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Introduction

If $M$ is a compact submanifold without boundary in $\mathbb{R}^n$ and $N_xM$ denotes the normal subspace to $M$ at $x$, the study of the Lagrangian system

$$\frac{d}{ds} \left( \nabla_v L(s, \gamma, \gamma') \right) - \nabla_q L(s, \gamma, \gamma') \in N_{\gamma(s)}M$$

has been carried out in [1], where the existence of infinitely many periodic solutions $\gamma$ is proved under quite general assumptions.

The corresponding problem on manifolds with boundary has been treated in [18], where the existence of a periodic solution is proved and in [5], where the existence of infinitely many periodic solutions is shown. The feature of unilateral constraints (cf. [6], [7], [14], [15], [16], [17], [18]) is that, even if the manifold $M$ is of class $C^\infty$, the corresponding variational problem does not have a smooth structure.

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For this reason, it seems to be natural to allow for the set $M$ itself a certain kind of irregularity. The aim of the paper is to treat the case in which $M$ is a $p$-convex set (see Def. 1.3) and $L$ is quadratic with respect to $\gamma'$, namely

$$L(s, q, v) = \frac{1}{2} (a(s, q)v, v) - V(s, q).$$

The particular case $a \equiv \text{Id}$, $V \equiv 0$, which leads to the study of geodesics, was already treated in [2] and [3].

The main tools are the techniques of non-smooth nonlinear analysis developed in [8], [9], [10] and [11]. Actually, the main part of the paper, the second section, is devoted to the proof that these techniques can be applied to our situation.

1. Recalls of non-smooth analysis and the main result

We will recall some notions of non-smooth analysis as developed in [6], [7], [8], [9], [11].

From now on, $H$ will be a real Hilbert space, $| \cdot |$ and $(\cdot, \cdot)$ its norm and scalar product, respectively.

If $u \in H$ and $r > 0$, we set $B(u, r) = \{v \in H \mid |v - u| < r\}$.

DEFINITION 1.1. — (see also [6] and [9]) Let $\Omega$ be an open subset of $H$ and $f : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ a map.

We set

$$D(f) = \{u \in \Omega \mid f(u) < +\infty\}.$$

Let $u$ belong to $D(f)$. The function $f$ is said to be subdifferentiable at $u$ if there exists $\alpha \in H$ such that

$$\liminf_{v \rightarrow u} \frac{f(v) - f(u) - (\alpha, v - u)}{|v - u|} \geq 0.$$

We denote by $\partial^- f(u)$ the (possibly empty) set of such $\alpha$'s and we set

$$D(\partial^- f) = \{u \in D(f) \mid \partial^- f(u) \neq \emptyset\}.$$

It is easy to check that $\partial^- f(u)$ is convex and closed $\forall u \in D(f)$.
If $u \in D(\partial^- f)$, $\text{grad}^- f(u)$ will denote the element of minimal norm of $\partial^- f(u)$.

Moreover, let $M$ be a subset of $H$. We denote by $I_M$ the function:

$$I_M(u) = \begin{cases} 0 & u \in M \\ +\infty & u \in H \setminus M. \end{cases}$$

It is easy to check that $\partial^- I_M(u)$ is a closed convex cone $\forall u \in M$.

We will call (outward) normal cone to $M$ at $u$ the set $\partial^- I_M(u)$ and tangent cone to $M$ at $u$ its negative polar $(\partial^- I_M(u))^-$, i.e.,

$$(\partial^- I_M(u))^- = \{ v \in H \mid (v, w) \leq 0, \forall w \in \partial^- I_M(u) \}.$$  

Remark 1.2. — Let us suppose that $g : \Omega \to \mathbb{R}$ is Fréchet differentiable at $u \in \Omega$. Then:

$$\partial^- (f + g)(u) \neq \emptyset \text{ if and only if } \partial^- f(u) \neq \emptyset$$

and

$$\partial^- (f + g)(u) = \{ \alpha + \text{grad} g(u) \mid \alpha \in \partial^- f(u) \}.$$  

Let us introduce the class of $p$-convex sets as defined in [2] and [3]. An other characterization of this class is in [4].

DEFINITION 1.3. — A subset $M$ of $H$ is said to be a $p$-convex set if there exists a continuous function $p : M \to \mathbb{R}^+$ such that

$$(\alpha, v - u) \leq p(u) |\alpha| |v - u|^2$$

whenever $u, v \in M$ and $\alpha \in \partial^- I_M(u)$.

Examples of $p$-convex sets are the following ones:

1. The $C^{1,1}_{\text{loc}}$-submanifolds (possibly with boundary) of $H$;
2. The convex subsets of $H$;
3. The images under a $C^{1,1}_{\text{loc}}$-diffeomorphism of convex sets;
4. The subset of $\mathbb{R}^n : \{ x \mid \max |x_i| \leq 1, \sum x_i^2 \geq 1 \}$ (note that it is not included in the classes (1), (2), (3)).
Now, we can state the main result of the paper.

Let $M$ be a $p$-convex subset of $\mathbb{R}^n$ and let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Lagrangian of the form

$$L(s, q, v) = \frac{1}{2} \left( a(s, q)v, v \right) - V(s, q)$$

where $a, V$ are of class $C^2$ on $\mathbb{R} \times \mathbb{R}^n$ and the matrix $a(s, q)$ is symmetric and positive definite, that is there exists a constant $\nu > 0$ such that

$$\left( a(s, q)v, v \right) \geq \nu |v|^2, \quad \forall s \in \mathbb{R}, \forall q, v \in \mathbb{R}^n \quad (1.1)$$

Moreover, let us suppose that $a$ and $V$ are 1-periodic in the first variable:

$$a(s + 1, q) = a(s, q); \quad V(s + 1, q) = V(s, q) \quad (1.2)$$

**Theorem 1.4.** — Let us suppose that $M$ is compact, connected and noncontractible in itself and that either

a) $\pi_1(M)$ has infinitely many conjugacy classes

or

b) $\pi_1(M)$ has a finite number of elements.

Then, there exists a sequence $\{\gamma_h\}_h \subset W^{2,\infty}(\mathbb{R}; \mathbb{R}^n)$ such that $\forall h \in \mathbb{N}$

i) $\gamma_h$ is 1-periodic and $\gamma_h(s) \in M$

ii) $\frac{d}{ds} \left( \nabla_v L(s, \gamma_h, \gamma'_h) \right) - \nabla_q L(s, \gamma_h, \gamma'_h) \in \partial^- I_M(\gamma_h)$ a.e. in $[0, 1[$

iii) $\lim_{h \to \infty} \int_0^1 L(s, \gamma_h, \gamma'_h) \, ds = +\infty$.

In order to apply the critical point theory for non-smooth functionals some other notions and results have to be recalled.

**Definition 1.5.** — Let $\Omega$ be an open subset of $H$ and $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ a function. A point $u \in D(f)$ is said to be a lower critical point for $f$ if $0 \in \partial^- f(u)$; $c \in \mathbb{R}$ is said to be a critical value of $f$ if there exists $u \in D(f)$ such that

$$0 \in \partial^- f(u) \quad \text{and} \quad f(u) = c.$$

**Definition 1.6.** — (see also [8], [11]) Let $\Omega$ be an open subset of $H$. A function $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ is said to have a $\varphi$-monotone subdifferential of order two if there exists a continuous function

$$\chi : D(f)^2 \times \mathbb{R}^2 \to \mathbb{R}^+$$

such that

$$(\alpha - \beta, u - v) \geq -\chi(u, v, f(u), f(v)) \left( 1 + |\alpha|^2 + |\beta|^2 \right) |u - v|^2$$
whenever
\[ u, v \in D(\partial^- f), \quad \alpha \in \partial^- f(u) \quad \text{and} \quad \beta \in \partial^- f(v). \]

The notion of $p$-convex set is actually a particular case of the previous notion. In fact, it turns out (see [2]) that a subset $M$ of $H$ is $p$-convex if and only if $I_M$ has a $\varphi$-monotone subdifferential of order two.

