that the new complement coincides with KRASNER complement. To each lower class A of a f^*–cut (A, B), as well as to each upper one, which is cofinal to a chain L, corresponds the point of E_{K_τ} which has as a representative the chain L, and inversely.

3. The completeness of the f^*–complement

Henceforth, E^* symbolizes the f^*–complement of an ordered set E. The first result concerning to the completeness of E^* is the next.

THEOREM 1.— $(E^*)^* = E^*$.

The proof of theorem 1 follows immediately from the lemmas 1 and 2.

LEMMA 1.— *([5], lem. 1). Between two comparable elements of the f^*–complement E^* lies at least one element of the set E. Moreover the trace on E of a f^*–cut of E^* is a f^*–cut of E.*

LEMMA 2.— *The trace on E of an f^*–gap of E^* is also a f^*–gap of E.*

Proof.— Let (A^*, B^*) and (A, B) be the f^*–gap of E^* and its trace on E, respectively. Suppose the classes A and B have not end points, otherwise these points are the end points of A^* and B^* as well. The statements (1), (2), (3) and (5) of Def. 1 are evidently fulfilled by A and B.

About the statement (4) : let L^* be a chain of E cofinal to A^*. The choice axiom implies that each chain has a well ordered subset which is cofinal to it. If $I^* = (x_i^*)$ is this subset of L^*, we'll assign to it a well ordered family $I = (x_i)$ of elements of A cofinal to I^*. So, for any successive elements x_i^*, x_{i+1}^* of E^* pick up an element $x_i \in A$, which is greater than x_i^* and smaller or equal to x_{i+1}^*. Such an element exists because of lemma 1.

The procedure goes on inductively, with respect to the element x^* of I^* : regardless of x^* being next to an element of I^* or being a limit point of a family of points of I^*, assign to x^* an element of A which lies between x^* and the element of I^*, which is next to x^*. The assigned x consist a chain cofinal to A. It is evident that another chain of elements of B, is cofinal (for the opposite direction) to B.

The proof of the theorem is an easy consequence : if (A, B) is a f^*–gap of E^*, then its trace (A, B) on E is also a f^*–gap in E, which defines an element in E^* not belonging to A^* or B^*, absurd.

Remark.— If a is an element of $E_{K_T} \backslash E$ and it has as a representative a monotone chain, say the increasing chain $A \subset E$, then a chain in the set E_{K_T} cofinal to A, gives in the set $(E_{K_T})_{K_T}$ an element different than a (actually it is smaller than a). Thus from this point of view the KRASNER complement is not a complete one.

We now proceed to give two results referring to the completeness of the chains which are maximal with respect to the inclusion into the complement E^*.

PROPOSITION 1.— *Every chain in the f^*–complement, maximal with respect to inclusion, has not gaps.*

Hence the statement asserts that chains into the f^*–complement are complete subsets.

Proof.— Consider a chain I^* maximal into the E^*, and a cut (A_1, B_1) of it. Suppose that the cut is a gap. We proceed to construct a cut (A^*, B^*) of E^*, in the meaning of Mac Neille's complement; then either A^* or B^* has an end point a and the subset $I^* \cup \{a\}$ will be a chain properly greater than I^*, which is absurd.

In fact. Put : $B^* = \{y \in E^* : (\forall x \in A_1)[x < y]\}$ and $A^* = \{x \in E^* : (\forall y \in B^*)[x < y]\}$.

The couple (A^*, B^*) is a cut and, on the other hand, there exists a f^*–cut (A_i, B_i) such that $A_1 \subset A_i$ and $B_1 \subset B_i$, whilst it is $A_i \subset A^*$ and $B_i \subset B^*$. Because of the non-existence of f^*–gaps in E^*, at least one of the classes A_i and B_i has an end point a.

