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TRAN NGOC GIAO

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Removable singularities and Liouville-type property of analytic multivalued functions

TRAN NGOC GIAO[1]

RÉSUMÉ. — Le but de cet article est l'étude du prolongement des fonctions analytiques à valeurs multiples. Nous obtenons l'équivalence entre une propriété du genre Liouville et les ensembles pour lesquels on peut prolonger ces fonctions.

ABSTRACT. — The purpose of this note is to study removable singularities for analytic multivalued functions. Moreover, the equivalence between Liouville-type properties and removable singularities results is proved.

Introduction

Let X a complex space. By $F_c(X)$ we denote the hyperspace of non-empty compact subsets of X.

As in [8] we say that an upper semi-continuous multivalued function $K : X \to F_c(Y)$, where X and Y are complex spaces, is analytic if for every open subset W of X and every plurisubharmonic function ψ on a neighbourhood of $\Gamma_K \restriction_W$, the graph of K on W, the function

$$\varphi(x) = \sup\{\psi(x, y) \mid y \in K(x)\}$$

is plurisubharmonic on W.

Analytic multivalued functions (for short: A.M.V. functions) have been investigated by several authors, in particular by Slodkowski [8, 9] and Ransford [5, 6, 7].

(1) Department of Mathematics, Pedagogical University of Vinh, Vietnam

In [7], Ransford has proved that every A.M.V. function

$$K : D \to F_c(V)\,,$$

where $D = \{z \in \mathbb{C} \mid |z| < 1\}$, $D^* = D \setminus \{0\}$ and V is either D or $D_{rs} = \{z \in \mathbb{C} \mid r < |z| < s\}$, $0 < r < s$, can be extended analytically to D.

This note considers a removable-singularity result for A.M.V. functions. Moreover, the equivalence between a Liouville-type property and extendibility of A.M.V. functions is proved.

1. Removable-singularities for analytic multivalued functions

An A.M.V. function $K : G \to F_c(Y)$ is said to be locally compact if for every $x \in X$ there exists a neighbourhood U of x such that $K(U \cap G)$ is relatively compact in Y, where G is an open subset of X.

THEOREM 1.1.— *Let G be an open set in $\mathbb{C}^n$, S a closed subset of G, Y is a Stein space. Then every A.M.V. function $K : G \setminus S \to F_c(Y)$ can be extended analytically to G if one of the following conditions is satisfied*

- *a) $S = H \cap (G \setminus U)$, where H is an analytic set in G, U is an open subset of G such that U meets every component of H;*
- *b) S is a set of zero $(2n-2)$-Hausdorff measure in G;*
- *c) S is a pluripolar set in G and K is locally compact.*

We first need the following, which is a generalization of the important result of Wermer [10].

LEMMA 1.2.— *Let A be a uniform algebra with Shilov boundary ∂_A^0 and U an open subset of $\mathbb{C}$. Let $h : U \to A$ be a holomorphic map. Then for every $f \in A$ such that $\sigma(f) \setminus f(\partial_A^0) \subset U$, where $\sigma(f)$ is the spectrum of f, the form*

$$K(\lambda) = \{\widehat{h}(\lambda, w) = \widehat{h(\lambda)}(w) \mid w \in \widehat{f}^{-1}(\lambda)\}$$

defines an A.M.V. function on $\sigma(f) \setminus f(\partial_A^0)$.

Proof.— This is basically Slodkowski's argument [8]. It is enough to show that $K(\lambda)$ satisfies condition (ii) of [8, theorem 3], i.e. for every polynomial $p(\lambda)$ and for every $a, b \in \mathbb{C}$ the function $\lambda \to \max\left|f_\lambda\big(K(\lambda)\big)\right|$, where $f_\lambda(z) = (z - \lambda a - b)^{-1}\exp\big(p(\lambda)\big)$, has local maximum property in $G = \big\{\lambda \in \sigma(f) \setminus \widehat{f}(\partial_A^0) \mid a\lambda + b \notin K(\lambda)\big\}$. Let D be a disc such that $\mathrm{cl}D \subset G$. Put $N = \widehat{f}^{-1}(D) \subset M_A$, where M_A is maximal ideal space of A, and let B denote the uniform closure of $A{\restriction}_{\mathrm{cl}N}$ on $\mathrm{cl}N$ and the form $k = \big(h(y) - af - b\big)^{-1}\exp\big(p(f)\big)$, where $a, b \in \mathbb{C}$ and p is a polynomial, defines an element of B. Denote