**Theorem 1.7.** (see [10]) Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function with a $\varphi$-monotone subdifferential of order two. We set
\[ d^*(u,v) = |u - v| + |f(u) - f(v)|, \quad \forall \ u, v \in D(f). \]

Let us suppose that:

1. $\inf_H f > -\infty$
2. every sequence $(u_h) \subset D(\partial^- f)$ with $\sup_h f(u_h) < +\infty$ and $\lim_h \text{grad}^- f(u_h) = 0$ has a subsequence converging in $H$.

Then, $f$ has at least $\text{cat}(D(f), d^*) = +\infty$, then
\[ \sup\{f(u) \mid u \in D(\partial^- f), 0 \in \partial^- f(u)\} = +\infty. \]

Let $M$ be a $p$-convex subset of $H$.

**Definition 1.8.** Let us denote by $\hat{A}$ the set of $u$'s $\in H$ with the two properties:

1. $\delta_p(u, M) < 1$ where $\delta_p(u, M) = \limsup_{|u-w| \to d(u,M)} 2p(w)|u - w|.
2. $\exists r \geq 0$ such that $M \cap \{v \in H \mid |v - u| \leq r\}$ is closed in $H$ and not empty.

Obviously, $M \subset \hat{A}$ and:

**Proposition 1.9.** (see prop. 2.9 in [2]) Let $M \subset H$ be $p$-convex and locally closed. Then $\hat{A}$ is open and $\forall u \in \hat{A}$ there exists one and only one $w \in M$ such that $|u - w| = d(u, M)$.

Moreover, if we set $\pi(u) = w$, then

1. $(u - \pi(u)) \in \partial^- I_M(\pi(u))$ and $2p(\pi(u))|u - \pi(u)| < 1$, $\forall u \in \hat{A}$
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ii) \( |\pi(u_1) - \pi(u_2)| \leq \left( 1 - p(\pi(u_1)) |u_1 - \pi(u_1)| + 
- p(\pi(u_2)) |u_2 - \pi(u_2)| \right)^{-1} |u_1 - u_2|, \)
\( \forall u_1, u_2 \in \tilde{A} \)

iii) \((t \pi(u) + (1 - t)u) \in \tilde{A}, \forall u \in \tilde{A}, \forall t \in [0, 1].\)

**Remark 1.10.** — Let us set \( A = \left\{ u \in \tilde{A} \mid 4p(\pi(u)) |u - \pi(u)| < 1 \right\}. \)
Then \( A \) is an open set containing \( M \) and, by proposition 1.9 ii), \( \pi : A \to M \)
is Lipschitz continuous of constant two.

**Proposition 1.11.** — (see prop. 2.2 in [2]) Let \( M \subset H \) be p-convex.
If \( \{u_h\}_h \subset M \) is a sequence converging to \( u \in M \) and \( \{\alpha_h\}_h \subset H \) is a sequence converging weakly to \( \alpha \) with \( \alpha_h \in \partial^- I_M(u_h) \), then \( \alpha \in \partial^- I_M(u) \).

**Proposition 1.12.** — (see prop. 2.12 in [2]) Let \( M \subset H \) be locally closed and p-convex. Then
\[
\lim_{t \to 0^+} \frac{\pi(u + tv) - u}{t} = P_u v, \quad \forall u \in M \quad \text{and} \quad \forall v \in H
\]
where \( P_u \) is the projection on the tangent cone to \( M \) at \( u \).

**Proposition 1.13.** — Let \( M \subset H \) be locally closed and p-convex. Let \( \{u_h\}_h \) be a sequence in \( M \) converging to \( u \in M \) and let \( \tau \in (\partial^- I_M(u))^- \).
Then
\[
\lim_{h} P_{u_h} \tau = \tau.
\]

**Proof.** — Since \( \{P_{u_h} \tau\}_h \) is bounded, up to a subsequence \( \{P_{u_h} \tau\}_h \) is weakly convergent to some \( \xi \in H \). Since \( (\tau - P_{u_h} \tau) \in \partial^- I_M(u_h) \), by proposition 1.11 we have \( (\tau - \xi) \in \partial^- I_M(u) \). Therefore \( (\tau - \xi, \tau) \leq 0 \),
which implies \( |\tau| \leq |\xi| \), hence
\[
\liminf_{h} |P_{u_h} \tau| \geq |\tau|.
\]
From the equality
\[
|\tau|^2 = |P_{u_h} \tau|^2 + |\tau - P_{u_h} \tau|^2
\]
the thesis follows. □
2. The variational structure of the problem

In this section, we want to supply our problem, that is the research of periodic orbits of the considered Lagrangian system, with a variational structure. Our aim is to characterize such periodic orbits as lower critical points of the functional

\[ f : L^2(0, 1; \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\} \]

defined in the following way:

\[
\begin{align*}
f(\gamma) &= \begin{cases} \\
\frac{1}{2} \int_0^1 (a(s, \gamma) \gamma', \gamma') \, ds - \int_0^1 V(s, \gamma) \, ds & \gamma \in X \\
+\infty & \gamma \in L^2(0, 1; \mathbb{R}^n) \setminus X
\end{cases}
\end{align*}
\]

where the space of the admissible paths is:

\[ X = \{ \gamma \in W^{1,2}(0, 1; \mathbb{R}^n) \mid \gamma(s) \in M, \, \gamma(0) = \gamma(1) \} . \]

Since \( M \) is compact, we shall assume the function \( p \) of definition 1.3 to be constant.

Moreover, if \( \gamma \in W^{1,2}(0, 1; \mathbb{R}^n) \) with \( \gamma(s) \in M \) and \( \delta \in L^2(0, 1; \mathbb{R}^n) \), we set

\[(P_{\gamma(s)}\delta)(s) = P_{\gamma(s)}\delta(s)\]

where \( P_{\gamma(s)} \) is the projection on the tangent cone to \( M \) at \( \gamma(s) \), according to the scalar product

\[(u, v)_s = (a(s, \gamma(s))u, v) .\]

Let us also denote by \( \pi_s \) the projection on \( M \) according to the scalar product \((u, v)_s\). By remark 1.8 and the assumptions on \( a \), there exists an open set \( A \) containing \( M \) such that each \( \pi_s \) is defined on \( A \) and is Lipschitz continuous of constant 2.

Let us begin with a regularity result.

**Theorem 2.1.** — Let \( \gamma \in X \). If \( \partial^- f(\gamma) \neq \emptyset \) then

\[ \gamma \in W^{2,2}(0, 1; \mathbb{R}^n), \quad \gamma'_+(0) = \gamma'_-(1) \]
Moreover, if \( 0 \in \partial^{-} f(\gamma) \) then \( \gamma \in W^{2,\infty}(0,1;\mathbb{R}^{n}) \).

For the proof of this theorem, we need some lemmas.

**Lemma 2.2.** Let us take \( \delta \in W^{1,2}(0,1;\mathbb{R}^{n}) \) and \( \gamma \in W^{1,2}(0,1;\mathbb{R}^{n}) \) such that \( \gamma(s) \in M, \forall s \in [0,1] \).

