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A new method of exact controllability in short time and applications

VILMOS KOMORNIK[(*)(1)]

RÉSUMÉ. — On introduit une méthode nouvelle, générale et constructive pour obtenir des estimations optimales ou presque optimales du temps de contrôlabilité exacte des systèmes d'évolution linéaires. La méthode est basée sur la méthode d'unicité hilbertienne (HUM) introduite par J.-L. Lions et sur une méthode d'estimation due à A. Haraux. On applique cette méthode pour plusieurs problèmes concrets concernant l'équation des ondes et de diverses modèles de plaques.

ABSTRACT. — We introduce a new general and constructive method which allows us to obtain the best or almost the best estimates of the time of exact controllability of linear evolution systems. The method is based on the Hilbert Uniqueness Method of J.-L. Lions and on an estimation method introduced by A. Haraux. Our method is applied for several concrete problems related to the wave equation and to various plate models.

0. Introduction

In order to motivate the investigations of the present paper, consider the following typical problem : let Ω be a regular, bounded domain in $R^N (N \geq 1)$ with boundary Γ, let T be a positive number and consider the system

$$(0.1) \qquad \left| \begin{array}{l} y'' - \Delta y = 0 \quad \text{in} \quad \Omega \times (0,T), \\ y(0) = y^\circ \quad \text{and} \quad y'(0) = y^1, \\ y = v \quad \text{on} \quad \Gamma \times (0,T) \end{array} \right.$$

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We recall the following exact controllability result due to L.F. Ho [1] and J.-L. Lions [2] : there exists a positive number T_0 such that if $T > T_0$, then for every initial data $(y^\circ, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control $v \in L^2(0,T;\Gamma)$ driving the system (0.1) to rest at time T i. e. such that the solution of (0.1) satisfies $y(T) = y'(T) = 0$.

According to the Hilbert Uniqueness Method introduced in J.-L. Lions [1], this result was a consequence of the following a priori estimates concerning the homogeneous system

$$
(0.2) \qquad \left|\begin{array}{l} z'' - \Delta z = 0 \quad \text{in} \quad \Omega \times R, \\ z(0) = z^\circ \quad \text{and} \quad z'(0) = z^1, \\ z = 0 \quad \text{on} \quad \Gamma \times R : \end{array}\right.
$$

there exists a positive number T_0 such that if $T > T_0$, then for every initial data $(z^\circ, z^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ the solution of (0.2) satisfies the inequalities

$$
(0.3) \qquad \left|\begin{array}{l} c\int_\Omega |\nabla z^\circ|^2 + |z^1|^2 \mathrm{d}x \le \int_0^T \int_\Gamma |\partial_y z|^2 \mathrm{d}\Gamma \mathrm{d}t \\ \\ \le C\int_\Omega |\nabla z^\circ|^2 + |z^1|^2 \mathrm{d}x \end{array}\right.
$$

where $\partial_y z$ denotes the normal derivative of z and where c and C are positive constants, independent of the choice of (z°, z^1).

The estimates (0.3) were proven by a multiplier method. This provided an explicit value for T_0. This value was not the best one; several methods were developed to improve the value of T_0 in this and similar problems. The main purpose of this paper is to introduce a new *general* and *constructive* method which allows one to obtain the best or almost the best value of T_0 in most exact controllability problems.

Let us explain briefly the main ideas of this method. The first observation is that (0.3) holds under a weaker assumption on T if the initial data belong to a suitable *subspace* of finite codimension of $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

To make this more precise, let us denote by $\eta_1 < \eta_2 < \dots$ the eigenvalues of $-\Delta$ in $H^2(\Omega) \cap H_0^1(\Omega)$, let Z_1, $Z_2, \dots$ be the corresponding eigenspaces and let $\omega_1, \omega_2, \dots$ be a sequence of the numbers $\pm(\eta_1)^{1/2}, \pm(\eta_2)^{1/2}, \dots$. It is well-known that every solution of (0.2) may be written in the form

$$
(0.4) \qquad z(t) = \sum_{j \ge 1} z_j e^{i\omega_j t}
$$

with suitable vector coefficients z_j. Furthermore, introducing in $H^2(\Omega) \cap H_0^1(\Omega)$ the norm

$$\|v\| = \left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right)^{1/2}$$

and the semi norm

$$|v| = \left(\int_\Gamma |\partial_y v|^2 \mathrm{d}\Gamma\right)_\Gamma^{1/2},$$

(0.3) may be written in the form

$$c \sum_{j \geq 1} \|z_j\|^2 \leq \int_0^T |z(t)|^2 \mathrm{d}t < C \sum_{j \geq 1} \|z_j\|^2.$$

Now the multiplier methode allows us to prove the following result : there exists a positive number $T_0'(< T_0)$ such that for every $T > T_0'$ there is a positive integer k and there are positive constants c and C such that (0.5) holds for all solutions $z(t)$ of (0.2) satisfying $z_j = 0$, $\forall j > k$.

The second observation is that (0.5) then extends *automatically* for all solutions of (0.2), without the assumption $z_j = 0$, $\forall_j < k$ (with possibly different constants c and C). Indeed, imagine for the moment that the coefficients z_j in (0.4) are not vectors scalars. Then (0.5) means that the system of exponential functions $e^{i\omega_j t}$ is a Riesz basis (in its closed linear hull) on the interval (0, T). It is well-known from the theory of nonharmonic Fourier series that if we have a Riesz basis of exponential functions on an interval (0, T) and if we enlarge it with a finite number of new exponential functions, then the resulting system will be a Riesz basis on every interval with length greater than T; cf. e. g. R. M. Young [1]. Recently, in order to prove some interior controllability theorems, an elementary proof was given for this result by A. Haraux [1] and, fortunately, this proof may easily be adapted to the more general case with vector coefficients. This proves (0.3) for all $T > T_0'$.

The plan of this paper is the following :

In section 1 we prove the basic abstract results of the new method.

In section 2 we introduce our method by improving two results of J.-L. Lions [2] concerning the wave equation. First we show that *lower-order* terms of the evolution equation do not affect in general the time of exact controllability. Secondly we give an application to a problem with a "*strenghtened norm*"; this is a typical situation for example in exact controllability problems with Neumann action.

Let us mention that the results of this section may also be obtained (at least if the boundary of the domain and the potential function is very smooth) by a powerful method of C. Bardos, G. Lebeau and J. Rauch [1] (based on microlocal analysis) developed for hyperbolic systems.

Sections 3 and 4 are devoted to two problems raised in J. Lagnese and J.-L. Lions [1]. Section 3 is devoted to the exact controllability of Kirchoff plates. We prove the *convergence of the* optimal exact *control functions* when the uniform thickness of the plate tends to zero. In Section 4 we prove the *uniform exact controllability* of Mindlin-Timoshenko plates with respect to the shear modulus.

Other applications are given in V. Komornik [5], [6].

The author is grateful to A. Haraux, J.-L. Lions and E. Zuazua for fruitful discussions.

1. Norm estimates

I.1. — Formulation of the results

Let H and V be two real or complex, infinite-dimensional *Hilbert spaces* with a dense and continuous imbedding $V \subset H$. Identifying H with its (anti)dual H' we obtain $V \subset H \subset V'$.

Let $a(u,v)$ be *continuous*, *symmetric* (rep. hermitian) *bilinear form* on V. Then there exists a unique bounded linear operator $A' : V \to V'$ such that $a(u,v) = < A'u, v > \quad , \quad \forall u,\ v \in V$. Let us introduce in H the operator A given by $D(A) := \{v \in V | A'v \in H\},\ Av := A'v$ for all $v \in D(A)$.

Assume that there exist two positive constants a_0 and a_1 such that

$$a(v,v) \geq a_0 \|v\|_V^2 - a_1 \|v\|_H^2,\ \forall v \in V, \tag{1.1}$$

and let s be an arbitrary nonnegative real number. Let us introduce for brevity the notation

$$W^s := D((A + a_1 I)^{s+1/2}) \times D((A + a_1 I)^s).$$

Then it is well-known (cf. J.-L. Lions and E. Magenes [1]) that for every initial data $(u^0, u^1) \in W^s$ the Cauchy problem

$$\left| \begin{array}{l} u'' + Au = 0 \quad \text{in} \quad R, \\ u(0) = u^\circ \quad \text{and} \quad u'(0) = u^1 \end{array} \right. \tag{1.2}$$

has a unique solution $u = u(t)$ in $C(R; D((A + a_1 I)^{s+1/2})) \cap C^1(R; D((A + a_1 I)^s))$. (The equation in (1.2) is understood in the following sense :

$$d^2/dt^2(u(t), v)_H + a(u(t), v) = 0 \quad \text{in} \quad \mathcal{D}'(R), \ \forall v \in V.$$

Furthermore, we have the conservation of energy :

$$(1.3) \qquad \left| \begin{array}{l} E(u^\circ, u^1) := (1/2(\|u^1\|_H^2 + a(u_0, u_0)) \\ = (1/2)(\|u'(t)\|_H^2 + a(u(t), u(t)), \ \forall t \in R. \end{array} \right.$$

Let p be a *semi norm* on W^s. For every compact interval $I \subset R$ we may define a new semi norm on W^s by the formula

$$(1.4) \qquad (u^\circ, u^1) \mapsto (\int_I p(u(t), u'(t))^2 dt)^{1/2}$$

The purpose of this section is to find conditions ensuring that this semi norm is equivalent to (or stronger than) the *norm*

$$(1.5) \qquad (u^\circ, u^1) \mapsto (\|u^\circ\|_v^2 + \|u^1\|_H^R$$

induced by $V \times H$ on W^s.

Let us assume that *the imbedding* $V \subset H$ *is compact.* Then the eigenvalue problem

$$(1.6) \qquad z \in V, a(z, v) = \eta(z, v)_H, \ \forall v \in V$$

has an infinite sequence of eigenvalues

$$(1.7) \qquad (-a_1 <)\eta_1 < \eta_2 < \ldots, \eta_j \to \infty,$$

and a corresponding sequence $Z_1, Z_2, \ldots$ of pairwise orthogonal (in H), finite-dimensional eigenspaces whose orthogonal direct sum is equal to H.

For every positive integer k, let us denote by W_k^s the subspace of W^s consisting of those elements (u, v), for which both u and v are orthogonal in H to the subspaces Z_j for all $j < k$. (If $k = 1$, then $W_k^s = W^s$.) It is clear that if $(u^\circ, u^1) \in W_k^s$, then the solution of (1.2) satisfies $(u(t), u'(t)) \in W_k^s$ for all $t \in R$.

We shall prove the following result.

THEOREM A.— *Let k_0 be a positive integer. Assume that for every integer $k \geq k_0$ there is a compact interval $I_k \subset R$ and there are positive constants c_k and C_k such that*

$$(1.8)\qquad \left| \begin{array}{l} c_k(\|u^\circ\|_V^2 + \|u^1\|_H^2 \leq \int_{I_k} p(u(t), u'(t))^2 dt \\ \leq C_k(\|u^\circ\|_V^2 + \|u^1\|_H^2), \quad \forall (u^\circ, u^1) \in W_k^s. \end{array} \right.$$

Assume also that for every positive integer $k < k_0$ there is a compact interval $I_k \subset R$ and there are positive constants c_k and C_k such that

$$(1.9)\qquad \left| \begin{array}{l} c_k(\|u^\circ\|_V^2 + \|u^1\|_H^2) \leq \int_{I_k} p(u(t), u'(t))^2 dt \\ \leq C_k(\|u^\circ\|_V^2 + \|u^1\|_H^2), \quad \forall (u^\circ, u^1) \in Z_k \times Z_k \end{array} \right.$$

Let us denote by $|I_k|$ the lengh of I_k and set

$$T_0 = \ inf\{|I_k| \ : \ k \geq k_0\}.$$

Then for every compact interval $I \subset R$ of length $> T_0$, there are two positive constants c and C such that

$$(1.10)\qquad \left| \begin{array}{l} c(\|u^\circ\|_V^2 + \|u^1\|_H^2) \leq \int_I p(u(t), u'(t))^2 dt \\ \leq C(\|u^\circ\|_V^2 + \|u^1\|_H^2), \quad \forall (u^\circ, u^1) \in W^s. \end{array} \right.$$

Moreover, if we denote by k the least positive integer $k \geq k_0$ such that $|I_k| < |I|$, then the constants c and C may be chosen to be continuous functions of the numbers c_j, C_j, η_j and of the end-points of the intervals I_j for $j = 1, \ldots, k$. □

Remark 1.1.— It is clear if condition (1.8) is satisfied for some positive integer k, then it is satisfied automatically for every greater value of k with the same interval and the same constant. In the applications, however, we shall usually obtain a sequence $|I_k|$ of decreasing length : $|I_1| > |I_2| > \ldots$ □

Remark 1.2.— Let us emphasize that T_0 does not depend on I_k for $k < k_0$. In fact, as we shall see later (cf. Proposition C), if (1.9) is satisfied for some compact interval I_k, then it is satisfied for *every* compact subinterval of R (with different constants c, (in general). □

Remark 1.3.— Theorem A implies in particular that for $|I| > T_0$ the semi norm (1.4) is in fact a norm. As a consequence, we obtain the following uniqueness theorem : if $u(t)$ is a solution of the system (1.2) such that $p(u(t), u'(t))$ vanishes in some interval of lenght $> T_0$, then $u(t)$ vanishes identically on R. □

Remark 1.4. — The last assertion of the theorem will follow from the constructive character of the proof; hence we shall obtain explicit constants c and C. This will be important when dealing with systems depending on a parameter, cf Section 3 below.

Remark 1.5. — If k is a positive integer such that the corresponding eigenvalue η_k is positive, then the natural norm of $V \times H$ is equivalent to the square root of the energy (1.3) on the subspace W_k^s (and therefore also on its subspace $Z_k \times Z_k$). In this case we may (and we shall) replace $\|u^\circ\|_v^2 + \|u^1\|_H^2$ by $E(u^\circ, u^1)$ in (1.8) and (1.9). □

We shall encounter several problems where the right side estimate of condition (1.8) of Theorem A will not be satisfied. (This will occur for example when we "strenghten" the semi norm p.) Then we shall apply the following result.