$$f_\lambda(z) = \big(z - \lambda a - b\big)^{-1}\exp\big(p(\lambda)\big)\,.$$

For $\lambda_* \in D$, we have

$$\begin{aligned}\max f_{\lambda_*}\big(K(\lambda_*)\big) &= \max\big|\widehat{k}\widehat{f}^{-1}(\lambda_*)\big| \\ &\leq \max|\widehat{k}|{\restriction}_N(\text{by Rossi's local maximum principle}) \\ &\leq \max\Big\{\max\big|\widehat{k}\big(\widehat{f}^{-1}(\lambda_*)\big|\ \Big|\ \lambda \in \partial D\Big\} \\ &= \max\Big\{\max\big|f_\lambda\big(K(\lambda)\big|\ \Big|\ \lambda \in \partial D\Big\}\,.\end{aligned}$$

Thus the function $\lambda \to \max\left|f_\lambda\big(K(\lambda)\big)\right|$ has the local maximum property.

The lemma is proved. □

LEMMA 1.3 (Slodkowski's theorem [9]).— *Let G be a bounded planar domain and $K : G \to F_c(\mathbb{C}^k)$ be an A.M.V. function such that $\sup\max_{x\in G}\left|K(x)\right| < \infty$. Then there exists a uniform algebra A and functions $f, g_1, \ldots, g_k \in A$ such that*

- *i) $\widehat{f}(M_A) \setminus \widehat{f}(\partial_A^0) = G$, where $\widehat{f}$ denotes the Gelfand transformation of f, M_A and ∂_A^0 are the maximal ideal space and the Shilov boundary respectively of A.*
- *ii) $\widehat{g}\big(\widehat{f}^{-1}(x)\big) = K(x)$ for every $x \in G$, where $\widehat{g} = (\widehat{g}_1, \ldots, \widehat{g}_k)$.*

LEMMA 1.4.— *Let $K : G \to F_c(Y)$ be an upper semi-continuous multivalued function, where G is an open subset of $\mathbb{C}^n$ and Y an analytic set in $\mathbb{C}^k$. If $K : F \to F_c(\mathbb{C}^k)$ is analytic, then $K : G \to F_c(Y)$ is also analytic.*

Proof.— We can assume that $n = 1$. Given φ a plurisubharmonic function on a neighborhood W of $\Gamma_K{\restriction}_U$, where U is an open subset of G, consider the plurisubharmonic function $\widetilde{\varphi}(z, w) = \varphi\big(z, \widehat{g}(w)\big)$ on

$(\mathrm{id}\times\widehat{g})^{-1}(W)$, where f, g, A are constructed as in lemma 1.3. By [3] we have

$$\widehat{\varphi}(z,w)=\lim\max\left\{c_j^n\log\left|\widehat{h}_j^n(z,w)\right|\right\}$$

for all $(z,w)\in(\mathrm{id}\times\widehat{g})^{-1}(W)$, where h_j^n are holomorphic maps from U into A.

Since $(\mathrm{id}\times\widehat{g})$ is continuous and W is open, it implies that

$$\begin{gathered}\overline{(\mathrm{id}\times\widehat{g})^{-1}(W)}\subset(\mathrm{id}\times\widehat{g})^{-1}(\overline{W})\Rightarrow\\ \partial(\mathrm{id}\times\widehat{g})^{-1}(W)\cup(\mathrm{id}\times\widehat{g})^{-1}(W)\subset(\mathrm{id}\times\widehat{g})^{-1}(W)\cup(\mathrm{id}\times\widehat{g})^{-1}(\partial W)\Rightarrow\\ \partial(\mathrm{id}\times\widehat{g})^{-1}(W)\subset(\mathrm{id}\times\widehat{g})^{-1}(\partial W).\end{gathered}$$

By lemma 1.2, the multivalued function

$$L(z)=\left\{\widehat{h}_j^n(z,w)\mid w\in\widehat{f}^{-1}(z)\right\}$$

is analytic on $\sigma(f)\setminus\widehat{f}(\partial_A^0)$. On the other hand $\widehat{f}^{-1}(\partial G)\supset\partial_A^0$, by Rossi's local maximum principle we have

$$\max\left|\widehat{h}_j^n(z,w)\right|_{\partial(\mathrm{id}\times\widehat{g})^{-1}(W)}=\max\left|\widehat{h}_j^n(z,w)\right|_{(\mathrm{id}\times\widehat{g})^{-1}(\partial W)}.$$

Since for every sequence of upper semi-continuous function ψ_n, $\psi=\lim\psi_n$ point-wise, $\lim\max(\psi_n\restriction_F)=\max(\psi\restriction_F)$ on every compact subset F [8], and since $(\mathrm{id}\times\widehat{g})^{-1}(\partial W)\supset(\mathrm{id}\times\widehat{g})^{-1}(W)$, it follows that the function γ given by

$$\begin{aligned}\gamma(z)&=\max\left\{\varphi(z,y)\mid y\in K(z)=\widehat{g}\widehat{f}^{-1}(z)\right\}\\&=\max\left\{\widetilde{\varphi}(z,y)\mid w\in\widehat{f}^{-1}(z)\right\}\end{aligned}$$

is plurisubharmonic on U. Hence the multivalued function $K:G\to F_c(Y)$ is analytic.