Then the following facts hold:

a) \( \lim_{t \to 0^{+}} \frac{\pi_{s}(\gamma + t\delta) - \gamma}{t} = P_{\gamma}\delta, \quad \forall s \in [0,1] \)

and

\[
\lim_{t \to 0^{+}} \sup_{s} \left\| \frac{\pi_{s}(\gamma + t\delta) - \gamma}{t} \right\|_{L^{\infty}} < \infty
\]

b) for every sufficiently small \( t > 0 \), we have

\( \pi_{s}(\gamma + t\delta) \in W^{1,2}(0,1;\mathbb{R}^{n}) \)

and \( a.e. \) in \( [0,1] \)

\[
\left| (\pi_{s}(\gamma + t\delta))' \right|_{s} \leq |\gamma' + t\delta'|_{s} + \text{const} |(\gamma + t\delta) - \pi_{s}(\gamma + t\delta)| (1 + |\gamma'| + t|\delta'|)
\]

\( c) \lim_{t \to 0^{+}} (\pi_{s}(\gamma + t\delta))' = \gamma' \) in \( L^{2}(0,1;\mathbb{R}^{n}) \).

**Proof.** a) First of all, let us remark that \( P_{\gamma}\delta \) is measurable. By proposition 1.12, we have

\[
\lim_{t \to 0^{+}} \frac{\pi_{s}(\gamma + t\delta) - \gamma}{t} = P_{\gamma}\delta
\]

and, also

\[
\left| \frac{\pi_{s}(\gamma + t\delta) - \gamma}{t} \right|_{s} \leq 2 \left| \frac{\gamma + t\delta - \gamma}{t} \right|_{s} \leq 2|\delta|_{s} \quad (2.2.1)
\]

Then, by (2.2.1), we get

\[
\left| \frac{\pi_{s}(\gamma + t\delta) - \gamma}{t} \right| \leq \text{const} \left| \frac{\pi_{s}(\gamma + t\delta) - \gamma}{t} \right|_{s} \leq \text{const} 2|\delta|_{s} \leq \text{const} |\delta| \quad (2.2.2)
\]

so, the proof of a) is over.
b) Let us consider the two scalar products

\[(u, v)_{s_i} = (A_{s_i} u, v) = (a(s_i, \gamma(s_i)) u, v), \quad i = 1, 2.\]

For every \(u \in A\), let \(w_{s_i} = \pi_{s_i} u\) the projection of \(u\) according to the scalar product \((u, v)_{s_i}\), \(i = 1, 2\). We want to prove that

\[|w_{s_1} - w_{s_2}| \leq \frac{2}{\nu} |u - w_{s_2}| |A_{s_1} - A_{s_2}|.\]  

(2.2.3)

Let us observe that, by proposition 1.9, \(u - w_{s_1} \in \partial^{-} I_M(w_{s_1})\), that is

\[
\liminf_{v \to u, v \in M} \frac{(u - w_{s_1}, w_{s_1} - v)_{s_1}}{|w_{s_1} - v|_{s_1}} \geq 0.
\]  

(2.2.4)

Passing to the usual metric in \(\mathbb{R}^n\), (2.2.4) is equivalent to

\[
\liminf_{v \to u, v \in M} \frac{(A_{s_1}(u) - A_{s_1}(w_{s_1}), w_{s_1} - v)}{|w_{s_1} - v|} \geq 0.
\]  

(2.2.5)

By (2.2.5), it is easy to deduce that

\[A_{s_1}(u) - A_{s_1}(w_{s_1}) \in \partial^{-} I_M(w_{s_1}).\]

Analogously,

\[A_{s_2}(u) - A_{s_2}(w_{s_2}) \in \partial^{-} I_M(w_{s_2}).\]

Since \(M\) is a p-convex set, we have:

\[
\begin{align*}
(A_{s_1}(u) - A_{s_1}(w_{s_1}) - A_{s_2}(u) + A_{s_2}(w_{s_2}), w_{s_1} - w_{s_2}) & \geq \\
& \geq -p \left( |A_{s_1}(u) - A_{s_1}(w_{s_1})| + |A_{s_2}(u) - A_{s_2}(w_{s_2})| \right) |w_{s_1} - w_{s_2}|^2. 
\end{align*}
\]  

(2.2.6)

On the other hand, we have

\[
\begin{align*}
(A_{s_1}(u) - A_{s_1}(w_{s_1}) - A_{s_2}(u) + A_{s_2}(w_{s_2}), w_{s_1} - w_{s_2}) & \leq \\
& \leq (A_{s_1}(u) - A_{s_1}(w_{s_1}) + A_{s_2}(w_{s_2}) - A_{s_1}(w_{s_2}), w_{s_1} - w_{s_2}) + \\
& \quad + (A_{s_1}(w_{s_2}) - A_{s_1}(w_{s_1}), w_{s_1} - w_{s_2}) \leq \\
& \leq ((A_{s_1} - A_{s_2})(u - w_{s_2}), w_{s_1} - w_{s_2) - |w_{s_1} - w_{s_2}|^2. 
\end{align*}
\]  

(2.2.7)
By (2.2.6) and (2.2.7), we obtain
\[ |\pi_{s_1}^s (\gamma + t\delta)(s_1) - \pi_{s_2}^s (\gamma + t\delta)(s_2) |_{s_1} \leq | s_2 - s_1 |_{s_1} \]
and then
\[ \frac{|\pi_{s_1}^s (\gamma + t\delta)(s_1) - \pi_{s_1}^s (\gamma + t\delta)(s_2) |_{s_1}}{| s_2 - s_1 |_{s_1}} + \frac{|\pi_{s_1}^s (\gamma + t\delta)(s_2) - \pi_{s_2}^s (\gamma + t\delta)(s_2) |_{s_1}}{| s_2 - s_1 |_{s_1}}. \]

By substituting \( A \) with a smaller open set containing \( M \), we can assume that
\[ [\nu - p\left( |A_{s_1}(u) - A_{s_1}(w_{s_1})| + |A_{s_2}(u) - A_{s_2}(w_{s_2})| \right)] |w_{s_1} - w_{s_2}|^2 \leq \left| u - w_{s_2} \right| |A_{s_1} - A_{s_2}| |w_{s_1} - w_{s_2}|. \]

By hypothesis (1.1), it follows
\[ |w_{s_1} - w_{s_2}|_{s_1}^2 \geq \nu |w_{s_1} - w_{s_2}|^2 \]
and then
\[ [\nu - p\left( |A_{s_1}(u) - A_{s_1}(w_{s_1})| + |A_{s_2}(u) - A_{s_2}(w_{s_2})| \right)] |w_{s_1} - w_{s_2}|^2 \leq \left| u - w_{s_2} \right| |A_{s_1} - A_{s_2}| |w_{s_1} - w_{s_2}|. \]

By substituting \( A \) with a smaller open set containing \( M \), we can assume that
\[ \left[ \nu - p\left( |A_{s_1}(u) - A_{s_1}(w_{s_1})| + |A_{s_2}(u) - A_{s_2}(w_{s_2})| \right) \right] \geq \frac{\nu}{2}, \]

hence (2.2.3) follows.