THEOREM B. — *Let k be a positive integer. Assume that for every integer $k \geq k_0$ there is a compact interval $I_k \subset R$ and a positive constant c_k such that the solution of (1.2) satisfies*

$$(1.11) \qquad \left| \begin{array}{l} \int_{I_k} p(u(t), u'(t))^2 dt \geq c_k(\|u^\circ\|_V^2 + \|u^1\|_H^2), \\ \forall (u^\circ, u^1) \in W_k^s \end{array} \right.$$

Assume also that for every positive integer $k < k_0$ there is a compact interval $I_k \subset R$ such that

$$(1.12) \qquad \int_{I_k} p(u(t), u'(t))^2 dt > 0, \quad \forall (u^\circ, u^1) \in Z_k \times Z_k \backslash \{(0,0)\}.$$

Set

$$T_0 = \inf\{|I_k| \ : \ k \geq k_0\}.$$

Then for every compact interval $I \subset R$ of length $> T_0$, there is a positive constant c such that

$$(1.13) \qquad \int_I p(u(t), u'(t))^2 dt \geq c(\|u^\circ\|_V^2 + \|u^1\|_H^2), \quad \forall (u^\circ, u^1) \in W^s. \quad □$$

Remark 1.6.— Remark that T_0 does not depend on $|I_k|$ for $k < k_0$. Indeed, as we shall see in Proposition C below, if (1.12) is satisfied for some compact interval I_k, then it is satisfied for every compact subinterval of R. □

Remark 1.7.—We have the same uniqueness theorem as in Remark 1.3. □

Remark 1.8.—We remark that the proof of Theorem B, given below, is not constructive. As a consequence, we do not obtain explicitly a constant c satisfying (1.13). □

Remark 1.9.—If $\eta_k > 0$ for some k, then we may (and we shall) replace $\|u^\circ\|_v^2 + \|u^1\|_H^2$ by $E(u^\circ, u^1)$ in the corresponding condition (1.11). □

Let us note that the real case of both theorems follows easily from the complex one by a standard complexification argument. Indeed, let us introduce the complexification $\widetilde{H}$, $\widetilde{V}$, $\widetilde{W^s}$, $\widetilde{W_k^s}$, $\widetilde{a}(.,.)$ by the usual way (cf. P. R. Halmos [1]), and define on $\widetilde{W^s}$ a semi norm $\widetilde{p}$ by the formula

$$\widetilde{p}((u^\circ, v^\circ), (u^1, v^1)) = (p(u^\circ, u^1)^2 + p(v^\circ, v^1)^2)^{1/2}$$

It is easy to verify that if the hypotheses of Theorem A or Theorem B are satisfied in the real case, then the corresponding conditions are satisfied in the complexified case, too. Applying the corresponding theorem in the complex case, our conclusion contains the desired real estimate (1.10) or (1.13) as a special case.

In view of this remark we shall restric ourselves in the sequel to the complex case.

Finally we shall prove the following result which permits to replace condition (1.9) of Theorem A or condition (1.12) of Theorem B by a stationary condition. Here we denote by $\widetilde{p}$ the complexification of p. (Naturally, $\widetilde{p} = p$ in the complex case.)

PROPOSITION C.— *Let k be a positive integer. If there is a compact interval $I \subset R$ such that*

$$\int_I p(u(t), u((t))^2 dt > 0, \quad \forall (u^\circ, u^1) \in Z_k \times Z_k \backslash \{(0,0)\}.$$

then

$$\widetilde{p}(v, \pm i\omega v) \neq 0, \quad \forall v \in Z_k \backslash \{0\}$$

where $\omega = (\eta_k)^{1/2}$ *if* $\eta_k \geq 0$ *and* $\omega = i(-\eta_k)^{1/2}$ *if* $\eta_k < 0$.

Conversely, if this last property is satisfied, then for every compact interval $I \subset R$ *there exist two positive constants* c *and* C *satisfying*

$$\left| \begin{array}{l} c(\|u^\circ\|_V^2 + \|u^1\|_H^2) \leq \int_I p(u(t), u'(t))^2 dt \\ \leq C(\|u^\circ\|_V^2 + \|u^1\|_H^2), \forall (u^\circ, u^1) \in Z_k \times Z_k. \quad \square \end{array} \right.$$

In subsection 1.2 we prove Theorem A in the special case where all the eigenvalues are positive. This is sufficient for many applications of the present paper. Our proof is essentially a vector extension of a method introduced in A. Haraux [1]. The proof of Theorem A in the general case is given in Subsection 1.3. Subsection 1.4 is devoted to the proof of Theorem B, and Proposition C is proved in Subsection 1.5.

I.2.— Proof of theorem A when all the eigenvalues are positive

First of all, in this case we may (and we shall) take $a_1 = 0$ in (1.1).

In order to simplify the calculations we set

$$\omega_{2j-1} = -\omega_{2j} = (\eta_j)^{1/2}, \; X_{2j-1} = X_{2j} = Z_j, \quad j = 1, 2, \ldots$$

Then for every $(u^\circ, u^1) \in W^s$ the solution $u = u(t)$ of (1.2) is represented by a unique convergent (in $C(R; D(A^{s+1/2})) \cap C^1(R; D(A^s))$) series

$$u(t) = \sum_{j \geq 1} u_j e^{i\omega_j t}, \; u_j \in X_j. \tag{1.14}$$

Conversely, every such function is a solution of (1.2) for some initial data $(u^\circ, u^1) \in W^s$.

In the sequel by solution we shall mean an arbitrary function $u \in C(R; D(A^{s+1/2})) \cap C^1(R; D(A^s))$ of the form (1.14).

We shall use for brevity the notation

$$\|v\| = a(v, v)^{1/2} \quad , v \in V \tag{1.15}$$

It is easy to verify that the norm $(\|u^\circ\|_V^2 + \|u^1\|_H^2)^{1/2}$ is equivalent to

$$\left(\sum_{j \geq 1} \|u_j\|^2 \right)^{1/2}$$

We may (and we shall) therefore replace $\|u^\circ\|_V^2 + \|u^1\|_H^2$ by $\sum_{j\geq 1} \|u_j\|^2$ in (1.8) - (1.10).

We have the following lemma.

LEMMA 1.10. — *Assume that for some integer $k \geq 2$ and for some positive numbers T and d_1 we have*

$$\int_{-T}^{T} p(u(t), u'(t))^2 dt \geq d_1 \sum_{j\geq k} \|u_j\|^2 \tag{1.16}$$

for all solutions $u(t)$ such that $u_1 = u_2 = \ldots = u_{k-1} = 0$.

Then for every $\varepsilon > 0$ there is a positive constant d_2 such that

$$\int_{-T-\varepsilon}^{T+\varepsilon} p(u(t), u'(t))^2 dt \geq d_2 \sum_{j\geq k} \|u_j\|^2 \tag{1.17}$$

for all solutions $u(t)$ with $u_1 = u_2 = \ldots = u_{k-2} = 0$.

The constant d_2 dépends only on d_1, ε and on $\min \{|\omega_j - \omega_{k-1}| \ : \ j \geq k\}$.

Proof. — Let $u(t)$ be an arbitrary solution satisfying $u_1 = u_2 = \ldots = u_{k-2} = 0$. Let us introduce, following A. Haraux [1], the auxiliary function

$$v(t) = u(t) - \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} e^{-i\omega_{k-1}s} u(t+s) ds, \ t \in R.$$

An easy calculation shows that

$$v(t) = \sum_{j\geq k} \left(1 - \frac{\sin(\omega_j - \omega_{k-1})\varepsilon}{(\omega_j - \omega_{k-1})\varepsilon}\right) u_j e^{i\omega_j t}$$

so that we may apply the estimate (1.16) for $v(t)$. Since

$$1 - \frac{\sin(\omega_j - \omega_{k-1})\varepsilon}{(\omega_j - \omega_{k-1})\varepsilon} \neq 0, \ \forall j \geq k \tag{1.18}$$

and

$$\frac{\sin(\omega_j - \omega_{k-1})\varepsilon}{(\omega_j - \omega_{k-1})\varepsilon} \to 0 \quad \text{as} \quad j \to \infty,$$

there existes a positive constant d_3, independent of the choice of $u(t)$, such that

$$\int_{-T}^{T} p(v(t), v'(t))^2 dt \geq d_1 d_3 \sum_{j\geq k} \|u_j\|^2 \tag{1.19}$$

On the other hand, from the definition of $v(t)$ we deduce the estimates

$$\int_{-T}^{T} p(v(t), v'(t))^2 dt \leq 2 \int_{-T}^{T} p(u(t), u'(t))^2 dt$$
$$+2 \int_{-T}^{T} \left(\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} p(u(t+s)),\ u'(t+s))ds \right)^2 dt$$
$$\leq 2 \int_{-T}^{T} p(u(t), u'(t))^2 dt$$
$$+\frac{1}{\varepsilon} \int_{-T}^{T} \int_{-\varepsilon}^{T} p(u(t+s), u'(t+s))^2 ds dt$$
$$\leq 4 \int_{-T-\varepsilon}^{T+\varepsilon} p(u(t), u'(t))^2 dt.$$

Combining with (1.19), the lemma follows by taking $d_2 = d_1 d_3/4$ □

Next we prove the following simple lemma.

LEMMA 1.11.— *Let k be a positive integer.*

a) *Assume that for some compact interval $I \subset R$ and for some positive constant c we have*

$$\int_I p(u(t), u'(t))^2 dt \leq c \sum_{j \geq 1} \|u_j\|^2,\ \forall (u^\circ, u^1) \in W_k^s.$$

Then the same inequality holds for every compact interval $I \subset R$ with a suitable positive constant c depending on $|I|$.

b) *Assume that for some compact interval $I \subset R$ and for some positive constant c we have*

$$\int_I p(u(t), u'(t))^2 dt \geq c \sum_{j \geq 1} \|u_j\|^2,\ \forall (u^\circ, u^1) \in W_k^s.$$

Then the same inequality holds for every compact interval $I \subset R$ having at least the same length as the above one, with the same constant c.

Proof.— It is sufficient to prove the lemma for the translates of the interval I; the general case of **a)** hence follows by the triangle inequality, while the general case of **b)** is then obvious.

If for example the inequality in **a)** is satisfied for some interval $I = [a, b]$, then for every $s \in R$ on the interval $I' = [a+s, b+s]$ we have

$$\int_{I'} p(u(t), u'(t))^2 dt = \int_I p(u(t+s))^2 u'(t+s))^2 dt.$$

Applying our hypothesis we obtain

$$\int_{I'} p(u(t), u'(t))^2 dt \le c \sum_{j\ge 1} \|e^{i\omega_j s} u_j\|^2, \ \forall (u^\circ, u^1) \in W_k^s.$$

Since $|e^{i\omega_j s}| = 1$, thise yields the desired inequality for I' with the same constant c.

The proof of the case **b)** is analogous. □

Now it is easy to establish the following proposition.

LEMMA 1.12.— *Assume that for some integer $m \ge 2$ and for some positive numbers T, d_1 and d_4 we have*

$$d_1 \sum_{j\ge 1} \|u_j\|^2 \le \int_{-T}^{T} p(u(t), u'(t))^2 dt \le d_4 \sum_{j\ge 1} \|u_j\|^2 \tag{1.20}$$

for all solutions $u(t)$ such that $u_1 = \ldots = u_{m-1} = 0$, and for all solutions $u(t)$ such that $u_j = 0$ for all $j \ne m-1$.

Then for every $\varepsilon > 0$ there exist two positive constants d_5 and d_6 such that

$$d_5 \sum_{j\ge 1} \|u_j\|^2 \le \int_{-T-\varepsilon}^{T+\varepsilon} p(u(t), u'(t))^2 dt \le d_6 \sum_{j\ge 1} \|u_j\|^2 \tag{1.21}$$

for all solutions $u(t)$ such that $u_1 = \ldots = u_{m-2} = 0$

Furthermore, d_5 and d_6 depend only on T, ε, d_1, d_4 and on $\min\{|\omega_j - \omega_{m-1}| \ : \ j \ge m\}$.

Proof.— Given a solution $u = u(t)$ with $u_1 = \ldots = u_{m-2} = 0$, let us put for brevity

$$u^a(t) = u_{m-1}\, e^{i\omega_{m-1} t},$$
$$u^b(t) = u(t) - u^a(t) = \sum_{j\ge m} u_j e^{i\omega_j t}$$

and

$$|u|_0 = \left(\int_{-T}^{T} p(u(t), u'(t))^2 dt\right)^{1/2},$$

$$|u|_\varepsilon = \left(\int_{-T-\varepsilon}^{T+\varepsilon} p(u(t), u'(t))^2 dt\right)^{1/2},$$

$$\|u\|_* = \left(\sum_{j\ge 1} \|u_j\|^2\right)^{1/2}.$$

Using the triangle inequality, (1.20) and applying Lemma 1.10 we obtain that

$$\begin{aligned}
\|u\|_* &< \|u^a\|_* + \|u^b\|_* \\
&\le d_1^{-1/2}|u^a|_0 + \|u^b\|_* \\
&\le d_1^{-1/2}(|u|_0 + |u^b|_0) + \|u^b\|_* \\
&\le d_1^{-1/2}(|u|_0 + d_4^{1/2}\|u^b\|_*) + \|u^b\|_* \\
&\le d_1^{-1/2}|u|_0 + d_2^{-1/2}(1 + d_1^{-1/2}d_4^{1/2})|u|_\varepsilon \\
&\le (d_1^{-1/2} + d_2^{-1/2} + d_1^{-1/2}d_4^{1/2}d_2^{-1/2})|u|_\varepsilon
\end{aligned}$$

i. e. the left side inequality of (1.21) is satisfied with

$$d_5 = (d_1^{-1/2} + d_2^{-1/2} + d_1^{-1/2}d_4^{1/2}d_2^{1/2})^{-2}$$

The right side inequality of (1.21) follows easily from (1.20), first applying the triangle inequality for the semi norm $|\cdot|_\varepsilon$ and then using Lemma 1.11. We may take for example

$$d_6 = (2 + \varepsilon/T)d_4 \quad \square$$

Finally we need the following variant of Lemma 1.11 :

LEMMA 1.13.— *Let k be a positive integer. Assume that there exists a compact interval $I \subset R$ and positive constants d_7, d_8 such that*

$$d_7\|u_k\|^2 \le \int_I p(u(t), u'(t))^2 dt \le d_8\|u_k\|^2 \tag{1.22}$$

for all solutions $u(t)$ with $u_j = 0$ for all $j \neq k$. Then the same inequality holds for every compact interval $I' \subset R$ instead of I, with suitable positive constants d_7' and d_8'.

Proof.— Since $\omega_j \in R$, $\forall j \geq 1$, the function $p(u(t), u'(t))$ is in fact independant of t :

$$p(u(t), u'(t)) = p(u_k, i\omega_k u_k), \ \forall t \in R.$$

Hence, if $I' \subset R$ is an arbitrary compact interval, then the desired inequality holds with $d_7' = (|I'|/|I|)d_7$ and $d_8' = (|I'|/|I|)d_8$. $\square$

Now we turn to the proof of the theorem. Let $n \geq k_0$ be the least integer satisfying $|I_n| < |I|$. It follows from (1.8) and from Lemma 1.11 that for every compact interval $J \subset R$ of length $|J| > |I|$ there exist two positive constants c and C such that

$$c\sum_{j\geq 1}\|u_j\|^2 \leq \int_I p(u(t),u'(t))^2 dt < C\sum_{j\geq 1}\|u_j\|^2 \tag{1.23}$$

for all solutions $u(t)$ satisfying $u_1 = \ldots = u_{2n-2} = 0$. If $n = 1$, then Theorem A follows by taking $J = I$.

If $n > 1$, then we apply Lemma 1.12 with $m = 2n-1$ and with $T > |I_n|/2$ arbitrary. Condition (1.20) is satisfied by (1.23), (1.8), (1.9) and Lemma 1.13 (we use (1.8) if $m - 1 \geq k_0$ and (1.9) if $m - 1 < k_0$). Applying Lemma 1.13 and then using Lemma 1.11 we find that for every compact interval $J \subset R$ of length $|J| > |I_n|$, the estimates (1.23) hold for all solutions $u(t)$ satisfying $u_1 = \ldots = u_{2n-3} = 0$ (the constants c, C are not necessarily the same as before).