Proof of theorem 1.1

Without loss of generality we may assume that Y is an analytic set in $\mathbb{C}^k$. Then the function

$$\theta(x)=\sup\left\{\|y\|\mid y\in K(x)\right\}$$

is plurisubharmonic on $G_0=G\setminus S$, where S satisfies one of the conditions a) or b) or c) of the theorem. By [4], θ can be extended to a plurisubharmonic function on C. This implies that for every $x_0\in S$ there exists a neighbourhood U of x_0 such that $K(U \cap G_0)$ is relatively compact. Define a upper semi-continuous extension of K by

$$\widehat{K}(x) = \begin{cases} K(x) & \text{for } x \in G_0 \\ \left\{ y \in Y \;\middle|\; \exists \{(x_n, y_n)\} \subset \Gamma_K ,\ (x_n, y_n) \to (x, y) \right\} & \text{for } x \in S. \end{cases}$$

We prove that $\widehat{K}$ is analytic at every $x_0 \in S$. Let G' be an open ball around x_0, $G' \subset G$. It suffices to show that $\widehat{K}\restriction_{L \cap G'}$ is analytic for every complex line L in $\mathbb{C}^n$. Using the Slodkowski theorem we can find a uniform algebra A and $f, g_1, \ldots, g_k \in A$ such that

i) $\widehat{g}\widehat{f}^{-1}(x) = \widehat{K}(x)$ for all $x \in L \cap (G' \setminus S)$;

ii) $f(\partial_A^0) = \partial\big(L \cap (G' \setminus S)\big)$.

We have to prove that $f(\partial_A^0) \cap (L \setminus G') = \emptyset$.

Suppose the contrary. Then there exists a complex line L in $\mathbb{C}^n$ such that $f(\partial_A^0) \cap (L \cap G') \neq \emptyset$. Since $\widehat{K}$ is analytic on $G' \setminus S$, it follows that $\widehat{f}(\partial_A^0) \cap \big(L \cap (G' \setminus S)\big) = \emptyset$. Hence there exists $w_0 \in \partial_A^0$ such that $\widehat{f}(w_0) = x_0$. Since G' is open and set of peak points of A is dense in ∂_A^0, we may assume that w_0 is a peak point. Hence there exists $h \in A$ such that $\big|\widehat{h}(w_0)\big| = 1$ and $\big|\widehat{h}(w)\big| < 1$ for $w \in M_A \setminus \{w_0\}$.

Consider the plurisubharmonic function

$$\varphi(x) = \log \max \big|\widehat{h}\widehat{f}^{-1}(x)\big| \quad \text{on} \quad G' \setminus S .$$

Then φ is plurisubharmonic on $G' \cap L$. Since

$$\log \max \big|\widehat{h}\widehat{f}^{-1}(x)\big| \leq 0 = \log \max \big|\widehat{h}\widehat{f}^{-1}(x_0)\big|$$

for every $x \in G'$, it follows that $\varphi =$ constant, which is impossible.

Thus $f(\partial_A^0) \cap (G' \cap L) = \emptyset$.

Theorem 1.1 is proved. □

2. Liouville-type property for analytic mulivalued functions

In the section we study the relation between a Liouville-type property and removable singularities of A.M.V. functions with values in convex domains.

THEOREM 2.1. — *Let D be a convex domain in $\mathbb{C}^n$. Then the following conditions are equivalent*

a) for every A.M.V. function $K : \mathbb{C} \to F_c(D)$, the multivalued function $\widehat{K} : \mathbb{C} \to F_c(D)$ given by $\widehat{K}(x) = \widehat{K(x)}$, where $\widehat{K(x)}$ is polynomial convex hull of $K(x)$, is constant;

b) every A.M.V. function $K : \Delta^* \to F_c(D)$ can be extended analytically on Δ, where Δ is the unit disc, $\Delta^* = \Delta \setminus \{0\}$;

c) every A.M.V. function $L : \Delta \setminus S \to F_c(D)$ can be extended analyticaly on Δ, where S is a polar set in Δ.