Now, let us consider \( s_2 \in ]0, 1[. \) We have
\[ \frac{|\pi_{s_1}^s (\gamma + t\delta)(s_1) - \pi_{s_2}^s (\gamma + t\delta)(s_2) |_{s_1}}{| s_2 - s_1 |_{s_1}} \leq \frac{|\pi_{s_1}^s (\gamma + t\delta)(s_1) - \pi_{s_1}^s (\gamma + t\delta)(s_2) |_{s_1}}{| s_2 - s_1 |_{s_1}} + \frac{|\pi_{s_1}^s (\gamma + t\delta)(s_2) - \pi_{s_2}^s (\gamma + t\delta)(s_2) |_{s_1}}{| s_2 - s_1 |_{s_1}}. \]

By i) and ii) of proposition 1.9, we have
\[ |\pi_{s_1}^s (\gamma + t\delta)(s_1) - \pi_{s_1}^s (\gamma + t\delta)(s_2) |_{s_1} \leq \frac{|(\gamma + t\delta)(s_1) - (\gamma + t\delta)(s_2) |_{s_1}}{\mathcal{D}} \]
where
\[ \mathcal{D} = \left( 1 - p |(\gamma + t\delta)(s_1) - \pi_{s_1}^s (\gamma + t\delta)(s_1) |_{s_1} + \right. \]
\[ - \left. p |(\gamma + t\delta)(s_2) - \pi_{s_1}^s (\gamma + t\delta)(s_2) |_{s_1} \right). \]
Hence
\[ |\pi_{s_1}(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_2)|_{s_1} \leq \]
\[ \leq \text{const} \left( |\gamma(s_1) - \gamma(s_2)| + t|\delta(s_1) - \gamma(s_2)| \right). \]

Moreover, by applying (2.2.3), we get
\[ |\pi_{s_2}(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)|_{s_1} \leq \]
\[ \leq \frac{2}{\nu} |(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)| a(s_1, \gamma(s_1)) - a(s_2, \gamma(s_2))| \leq \]
\[ \leq \text{const} |(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)| (|s_2 - s_1| + |\gamma(s_2) - \gamma(s_1)|) \leq \]
\[ \leq \text{const} (|s_2 - s_1| + |\gamma(s_2) - \gamma(s_1)|). \]

Therefore
\[ \pi_s(\gamma + t\delta) \in W^{1,2}(0, 1; \mathbb{R}^n) \]
and we have a.e. in \( ]0, 1[ \)
\[ \lim_{s_2 \to s_1} \frac{|\pi_{s_1}(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_2)|_{s_1}}{|s_2 - s_1|_{s_1}} \leq \]
\[ \leq \frac{|(\gamma + t\delta)'(s_1)|_{s_1}}{1 - 2\nu |(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_1)|_{s_1}} \leq \]
\[ \leq |(\gamma + t\delta)'(s_1)|_{s_1} + \]
\[ + \text{const} |(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_1)| (|\gamma + t\delta)'(s_1)| \quad (2.2.9) \]

and
\[ \lim_{s_2 \to s_1} \frac{|\pi_{s_1}(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)|_{s_1}}{|s_2 - s_1|_{s_1}} \leq \]
\[ \leq \text{const} \lim_{s_2 \to s_1} |(\gamma + t\delta)(s_2) - \pi_{s_2}(\gamma + t\delta)(s_2)| \times \]
\[ \times \left( \frac{|s_2 - s_1| + |\gamma(s_2) - \gamma(s_1)|}{|s_2 - s_1|} \right) \leq \]
\[ \leq \text{const} |(\gamma + t\delta)(s_1) - \pi_{s_1}(\gamma + t\delta)(s_1) - (1 + |\gamma'(s_1)|). \quad (2.2.10) \]

Hence we have a.e. in \( ]0, 1[ \)
\[ |(\pi_s(\gamma + t\delta))'|_s \leq |\gamma' + t\delta'|_s + \text{const} |(\gamma + t\delta) - \pi_s(\gamma + t\delta)| (1 + |\gamma'| + t|\delta'|). \]
c) In \( L^2(0, 1; \mathbb{R}^n) \) we can consider the following scalar product
\[
(\eta, \xi) = \int_0^1 a(s, \gamma(s)) \eta(s) \xi(s) \, ds = \int_0^1 (\eta(s), \xi(s))_s \, ds.
\]
Since, by a) \( \pi_s(\gamma + t\delta) \to \gamma \) in \( L^\infty \) and \( (\pi_s(\gamma + t\delta))' \) is bounded in \( L^2(0, 1; \mathbb{R}^n) \) as \( t \to 0 \), we have that
\[
(\pi_s(\gamma + t\delta))' \text{ weakly converges to } \gamma' \text{ in } L^2(0, 1; \mathbb{R}^n).
\] (2.2.11)
Moreover
\[
\lim_{t \to 0^+} \left| (\gamma + t\delta) - \pi_s(\gamma + t\delta) \right| = 0 \text{ uniformly on } [0, 1],
\]
and by b) we have
\[
\limsup_{t \to 0} \int_0^1 \left| (\pi_s(\gamma + t\delta))' \right|_s^2 \, ds \leq \int_0^1 |\gamma'|_s^2 \, ds. \tag{2.2.12}
\]
Combining (2.2.11) and (2.2.12), we get
\[
\lim_{t \to 0^+} \left( \pi_s(\gamma + t\delta) \right)' = \gamma' \text{ in } L^2(0, 1; \mathbb{R}^n). \Box
\]

**Lemma 2.3.** — Let us take \( \delta \in W^{1,2}(0, 1; \mathbb{R}^n) \) and \( \gamma \in W^{1,2}(0, 1; \mathbb{R}^n) \) such that \( \gamma(s) \in M, \forall \ s \in [0, 1] \).

Then
\[
\liminf_{t \to 0^+} \frac{1}{t} \left\{ \frac{1}{2} \int_0^1 (a(s, \gamma)(\gamma + t\delta)', (\gamma + t\delta))' \, ds + \frac{1}{2} \int_0^1 (a(s, \gamma)(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))') \, ds \right\} \geq -\text{const} \int_0^1 |\gamma'|(1 + |\gamma'|)\delta - P_\gamma \delta | \, ds. \tag{2.3.1}
\]

**Proof.** — Let us fix \( t > 0 \) and let us take the path \( \gamma + t\delta \). If \( t \) is small, \( (\gamma + t\delta)(s) \in A, \forall \ s \in [0, 1] \).

By lemma 2.2 b), we have \( \pi_s(\gamma + t\delta) \in W^{1,2}(0, 1; M) \) and
\[
\frac{1}{2} \int_0^1 \frac{1}{t} \left\{ \left| (\gamma + t\delta)' \right|_s^2 - \left| (\pi_s(\gamma + t\delta))' \right|_s^2 \right\} \, ds \geq \nonumber
\]
\[
\geq \frac{1}{2} \int_0^1 \frac{1}{t} \left\{ -\text{const} |(\gamma + t\delta) - \pi_s(\gamma + t\delta)|^2 (1 + |\gamma'| + t|\delta'|)^2 + \right. \nonumber
\]
\[
-\text{const} |(\gamma + t\delta)'| |(\gamma + t\delta) - \pi_s(\gamma + t\delta)| (1 + |\gamma'| + t|\delta'|) \right\} \, ds.
\]

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Combining lemma 2.2 c) with Lebesgue theorem, we get:

\[
\liminf_{t \to 0^+} \frac{1}{t} \int_0^1 \frac{1}{t} \left\{ \left| (\gamma + t\delta)' \right|^2_s - \left| (\pi_s(\gamma + t\delta))' \right|^2_s \right\} \, ds \\
\geq - \text{const} \int_0^1 |\gamma'|(1 + |\gamma'|) \delta - P_\gamma \delta | \, ds. \quad \square
\]