Repeating this argument $2n - 3$ times we obtain that for every compact interval $J \subset R$ of length $|J| > |I_n|$, the estimates (1.23) hold for all solutions. Putting $J = I$ hence the estimates (1.10) of Theorem A follow

The last assertion of Theorem A follows from the analysis of the explicit constants obtained during the proof □

I.3. — Proof of theorem A in the general case

We shall use the following notations : if $\eta_{j'} = 0$ for some j', then set

$\omega_1 = \omega_2 = 0$, $X_1 = X_2 = Z_{j'}$,

$\omega_{2j+1} = -\omega_{2j+2} = i(-\eta_j)^{1/2}$, $X_{2j+1} = X_{2j+2} = Z_j$ if $j < j'$

and

$\omega_{2j-1} = -\omega_{2j} = (\eta_j)^{1/2}$, $X_{2j-1} = X_{2j} = Z_j$ if $j > j'$.

If all the eigenvalues are different from zero, then let j' be the first index such that $\eta_{j'} > 0$ and set

$\omega_1 = \omega_2 = 0$, $X_1 = X_2 = \{0\}$,

$\omega_{2j+1} = -\omega_{2j+2} = i(-\eta_j)^{1/2}$, $X_{2j+1} = Z_j$ if $j < j'$

and

$\omega_{2j+1} = -\omega_{2j+2} = (\eta_j)^{1/2}$, $X_{2j+1} = X_{2j-2} = Z_j$ if $j \geq j'$.

Let us note that

$$S := \sup\{|\mathrm{Im}\omega_j| \,:\, j \geq 1\} \leq |\eta_1|^{1/2} < \infty. \tag{1.24}$$

Using this notation, there is a one-to-one correspondance between the elements (u°, u^1) of W^s and the convergent series of the form

$$(1.25) \qquad u(t) = u_1 + u_2 t + \sum_{j\geq 3} u_j e^{i\omega_j t}\ ,\ u_j \in X_j$$

in $C(R;\ D((A+a_1 I)^{s+1/2})) \cap C^1(R; D((A+a_1 I)^s))$, the latter being the solution of (1.2). In the sequel by solution we shall mean an arbitrary function $u \in C(R; D((A+a_1 I)^{s+1/2})) \cap C^1(R; D((A+a_1 I)^s))$ of the form (1.25). Introducing in V the norm $\|v\| = (a(v,v) + a_1 \|v\|_H^2)^{1/2}$ (cf. (1.15)), we have again the equivalence of

$$\|u^\circ\|_V^2 + \|u^1\|_H^2 \quad \text{and} \quad \sum_{j\geq 1} \|u_j\|^2.$$

The proof of Theorem A in the general case follows the same line as in the special case considered before :

Lemma 1.10 remains valid with the only modification that d_2 depends now also on S (cf. (1.24)). There are two small changes in the proof :

– Since the numbers ω_j are not necessarily real, (1.18) is not obvious. However, thanks to (1.24), it is satisfied if we choose $\varepsilon > 0$ sufficiently small.

– In the last estimates of the proof we obtain the factor $2 + 2e^{2S\varepsilon}$ instead of 4; hence (1.17) follows with $d_2 = d_1 d_3/(2 + 2e^{2S\varepsilon})$.

Lemma 1.11 remains valid, too, with the only change that the constants c now depend on I. In case **a)** we obtain the inequality

$$\int_{I'} p(u(t), u'(t))^2 dt \leq c(\|u_1 + u_2 s\|^2 + \sum_{j\geq 2} \|e^{i\omega_j s} u_j\|^2)$$
$$\leq ce^{2S|s|}(\|u_1 + u_2 s\|^2 + \sum_{j\geq 2} \|u_j\|^2),$$

while in case **b)** we have

$$\int_{I'} p(u(t), u'(t))^2 dt \geq c(\|u_1 + u_2 s\|^2 + \sum_{j\geq 2} \|e^{i\omega_j s} u_j\|^2)$$
$$\geq ce^{2S|s|}(\|u_1 + u_2 s\|^2 + \sum_{j\geq 2} \|u_j\|^2).$$

Therefore it is sufficient to find two positive constants c' and c'' such that

$$c'(\|u_1\|^2 + \|u_2\|^2) \leq \|u_1 + u_2 s\|^2 + \|u_2\|^2$$
$$\leq c''(\|u_1\|^2 + \|u_2\|^2),\ \forall u_1, u_2 \in X_1 = X_2.$$

This is satisfied for example if we take

$$c' = 1/(4s^2 + 4) \quad \text{and} \quad c'' = 2s^2 + 2$$

(to obtain c' we may distinguish the cases $\|u_1\| > 2|s| \quad \|u_2\|$ and $\|u_1\| \leq 2|s| \quad \|u_2\|$.)

Lemma 1.12 and its proof remain valid except that we have to write $u_2 t$ instead of $u_2 e^{i\omega_2 t}$ everywhere. Again, since we apply Lemma 1.10, the constants d_5, d_6 will now depend also on S.

Finally, Lemma 1.13 remains valid, too, but the proof is slightly different if ω_k is imaginary. If $k \neq 2$, then we have the formula

$$p(u(t), u'(t)) = |e^{i\omega_k t}| p(u_k, i\omega_k u_k),\ \forall t \in R$$

whence the desired inequality follows easily with

$$d_7' = \left(\int_{I'} e^{-2S|t|} dt / \int_I e^{2S|t|} dt\right) d_7$$

and

$$d_8' = \left(\int_{I'} e^{2S|t|} dt / \int_I e^{-2S|t|} dt\right) d_8.$$

The case $k = 2$ is more technical. Clearly it is sufficient to find two positive, continuous real functions defined in $\{(a,b) \in R^2 \ : \ a < b\}$ such that

$$(1.26) \qquad \left| \begin{array}{l} f(a,b)(p(u_2,0)^2 + p(0,u_2)^2) \leq \int_a^b p(u(t), u'(t))^2 dt \\ \leq g(a,b)(p(u_2,0)^2 + p(0,u_2)^2),\ \forall u_2 \in X_2,\ u(t) = u_2 t. \end{array} \right.$$

The right side of (1.26) follows at once from the triangle inequality and from the Cauchy-Schwary inequality. We have

$$\int_a^b p(u(t), u'(t))^2 dt \leq 2 \int_a^b t^2 p(u_2, 0)^2 + p(0, u_2)^2 dt$$
$$\leq 2 \int_a^b 1 + t^2 dt (p(u_2,0)^2 + p(0,u_2)^2),$$

whence the right side inequality of (1.26) is satisfied with

$$g(a,b) = 2\int_a^b 1 + t^2 dt.$$

To prove the reverse inequality first we write the inequality

$$s^2 p(u_2,0)^2 \le 2p(u_2 t, u_2)^2 + 2p(u_2(t+s), u_2)^2, \ s,t \in R;$$

this may be obtained again by applying the triangle inequality and the Cauchy-schwarz inequality. Choosing $s = (b-a)/2$ and integrating by t from a to $a+s$ we find

$$p(u_2,0)^2 \le 16(b-a)^{-3} \int_a^b p(u(t),u'(t))^2 dt. \tag{1.27}$$

Next we write the inequality

$$p(0,u_2)^2 \le 2p(u_2 t, u_2)^2 + 2t^2 p(u_2,0)^2, \ t \in R.$$

Integrating by t from a to b and then using (1.27) we obtain

$$p(0,u_2)^2 \le \Big(\frac{2}{b-a} + \frac{32}{3}\frac{b^3-a^3}{(b-a)^3}\Big) \int_a^b p(u(t),u'(t))^2 dt. \tag{1.28}$$

The left side inequality of (1.26) now follows from (1.27) and (1.28) by taking

$$f(a,b) = \Big(16(b-a)^{-3} + \frac{2}{b-a} + \frac{32}{3}\frac{b^3-a^3}{(b-a)^3}\Big)^{-1}. \quad \square$$

1.4. — Proof of theorem B

We use the same notations as in the preceeding subsection. First we note that condition (1.12) implies in fact condition (1.9) because $Z_k \times Z_k$ if finite-dimensional. Using this remark, we may repeat the proof of Theorem A, replacing Lemma 1.12 by the following one :

LEMMA 1.14. — *Assume that for some integer $m \ge 2$ and for some positive numbers T and d_1 we have*

$$d_1 \sum_{j\ge 1} \|u_j\|^2 \le \int_{-T}^{T} p(u(t),u'(t))^2 dt \tag{1.29}$$

for all solutions $u(t)$ such that $u_1 = \ldots = u_{m-1} = 0$, and for all solutions $u(t)$ such that $u_j = 0$ for all $j \neq m-1$.

Then for every $\varepsilon > 0$ there exists a positive constant d_5 such that

$$d_5 \sum_{j\geq 1} \|u_j\|^2 \leq \int_{-T-\varepsilon}^{T+\varepsilon} p(u(t), u'(t))^2 dt \tag{1.30}$$

for all solutions $u(t)$ such that $u_1 = \ldots = u_{m-2} = 0$.

Proof. — Let us denote by Y (resp. by Y^b) the vector space of all solutions satisfying $u_1 = \ldots = u_{m-2} = 0$ (resp. $u_1 = \ldots = u_{m-1} = 0$) and let Y^a be finite-dimensional vector space of all solutions satisfying $u_j = 0$ for all $j \neq m-1$. Then Y is the direct sum of Y^a and Y^b. Given $u \in Y$ arbitrarily, we introduce the same notations u^a, u^b, $|u|_0$, $|u|_\varepsilon$, $\|u\|_*$ as in the proof Lemma 1.12. (If $m = 3$, then according to the preceeding subsection we set $u^a(t) = u_2 t$ instead of $u_2 e^{i\omega_2 t}$.) Then we have the decomposition $u = u^a + u^b$, $u^a \in Y^a$, $u^b \in Y^b$.

Let us observe that the semi norm $|\cdot|_\varepsilon$ is in fact a *norm* on Y. Indeed, if $u \in Y$ and $|u|_\varepsilon = 0$, then applying Lemma 1.10 to (1.29) we obtain $\|u^b\|_* \leq |u|_\varepsilon$ whence $u^b = 0$ and $u = u^a$.

Now applying (1.29) for $u = u^a$ we find $\|u\|_* = \|u^a\|_* = 0$ i. e. $u = 0$.

We shall prove the lemma by contradiction. Assuming that (1.30) does not hold, in view of Lemma 1.10 there exists a sequence $(u_n)_{n\geq 1} \subset Y$ such that, using the decompositions $u_n = u_n^a + u_n^b$, $u_n^a \in Y^a$, $u_n^b \in Y^b$, we have

$$\|u_n^a\|_* = 1, \quad \forall n \geq 1, \tag{1.31}$$

$$|u_n|_\varepsilon \to 0 \tag{1.32}$$

and

$$\|u_n^b\|_* \to 0 \tag{1.33}$$

Being Y^a finite-dimensional, we may also assume (taking a subsequence if needed) that (u_n^a) converges :

$$\|u_n^a - u^a\|_* \to 0 \tag{1.34}$$

Since $|\cdot|_\varepsilon$ is also a norm on the finite-dimensional space Y^a, (1.34) implies that

$$(1.35) \qquad |u_n^a - u^a|_\varepsilon \to 0$$

From (1.32) and (1.35) we conclude that u^a belongs to the closure of Y^b with respect to the norm $|\cdot|_\varepsilon$ in Y. Our hypothesis (1.29) implies that

$$d_1^{1/2}\|v\|_* \leq \|v\|_\varepsilon, \ \forall v \in Y^b.$$

This inequality extends by continuity for all $u \in Y$ in the closure of Y^b with respect to $|\cdot|_\varepsilon$. In particular we have

$$d_1^{1/2}\|u^a + u_n^b\|_* \leq |u^a + u_n^b|_\varepsilon, \ \forall n \geq 1.$$

Using (1.32), (1.33), (1.35) and letting $n \to$, we obtain that $\|u^a\|_* = 1$.

On the other hand, it follows at once from (1.31) and (1.34) that $\|u^a\|_* = 1$. This contradiction proves the lemma. □

1.5. — Proof of proposition C

Applying the same complexification argument as for Theorems A and B, it is sufficient to consider the complex case.

The first part of the proposition is immediate. Assume that there is a compact interval $I \subset R$ such that

$$(1.36) \qquad \int_I p(u(t), u'(t))^2 dt > 0, \ \forall 0 \neq (u^\circ, u^1) \in Z_k \times Z_k,$$

and let $0 \neq v \in Z_k$ be given arbitrarily. Applying (1.36) with $u(t) = ve^{\pm i\omega t}$ and using the obvious identity

$$\int_I p(u(t), u'(t))^2 dt = \int_I |e^{\pm i\omega t}|^2 dt \ p(v, \pm i\omega v)^2$$

we obtain

$$(1.37) \qquad p(v, \pm i\omega v) > 0.$$

Conversely, assuming that (1.37) holds for all $0 \neq v \in Z_k$, it is sufficient to prove that $(u^\circ, u^1) \in Z_k \times Z_k$ and $\int_I p(u(t), u'(t))^2 dt = 0$ imply $u^\circ = u^1 = 0$.

Indeed, being $Z_k \times Z_k$ finite-dimensional, then there exist two positive constants c and C such that

$$c\|u^\circ\|_V^2 + \|u^1\|_H^2) \le \int_I p(u(t), u'(t))^2 dt$$
$$\le C(\|u^\circ\|_V^2 + \|u^1\|_H^2),\ \forall (u^\circ, u^1) \in Z_k \times Z_k.$$

Consider first the case $\eta_k \ne 0$. Then $u(t)$ has the form

$$u(t) = ve^{i\omega t} + we^{-i\omega t}, v \quad \text{and} \quad w \in Z_k,$$

and from the hypothesis $\int_I p(u(t), u'(t))^2 dt = 0$ we deduce that

$$p(ve^{i\omega t} + we^{-i\omega t}, i\omega(ve^{i\omega t} - we^{-i\omega t})) = 0,\ \forall t \in I.$$

Using the homogenity of p and applying the triangle inequality hence we obtain

$$p(v(e^{2i\omega t} - e^{2i\omega s}),\ i\omega v(e^{2i\omega t} - e^{2i\omega s})) = 0$$

for all $t, s \in I$. Choosing t, s such that $e^{2i\omega t} - e^{2i\omega s} \ne 0$, using the homogenity of p and finally applying (1.37) we obtain $v = 0$. We may prove by the same way that $w = 0$. Hence $u(t) \equiv 0$ and $u^\circ = u^1 = 0$.

Consider now the case $\eta_k = 0$. Then $u(t)$ has the form

$$u(t) = vt + w,\ v \quad \text{and} \quad w \in Z_k.$$

Repeating the above argument now we obtain

$$p(vt + w, v) = 0,\ \forall t \in I \tag{1.38}$$

and

$$p(v(t-s), 0) = 0,\ \forall t, s \in I.$$

Choosing $t \ne s$, using the homogenity of p and applying (1.37) we obtain $v = 0$. Substituting this result into (1.38) we obtain $p(w, 0) = 0$ whence, using (1.37) again, $w = 0$. Hence $u(t) = 0$ and $u^\circ = u^1 = 0$. □

2. Applications to the exact controllability of the wave equation

The main purpose of this section is to introduce the new estimation method. We shall improve some results obtained in J. -L. Lions [2], [3].