PROPOSITION 2.— *The Dedekind complement of a chain I of E and every maximal chain of the f^*–complement E^* whose trace on E is the chain I are isomorphic (that is, there exists a surjective and isotone map of the former onto the last chain).*

Proof. — By Dedekind complement of I we mean the simple complement of a totally ordered set. If I_D is the Dedekind complement of I and I^* is a maximal (with respect to the inclusion relation) chain of E^* whose trace on E is the chain I, define a map $f : I_D \to I^*$ as follows :

for each $x \in I$, put $f(x) = x$.

Let $e = (I_1, I_2)$ be a gap of I. Consider the subsets :

$I_2^* = \{y \in I^* : (\forall x \in I_1)\ [x < y]\}$ and $I_1^* = I^* \backslash I_2^*$. The set I_1^* has not an end point, because if $max\ I_1^* = a$, then for $x \in I_1$, $x < a$, hence $a \in I_2^*$, false. On the other hand, the class I_2^* – because of prop. 1 – has a minimum e^*.. Then put $f(x) = x$.

The map f is injective and preserves the order. In fact; let $e = (I_1, I_2)$ and $e' = (I_1', I_2')$ be two gaps of I with $e < e'$. Then there is $x \in I_2 \cap I_1'$, hence $x \in I^*$ and $f(e) < x < f(e')$. It is evident that the last result is valid, when e or e' is an element of I.

Finally f is surjective : let $e^* \in I^* \backslash I$; consider the subsets $I_1 = \{x \in I : x < e^*\}$ and $I_2 = I \backslash I_1$. The subsets I_1 and I_2 consist a partition of I without any point $x \in E$ between the elements of these two classes. On the other hand, neither of the two classes has an end point, because if, say, $a = \max\ I_1$ and $e^* = (A, B) \in L^*(E)$, then $a \in A$ and there exists $x \in A$, $x > a$, that is the element x lies between the classes I_1 and I_2, absurd. Thus $e = (I_1, I_2)$ is a gap of I and – by the definition of f – it is $f(e) = e^*$.

4. The f^* and Krasner complements of $\mathbf{N}_0$

The last paragraph is devoted to a roughly speaking algebraic application of the mentioned complements.

Consider the set $\mathbf{N}_0$ of non-negative integers ordered by divisibility. Put m/n if m divides $n (m, n \in \mathbf{N}_0)$. The least common multiple and the greatest common divisor of the natural numbers $m_1, m_2, \ldots, m_n$ are symbolized by $[m_1, m_2, \ldots, m_n]$ and $(m_1, m_2, \ldots, m_n)$ respectively.

The two complements of $\mathbf{N}_0$ coincide, so we refer exclusively to the structure $(\mathbf{N}_0)_{Kr}$.

PROPOSITION 1.— *The lattice* $(\mathbf{N}_0, l)$ *has not any gap.*

Proof.— Consider a cut (E_1, E_2) in $\mathbf{N}_0$. If $m_1, m_2, \ldots, m_i$ belong to E_1, their *l.c.m.* $[m_1, m_2, \ldots, m_i]$ divides each $n \in E_2$ and if $n_1, n_2, \ldots, n_j$ belong to E_2, then each $m \in E_1$ divides the *g.c.d.* of $n_1, n_2, \ldots, n_j$.

Three cases are possible.

(1) $E_2 = \emptyset$; then $E_1 = N_0$, E_1 has a maximum point 0, hence (E_1, E_2) is not a gap.

(2) $E_2 = \{0\}$; then each $m \neq 0$ divides 0, hence m belongs to E_1. The class E_2 has a minimum and the cut (E_1, E_2) is not a gap.

(3) There exists $n \in E_2$, $n \neq 0$. Then each element $m \in E_1$ divides n and the class E_1 has a finite number of elements, say $E_1 = \{m_1, m_2, \ldots, m_i\}$. If $m = [m_1, m_2, \ldots, m_i]$, each proper divisor of m belongs to E_1 and each multiple of it to E_2. The point m itself either belongs to E_1 or to E_2 and in each case it is the end point of the respective class. Thus the couple (E_1, E_2) is non-gap, even if $m \in E_2$ is of the form $p^e \neq 1$, where p is a prime. In fact, in this exceptional case, the class E_1 is the set of the proper divisors of p^e, (that is the divisors of p^{e-1}), so each $n \in E_2$ is a multiple of p^{e-1} and it does not need to be a multiple of p^e.