**Lemma 2.4.** Let us take \( \delta \in W^{1,2}(0, 1; \mathbb{R}^n) \) with \( \delta(0) = \delta(1) \) and \( \alpha \in \partial^- f(\gamma) \). Then

\[
\int_0^1 (a(s, \gamma) \gamma', \delta') \, ds - \int_0^1 (\alpha, P_\gamma \delta) \, ds \geq \\
\geq - \text{const} \int_0^1 |\gamma'|(1 + |\gamma'|) \delta - P_\gamma \delta | \, ds + \\
+ \int_0^1 (\nabla_q V(s, \gamma), P_\gamma \delta) \, ds - \frac{1}{2} \int_0^1 \left( \frac{\partial a(s, \gamma)}{\partial q} (P_\gamma \delta) \gamma', \gamma' \right) \, ds. \tag{2.4.1}
\]

**Proof.** Let us take \( \delta \in W^{1,2}(0, 1; \mathbb{R}^n) \) with \( \delta(0) = \delta(1) \) and \( t > 0 \) small enough that

\[
\pi_s(\gamma + t\delta) \in M.
\]

Let us observe that, by setting

\[
f_1(\gamma) = \frac{1}{2} \int_0^1 (a(s, \gamma) \gamma', \gamma') \, ds,
\]
from remark 1.2, we deduce that \( \alpha \in \partial^- f(\gamma) \) if and only if \( \alpha = \tilde{\alpha} - \nabla_q V(s, \gamma) \) where \( \tilde{\alpha} \in \partial^- f_1(\gamma) \).

Now, let us take \( \alpha \in \partial^- f(\gamma) \). By proposition 1.12, we have:

\[
\int_0^1 (a(s, \gamma) \gamma', \delta') \, ds - \int_0^1 (\alpha, P_\gamma \delta) \, ds = \\
= \lim_{t \to 0^+} \frac{1}{t} \int_0^1 \left\{ \frac{1}{2} (a(s, \gamma)(\gamma + t\delta)', (\gamma + t\delta)') - \frac{1}{2} (a(s, \gamma) \gamma', \gamma') + \\
- \tilde{\alpha}(\pi_s(\gamma + t\delta) - \gamma) \right\} \, ds + \int_0^1 (\nabla_q V(s, \gamma), P_\gamma \delta) \, ds \geq \\
\geq \liminf_{t \to 0^+} \frac{1}{t} \int_0^1 \left\{ \frac{1}{2} (a(s, \pi_s(\gamma + t\delta))(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))') + \\
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\]

Recalling that \( (\pi_s(\gamma + t\delta) - \gamma) / t \) is bounded in \( L^2(0,1; \mathbb{R}^n) \), by lemma 2.2 c), proposition 1.12 and Lebesgue theorem, we get

\[
-\frac{1}{2} (a(s, \gamma)\gamma', \gamma') - \alpha(\pi_s(\gamma + t\delta) - \gamma) ds + \liminf_{t \to 0^+} \frac{1}{t} \int_0^1 \left\{ \frac{1}{2} \left( a(s, \gamma)(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))' \right) + \frac{1}{2} \left( a(s, \pi_s(\gamma + t\delta))(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))' \right) ds + \right. \\
+ \int_0^1 (\nabla_q V(s, \gamma), P, \gamma) ds.
\]

Then, by definition 1.1 and lemma 2.3, we get the thesis. □

**Lemma 2.5.** Let us take \( \alpha \in L^2(0,1; \mathbb{R}^n) \) and \( \gamma \in W^{1,2}(0,1; \mathbb{R}^n) \). Let us suppose that \( (\pi_t(s))_t \) holds \( \forall t \in W^{1,2}(0,1; \mathbb{R}^n) \) with \( \delta(0) = \delta(1) \).

Then, by definition 1.1 and lemma 2.3, we get the thesis. □

\[
\liminf_{t \to 0^+} \frac{1}{t} \int_0^1 \left\{ \frac{1}{2} \left( a(s, \gamma)(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))' \right) + \frac{1}{2} \left( a(s, \pi_s(\gamma + t\delta))(\pi_s(\gamma + t\delta))', (\pi_s(\gamma + t\delta))' \right) ds \geq \right. \\
\geq -\frac{1}{2} \int_0^1 \left( \frac{\partial a(s, \gamma)}{\partial q} (P, \gamma)\gamma', \gamma' \right) ds.
\]

---

**Lemma 2.5.** — Let us take \( \alpha \in L^2(0,1; \mathbb{R}^n) \) and \( \gamma \in W^{1,2}(0,1; \mathbb{R}^n) \) such that \( \gamma(s) \in M, \forall s \in [0,1] \). Let us suppose that (2.4.1) holds \( \forall \delta \in W^{1,2}(0,1; \mathbb{R}^n) \) with \( \delta(0) = \delta(1) \).

Then

\( i) \ \gamma \in W^{2,2}(0,1; \mathbb{R}^n); \quad \gamma_+'(0) = \gamma_-'(1) \)

\( ii) \ \alpha + \frac{d}{ds} (a(s, \gamma)\gamma') + \nabla_q V(s, \gamma) - \frac{1}{2} \frac{\partial a(s, \gamma)}{\partial q} (\gamma', \gamma') \in \partial^- I_M(\gamma(s)) \ a.e. \)

\( iii) \ |\gamma''(s)| \leq \text{const} \left( 1 + |\alpha(s)| + |\gamma'(s)|^2 \right) \ a.e. \)
Proof. — Since $|\delta - P_\gamma \delta| \leq |\delta|$, by applying Cauchy-Schwartz inequality to (2.4.1), we obtain $\forall \delta \in W^{1,2}(0,1; \mathbb{R}^n)$ with $\delta(0) = \delta(1)$

$$
\int_0^1 (a(s, \gamma)\gamma', \delta') \, ds \geq -\|\alpha\|_{L^2} \|\delta\|_{L^2} - \text{const} \int_0^1 |\gamma'| (1 + |\gamma'|) |\delta| \, ds +
- \int_0^1 |\nabla q V(s, \gamma)| |\delta| \, ds - \frac{1}{2} \int_0^1 \left| \frac{\partial a(s, \gamma)}{\partial q} \right| |\delta| |\gamma'|^2 \, ds
$$

and then

$$
\left| \int_0^1 (a(s, \gamma)\gamma', \delta') \, ds \right| \leq \left( \|\alpha\|_{L^2} + \text{const} \int_0^1 |\gamma'| \, ds +
+ \text{const} \int_0^1 |\gamma'|^2 \, ds + \text{const} \right) \|\delta\|_{L^\infty}. \quad (2.5.1)
$$

By (2.5.1), we deduce that $a(s, \gamma)\gamma' \in L^\infty$ and

$$
\|a(s, \gamma)\gamma'\|_{L^\infty} \leq \|a(s, \gamma)\gamma'\|_{L^1} + \|\alpha\|_{L^2} +
+ \text{const} \int_0^1 |\gamma'| \, ds + \text{const} \int_0^1 |\gamma'|^2 \, ds + \text{const}. \quad (2.5.2)
$$

On the hand, by hypothesis (1.1)

$$
\nu |\gamma'|^2 \leq (a(s, \gamma)\gamma', \gamma') \leq |a(s, \gamma)\gamma'| |\gamma'|
$$

which implies

$$
\nu \|\gamma'\|_{L^\infty} \leq \|a(s, \gamma)\gamma'\|_{L^\infty}.
$$

Thus, $\gamma' \in L^\infty$ and

$$
\|\gamma'\|_{L^\infty} \leq \frac{1}{\nu} \left( \|a(s, \gamma)\gamma'\|_{L^1} + \|\alpha\|_{L^2} + \text{const} \int_0^1 |\gamma'| \, ds +
+ \text{const} \int_0^1 |\gamma'|^2 \, ds + \text{const} \right). \quad (2.5.3)
$$