In the first subsection pure Dirichlet action is considered. We show that Theorem A allows us to treat equations containing a potential term by constructive way. The same method may be applied in other situations as well, to prove that the *lower-order terms* in the evolution equation do not change the minimal time of exact controllability, cf. V. Komornik [5], [6].

In the second subsection pure Neumann action is considered. In this case the "natural" semi norm p is not a norm; this motivates the introduction of *strenghtened norms*. This is a typical situation for the application of Theorem B.

2.1. — A problem with a lower order term

Let Ω be a non-empty, bounded, connected open set in $R^N (N = 1, 2, \ldots)$ having a boundary Γ of class C^2, and denote by $\nu(x) = (\nu_1(x), \ldots, \nu_N(x))$ the unit normal vector to Γ, directed towards the exterior of Ω. Fix a point $x^\circ \in R^N$ arbitrarily and set

$$\Gamma_+ = \{x \in \Gamma \ : \ m(x) \cdot \nu(x) > 0\},$$
$$\Gamma_- = \{x \in \Gamma \ : \ m(x) \cdot \nu(x) \leq 0\},$$

and

$$R_0 = \sup\{|m(x)| \ : \ x \in \Gamma\},$$

where $m(x) = x - x^\circ$, $x \in R^N$, and $\cdot$ denotes the scalar product in R^N.

Let $q : \Omega \to R'$ be a function satisfying

$$\text{(2.1)} \qquad \left| \begin{array}{l} q \in L^p(\Omega) \quad \text{with} \quad p > N \quad \text{if} \quad N > 2 \quad \text{and} \\ \text{with} \quad p > 2 \quad \text{if} \quad N = 1. \end{array} \right.$$

Then, applying the Sobolev imbedding theorem and the interpolational inequality, there are positive constants a_0, a_1 such that

$$\text{(2.2)} \qquad \int_\Omega |\nabla u|^2 + qu^2 dx \geq \int_\Omega a_0 |\nabla u|^2 - a_1 u^2 dx, \ \forall u \in H_0^1(\Omega)$$

Consider the following system :

$$\text{(2.3)} \qquad \left| \begin{array}{l} u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times R, \\ u(0) = u^\circ \quad \text{and} \quad u'(0) = u^1 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \Gamma \times R. \end{array} \right.$$

(As usual, we write u', u'' for $\partial u/\partial t$, $\partial^2 u/\partial t^2$). It is well-known that for every initial date $(u^\circ, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, (2.3) has a unique solution

$$u \in C(R; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(R; H_0^1(\Omega)).$$

The purpose of this subsection is to prove the following estimate :

THEOREM 2.1.— *Assume (2.1), (2.2) and let $I \subset R$ be a compact interval of length $|I| > 2R_0$. Then there exist two positive constants c and C such that for every $(u^\circ, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, solution of (2.3) satisfies the following inequalities :*

$$(2.4) \qquad \left| \begin{aligned} & c(\|u^\circ\|^2_{H_0^1(\Omega)} + \|u^1\|^2_{L^2(\Omega)}) \leq \int_I \int_{\Gamma_+} (\partial_y u)^2 d\Gamma dt \\ & \leq C(\|u^\circ\|^2_{H_0^1(\Omega)} + \|u^1\|^2_{L^2(\Omega)}). \quad \square \end{aligned} \right.$$

(In (2.4) $\partial_y u$ denotes the normal derivative of u.)

Remark 2.2.— In the special case $q = 0$ these estimates were obtained by L. F. Ho [1] under a stronger hypothesis on $|I|$, and then by J. -L. Lions [2] under the above hypothesis $|I| > 2R_0$, using a non-constructive compactness-uniqueness argument. Later a constructive proof was given in V. Komornik [1], [2] (cf. also J.-L. Lions [3]). The case $q \geq 0$ of Theorem 2.1 was proved earlier in V. Komornik and E. Zuazua [2], using an indirect argument. Applying the same method, E. Zuazua [3] proved also a variant of Theorem 2.1, under the assumption $q \in L^\infty(\Omega \times I)$. $\square$

Remark 2.3.— Applying HUM as in J.-L. Lions [2], [3], Theorem 2.1 implies the exact controllability of the system

$$(2.5) \qquad \left| \begin{array}{l} y'' - \Delta y + qy = 0 \quad \text{in} \quad \Omega \times (0,T), \\ y(0) = y^\circ \quad \text{and} \quad y'(0) = y^1 \quad \text{in } \Omega, \\ y = 0 \quad \text{on} \quad \Gamma_- \times (0,T) \quad \text{and} \quad y = v \quad \text{on} \quad \Gamma_+ \times (0,T) \end{array} \right.$$

in the following sense : if $T > 2R_0$, then for every initial data $(y^\circ, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ we can find a corresponding control $v \in L^2(\Gamma_+ \times (0,T))$ driving the system (2.5) to rest in time T, i. e. such that $y(T) = y'(T) = 0$. For the presentation of the HUM method (Hilbert Uniqueness Method) we refer to J.-L. Lions [1], [2], [3]. $\square$

Proof of Theorem 2.1.— We are going to apply Theorem A with $V = H_0^1(\Omega), H = L^2(\Omega)$, $a(u,v) = \displaystyle\int_\Omega \nabla u \cdot \nabla v + quv \, dx$, $s = 1/2$ and

$$p(u^\circ, u^1) = \left(\int_{\Gamma_+} (\partial_y u)^2 d\Gamma \right)^{1/2} ;$$

the integer k_0 will be chosen later. Then we introduce the eigenvalues η_k, the eigenspaces Z_k, the spaces $W^{1/2}$, $W_k^{1/2}$ and the energy E according to the general scheme described in Subsection 1.1. We have in particular

$$E = E(u;t) = (1/2)\int_\Omega (u')^2 + |\nabla u|^2 + qu^2 dx, \quad \forall t \in R.$$

It follows from (2.1) and (2.2) that the bilinear form $a(u,v)$ is continuous on V and satisfies the condition (1.1).

To prove the properties (1.8) and (1.9) we adapt the method in J.-L. Lions [2], [3], with an extra argument if we cannot choose $a_1 = 0$ in (2.2).

First we choose an arbitrary function $h \in C^1(\overline{\Omega}; R^N)$ such that $h = \nu$ on Γ. Multiplying the equation (2.3) by $h \cdot \nabla u$ and integrating by parts on $\Gamma \times J$ where $J \subset R$ is an arbitrary compact interval, we obtain easily the estimates

$$\left| \begin{array}{l} \displaystyle\int_J p(u(t), u'(t))^2 dt \leq C'(\|u^\circ\|_v^2 + \|u^1\|_H^2), \\ \forall (u^\circ, u^1) \in W^{1/2} \end{array} \right.$$

with a constant C' depending on J. In particular, the right side inequalities in (1.8) and (1.9) are satisfied for every compact interval I_k.

To prove the reverse inequality we replace the multiplier $h \cdot \nabla u$ by $2m \cdot \nabla u + (N-1)u$. We obtain for every $T > 0$ the identity

$$(2.6)\qquad \left| \begin{array}{l} \displaystyle\int_0^T \int_\Gamma (m \cdot \nu)|\partial_y u|^2 d\Gamma dt \\ \displaystyle= \left[\int_\Omega (2m \cdot \nabla u + (N-1)u)u' dx \right]_0^T \\ \displaystyle+ \int_0^T \int_\Omega |u'|^2 + |\nabla u|^2 dx dt \\ \displaystyle+ \int_0^T \int_\Omega qu(2m \cdot \nabla u + (N-1)u) dx dt. \end{array} \right.$$

Let k be a positive integer and assume that $(u^\circ, u^1) \in W_k^{1/2}$. In the following estimates $0(1)$ and $0(1)$ will denote diverse constants depending only on k (i. e. independent of $t \in R$ and of the choice of $(u^\circ, u^1) \in W_k^{1/2}$) and being bounded resp. converting to zero as $k \to \infty$. Furthermore, in order to simplify the notations we shall write $\|\cdot\|_*$ instead of $\|\cdot\|_{L^*(\Omega)}$, $\forall r \in [1, \infty]$.

We needs some inequalities. First, it follows from (2.1), applying the inequalities of Sobolev and Poincaré, that there exists a constant $r_0 > 2p/(p-2)$ satisfying

$$\|u\|_* \le o(1)\|\nabla u\|_2, \ \forall r \in [1, r_0]. \tag{2.7}$$

Applying (2.7) to the right side of the inequality (Hölder)

$$\left| \int_\Omega qu^2 dx \right| \le \|q\|_p \|u\|_{2p/(p-2)\|u\|_2} \tag{2.8}$$

and using the definition of η_k we find

$$\eta_k \int_\Omega u^2 dx \le \int_\Omega |\nabla u|^2 + qu^2 dx \le o(1) \int_\Omega |\nabla u|^2 dx$$

whence

$$\|u\|_2 \le o(1)\|\nabla u\|_2. \tag{2.9}$$

Interpolating between (2.7) and (2.9) we deduce

$$\|u\|_{2p/(p-2)} = o(1)\|\nabla u\|_2 \,, \tag{2.10}$$

and then combining (2.7), (2.8), (2.10) we obtain

$$\left| \begin{aligned} &(1 - o(1)) \int_\Omega |\nabla u|^2 dx \le \int_\Omega |\nabla u|^2 + qu^2 dx \\ &\le (1 + o(1)) \textstyle\int_\Omega |\nabla u|^2 dx. \end{aligned} \right. \tag{2.11}$$

Let us finally recall the inequality

$$\|2m \cdot \nabla u + (N-1)u\|_2 \le \|2m \cdot \nabla u\|_2 \le 2R_0 \|\nabla u\|_2 \tag{2.12}$$

obtained in V. Komornik [1], [2].

Using (2.10) and (2.12) in (2.6) we obtain the following estimate :

$$\left| \begin{aligned} &R_0 \int_0^T \int_{\Gamma_+} (\partial_y u)^2 d\Gamma dt \\ &\ge -2R_0(\|\nabla u(T)\|_2 \|u'(T)\|_2 + \|\nabla u(0)\|_2 \|u'(0)\|_2) \\ &+ \int_0^T \int_\Omega (u')^2 + |\nabla u|^2 dx\ dt - o(1) \int_0^T \|\nabla u(t)\|_2^2 dt. \end{aligned} \right.$$

Using (2.11) and the definition of the energy hence we conclude that

$$R_0 \int_0^T \int_{\Gamma_+} (\partial_\nu u)^2 d\Gamma dt \geq 2T(1 - 0(1))E - 4R_0(1 + o(1))E$$

i.e.

$$\int_0^T \int_{\Gamma_+} (\partial_\nu u)^2 d\Gamma dt \geq (2/R_0)(T - 2R_0 - o(1))E. \tag{2.13}$$

Since $\eta_k \to \infty$, in view of Remark 1.5 (2.13) implies the existence of a positive integer k_0 such that the left side inequality in (1.8) is satisfied for all $k \geq k_0$. Consequently, the hypothesis (1.8) of Theorem A is satisfied with a sequence of intervals I_k such that $T = 2R_0$.

Finally we prove that hypothesis (1.9) of Theorem A is satisfied for all $k < k_0$. According to Proposition C, it is sufficient to verify that $v \in Z_k$ and $\in_{\Gamma_+} |\partial_y v|^2 d\Gamma = 0$ imply $v = 0$. In other words, it is sufficient to prove that the problem

$$\left| \begin{array}{l} -\nabla v + qv = \eta_k v \quad \text{in} \quad \Omega \\ v = 0 \quad \text{on} \quad \Gamma \\ \partial_\nu v = 0 \quad \text{on} \quad \Gamma_+ \end{array} \right.$$

has no nontrivial solutions in $H_0^1(\Omega)$. This is, however, a well-known unique continuation theorem, cf. e.g. C.E. Kenig, A. Ruiz and C.D. Sogge [1].

We may therefore apply Theorem A and hence Theorem 2.1 follows. □

Remark 2.4.— The application of inequality (2.12) has simplified the proof, but its use is not necessary. Therefore the above method may be also applied in other problems of this type where we do not have an inequality analogous to (2.12). □

Remark 2.2.bis.— For $q \geq 0$ Theorem 2.1 was proven earlier in V. Komornik and E. Zuazua [2] by an indirect compactness-uniqueness argument. Applying this method E. Zuazua [3] proved a variant of Theorem 2.1 under the assumption $q \in L^\infty(I \times \Omega)$. □

2.2.— A problem with a strenghtened norm

Consider here instead of (2.3) the following system :

$$\left| \begin{array}{l} u'' - \Delta u = 0 \quad \text{in} \quad \Omega \times R, \\ u(0) = u^\circ \quad \text{and} \quad u'(0) = u^1 \quad \text{in } \Omega, \\ \partial_\nu u = 0 \quad \text{on} \quad \Gamma \times R. \end{array} \right. \tag{2.14}$$

It is well-known that for every initial data $(u^\circ, u^1) \in H^2(\Omega) \times H^1(\Omega)$ satistying the compatibility condition $\partial_\nu u^\circ = 0$ on Γ, (2.14) has a unique solution in

$$C(R; H^2(\Omega)) \cap C^1(R; H^1(\Omega))$$

Fix $x^\circ \in R^N$ arbitrarily and introduce the same notations Γ_+, Γ_-, $m(x)$, R_0 as in the preceeding subsection. We shall prove the following theorem.

THEOREM 2.5.— *Let Γ_0 be an arbitrary subset of positive measure in Γ and let $I \subset R$ be an arbitrary compact interval of length $|I| > 2R_0$. Then there exists a positive constant c such that for every initial data $(u^\circ, u^1) \in H^2(\Omega) \times H^1(\Omega)$, $\partial_\nu u^\circ = 0$ on Γ, the solution of (2.14) satisfies the following inequality :*

$$(2.15) \qquad \left| \begin{array}{l} \displaystyle\int_I \int_{\Gamma_+} |u'|^2 d\Gamma dt + \int_I \int_{\Gamma_-} |\nabla u|^2 d\Gamma\, dt \\ \displaystyle\int_I \int_{\Gamma_0} u^2 d\Gamma\, dt \geq c(\|u^0\|^2_{H^1(\Omega)} + \|u^1\|^2_{L^2(\Omega)}). \quad \Box \end{array} \right.$$

Remark 2.6.— In the special case $\Gamma_0 = \Gamma$ Theorem 2.5 was proven in J.-L. Lions [2], by an indirect argument. It was later improved in V. Komornik [1], [2] by taking $\Gamma_0 = \Gamma_+$ and by giving a constructive proof; cf. also J.-L. Lions [3].