The *Prüfer group* (p^∞) is a famous example of an additive group whose each extension to a ring is a zero ring (e.g. [4], p. 60). If a_k is any root of the unity with amplitude $\frac{2\pi}{p^k}$, p is a prime, $k \in \mathbf{N}$, then each number $a_1^{\sigma_1}\, a_2^{\sigma_2} \ldots a_n^{\sigma_n}$, where $\sigma_i \in \mathbf{N}_0$, $\sigma_i \leq p-1$, $i \in \{1, 2, \ldots, n\}$, written in the additive form $\sigma_1 a_1 + \sigma_2 a_2 + \cdots + \sigma_n a_n$, belongs to (p^∞). The product of two such elements is zero. *The set of Prüfer's numbers* is the set of all numbers of the form

$$\prod_{p \in P} P^{\omega_p}$$

where ω_p is non-negative integer or infinity and P is the set of prime numbers.

The set is closed under the operations of the product, of the *g.c.d.* as well as of *l.c.m.* It could be characterized as a generalization of the set of the coefficients of Prüfer rings, when the prime p goes through the set P of prime numbers (e.g. [3], §§35 and 81).

THEOREM 2.— *The* KRASNER *complement of* $(\mathbf{N}_0/)$ *is the set of Prüfer's numbers.*

Proof.— Let e^* be an element of $(\mathbf{N}_0, /)_{kr}$ and $\widetilde{a} = (A, /)$ a representative of e, which we suppose to be monotone (considering $\widetilde{a}$, if it is needed, as a final section). If α and $\widetilde{\alpha}$ are of opposite kind, then either $A = \{0\}$, in which case the support A has a last element $a^* = 0$, or $A \neq \{0\}$ in which case there exists an element $a \neq 0$ and because a has a finite number of divisors, A has a last element a^*, hence $e^* = a^*$. On the other hand, if α and $\widetilde{\alpha}$ coincide and A has a last element a^*, we also have $e^* = a^*$.

Suppose now that α and $\widetilde{\alpha}$ coincide and that A has not a last element. If a, a', x are non zero elements of $\widetilde{a}$ and $a/x/a'$, then x divides a' and the number of these $x's$ is finite. Thus the chain A can be written as $\{a_1, a_2, \ldots, a_i, \ldots\}$ where a_i/a_{i+1} for each $i \in \mathbf{N}$. Inversely each sequence of the above form is the representative of an element non-belonging to $\mathbf{N}_0$. In fact, let p be a prime number and $\omega_p(a_i)$ the order of a_i with respect to p. Then $\omega_p(a_{i+1}) \geq \omega_p(a_i)$ and the function ω_p is an increasing function of i. If ω_p is bounded, it finally gets a fixed value, otherwise it is increasing infinitely. Define $\omega_p(A)$ as the fixed value in the former case and the value $+\infty$ in the last one, (put $n < +\infty$ for each $n \in \mathbf{N}$). If $\widetilde{b} = (B, /)$ is a monotone structure and $\widetilde{a} \equiv \widetilde{b}$, then $\widetilde{b}$ is increasing without a last element, thus $B = \{b_1, b_2, \ldots, b_i, \ldots\}$, where $b_1/b_2/\ldots/b_i/b_{i+1}/\ldots$ for any $a_i \in A$ and any index j_0, there exist $b_j \in B$, $j > j_0$ and $a_k \in A$ such that $a_i \leq b_j \leq a_k$. This shows that $\omega_p(a_i) \leq \omega_p(b_j) \leq \omega_p(a_k)$, hence $\omega_p(A) \leq \omega_p(B) \leq \omega_p(A)$ or $\omega_p(A) = \omega_p(B)$.