By using (2.5.3), we have

$$
\int_0^1 |\gamma'|(1 + |\gamma'|)|\delta| \, ds \leq \|\delta\|_{L^2} \|\gamma'\|_{L^2} + \|\delta\|_{L^2} \|\gamma'|_{L^2}^2 \leq
\leq \|\delta\|_{L^2} \|\gamma'\|_{L^2} + \|\delta\|_{L^2} \|\gamma'|_{L^2} \|\gamma'|_{L^\infty} \leq
\leq \|\gamma'\|_{L^2} \left( 1 + \|\gamma'|_{L^\infty} \right) \|\delta\|_{L^2} \leq
\leq \|\gamma'\|_{L^2} \left( 1 + \frac{1}{\nu} \left( \text{const} \|\gamma'\|_{L^1} + \|\alpha\|_{L^2} +
+ \text{const} \|\gamma'||_{L^2}^2 + \text{const} \right) \right) \|\delta\|_{L^2}. \quad (2.5.4)
$$
Since,
\[ \frac{1}{2} \int_0^1 \left| \frac{\partial a(s, \gamma)}{\partial q} \right| |\delta| |\gamma'|^2 \, ds \leq \text{const} \|\delta\|_{L^2} \|\gamma'\|_{L^4}^2 \]

by (2.4.1) and (2.5.4), we have
\[
\left| \int_0^1 (a(s, \gamma) \gamma', \delta') \, ds \right| \leq \\
\leq \left[ \|\alpha\|_{L^2} + \text{const} \|\gamma'\|_{L^2} \left( 1 + \|\gamma'\|_{L^1} + \|\alpha\|_{L^2} + \|\gamma'\|_{L^2}^2 \right) + \\
\text{const} \|\delta\|_{L^2} \right] \leq \\
\leq \left[ (1 + \text{const} \|\gamma'\|_{L^2}) \|\alpha\|_{L^2} + \\
+ \text{const} \|\gamma'\|_{L^2} \left( 1 + \|\gamma'\|_{L^1} + \|\gamma'\|_{L^2}^2 \right) + \text{const} \right] \|\delta\|_{L^2}.
\]

So, we can conclude that
\[ a(s, \gamma) \gamma' \in W^{1,2}(0, 1; \mathbb{R}^n). \]

Moreover, by (2.4.1), we deduce that
\[ \left| \frac{d}{ds} (a(s, \gamma) \gamma') \right| \leq \text{const} \left( 1 + |\alpha| + |\gamma'|^2 \right) \quad \text{a.e. in } [0, 1[. \]

It remains to prove that \( \gamma' \in W^{1,2}(0, 1; \mathbb{R}^n) \). By hypothesis (1.1), we have
\[
\nu^2 \frac{|\gamma'(s_2) - \gamma'(s_1)|^2}{|s_2 - s_1|} \leq \\
\leq \frac{|a(s_1, \gamma(s_1)) (\gamma'(s_2) - \gamma'(s_1))|^2}{|s_2 - s_1|} = \frac{\mathcal{N}}{|s_2 - s_1|} \leq \\
\leq 2 \frac{|a(s_2, \gamma(s_2)) \gamma'(s_2) - a(s_1, \gamma(s_1)) \gamma'(s_1)|^2}{|s_2 - s_1|} + \\
+ 2 \frac{|a(s_2, \gamma(s_2)) - a(s_1, \gamma(s_1))| \gamma'(s_2)^2}{|s_2 - s_1|}. \tag{2.5.6}
\]

where
\[ \mathcal{N} = |a(s_2, \gamma(s_2)) \gamma'(s_2) - a(s_1, \gamma(s_1)) \gamma'(s_1) + \\
- [a(s_2, \gamma(s_2)) - a(s_1, \gamma(s_1))] \gamma'(s_2)^2|. \]
Since $a(y), a(s, y)y' \in W^{1,2}(0,1; \mathbb{R}^n)$ and $y' \in L^{\infty}(0,1; \mathbb{R}^n)$, we deduce that

$$y' \in W^{1,2}(0,1; \mathbb{R}^n)$$

and

$$|y''| \leq \frac{1}{\nu} \left( \left| \frac{d}{ds} \left( a(s, y) \gamma' \right) \right| + \left| \frac{d}{ds} \left( a(s, y) \right) \right| \right) \leq \text{const} \left( 1 + |\alpha| + |\gamma'|^2 \right) \text{ a.e. in } ]0,1[.$$

If we set

$$\tilde{\gamma}(s) = \begin{cases} 
\gamma(s + \frac{1}{2}) & 0 \leq s \leq \frac{1}{2} \\
\gamma(s - \frac{1}{2}) & \frac{1}{2} \leq s \leq 1 
\end{cases}$$

it turns out that $\tilde{\gamma}$ also satisfies (2.4.1) with $a, V$ and $\alpha$ substituted by other suitable maps. It follows that

$$\tilde{\gamma} \in W^{2,2}(0,1; \mathbb{R}^n),$$

hence

$$\gamma'_+(0) = \gamma'_-(1).$$

Since $\partial^- I_M(\gamma(s))$ is a closed convex cone, to prove ii) is equivalent to prove that a.e. $\forall \eta \in (\partial^- I_M(\gamma(s)))^-$

$$\left( \alpha + \frac{d}{ds} \left( a(s, y) \gamma' \right) + \nabla_q V(s, \gamma) - \frac{1}{2} \frac{\partial a(s, y)}{\partial q}(\gamma', \gamma'), \eta \right) \leq 0.$$

Let us define the following functions:

$$\delta_n(s) = \rho_n(s - s_0)\eta, \quad \forall n \in \mathbb{N}$$

where

$$\begin{cases} 
s_0 \in ]0,1[ \text{ is a Lebesgue point for } \alpha \text{ and for } \frac{d}{ds} (a(s, y)\gamma'); \\
\eta \in (\partial^- I_M(\gamma(s_0)))^-; \\
\rho_n \in C_0^\infty(\mathbb{R}), \quad \rho_n \geq 0, \quad \int_0^1 \rho_n \, ds = 1 \quad \text{and} \quad \text{supt } \rho_n \subset \left[ -\frac{1}{n}, \frac{1}{n} \right].
\end{cases}$$
Clearly, $\delta_n \in W^{1,2}_0(0, 1; \mathbb{R}^n)$ and then by (2.4.1) we get
\[
- \left( \eta, \int_0^1 \rho_n(s_0 - s) \frac{d}{ds} (a(s, \gamma) \gamma') ds \right) \geq 
\geq \int_0^1 \rho_n(s_0 - s)(\alpha(s), P_\gamma \eta)(s) ds + 
- \text{const} \int_0^1 \rho_n(s_0 - s) |\eta - P_\gamma \eta| |\gamma'| ds + 
- \text{const} \int_0^1 \rho_n(s_0 - s) |\eta - P_\gamma \eta| |\gamma'|^2 ds + 
+ \int_0^1 \rho_n(s_0 - s)(\nabla_q V(s, \gamma), P_\gamma \eta)(s) ds + 
- \frac{1}{2} \int_0^1 \rho_n(s_0 - s) \left( \frac{\partial a(s, \gamma)}{\partial q} (P_\gamma \eta) \gamma', \gamma' \right) ds. \tag{2.5.7}
\]