Remark 2.7.— Applying HUM we may deduce from Theorem 2.5 the exact controllability of the system

$$\left| \begin{array}{l} y'' - \Delta y = 0 \quad \text{in} \quad \Omega \times (0, T), \\ y(0) = y^\circ \quad \text{and} \quad y'(0) = y^1 \quad \text{in} \quad \Omega, \\ \partial_\nu y = v \quad \text{on} \quad \Gamma \times (0, T) \end{array} \right.$$

in suitable function spaces if $T > 2R_0$, cf. J.-L. Lions [2], [3]. $\Box$

Proof of Theorem 2.5.— We shall apply Theorem B with $H = L^2(\Omega)$, $V = H^1(\Omega)$, $a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$, $s = 1/2$, $k_0 = 2$ and

$$p(u^\circ, u^1) = \left(\int_{\Gamma_+} |u^1|^2 d\Gamma + \int_{\Gamma_-} |\nabla u^\circ|^2 d\Gamma + \int_{\Gamma_0} |u^\circ|^2 d\Gamma \right)^{1/2}.$$

The energy of the solutions is now given by the formula

$$E = E(u; t) = (1/2) \int_\Omega |u'(t)|^2 + |\nabla u(t)|^2 dx.$$

We recall the following identity obtained by the same multiplier $2m \cdot \nabla u + (N-1)u$ as in the preceeding subsection :

$$(2.16) \qquad \left| \begin{array}{l} \displaystyle\int_0^T \int_\Gamma (m \cdot \nu)(u'|^2 - |\nabla u|^2) d\Gamma \, dt \\ \displaystyle = \left[\int_\Omega (2m \cdot \nabla u + (N-1)u) u' dx \right]_0^T \\ \displaystyle + \int_0^T \int_\Omega |u'|^2 + |\nabla u|^2 dx \, dt. \end{array} \right.$$

Let $k \geq 2$ be an arbitrary integer and assume that $(u^\circ, u^1) \in W_k^{1/2}$. Then we obtain easily from this identity the following estimate :

$$(2.17) \qquad \left| \begin{array}{l} \displaystyle\int_0^T \int_{\Gamma_+} |u'|^2 d\Gamma \, dt + \int_0^T \int_{\Gamma_-} |\nabla u|^2 d\Gamma \, dt \\ \geq (2/R_0)(T - 2R_0 - (N-1)\eta_k^{-1/2})E. \end{array} \right.$$

If $T > 2R_0$, then for k sufficiently large the factor of E is positive. Furthermore, for k arbitrary, the factor of E is positive again if T is sufficiently large. Finally it is clear that in the present case we have $\eta_k > 0$, $\forall k \geq 2$. Using Remark 1.9 we conclude that the hypothesis (1.11) of Theorem B is satisfied with a sequence intervals I_k such that $T_0 = 2R_0$.

Now we verify the condition (1.12). Since $k_0 = 2$ and $\eta_1 = 0$, in view of Proposition C it is sufficient to prove that

$$p(v, 0) \neq 0, \quad \forall 0 \neq v \in Z_1.$$

This is easy to verify. Indeed, the elements of Z_1 are the constant functions. If v is a non-zero constant function, then we have

$$p(v, 0) = \left(\int_{\Gamma_0} v^2 d\Gamma \right)^{1/2} = |v| \text{meas}(\Gamma_0) > 0$$

since Γ_0 is of positive measure in Λ by hypothesis. This proves (1.12).

We may apply Theorem B and hence Theorem 2.5 follows. $\square$

3. Applications to the exact controllability of Kirchoff plates

Our standard reference in this (and in the next) section is J. Lagnese and J.-L. Lions [1], referred in the sequel by [LL]; it contains also the physical interpretation of the models considered below. The purpose of this section is to complete the results of [LL] concerning the exact controllability of the Kirchoff plate models.

Let Ω be a non-empty, bounded domain in R^2 having a boundary Γ of class C^2, let T be a positive number and fix a point $x_0 \in R^2$ arbitrarily. We introduce the same notations $\nu(x) = (\nu_1(x), \nu_2(x))$, $m(x) = x - x^\circ$, Γ_+, Γ_- and R_0 as in the preceeding section. After recaling, the state equation of a Kirchoff plate is given by

$$w'' - h^2 \Delta w'' + \Delta^2 w = 0 \quad \text{in} \quad \Omega \times (0,T) \tag{3.1}$$

where h denotes the width of the plate, and we have the usual initial conditions

$$w(0) = w^\circ \quad \text{and} \quad w'(0) = w^1 \quad \text{in} \quad \Omega. \tag{3.2}$$

Following [LL], three kinds of boundary conditions will be considered, given by (3.3), (3.4) and (3.5), respectively (we shall denote the normal derivative by ∂_ν) :

$$\left| \begin{array}{l} w = \partial_y w = 0 \quad \text{on} \quad \Gamma_- \times (0,T), \\ w = 0 \quad \text{and} \quad \partial_\nu w = v \quad \text{on} \quad \Gamma_+ \times (0,T), \end{array} \right. \tag{3.3}$$

$$\left| \begin{array}{l} w = \Delta w = 0 \quad \text{on} \quad \Gamma_- \times (0,T), \\ w = v_0 \quad \text{and} \quad \Delta w = v_1 \quad \text{on} \quad \Gamma_+ \times (0,T), \end{array} \right. \tag{3.4}$$

$$\left| \begin{array}{l} w = 0 \quad \text{and} \quad \partial_y w = v_0 \quad \text{on} \quad \Gamma_+ \times (0,T), \\ \Delta w + (1-\mu) B_1 w = v_1 \quad \text{on} \quad \Gamma_- \times (0,T), \\ \partial_y \Delta w + (1-\mu) B_2 w - h^2 \partial_\nu w'' = v_2 \quad \text{on} \quad \Gamma_- \times (0,T); \end{array} \right. \tag{3.5}$$

in the last case μ denotes the Poisson's ratio and $B_1 w$, $B_2 w$ are given by

$$B_1 w = 2\nu_1 \nu_2 \partial_1 \partial_2 w - \nu_1^2 \partial_2^2 w - \nu_2^2 \partial_1^2 w, \tag{3.6}$$

$$B_2 w = \partial_\tau [(\nu_1^2 - \nu_2^2) \partial_1 \partial_2 w + \nu_1 \nu_2 (\partial_2^2 w - \partial_1^2 w)] \tag{3.7}$$

where $\quad \tau = (-\nu_2, \nu_1)$.

Our main goal is to prove the convergence of the optimal exact control functions v (respectively v_i) if $h \to 0$.

3.1. — The first case

We shall consider in this subsection the system (3.1), (3.2), (3.3). Let us denote by a the least positive constant satisfying the condition

$$\int_\Omega \sum_{i,k=1}^{2} (\partial_i \partial_k z)^2 dx \leq a^2 \int_\Omega (\Delta z)^2 dx, \ \forall z \in H_0^2(\Omega). \tag{3.8}$$

We are going to prove the following theorems :

THEOREM 3.1.— *Assume that $T > aR_0h$. Then for every initial data $(w^\circ, w^1) \in H_0^2(\Omega) \times L^2(\Omega)$ there exists a control $v \in L^2(\Gamma \times (0,T))$ such that the solution of (3.1), (3.2), (3.3) satisfies $w(T) = w'(T) = 0$ in Ω.* □

Theorem 3.1 will be proved by the HUM method. Hence we obtain in particular that among the control functions v there is a unique one, denoted by v_h, which minimizes the integral

$$\int_0^T \int_\Gamma v^2 d\Gamma \, dt.$$

THEOREM 3.2.— *Let $T > 0$ and $(\omega^0, \omega^1 \in H_0^2(\Omega) \times L^2(\Omega)$ be given arbitraly. Then $v_h \to v$ in $L^2(\Gamma \times (0,T))$ weakly as $h \to 0$.* □

Remark 3.3.— The case $h = 0$ of Theorem 3.1 was already proven earlier (by a nonconstructive way) by E. Zuazua [2]. □

Remark 3.4.— Theorems 3.1 and 3.2 were proven earlier in [LL] under stronger hypotheses on T. □

For the proof of these theorems consider, following [LL], the homogeneous system (3.1), (3.2) and

$$\omega = \partial_\nu \omega = 0 \quad \text{on} \quad \Gamma \times (0,T). \tag{3.9}$$

For every $(\omega^0, \omega^1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ this system has a unique solution in $C([0,T];\ H_0^2(\Omega)) \cap C^1([0,T];\ H_0^1(\Omega))$. Defining the energy of the solution by

$$E = (1/2) \int_\Omega (\omega')^2 + h^2 |\nabla \omega'|^2 + (\Delta \omega)^2 \, dx, \tag{3.10}$$

it is independent of $t \in [0, T]$.

According to HUM, Theorem 3.1 will be proven if we establish the estimates

$$cE \leq \int_0^T \int_\Gamma (\Delta\omega)^2 d\Gamma dt \tag{3.11}$$

for all initial date (ω^0, ω^1) from some dense subspace of $H_0^2(\Omega) \times H_0^1(\Omega)$, where c is a positive constant, depending on $T(> aR_0 h)$ but independent of the choice of (ω^0, ω^1).

Furthermore, Theorem 3.2 will follow from the same estimates (3.11) if we prove that c may be chosen uniformly with respect to $h \to 0$. Thus both theorems will follow from the following stronger result :

THEOREM 3.5.— *Fix a positive number h_0 satisfying $T > aR_0 h_0$. Then there are two positive constants c and C such that for every $0 \leq h \leq h_0$ and for every $(\omega^0, \omega^1) \in (H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega)$ the solution of (3.1), (3.2), (3.9) satisfies the inequalities*

$$cE \leq \int_0^T \int_\Gamma (\Delta\omega)^2 d\Gamma\, dt \leq CE. \tag{3.12}$$

Proof.— We are going to apply Theorem A with the following choice : we take $H = L^2(\Omega)$ if $h = 0$ and $H = H_0^1(\Omega)$ if $h > 0$, endowed with the norm

$$\|v\|_H = \left(\int_\Omega v^2 + h^2 |\nabla v|^2 dx \right)^{1/2}$$

and we set $V = H_0^2(\Omega)$, $a(u, v) = \int_\Omega \Delta u \Delta v dx$, $s = 1/2$, $k_0 = 1$ and

$$p(\omega^0, \omega^1) = \left(\int_\Gamma (\Delta\omega^0)^2 d\Gamma \right)^{1/2}.$$

We introduce the notations η_j and Z_j ($j \geq 1$) as in Theorem A (they depend on h); note that $\eta_1 > 0$.

Let $k \in C^1(\overline{\Omega}; R)$ be arbitrary function such that $k = \nu$ on Γ. Multiplying the equation (3.1) by $k \cdot \nabla\omega$ and integrating by parts on $\Omega \times (0, T)$, it is easy to obtain the right side inequality of (3.12) with a constant C, independent of $h \in [0, h_0]$; cf. [LL]. In other words, using also Remark 1.5, the right side inequality (1.8) in Theorem A is satisfied.

To prove the reverse inequality, let us recall the following basic identity from [LL] :

$$
(3.13)\qquad \left|\begin{aligned}
&\int_0^T \int_\Gamma (m\cdot\nu)(\Delta\omega)^2 d\Gamma\, dt\\
&= 2TE + \int_0^T \int_\Omega 2(\omega')^2 dx\, dt\\
&+ \left[\int_\Omega \omega'(2m\cdot\nabla\omega - \omega) + h^2\nabla\omega'\cdot\nabla(2m\cdot\nabla\omega-\omega)dx\right]_0^T ;
\end{aligned}\right.
$$

this is obtained by the same multiplier method as above, using instead of k the multiplier $2m\cdot\nabla\omega - \omega$.

If $(\omega^0, \omega^1) \in W_k^{1/2}$ for some integer $k \geq 1$, then we have

$$
(3.14)\qquad \int_\Omega (\Delta w)^2 dx \geq \eta_k \int_\Omega \omega^2 + h^2|\nabla\omega|^2 dx,\ \forall t \in [0,T].
$$

We also need the following estimates :

$$
(3.15)\qquad \|\nabla\omega\|_2 \leq \eta_k^{-1/4}\|\Delta\omega\|_2 ,\ \forall t \in [0,T].
$$

For proving this we remark that

$$
\int_\Omega |\nabla\omega|^2 dx = -\int_\Omega \omega\Delta\omega dx \leq \|\omega\|_2\|\Delta\omega|_2 ;
$$

dividing by $\int_\Omega (\Delta\omega)^2 dx = \|\Delta\omega\|_2^2$ and applying (3.14), hence (3.15) follows.

Furthermore, using (3.8) and (3.15) we obtain

$$
\int_\Omega |\nabla(2m\cdot\nabla\omega - \omega)|^2 dx = \sum_{i=1}^2 \int_\Omega (\partial_i\omega)^2
$$

$$
+4(\partial_i\omega)\left(\sum_{k=1}^2 m_k\partial_k\partial_i\omega\right) + 4\left(\sum_{k=1}^2 m_k\partial_k\partial_i\omega\right)^2 dx
$$

$$
\leq (\eta_k^{-1/2} + 4\eta_k^{-1/4}aR_0 + 4a^2R_0^2)|\Delta\omega\|_2^2
$$

i. e.

$$
(3.16)\qquad \left|\begin{aligned}
&\|\nabla(2m\cdot\nabla\omega - \omega)\|_2\\
&\leq (\eta_k^{-1/2} + 4\eta_k^{-1/4}aR_0 + 4a^2R_0^2)^{1/2}\|\Delta\omega\|_2.
\end{aligned}\right.
$$

From (3.14)-(3.16) we deduce for every $t \in [0,T]$ the following estimates :

$$|\int_\Omega \omega'(2m\cdot\nabla\omega-\omega)+h^2\nabla\omega'\cdot\nabla(2m\cdot\nabla\omega-\omega)dx|$$
$$\leq \|\omega'\|_2(2R_0\eta_k^{-1/4}+\eta_k^{-1/2})\|\Delta\omega\|_2$$
$$+h^2\|\nabla\omega'\|_2(\eta_k^{-1/2}+4\eta_k^{-1/4}aR_0+4a^2R_0^2)^{1/2}\|\Delta\omega\|_2$$
$$\leq (2R_0\eta_k^{-1/4}+\eta_k^{-1/2}+$$
$$h(\eta_k^{-1/2}+4\eta_k^{-1/4}aR_0+4a^2R_0^2)^{1/2})E.$$

Using these estimates we conclude from (3.13) that

$$\text{(3.17)}\qquad \left|\begin{aligned}&\int_0^T\int_\Gamma(\Delta\omega)^2d\Gamma\,dt\geq(2/R_0)(T-2R_0\eta_k^{-1/4}-\eta_k^{-1/2}\\&\qquad -h(\eta_k^{-1/2}+4\eta_k^{-1/4}aR_0+4a^2R_0^2)^{1/2})E.\end{aligned}\right.$$

Here the eigenvalues η_k depend on h. Using the variational characterization of the eigenvalues it is easy to show that the eigenvalues are decreasing functions of $h>0$. Let us denote by $\widetilde{\eta}_k$ the eigenvalue η_k corresponding to $h=h_0$. Then it follows from (3.17) that the hypothesis (1.9) of Theorem A is satisfied for example by choosing

$$I_k=[0,2R_0\widetilde{\eta}_k^{-1/4}+\widetilde{\eta}_k^{-1/2}+h(\widetilde{\eta}_k^{-1/2}$$
$$+4\widetilde{\eta}_k^{-1/4}aR_0+4a^2R_0^2)^{1/2}+1/k].$$

and $c_k=2/(R_0k)$ (cf. Remark 1.5).