Inversely, if for each prime p, $\omega_p(B) = \omega_p(A)$, it is $\widetilde{a} \equiv \widetilde{b}$. In fact; let $a_i \in A$, and $a_i = p_1^{\sigma_1}\ p_2^{\sigma_2} \ldots p_s^{\sigma_s}$ be the prime factor decomposition of a_i. It is $\sigma_q \leq \omega_{p_q}(A) = \omega_{p_q}(B)$, $q = 1, 2, \ldots, s$. Hence there exist indices $j_1, j_2, \ldots, j_s$, such that $\omega_{p_q}(b_{j_q}) \geq \sigma_q$. But then, for a given j_0, it is $j \geq j_0$ where $j = \max\limits_{0 \leq q \leq s} j_q$ and for all values of q, there holds : $\omega_{p_q}(b_j) \geq \omega_{p_q}(b_{j_q}) \geq \sigma_q$, hence $a_i < b_j$. Changing the roles of $\widetilde{a}$ and $\widetilde{b}$, it is evident that there exists an a_k such that $b_j < a_k$, thus $\widetilde{a} \equiv \widetilde{b}$. So the class of equivalence e^* of the structure $\widetilde{a} = (A, /)$ which is monotone and has no last element, is completely defined by the assignment :

$$p \to \omega_p(A),\ p \in P, \quad P \text{ the set of prime numbers.}$$

The element e^* can be written under the form $\prod\limits_{p \in P} p^{\omega_p(A)}$.

Consider now a number of the form $\prod\limits_{p \in P*} p^{\sigma_P}$, where $\sigma_p \in \mathbf{N}_0 \cup \{+\infty\}$ and p^* is an infinite subset of P, or p^* is finite but one of σ_p is $+\infty$. Suppose that all of the prime numbers $p_1, p_2, \ldots, p_i, \ldots i < L \leq +\infty$ have been put in an increasing order. Put

$$a_i = \Pi_{p_q}^{\min\{i,\sigma_{p_q}\}}$$
$$q < \min\{i+1, 1\}$$

We have to define the value of $\sigma_{p_q}(a_i)$. Put $\sigma_{p_q}(a_i) = 0$ or Min $\{i, \sigma_{p_q})$ depending on whether $q > L$ or $i \geq q$. If $A = \{a_1, a_2, \ldots a_i, \ldots\}$ then we get $\omega_p(A) = \sigma_p$.

The elements of the last form are the elements of $(\mathbf{N}_0)_{k\tau} \backslash \mathbf{N}_0$.

Remark. — It is evident that the operation of the product and the operations of finding the least common multiple and the greatest common divisor in the set $(\mathbf{N}_0)_{k\tau}$, is an extension of the respective habitual operations in the set $\mathbf{N}_0$, that is, these operations can be defined for any subset of them and not only for finite subsets. So the least common multiple of any subset of elements of the set $(\mathbf{N}_0)_{k\tau} \backslash \mathbf{N}_0$ is zero.

References

[1] DOKAS (L.). — *Certains complétés des ensembles ordonnés munit d'opérations.* Complété de KRASNER, *C.R.Acad. Paris*, t. **256**, 1963, p. 3937-39.

[2] ERNE (M.). — Posets isomorphic to their extensions, *Order* , t. **2**, 1985, p. 199-210.

[3] FUCHS (L.). — Infinite Abelian Groups, Tomes I, II, Academic Press *N.York London* 1973.

[4] REDEI (L.). — Algebra., Pergamon Press. Oxford, 1967.

[5] STRATIGOPOULOS (D.), STABAKIS (J.). — Sur les ensembles ordonnés (f^*−coupes), *C.R.Acad. Paris*, t. **285**, 1977, p. 81-84.

(Manuscrit reçu le 18 avril 1988)