By proposition 1.13, $P_\gamma \delta$ is continuous at $s_0$, hence passing to the limit as $s \to s_0$ in (2.5.7), we obtain
\[
- \left( \eta, \frac{d}{ds} (a(s_0, \gamma(s_0)) \gamma'(s_0)) \right) \geq 
\geq (\alpha(s_0), (P_\gamma \eta)(s_0)) - \text{const} |\eta - P_\gamma \eta|(s_0) |\gamma'|^2(s_0) + 
- \text{const} |\eta - P_\gamma \eta|(s_0) |\gamma'|(s_0) + (\nabla_q V(s_0, \gamma(s_0)), P_\gamma \eta)(s_0) + 
- \frac{1}{2} \left( \frac{\partial a(s_0, \gamma(s_0))}{\partial q} (P_\gamma \eta)(s_0) \gamma'(s_0), \gamma'(s_0) \right) = 
= (\alpha(s_0), \eta(s_0)) + (\nabla_q V(s_0, \gamma(s_0)), P_\gamma \eta)(s_0) + 
- \frac{1}{2} \left( \frac{\partial a(s_0, \gamma(s_0))}{\partial q} (P_\gamma \eta)(s_0) \gamma'(s_0), \gamma'(s_0) \right) \text{ a.e. in } ]0, 1[. \ Square
\]

Finally, we are able to prove theorem 2.1.

Proof of theorem 2.1
As a direct consequence of lemmas 2.4 and 2.5, we get
\[
\gamma \in W^{2,2}(0, 1; \mathbb{R}^n), \quad \gamma'_+(0) = \gamma'_-(1)
\]
and
\[
|\gamma''(s)| \leq \text{const} \left( 1 + |\alpha(s)| + |\gamma'(s)|^2 \right) \text{ a.e. in } ]0, 1[.
\]

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If $0 \in \partial^+ f(\gamma)$, it is evident that

$$\gamma'' \in L^\infty(0, 1; \mathbb{R}^n). \quad \square$$

Now, let us prove two properties of $f$.

**Theorem 2.6.** The functional $f : L^2(0, 1; \mathbb{R}^n) \to \mathbb{R} \cup \{\pm \infty\}$ is lower semicontinuous and there exists a continuous function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta) \, ds \geq -\theta(f(\gamma)) \left( 1 + \|\alpha\|_{L^2}^2 \right) \|\delta\|_{L^2}^2$$

whenever $\gamma, \gamma + \delta \in X$ and $\alpha \in \partial^+ f(\gamma)$.

In particular, $f$ has a $\varphi$-monotone subdifferential of order two.

**Proof.** Let us take a sequence $\{\gamma_n\}_n \subset X$ such that

$$\lim_{n} \gamma_n = \gamma \text{ in } L^2(0, 1; \mathbb{R}^n) \text{ and } f(\gamma_n) \leq c.$$

In order to prove that $f$ is lower semicontinuous, it is enough to prove that $f(\gamma) \leq c$. Since,

$$\frac{1}{2} \int_0^1 (a(s, \gamma_n)\gamma_n', \gamma_n') \, ds - \int_0^1 V(s, \gamma_n) \, ds \leq c$$

and by hypothesis (1.1)

$$\int_0^1 (a(s, \gamma_n)\gamma_n', \gamma_n') \, ds \geq \nu \int_0^1 |\gamma_n'|^2 \, ds$$

we can deduce that $\{\gamma_n\}_n$ is bounded in $L^2(0, 1; \mathbb{R}^n)$ and thus, by the compactness of $M$, $\gamma_n$ weakly converges to $\gamma$ in $W^{1,2}(0, 1; \mathbb{R}^n)$. Besides, $L$ is continuous in the three variables and convex in the third one, so it is weakly lower semicontinuous in $W^{1,2}(0, 1; \mathbb{R}^n)$. This implies

$$\frac{1}{2} \int_0^1 (a(s, \gamma)\gamma', \gamma') \, ds - \int_0^1 V(s, \gamma) \, ds \leq \liminf_{n} \frac{1}{2} \int_0^1 (a(s, \gamma_n)\gamma_n', \gamma_n') \, ds - \int_0^1 V(s, \gamma_n) \, ds \leq c.$$

Moreover, $\{\gamma_n\}_n$ converges uniformly to $\gamma$. Since $M$ is closed, then $\gamma \in X$. 

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Now, let us take \( \gamma \in X \cap W^{2,2}(0, 1; \mathbb{R}^n) \), with \( \gamma_+(0) = \gamma'_+(0) = 1 \) and \( \alpha \in \partial f(\gamma) \). Let \( \delta \in W^{1,2}(0, 1; \mathbb{R}^n) \) with \( \delta(0) = \delta(1) \) be such that \( \gamma + \delta \in X \). Then, by Taylor’s formula, we have:

\[
f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta) \, ds =
\]

\[
= \frac{1}{2} \int_0^1 \left( a(s, \gamma + \delta)(\gamma' + \delta'), (\gamma' + \delta') \right) \, ds - \frac{1}{2} \int_0^1 (a(s, \gamma)\gamma', \gamma') \, ds +
\]

\[
- \int_0^1 V(s, \gamma + \delta) \, ds + \int_0^1 V(s, \gamma) \, ds - \int_0^1 (\alpha, \delta) \, ds =
\]

\[
= \frac{1}{2} \int_0^1 \left( \frac{\partial a(s, \gamma)}{\partial q}(\delta)\gamma', \gamma' \right) \, ds - \int_0^1 \left( \nabla_q V(s, \gamma), \delta \right) \, ds +
\]

\[
+ \int_0^1 \left( a(s, \gamma)\gamma', \delta' \right) \, ds - \int_0^1 (\alpha, \delta) \, ds +
\]

\[
+ \frac{1}{4} \int_0^1 \left( \frac{\partial^2 a(s, \gamma)}{\partial q^2}(\delta)^2 \gamma', \gamma' \right) \, ds - \frac{1}{2} \int_0^1 \left( \nabla_{qq} V(s, \gamma) \delta, \delta \right) \, ds +
\]

\[
+ \frac{1}{4} \int_0^1 \left( \frac{\partial a(s, \gamma)}{\partial q}(\delta), \delta' \right) \, ds + \frac{1}{4} \int_0^1 \left( a(s, \gamma)\delta', \delta' \right) \, ds
\]

where \( \tilde{\gamma} = \gamma + t\delta \) for some \( t = t(s) \in ]0, 1[ \). Reordering terms, by hypothesis (1.1), we get:

\[
f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta) \, ds \geq
\]

\[
\geq \frac{1}{2} \int_0^1 \left( \frac{\partial a(s, \gamma)}{\partial q}(\gamma', \gamma') - \nabla_q V(s, \gamma) - \frac{d}{ds} (a(s, \gamma)\gamma') - \alpha, \delta \right) \, ds +
\]

\[
- \left( \frac{1}{4} \left\| \frac{\partial^2 a(s, \gamma)}{\partial q^2} \right\|_{L_\infty} \| \gamma' \|^2_{L_2} + \frac{1}{2} \left\| \nabla_{qq} V(s, \gamma) \right\|_{L_\infty} \| \delta \|^2_{L_\infty} \right) +
\]

\[
- \frac{1}{4} \left\| \frac{\partial a(s, \gamma)}{\partial q} \right\|_{L_\infty} \| \delta \|_{L_2} \| \delta' \|_{L^2} + \frac{\nu}{2} \| \delta' \|^2_{L_2}.
\]