Let us observe that I_k and c_k are both independent of $h\in[0,h_0]$. We may therefore apply Theorem A (observe that condition (1.9) is now empty) and Theorem 3.5 follows. □

3.2. — The second case

This subsection is devoted to the study of exact controllability of the system (3.1), (3.2) and (3.4). The following two theorems will be proved :

THEOREM 3.6.— *Assume that $T>R_0h$. Then for every intitial data $(\omega^0,\omega^1)\in L^2(\Omega)\times H^{-1}(\Omega)$ there exist control functions $v_0\in L^2(\Gamma_+\times(0,T))$ and $v_1\in(H^2(0,T;\ L^2(\Gamma_+)))'$ such that the solution of (3.1), (3.2) and (3.4) satisfies $\omega(T)=\omega'(T)=0$ in Ω.* □

The controls v_0, v_1 will be constructed by HUM; let us denote these special controls by $v_{0,h}$ and $v_{1,h}$.

THEOREM 3.7.— *Let $T > 0$ and $(\omega^0, \omega^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ be given arbitrarily. Then $v_{0,h} \to v_{0,0}$ in $L^2(\Gamma_+ \times (0,T))$ and $v_{1,h} \to v_{1,0}$ in $(H^2(0,T; L^2(\Gamma_+))'$ weakly as $h \to 0$.* □

Remark 3.8. — Theorems 3.6 and 3.7 were proven in [LL] under stronger hypotheses on T. □

Similarly to the preceeding subsection, both theorems will follow from estimates stated below concerning the homogeneous system (3.1), (3.2) and

$$\omega = \Delta\omega = 0 \quad \text{on} \quad \Gamma \times (0,T). \tag{3.18}$$

Set $H = H_0^1(\Omega)$ if $h = 0$ and $H = H^2(\Omega) \cap H_0^1(\Omega)$ if $h > 0$, the norm of H being defined by

$$\|v\|_H = \left(\int_\Omega |\nabla v|^2 + h^2 (\Delta v)^2 dx \right)^{1/2},$$

and set $V = \{v \in H^3(\Omega) : v = \Delta v = 0 \text{ on } \Gamma\}$. Then for every $(\omega^0, \omega^1) \in V \times H$ the system (3.1), (3.2), (3.18) admits a unique solution $\omega \in C([0,T]; V) \cap C^1([0,T]; H)$. If we define its energy by

$$E = (1/2) \int_\Omega |\nabla \omega'|^2 + h^2 (\Delta\omega')^2 + |\nabla \Delta \omega|^2 dx, \tag{3.19}$$

then E does not depend on $t \in [0,T]$.

Let us introduce on V the bounded, symmetric, coercive bilinear form $a(u,v) = \displaystyle\int_\Omega \nabla\Delta u \cdot \nabla \Delta v dx$ and then the operator A and the spaces W^s associated to H, V and $a(u,v)$ as in section 1.

THEOREM 3.9. — *Fix a positive number h_0 with $T > R_0 h_0$. Then there are two positive constants c and C such that for every $h \in [0, h_0]$ and for every $(\omega^0, \omega^1) \in W^{1/2}$ the solution of (3.1), (3.2), (3.18) satisfies the estimates*

$$cE \le \int_0^T \int_\Gamma (\partial_\nu \omega')^2 + (\partial_\nu \Delta\omega)^2 d\Gamma dt \le CE. \tag{3.20}$$

Proof. — We shall apply Theorem A with H, V, $a(u,v)$ as defined above and with $s = 1/2$, $k_0 = 1$ and

$$p(\omega^0, \omega^1) = \left(\int_\Gamma (\partial_\nu \omega^1)^2 + (\partial_\nu \Delta \omega^0)^2 d\Gamma \right)^{1/2}.$$

According to the notations of Theorem A, we denote by η_j and Z_j the eigenvalues and the eigenspaces of the eigenvalue problem

$$(3.21)\qquad \left| \begin{aligned} &\int\int_\Omega \nabla\Delta u \cdot \nabla\Delta v dx = \eta \int_\Omega \nabla u \cdot \nabla v + h^2 \Delta u \Delta v dx, \\ &\forall v \in V \end{aligned} \right.$$

We apply the multiplier method as in [LL]. Multiplying (3.1) by $k \cdot \nabla\omega$ where $k \in C^1(\overline{\Omega}; R)$ and $k = \nu$ on Γ, we obtain easily the right side estimate in (3.20) with C independent of $h \in [0, h_0]$.

Secondly, multiplying (3.1) by $2m \cdot \nabla\Delta\omega + \Delta\omega$, we obtain the following basic identity :

$$(3.22)\qquad \left| \begin{aligned} &\int_0^T \int_\Gamma (m \cdot \nu)((\partial_\nu \omega')^2 + (\partial_\nu \Delta\omega)^2) d\Gamma\, dt \\ &= -\left[\int_\Omega (\omega' - h^2 \Delta\omega')(2m \cdot \nabla\Delta\omega + \Delta\omega) dx \right]_0^T \\ &+2TE + \int_0^T \int_\Omega 2|\nabla\omega'|^2 dx\, dt. \end{aligned} \right.$$

Assume that $(\omega^0, \omega^1) \in W_k^{1/2}$ for some $k \geq 1$. Then we have

$$(3.23)\qquad \left| \begin{aligned} &\int_\Omega |\nabla\Delta\omega|^2 dx \geq \eta_k \int_\Omega |\nabla\omega|^2 + h^2 (\Delta\omega)^2 dx, \\ &\forall t \in [0, T]. \end{aligned} \right.$$

Observe that the eigenvalue problem

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \mu \int_\Omega uv\, dx,\ \forall v \in H_0^1(\Omega)$$

has the same eigenspaces as (3.21), and the corresponding eigenvalues μ_j are related to the $\eta_j's$ as follows :

$$(3.24)\qquad \eta_j = \mu_j^2 / (1 + h^2 \mu_j).$$

In particular, (3.24) implies that

$$(3.25)\qquad \mu_j \geq \eta_j^{1/2}.$$

Using (3.19, (3.23), (3.25) and the inequality (cf. [LL])

$$\|2m \cdot \nabla\Delta\omega + \Delta\omega\|_2 \leq \|2m \cdot \nabla\Delta\omega\|_2$$

of the type considered in V. Komornik [1], the second integral in (3.23) may be estimated as follows :

$$\begin{aligned}
&|\int_\Omega (\omega' - h^2\Delta\omega')(2m \cdot \nabla\Delta\omega + \Delta\omega)dx| \\
&\leq (\|\omega'\|_2 + h^2\|\Delta\omega'\|_2)\|2m \cdot \nabla\Delta\omega\|_2 \\
&\leq (\mu_k^{-1/2}\|\nabla\omega'\|_2 + h^2\|\Delta\omega'\|_2)2R_0\|\nabla\Delta\omega\|_2 \\
&\leq (h + 2\eta_k^{-1/4})R_0 R.
\end{aligned}$$

Using this inequality, from (3.22) we conclude that

$$(3.26) \qquad \left| \begin{aligned} &\int_0^T \int_\Gamma (\partial_\nu\omega')^2 d\Gamma\, dt \geq (2/R_0)(T - R_0 h \\ &-2R_0\eta_k^{-1/4})E. \end{aligned} \right.$$

Let us denote by $\widetilde{\eta}_k$ the eigenvalue η_k corresponding to $h = h_0$. Then $\eta_k \geq \widetilde{\eta}_k$ and so (3.26) remains valid if we replace η_k by $\widetilde{\eta}_k$. It follows that the hypothesis (1.9) of Theorem A is satisfied with

$$I_k = [0, R_0 h + 2R_0\widetilde{\eta}_k^{-1/4} + 1/k]$$

and

$$c_k = 2/(R_0 k).$$

Applying Theorem A, Theorem 3.9 follows. □

3.3. — The third case if $\Gamma_- \neq \emptyset$

Now we turn to the study of the system (3.1), (3.2) and (3.5). We shall assume here that

$$(3.27) \qquad \Gamma_- \neq \emptyset$$

and

$$(3.28) \qquad \overline{\Gamma_+} \cap \overline{\Gamma_-} = \emptyset.$$

The assumption (3.28) is made in order to avoid some lack of regularity of the solutions; for problems of this type we refer to P. Grisvard [1] and also to V. Komornik and E. Zuazua [1], [2]. The case where (3.27) is not satisfied will be considered in the next subsection.

First we introduce some notations. Let H be the Hilbert space $L^2(\Omega)$ if $h = 0$ and $\{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_-\}$ if $h > 0$, equipped in both cases with the norm

$$\|v\|_H = \left(\int_\Omega v^2 + h^2 |\nabla v|^2 dx \right)^{1/2}.$$

Let V be the subspace of the Hilbert space $H^2(\Omega)$ defined by

$$V = \{v \in H^2(\Omega) : v = \partial_\nu v = 0 \quad \text{on} \quad \Gamma_-\},$$

and for $u, v \in V$ let us set

$$\alpha(u,v) = \partial_1^2 u \partial_1^2 v + \partial_2^2 u \partial_2^2 v$$
$$+ \mu(\partial_1^2 u \partial_2^2 v + \partial_2^2 u \partial_1^2 v) + 2(1-\mu)\partial_1\partial_2 u \partial_1 \partial_2 v$$

and

$$a(u,v) = \int_\Omega \alpha(u,v) dx.$$

We recall that $a(u,v)$ is a continuous, symetric, coercive bilinear form on V. Let b the smallest constant satisfying

$$\int_\Omega \sum_{i,k=1}^{2} (\partial_i \partial_k v)^2 dx \le b^2 a(v,v), \ \forall v \in V. \tag{3.29}$$

Let us also introduce the spaces W^s as in Section 1.

$$\left| \begin{array}{l} \Delta\omega + (1-\mu)B_1\omega = 0 \quad \text{on} \quad \Gamma_+ \times (0,T), \\ \partial_\nu \Delta\omega + (1-\mu)B_2\omega - h^2 \partial_\nu \omega'' = 0 \quad \text{on} \quad \Gamma_+ \times (0,T), \\ \omega = \partial_\nu \omega = 0 \quad \text{on} \quad \Gamma_- \times (0,T) \end{array} \right. \tag{3.30}$$

where B_1, B_2 are defined by (3.6), (3.7). Given $(\omega^0, \omega^1) \in V \times H$ arbitrarily, this system has a unique solution

$$\omega \in C([0,T]; V) \cap C^1([0,T]; H);$$

furthermore, defining its energy by

$$E = (1/2) \int_\Omega (\omega')^2 + h^2 |\nabla \omega'|^2 dx + (1/2) a(\omega, \omega), \tag{3.31}$$

it does not depend on $t \in [0,T]$.

We shall prove the following estimate :

THEOREM 3.10.— *Assume that $T > 2bR_0h$. Then there exists a positive constant c such that for every $(\omega^0, \omega^1) \in W^{1/2}$, the solution of (3.1), (3.2) and (3.30) satisfies the inequality*

$$\int_0^T \int_{\Gamma_+} (\omega')^2 + h^2|\nabla\omega'|^2 d\Gamma\, dt \geq c(\|\omega^0\|_v^2 + \|\omega\|_H^2). \quad \square \tag{3.32}$$

Remark 3.11.— Theorem 3.10 improves an earlier result obtained in [LL], by weakening the assumption on T. Applying HUM as in [LL], it follows that the system (3.1), (3.2), (3.5) is exactly controllable (in suitable function classes) for every $T > 2bR_0h$. □

Remark 3.12.— Contrary to [LL], under this weaker assumption on T we are not able to prove the existence of a constant c in (3.32) which is independent of $h \to 0$. Therefore we cannot prove the convergence of the optimal controls as $h \to 0$. □

Proof of Theorem 3.10.— We shall apply Theorem B with $H, V, a(u,v)$ as defined above, with $s = 1/2$, $k_0 = 1$ and with

$$p(\omega^0, \omega^1) = \left(\int_{\Gamma_+} (\omega^1)^2 + h^2|\nabla\omega^1|^2 d\Gamma\right)^{1/2}.$$

We introduce the notations η_j and Z_j $(j \geq 1)$ as in Theorem B. Our starting point is the identity.

$$\left| \begin{aligned} &\int_0^T \int_{\Gamma_-} (m\cdot\nu)(\Delta\omega)^2 d\Gamma\, dt \\ &\int_0^T \int_{\Gamma_+} (m\cdot\nu)((\omega') + h^2|\nabla\omega'|^2 - \alpha(\omega,\omega))d\Gamma\, dt \\ &\left[\int_\Omega \omega'(2m\cdot\nabla\omega - \omega) + h^2\nabla\omega'\cdot\nabla(2m\cdot\nabla\omega - \omega)dx\right]_0^T \\ &+2TE + \int_0^T \int_\Omega 2(\omega')^2 dx\, dt, \end{aligned} \right. \tag{3.33}$$

proven in [LL] by multiplying the equation (3.1) by $2m\cdot\nabla\omega - \omega$.

Fix a positive integer k arbitrarily and assume that $(\omega^0, \omega^1) \in W_k^{1/2}$. Then we have

$$a(w,w) \geq \eta_k \int_\Omega \omega^2 + h^2 |\nabla \omega|^2 dx, \ \forall t \in [0,T]. \tag{3.34}$$

Furthermore, interpolating between the spaces $L(\Omega)$ and $H^2(\Omega)$ we obtain

$$\|\nabla \omega\|_2 \leq B\eta_k^{-1/4} a(\omega,\omega)^{1/2}, \ \forall t \in [0,T] \tag{3.35}$$

where B is a constant independent of k, ω and t.

Using (3.29) and (3.35) we deduce, as in subsection 3.1, the following estimates :

$$\left| \begin{array}{l} \|(2m \cdot \nabla \omega - \omega)\|_2 \leq (B^2 \eta_k^{-1/2} + 4BbR_0\eta_k^{-1/4} \\ +4b^2R_0^2)^{1/2} a(\omega,\omega)^{1/2}. \end{array} \right. \tag{3.36}$$

From (3.34)-(3.36) we conclude that for every $t \in [0,T]$,

$$\begin{aligned}
&|\int_\Omega \omega'(2m \cdot \nabla \omega - \omega) + h^2 \nabla \omega' \cdot \nabla (2m \cdot \nabla \omega - \omega) dx| \\
&\leq \|\omega'\|_2 (2R_0 B \eta_k^{-1/4} + \eta_k^{-1/2}) a(\omega,\omega)^{1/2} \\
&+h^2 \|\nabla \omega'\|_2 (B^2 \eta_k^{-1/2} + 4BbR_0 \eta_k^{-1/4} + 4b^2 R_0^2)^{1/2} a(\omega,\omega)^{1/2} \\
&\leq (2R_0 B \eta_k^{-1/4} + \eta_k^{-1/2} \\
&+h(B^2 \eta_k^{-1/2} + 4BbR_0 \eta_k^{-1/4} + 4b^2 R_0^2)^{1/2})E.
\end{aligned}$$

Using this inequality we obtain from (3.33) the following estimate :

$$\left| \begin{array}{l} \displaystyle\int_0^T p(\omega,\omega')^2 dt \geq (2/R_0)(T - 2R_0 B \eta_k^{-1/4} - \eta_k^{-1/2} \\ \qquad h(B^2 \eta_k^{-1/2} + 4BbR_0 \eta_k^{-1/4} + 4b^2 R_0^2)^{1/2})E. \end{array} \right. \tag{3.37}$$

Hence, using Remark 1.5 and the positivity of η_1 we obtain that the hypothesis (1.11) of Theorem B is satisfied with a sequence I_k such that $T_0 = 2bR_0 h$.

Since $k_0 = 1$, the condition (1.12) is empty.

We may therefore apply Theorem B and Theorem 3.10 follows. □

3.4. — The third case if $\Gamma_- = \emptyset$

We consider now the system (3.1), (3.2) and

$$\text{(3.38)} \qquad \left| \begin{array}{l} \Delta\omega + (1-\mu)B_1\omega = 0 \quad \text{on} \quad \Gamma \times (0,T), \\ \partial_\nu \Delta\omega + (1-\mu)B_2\omega - h^2 \partial_\nu \omega'' = 0 \quad \text{on} \quad \Gamma \times (0,T) \end{array} \right.$$

where B_1, B_2 are defined by (3.6), (3.7), and we assume that (3.39) $m \cdot \nu \geq 0$ on Γ.

Let H be the Hilbert space $L^2(\Omega)$ if $h = 0$ and $H^1(\Omega)$ if $h > 0$, equipped with the norm

$$\|v\|_H = \left(\int_\Omega v^2 + h^2 |\nabla v|^2 dx \right)^{1/2}$$

and set $V = H^2(\Omega)$. We introduce the same bilinear form $a(u,v)$ as in the preceeding subsection. We remark that

$$\text{(3.40)} \qquad a(u,v) \geq \mu \int_\Omega (\Delta v)^2 dx + (1-\mu) \int_\Omega |\nabla v|^2 dx, \ \forall v \in V.$$

It follows that condition (1.1) is satisfied.

Introducing the corresponding notations of Section 1, (3.40) implies that $\eta_1 = 0$ and Z_1 consists of the constant functions. Let us denote by b the least positive constant such that

$$\int_\Omega \sum_{i,k=1}^{2} (\partial_i \partial_k v)^2 dx \leq b^2 a(v,v)$$

for all $v \in V$, orthogonal to Z_1 in H.

Let Γ_0 be an arbitrary subset of positive measure in Γ. We shall prove the following result :

THEOREM 3.13. — *Assume that $T > 2bR_0h$. Then there exists a positive constant c such that for every $(\omega^0, \omega^1) \in W^{1/2}$, the solution of (3.1), (3.2) and (3.38) satifies the inequality*

$$\text{(3.41)} \qquad \left| \begin{array}{l} \displaystyle\int_0^T \int_\Gamma (\omega')^2 + h^2 |\nabla \omega'|^2 d\Gamma \, dt + \int_0^T \int_{\Gamma_0} \omega^2 d\Gamma dt \\ \geq c(\|\omega^0\|_V^2 + \|\omega^1\|_H^2. \end{array} \right. \qquad \square$$

Remark 3.14. — In [LL] Theorem 3.13 was proved under a stronger hypothesis $T > T_0' > 0$ with T_0' independent of h. Applying HUM, Theorem

3.13 implies the exact controllability of the system (3.1), (3.2), (3.5) in suitable function spaces if $T > 2bR_0h$. □

Proof of Theorem 3.13. — We shall apply Theorem B with $H, V, a(u,v)$ as defined above, $s = 1/2$, $k_0 = 2$ and with

$$p(\omega^0,\omega^1) = \left(\int_\Gamma (\omega^1)^2 + h^2|\nabla\omega^1|^2 d\Gamma + \int_{\Gamma_0} (\omega^0)^2 d\Gamma\right)^{1/2}$$

We have now instead of (3.33) the following identity :

$$\int_0^T \int_\Gamma (m\cdot\nu)((\omega')^2 + h^2|\nabla\omega'|^2 - a(\omega,\omega))d\Gamma\, dt$$
$$= \left[\int_\Omega \omega'(2m\cdot\nabla\omega - \omega) + h^2\nabla\omega'\cdot\nabla(2m\cdot\nabla\omega - \omega)dx\right]_0^T$$
$$+2TE + \int_0^T \int_\Omega 2(\omega')^2 dx\, dt$$

where E is defined by (3.31).

If $(\omega^0,\omega^1) \in W_k^{1/2}$ for some $k \geq 2$, then $\eta_k > 0$ and the calculations of Subsection 3.3 remain valid. We obtain

$$\int_0^T \int_\Gamma (\omega')^2 + h^2|\nabla\omega'|^2 d\Gamma\, dt \geq (2/R_0)(T - 2R_0B\eta_k^{-1/4}$$
$$-\eta_k^{-1/2} - h(B^2\eta_k^{-1/2} + 4BbR_0\eta_k^{-1/4} + 4b^2R_0^2)^{1/2})E;$$

hence the condition (1.11) of Theorem B is satisfied with a sequence of intervals I_k such that $T_0 = 2bR_0h$.

On other hand, condition (1.12) follows at once from the definition of Γ_0 and from the structure of Z_1. Indeed, we have obviously

$$p(\omega,0)^2 = \int_{\Gamma_0} \omega^2 d\Gamma = \text{meas}(\Gamma_0)\omega^2,\ \forall\omega\in Z_1$$

implying (1.12) in view of Proposition C.

Applying Theorem B, hence Theorem 3.13 follows. □

4. Applications to the exact controllability of Mindlin – Timoshenko plates

The purpose of this section is to prove the uniform exact controllability of some Mindlin – Timoshenko plate models with respect to the shear modulus. As in Section 3, our standard reference is J. Lagnese and J.-L. Lions [1], referred to by [LL] in the sequel; it contains also the interpretation of the models considered below.

Let Ω be a non-empty, bounded domain in R^2 having a boundary Γ of class C^2 and let $\Gamma = \Gamma_0 \cup \Gamma_1$ be a partition of Γ. In order to ensure sufficient regularity of the solutions of the systems considered in this section, we assume that

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset \tag{4.1}$$

In this section we shall use for brevity the following notation : we write $\nabla f = (f_x, f_y)$; the normal and tangential derivatives of a function f will be denoted by $\partial_\nu f$ and $\partial_\tau f$, respectively.

In the first subsection we shall study the exact controllability of the following system (modelling a plate clamped along Γ_0) :

$$\left| \begin{aligned} &\frac{\rho h^3}{12}\psi'' - D\Big(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy} + \frac{1+\mu}{2}\varphi_{xy}\Big) + K(\psi + \omega_x) = 0,\\ &\frac{\rho h^3}{12}\varphi'' - D\Big(\varphi_{yy} + \frac{1-\mu}{2}\varphi_{xx} + \frac{1+\mu}{2}\psi_{xy}\Big) + K(\varphi + \omega_y) = 0,\\ &\rho h\omega'' - K((\omega_x + \psi)_x + (\omega_y + \varphi)_y) = 0 \quad \text{in} \quad \Omega \times (0,T), \end{aligned} \right. \tag{4.2}$$

$$\left| \begin{aligned} &D\Big(\nu_1\psi_x + \mu\nu_1\varphi_y + \frac{1-\mu}{2}(\psi_y + \varphi_x)\nu_2\Big) = v_1,\\ &D\Big(\nu_2\varphi_y + \mu\nu_2\psi_x + \frac{1-\mu}{2}(\psi_y + \varphi_x)\nu_1\Big) = v_2,\\ &K(\omega_\nu + \nu_1\psi + \nu_2\varphi) = v_3 \quad \text{on} \quad \Gamma_1 \times (0,T), \end{aligned} \right. \tag{4.3}$$

$$\psi = h_1, \quad \varphi = h_2, \quad \omega = h_3 \quad \text{on} \quad \Gamma_0 \times (0,T), \tag{4.4}$$

$$\left| \begin{aligned} &\psi(0) = \psi^0, \quad \psi'(0) = \psi^1, \quad \varphi(0) = \varphi^0, \quad \varphi'(0) = \varphi^1,\\ &\omega(0) = \omega^0, \ \omega'(0) = \omega^1 \quad \text{in} \quad \Omega, \end{aligned} \right. \tag{4.5}$$

where $\rho,\ h,\ \mu,\ D,\ ,K,\ T$ are suitable positive constants and $\mu < 1/2$.

In the second subsection we shall study the exact controllability of the system (4.2), (4.3), (4.5) and

$$\tag{4.6} \left| \begin{aligned} &D\Big(\nu_1\psi_x + \mu\nu_1\varphi_y + \frac{1-\mu}{2}(\psi_y + \varphi_x)\nu_2\Big) = h_1, \\ &D\Big(\nu_2\varphi_y + \mu\nu_2\psi_x + \frac{1-\mu}{2}(\psi_y + \varphi_x)\nu_1\Big) = h_2, \\ &\omega = h_3 \quad \text{on} \quad \Gamma_0 \times (0,T), \end{aligned} \right.$$

modelling a plate hinged along Γ_0. (In fact we shall treat also the case where $\Gamma_0 = \emptyset$.)

We shall restrict ourselves to the proof of suitable a priori estimates. The corresponding exact controllability theorems may then be deduced by HUM as described for the cases considered here in [LL].

4.1. — Clamped plates

Let us assume that

$$\tag{4.7} \Gamma_0 \neq \emptyset$$

Consider the homogeneous system (4.2), (4.5),

$$\tag{4.8} \left| \begin{aligned} &D\Big(\nu_1\psi_x + \mu\nu_1\varphi_y + \frac{1-\mu}{2}(\psi_y + \varphi_x)\nu_2\Big) = 0, \\ &D\Big(\nu_2\varphi_y + \mu\nu_2\psi_x + \frac{1-\mu}{2}(\psi_y + \varphi_x)\nu_1\Big) = 0, \\ &K(\omega_\nu + \nu_1\psi + \nu_2\varphi) = 0 \quad \text{on} \quad \Gamma_1 \times (0,T), \end{aligned} \right.$$

and

$$\tag{4.9} \psi = \varphi = \omega = 0 \quad \text{on} \quad \Gamma_0 \times (0,T),$$

Fix a point $(x^0, y^0) \in R^2$ arbitrarily and set

$m(x,y) = (x - x^0 \cdot y - y^0),\ (x,y) \in R^2,$

$R_0 = \sup\{|m(x,y)| \ :\ (x,y) \in \Gamma\},$

$\Gamma_+ = \{(x,y) \in \Gamma \ :\ m(x,y) \cdot \nu(x,y) > 0\},$

$\Gamma_- = \{(x,y) \in \Gamma \ :\ m(x,y) \cdot \nu(x,y) \leq 0\}.$

We shall prove the following theorem :

THEOREM 4.1.— *There exists a constant T_0, independent of K, such that if $T > T_0$, then every sufficiently smooth solution of (4.2), (4.5), (4.8), (4.9) satisfies the inequality*

$$\left| \begin{aligned} &\int_0^T \int_{\Gamma_1 \cap \Gamma_+} \psi'^2 + \varphi'^2 + \omega'^2 d\Gamma dt \\ &+ \int_0^T \int_{\Gamma_1 \cap \Gamma_-} \psi^2 + \varphi^2 + \psi_\tau^2 + \varphi_\tau^2 + \omega_\tau^2 d\Gamma dt \\ &+ \int_0^T \int_{\Gamma_0 \cap \Gamma_+} \psi_\nu^2 + \varphi_\nu^2 + \omega_\nu^2 d\Gamma dt \\ &\geq c(\|(\psi^0, \varphi^0, \omega^0)\|^2_{(H^1(\Omega))^3} + \|(\psi^1, \varphi^1, \omega^1)\|^2_{(L^2(\Omega))^3}); \end{aligned} \right. \tag{4.10}$$

here c denotes a positive constant depending on K but independent of the initial data. □

Remark 4.2.— As we shall see, the precise assumption on the smoothness of the solution is that the initial data belong to the space $W^{1/2}$ defined below. □

Remark 4.3.— In [LL] the estimates (4.10) were proven under a stronger assumption of the form $T > T_0(K)$ with $T_0(K) \to \infty$ as $K \to \infty$. □

Remark 4.4.— We note that in the special case $\Gamma_1 = \emptyset$ the reverse inequality of (4.10) is also true. □

Remark 4.5.— Theorem 4.1 implies the following exact controllability theorem (cf. [LL]) :
if $T > T_0$, then for every initial data $((\psi^0, \varphi^0, \omega^0), (\psi^1, \varphi^1, \omega^1)) \in V \times H$ there are control functions h_1, h_2, h_3 and v_1, v_2, v_3 driving the system (4.2) – (4.5) to rest in time T. The controls h_i belong to $L^2(\Gamma_0 \times (0,T))$ and the controls v_i have the structure $v_i = \xi_i'' + \partial_\tau^2 \xi_i$ where $\xi_i \in L^2(0,T; H^1(\Gamma_1)) \cap H^1(0,T; L^2(\Gamma_1))$. □

Proof of Theorem 4.1.—Let H be the Hilbert space $(L^2(\Omega))^3$ endowed with the norm

$$\|(\psi, \varphi, \omega)\|_H := c(\psi, \varphi, \omega)^{1/2}$$

where

$$c(\psi, \varphi, \omega) = \rho h \int_\Omega (h^2/12)(\psi^2 + \varphi^2) + \omega^2 dx dy$$

and let us introduce the Hilbert space

$$V = (\{v \in H^1(\Omega) \ : \ v = 0 \quad \text{on} \quad \Gamma_0\})^3$$

considered as a subspace of $H^1(\Omega)^3$. For $(\psi,\varphi,\omega),(\widetilde{\psi},\widetilde{\varphi},\widetilde{\omega})\in V$ we set (cf. [LL]).

$$a_0((\psi,\varphi),(\widetilde{\psi},\widetilde{\varphi}))=D\int_\Omega(\psi_x+\varphi_y)(\widetilde{\psi}_x+\widetilde{\varphi}_y)$$
$$+\frac{1-\mu}{2}(\psi_y-\varphi_x)(\widetilde{\psi}_y-\widetilde{\varphi}_x)dxdy,$$
$$a_1((\psi,\varphi,\omega),(\widetilde{\psi},\widetilde{\varphi},\widetilde{\omega}))$$
$$=D\int_\Omega(\psi+\omega_x)(\widetilde{\psi}+\widetilde{\omega}_x)+(\varphi+\omega_y)(\widetilde{\varphi}+\widetilde{\omega}_y)dxdy,$$
$$a((\psi,\varphi,\omega),(\widetilde{\psi},\widetilde{\varphi},\widetilde{\omega}))=a_0((\psi,\varphi),(\widetilde{\psi},\widetilde{\varphi}))$$
$$+Ka_1((\psi,\varphi,\omega),(\widetilde{\psi},\widetilde{\varphi},\widetilde{\omega})),$$
$$a_0(\psi,\varphi)=a_0((\psi,\varphi)=a_0((\psi,\varphi),(\psi,\varphi)),$$
$$a_1(\psi,\varphi,\omega)=a((\psi,\varphi,\omega),(\psi,\varphi,\omega).$$

Using the Korn inequality it is easy to verify that the bilinear form $a(\cdot,\cdot)$ has the property (1.1) of Section 1. Let us introduce the corresponding notations $W^s, E, W_k^s, \eta_k, Z_k$ of Section 1. The system (1.2) is then equivalent to (4.2), (4.5), (4.8), (4.9). Furthermore, the Korn inequality shows also that $\eta_1>0$.