Thus, by \( p \)-convexity of \( M \), we get:

\[
f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta) \, ds \geq
\]

\[
\geq - \left( \frac{1}{2} \left\| \frac{\partial a(s, \gamma)}{\partial q} \right\|_{L_\infty} \| \gamma' \|_{L_2} + \| \nabla_q V(s, \gamma) \| +
\]

\[
+ \left\| \frac{d}{ds} (a(s, \gamma)) \gamma' + a(s, \gamma)\gamma'' \right\|_{L_1} + \| a \|_{L_1} \right) \| \delta \|_{L_\infty} +
\]

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Using the interpolation inequality:

\[ \| \delta \|_{L^\infty}^2 \leq \| \delta \|_{L^2}^2 + 2 \| \delta \|_{L^2} \| \delta' \|_{L^2} \]

we obtain:

\[
 f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta) \, ds \geq \\
 \geq -\text{const} \left( \| \gamma'' \|_{L^1} + \| \alpha \|_{L^1} \right) \left( \| \delta \|_{L^2}^2 + 2 \| \delta \|_{L^2} \| \delta' \|_{L^2} \right) + \\
 -\text{const} \| \gamma' \|_{L^2} \left( \| \delta \|_{L^2}^2 + 2 \| \delta \|_{L^2} \| \delta' \|_{L^2} \right) + \\
 -\text{const} \left( \| \delta \|_{L^2} \| \delta' \|_{L^2} + \| \delta \|_{L^2}^{1/2} \| \delta' \|_{L^2}^{3/2} \right) + \frac{\nu}{2} \| \delta' \|_{L^2}^2. 
\]

Thus, by theorem 2.1 and hypothesis 1.1, we are able to conclude that:

\[
 f(\gamma + \delta) - f(\gamma) - \int_0^1 (\alpha, \delta) \, ds \geq -\theta(f(\gamma)) \left( 1 + \| \alpha \|_2^2 \right) \| \delta \|_{L^2}^2. \quad \Box
\]

**Theorem 2.7.** — Let us consider \( \alpha \in L^2(0,1;\mathbb{R}^n) \) and \( \gamma \in X \cap W^{2,2}(0,1;\mathbb{R}^n) \) with \( \gamma^+(0) = \gamma^-(1) \). Then \( \alpha \in \partial^- f(\gamma) \) if and only if

\[
 \alpha(s) + \frac{d}{ds} (a(s, \gamma) \gamma') + \nabla_q V(s, \gamma) - \frac{1}{2} \frac{\partial a(s, \gamma)}{\partial q} (\gamma', \gamma') \in \partial^- I_M(\gamma(s)) \text{ a.e.}
\]

**Proof.** — If \( \alpha \in \partial^- f(\gamma) \), we have the thesis by lemmas 2.4 and 2.5. Viceversa, if

\[
 \alpha(s) + \frac{d}{ds} (a(s, \gamma) \gamma') + \nabla_q V(s, \gamma) - \frac{1}{2} \frac{\partial a(s, \gamma)}{\partial q} (\gamma', \gamma') \in \partial^- I_M(\gamma(s)) \text{ a.e.}
\]

the proof of theorem 2.6 shows that \( \alpha \in \partial^- f(\gamma) \). \( \Box \)

Finally, we can state the already quoted characterization.
THEOREM 2.8. — Let us consider \( \gamma \in X \). Then \( 0 \in \partial^- f(\gamma) \) if and only if \( \gamma \in W^{2,\infty}, \gamma'_+(0) = \gamma'_-(1) \) and

\[
\alpha(s) + \frac{d}{ds} (a(s, \gamma) \gamma') + \nabla_q V(s, \gamma) - \frac{1}{2} \frac{\partial a(s, \gamma)}{\partial q} (\gamma', \gamma') \in \partial^- I_M(\gamma(s)) \text{ a.e.}
\]

Proof. — If \( 0 \in \partial^- f(\gamma) \), from theorem 2.1 \( \gamma \in W^{2,\infty}(0, 1; \mathbb{R}^n) \) and \( \gamma^+(0) = \gamma^-(1) \). Moreover from theorem 2.7

\[
\alpha(s) + \frac{d}{ds} (a(s, \gamma) \gamma') + \nabla_q V(s, \gamma) - \frac{1}{2} \frac{\partial a(s, \gamma)}{\partial q} (\gamma', \gamma') \in \partial^- I_M(\gamma(s)) \text{ a.e.}
\]

Viceversa, it is enough to apply theorem 2.7 with \( \alpha = 0 \). \( \square \)

3. The category of the space of the admissible paths

After theorem 2.8, our goal is to prove the existence of infinitely many lower critical points for \( f \) on \( X \) by means to theorem 1.7. Therefore, let us investigate the topological properties of \( X \).

If \( Y \) is a topological space, we will denote by \( \Lambda(Y) \) the free loop space of \( Y \).

Let us recall that in [5], using results contained in [12], [13] and in [19], it is proved the following theorem.

THEOREM 3.1. — (see theorem 3.3 in [5]) Let \( A \) be an open subset of \( \mathbb{R}^n \), connected and non-contractible in itself. Moreover, let us suppose that either

i) \( \pi_1(A) \) has infinitely many conjugacy classes

or

ii) \( \pi_1(A) \) has a finite number of elements.

Then \( \text{cat } \Lambda(A) = +\infty \).

Now, let us consider \( X \) endowed with the \( W^{1,2} \)-topology and the space

\[
\Lambda(M) = \{ \gamma \mid [0, 1] \rightarrow M, \gamma \text{ is continuous and } \gamma(0) = \gamma(1) \}
\]

endowed with the uniform topology.
THEOREM 3.2. — (see theorem 4.5 in [4]) The inclusion map \( i : X \to \Lambda(M) \) is a homotopy equivalence.

Now, we are able to evaluate the category of \( X \).

THEOREM 3.3. — Let \( M \subset \mathbb{R}^n \) be a connected, non-contractible in itself, compact \( p \)-convex set. Let us suppose that either

i) \( \pi_1(M) \) has infinitely many conjugacy classes

or

ii) \( \pi_1(M) \) has a finite number of elements.

Then \( \text{cat}(X) = +\infty \).

Proof. — Let us consider \( A \), the open subset of \( \mathbb{R}^n \) defined in Remark 1.10. Clearly, \( M \) is a deformation retract of it. Then, \( A \) is homotopically equivalent to \( M \) and \( \Lambda(A) \) is homotopically equivalent to \( \Lambda(M) \). By applying theorems 3.1 and 3.2, the proof is over. □

Finally, we are able to prove the main theorem.

Proof of theorem 1.4

We want to apply theorem 1.7. Let us consider the functional \( f \) defined in section 2. By hypothesis (1.1) and theorem 2.6, \( f \) is a lower semicontinuous function, bounded below and it has a \( \varphi \)-monotone subdifferential of order two.

Moreover, let us observe that \( D(f) = X \) and that \( d^* \) induces the \( W^{1,2} \)-topology on \( X \). By theorem 3.3, \( \text{cat}(D(f), d^*) = +\infty \).

Now, we will consider a sequence \( \{\gamma_h\}_h \subset D(\partial^- f) \) with \( \sup_h f(\gamma_h) < +\infty \) and \( \lim_h \text{grad}^+ f(\gamma_h) = 0 \). Since \( M \) is compact, \( \{\gamma_h\}_h \) is bounded in \( L^2(0, 1; \mathbb{R}^n) \). But also \( \{\gamma'_h\}_h \) is bounded in \( L^2(0, 1; \mathbb{R}^n) \). Thus, by Rellich's theorem, \( \{\gamma_h\}_h \) has a subsequence converging in \( L^2(0, 1; \mathbb{R}^n) \).

So, applying theorem 1.7 and theorem 2.8, the thesis follows. □

References