We are going to apply Theorem B with $s=1/2$, $k_0=1$ and the semi norm p being defined by

$$p((\psi^0,\varphi^0,\omega^0),(\psi^1,\varphi^1,\omega^1))^2$$
$$=\int_{\Gamma_1\cap\Gamma_+}(\psi^1)^2+(\varphi^1)^2+(\omega^1)^2d\Gamma$$
$$+\int_{\Gamma_1\cap\Gamma_-}(\psi^0)^2+(\varphi^0)^2+(\psi^0_\tau)^2+(\varphi^0_\tau)+(\omega^0_\tau)^2d\Gamma$$
$$+\int_{\Gamma_0\cap\Gamma_+}(\psi^0_\nu)^2+(\varphi^0_\nu)^2+(\omega^0_\nu)^2d\Gamma$$

Fix the positive ε (to be precised later). We recall from [LL] the following identity, valid for all solutions with initial data in $W^{1/2}$:

$$\frac{\rho h}{2}\int_0^T\int_{\Gamma_1}(m\cdot\nu)\left[\frac{h^2}{12}((\psi')^2+(\varphi')^2)+(\omega')^2\right]d\Gamma dt$$
$$-\frac{1}{2}\int_{\Gamma_1}^T(m\cdot\nu)\Bigg\{D\left[\psi_x^2+\varphi_y^2+2\mu\psi_x\varphi_y+\frac{1-\mu}{2}(\psi_y+\varphi_x)^2\right]$$
$$+K[(\psi+\omega_x)^2+(\varphi+\omega_y)^2]\Bigg\}d\Gamma dt$$

$$+\frac{1}{2}\int_0^T\int_{\Gamma_0}(m\cdot\nu)\left\{D\left[(\nu_1\psi_2+\nu_2\varphi_2)^2+\frac{1-\mu}{2}(\nu_2\psi_2-\nu_1\varphi_\nu)^2\right]+K\omega_\nu^2 d\Gamma dt\right.$$

$$=(1-2\varepsilon)\rho h\int_0^T\int_\Omega(\omega')^2dxdydt+\varepsilon\int_0^T c(\psi',\varphi',\omega')dt$$

$$+(1-\varepsilon)\int_0^T a_0(\psi,\varphi)dt+\varepsilon K\int_0^T\int_\Omega\omega_x^2+\omega_y^2-\psi^2-\varphi^2dxdydt$$

$$+\left[\rho h\int_\Omega\frac{h^2}{12}(\psi'm\cdot\nabla\psi+\varphi'm\cdot\nabla\varphi)+\omega'm\cdot\nabla\omega dxdy\right.$$

$$\left.+(1-\varepsilon)\frac{\rho h^3}{12}\int_\Omega\psi'\psi+\varphi'\varphi dxdy+\varepsilon\rho h\int_\Omega\omega'\omega dxdy\right]_0^T.$$

As it was shown in [LL], this identity implies the following inequality :

$$(R_0/2)\int_0^T p((\psi,\varphi,\omega),(\psi',\varphi',\omega'))^2dt$$

$$\geq\int_0^T\left\{(1-2\varepsilon)\rho h\int_\Omega(\omega')^2dxdy+\varepsilon c(\psi',\varphi',\omega')\right.$$

$$\left.+(1-\varepsilon-2\varepsilon K\eta_1^{-1}c')a_0(\psi,\varphi)+\frac{\varepsilon K}{2}a_1(\psi,\varphi,\omega)\right\}dt-c''E$$

where c' and c'' are positive constants, independent of K, T and ε.

Now we observe that if the initial data belong to $W_k^{1/2}$ for some positive integer k, then the same reasoning yields the stronger estimate by replacing η_1 by η_k in the above inequality; furthermore the constants c', c'' do not depend on k. Chosing $\varepsilon_k=1/(2+2Jc'\eta_k^{-1})$ this gives

$$\int_0^T p((\psi,\varphi,\omega),(\psi',\varphi',\omega'))^2dt\geq(2/R_0)(\varepsilon_kT-c'')E.$$

In view of Remark 1.5 condition (1.11) of Theorem B is therefore satisfied with a sequence of intervals I_k such that $T_0=2/c''$.

Since $k_0=1$, condition (1.12) is now empty. We may apply Theorem B and Theorem 4.1 follows because (4.10) is a special case of the estimate (1.13). □

4.2. — Hinged plates

Let us fix a positive number γ satisfying $\gamma<(1-\mu)/2$ and consider the system (4.2), (4.5),

$$
(4.11) \qquad \left|\begin{array}{l}
\psi_\nu + (\mu+\gamma)\varphi_\tau + \dfrac{1+\mu}{2}(\varphi_x - \psi_y)\nu_2 = 0, \\
\varphi_\nu + (\mu+\gamma)\psi_\tau + \dfrac{1+\mu}{2}(\varphi_x - \psi_y)\nu_1 = 0, \\
\omega_\nu + \nu_1\psi + \nu_2\varphi = 0 \quad \text{on} \quad \Gamma_1 \times (0,T),
\end{array}\right.
$$

$$
(4.12) \qquad \left|\begin{array}{l}
\psi_\nu + (\mu+\gamma)\varphi_\tau + \dfrac{1+\mu}{2}(\varphi_x - \psi_y)\nu_2 = 0, \\
\varphi_\nu - (\mu+\gamma)\psi_\tau - \dfrac{1+\mu}{2}(\varphi_x - \psi_y)\nu_1 = 0, \\
\omega = 0 \quad \text{on} \quad \Gamma_0 \times (0,T).
\end{array}\right.
$$

Fix $(x^\circ, y^\circ) \in R^2$ arbitrarily and introduce the same notations $m(x,y)$, R_0, Γ_+, Γ_- as in Subsection 4.1. Let Γ_2 be an arbitrary subset of positive measure in Γ.

We shall prove the following result :

THEOREM 4.6.— *There exists a constant T_0, independent of K, such that if $T > T_0$, then every sufficiently smooth solution of (4.2), (4.5), (4.11), (4.12) satisfies the inequality*

$$
(4.13) \qquad \left|\begin{array}{l}
\displaystyle\int_0^T \int_{\Gamma_+} \psi'^2 + \varphi'^2 d\Gamma dt + \int_0^T \int_{\Gamma_1 \cap \Gamma_+} \omega'^2 d\Gamma dt \\
\displaystyle + \int_0^T \int_{\Gamma_-} \psi^2 + \varphi^2 + \psi_\tau^2 + \varphi_\tau^2 d\Gamma \int_0^T \int_{\Gamma_1 \cap \Gamma_-} \omega_\tau^2 d\Gamma dt \\
\displaystyle + \int_0^T \int_{\Gamma_0 \cap \Gamma_+} w_\nu^2 d\Gamma dt + \int_0^T \int_{\Gamma_2} \psi^2 + \varphi^2 + x^2 d\Gamma dt \\
\geq c(\|(\psi^\circ, \varphi^\circ, \omega^\circ)\|^2_{(H^1(\Omega))^3} + \|(\psi^1, \varphi^1, \omega^1)\|^2_{(L^2(\Omega))^3})
\end{array}\right.
$$

where c is a positive constant depending on K but independent of initial data. □

Remark 4.7.— The precise assumption on the smoothness of the solution is that the initial data belong to the space $W^{1/2}$ defined below. □

Remark 4.8.— In [LL] the estimates (4.13) were proven under a stronger assumption of the form $T > T_0(K)$ with $T_0(K) \to \infty$ as $K \to \infty$, and with $\Gamma_2 = \Gamma$. □

Remark 4.9. — In Theorem 4.6 the case $\Gamma_0 = \emptyset$ is not excluded; however, this is no more a "hinged" plate. □

Remark 4.10. — Theorem 4.6 implies the exact controllability of the system (4.2), (4.3), (4.5), (4.6) in suitable function spaces; cf. [LL]. □

Proof of Theorem 4.6. — Let H be the Hilbert space $(L^2(\Omega))^3$ endowed with the same norm as in the preceeding subsection and let V be the subspace of $(H^1(\Omega))^3$ defined by

$$V = \{'\psi, \varphi, \omega) \in (H^1(\Omega))^3 \ : \ \omega \doteq 0 \quad \text{on} \quad \Gamma_0\}.$$

Let us define the bilinear and quadratic forms a_0, a_1 as in the preceeding subsection. Furthermore, fix a positive number γ satisfying $\gamma < (1-\mu)/2$ and set

$$\begin{aligned}\widehat{a}((\psi,\varphi),(\widetilde{\psi},\widetilde{\varphi})) &= a_0((\psi,\varphi),(\widetilde{\psi},\widetilde{\varphi})) \\ &\quad + \gamma D \int_\Omega \psi_x \widetilde{\varphi}_y + \varphi_y \widetilde{\psi}_x - \psi_y \widetilde{\varphi}_x - \varphi_x \widetilde{\psi}_y dx\, dy, \\ \widehat{a}_0(\psi,\varphi) &= \widehat{a}_0((\psi,\varphi),(\psi,\varphi))\end{aligned}$$

and

$$a((\psi,\varphi,\omega),(\widetilde{\psi},\widetilde{\varphi},\widetilde{\omega})) = \widehat{a}_0((\psi,\varphi),(\widetilde{\psi},\widetilde{\varphi}))$$

$$+ K a_1((\psi,\varphi,\omega),(\widetilde{\psi},\widetilde{\varphi},\widetilde{\omega}))$$

($a(\cdot,\cdot)$ is not the same bilinear form as in Subsection 4.1). It is easy to verify, using Korn's inequality, that $a(\cdot,\cdot)$ has property (1.1). Introducing the corresponding general notations of Section 1, it is clear that the system (1.2) is now equivalent to (4.2), (4.5), (4.11), (4.12). Also, it is easy to show that $\eta_1 = 0$ and

$$Z_1 = \{(c_1, c_2, -c_1 x - c_2 y - c_3) \ : \ c_1, c_2, c_3 \in R\}. \tag{4.14}$$

We shall apply Theorem B with $s = 1/2$, $k_0 = 2$ and

$$\begin{aligned}&p((\psi^\circ,\varphi^\circ,\omega^\circ),(\psi^1,\varphi^1,\omega^1))^2 \\ &= \int_{\Gamma_+} \psi'^2 + \varphi'^2 d\Gamma + \int_{\Gamma_1 \cap \Gamma_+} \omega'^2 d\Gamma \\ &+ \int_{\Gamma_-} \psi^2 + \varphi^2 + \psi_\tau^2 + \varphi_\tau^2 d\Gamma + \int_{\Gamma_1 \cap \Gamma_-} \omega_\tau^2 d\Gamma\end{aligned}$$

$$+\int_{\Gamma_0\cap\Gamma_+}\omega_\nu^2 d\Gamma+\int_{\Gamma_2}\psi^2+\varphi^2+\omega^2 d\Gamma.$$

Let ε be a positive number (to be precised later) and consider a solution of (4.2), (4.5), (4.11) and (4.12) with initial data in $W^{1/2}$. We recall the following identity from [LL] :

$$\frac{\rho h}{2}\int_0^T\int_{\Gamma_1}(m\cdot\nu)\left[\frac{h^2}{12}((\psi')^2+(\varphi')^2)+(\omega')^2\right]d\Gamma dt$$

$$-\frac{D}{2}\int_\Gamma(m\cdot\nu)\left[\psi_x^2+\varphi_y^2+2\mu\psi_x\varphi_y+\frac{1-\mu}{2}(\psi_y+\varphi_x)^2\right.$$

$$\left.+2\gamma(\psi_x\varphi_y-\psi_y\varphi_x)\right]d\Gamma dt-\frac{K}{2}\int_0^T\int_{\Gamma_0}(m\cdot\nu)(\psi^2+\varphi^2)d\Gamma dt$$

$$-\frac{K}{2}\int_0^T\int_{\Gamma_1}(m\cdot\nu)[(\psi+\omega_x)^2+(\varphi+\omega_y)^2]d\Gamma\, dt$$

$$+\frac{1}{2}\int_0^T\int_{\Gamma_0}(m\cdot\nu)\left[\frac{\rho h^3}{12}({\psi'}^2+{\varphi'}^2)+K\omega_y^2\right]d\Gamma\, dt$$

$$=(1-2\varepsilon)\rho h\int_0^T\int_\Omega(\omega')^2 dxdydt+\varepsilon\int_0^T c(\psi',\varphi',\omega')dt$$

$$+(1-\varepsilon)\int_0^T\widehat{a}_0(\psi,\varphi)dt+\varepsilon K\int_0^T\int_\Omega\omega_x^2+\omega_y^2-\psi^2-\varphi^2 dxdydt$$

$$+\left[\rho h\int_\Omega\frac{h^2}{12}(\psi' m\cdot\nabla\psi+\varphi' m\cdot\nabla\varphi)+\omega' m\cdot\nabla\omega dxdy\right.$$

$$\left.+(1-\varepsilon)\frac{\rho h^3}{12}\int_\Omega\psi'\psi+\varphi'\varphi dxdy+\varepsilon\rho h\int_\Omega\omega'\omega dxdy\right]_0^T.$$

Contrary to the case of the clamped plates, this identity does not yield directly an estimate of the type (4.13). Indeed, in the proof of Theorem 4.1 the Poincaré inequality is used and here no such inequalities are available because $\eta_1=0$. This difficulty was solved in [LL] by strenghtening the semi norm p and applying an extra argument of the type used in V. Komornik [1], [2].

Now we give a simpler method. First we observe that if the initial data belong to $W_k^{1/2}$ for some $k\geq 2$, then the proof of Theorem 4.1 may be easily adapted, using an inequality of Poincaré-Wirtinger type instead of the Poincaré inequality. This leads to the estimate

$$\int_0^T p((\psi,\varphi,\omega),(\psi',\varphi',\omega'))^2 dt\geq(2/R_0)(\varepsilon_k T-c'')E$$

where $\varepsilon_k = 1/(2 + 2Kc'\eta_k^{-1})$ and c', c'' do not depend on k, K and T. It follows that condition (1.11) of Theorem B is satisfied. (The choice $k_0 = 2$ is important here : for $k = 1$ the corresponding condition does not follow by this way.)

We have to verify also condition (1.12) for $k = 1$. Let $(\psi, \varphi, \omega) = (c_1, c_2, -c_1x - c_2y - c_3)$ be an arbitrary non-zero element of Z_1 (cf. (4.14). Since Γ_2 is of positive measure, we have

$$p((\psi,\varphi,\omega),(0,0,0))^2 \geq \int_{\Gamma_2} c_1^2 + c_2^2 + (c_1x + c_2y + c_3)^3 d\Gamma > 0$$

Using Proposition C hence condition (1.12) follows. We may apply Theorem B and this completes the proof of Theorem 4.6 because (4.13) is a special case of (1.13). □

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