To prove the theorem we shall use the hyperboliticity of convex domains. In [1] Bath proved that a convex domain D is hyperbolic if and only if D does not contain complex lines (i.e. every holomorphic map $h : \mathbb{C} \to D$ is constant).

Proof of theorem 2.1

Consider the condition:

$$D \quad \text{is hyperbolic} \tag{1}$$

We shall prove that a) $\Leftrightarrow$ (1) $\Rightarrow$ c) $\Rightarrow$ b) $\Rightarrow$ (1).

We first write

$$D = \bigcap_{\alpha \in I} \{\operatorname{Re} x_\alpha^* < \varepsilon_\alpha\},$$

where $\{x_\alpha^*\}$ are linear forms on $\mathbb{C}^n$. Without loss of generality we may assume that $0 \in D$. Then $\varepsilon_\alpha > 0$ for all α.

Let $\{x_{\alpha_1}^*, \ldots, x_{\alpha_p}^*\}$ be a maximal linearly independent system of $\{x_\alpha^*\}$. Take $\theta_\alpha : H_\alpha \to \Delta$, where $H_\alpha = \{z \in \mathbb{C} : \operatorname{Re} z < \varepsilon_\alpha\}$, is a biholomorphism. Define a holomorphic map

$$\gamma : D_1 \to \Delta^p, \quad \text{where} \quad D_1 = \bigcap_{j=1}^{p} \{\operatorname{Re} x_{\alpha_j}^*\}$$

by

$$\gamma(x) = \Big(\theta_{\alpha_1}\big(x_{\alpha_1}^*(x)\big), \ldots, \theta_{\alpha_p}\big(x_{\alpha_p}^*(x)\big)\Big)\,.$$

Obviously, γ is a biholomorphism if and only if $\bigcap_{j=1}^{p} \operatorname{Ker} x_{\alpha_j}^* = \{0\}$ or, equivalently, D_1 does not contain C.

a) $\Rightarrow$ (1) Because every holomorphic map $h : \mathbb{C} \to D$ is an A.M.V. function and $\widehat{h(z)} = h(z)$, from a) we have $h = \text{const}$, thus D is hyperbolic.

(1) $\Rightarrow$ a) Let $K : \mathbb{C} \to F_c(D)$ be an A.M.V. function. Suppose $\widehat{K}(z_1) \neq \widehat{K}(z_2)$ for two points $z_1, z_2 \in \mathbb{C}$. Take a plurisubharmonic function φ on Δ^p such that

$$\sup\{\varphi(y) \mid y \in \gamma\widehat{K}(z_1)\} \neq \sup\{\varphi(y) \mid y \in \gamma\widehat{K}(z_2)\}.$$

Since K is analytic, the function

$$\begin{aligned}\widetilde{\varphi}(z) &= \sup\{\varphi(y) \mid y \in \gamma K(z)\}\\ &= \sup\{\varphi(y) \mid y \in \widehat{\gamma K}(z)\}\\ &= \sup\{\varphi(y) \mid y \in \gamma\widehat{K}(z)\}\end{aligned}$$

is subharmonic on $\mathbb{C}$. On the other hand, since $\gamma\widehat{K}(z) \subset \Delta^p$ for all $z \in \mathbb{C}$, $\widetilde{\varphi}$ is bounded on $\mathbb{C}$. This is impossible because of the subharmonicity of $\widetilde{\varphi}$ and of the relation $\widetilde{\varphi}(z_1) \neq \widetilde{\varphi}(z_2)$.

(1) $\Rightarrow$ c) By the hypothesis, D and hence D_1 is hyperbolic. By theorem 1.1, γL and hence L can be extended to an A.M.V. function $\widetilde{L} : \Delta \to F_c(D_1)$. It remains to show that $\widetilde{L}(z_0) \subset D$ for every $z_0 \in S$.

Let $\alpha \in I$ and $\widetilde{x_\alpha^* L}$ be an extension of $x_\alpha^* L$ with values in $F_c(H_\alpha)$.

Assume that $\widehat{\widetilde{x_\alpha^* L}}(z_0) \neq \widehat{x_\alpha^* \widetilde{L}}(z_0)$ for $z_0 \in S$. Take a plurisubharmonic function φ on $\mathbb{C}$ such that $\varphi_1(z_0) \neq \varphi_2(z_0)$, where

$$\varphi_1(z) = \sup\{\varphi(y) \mid y \in \widehat{\widetilde{x_\alpha^* L}}(z)\} = \sup\{\varphi(y) \mid y \in \widetilde{x_\alpha^* L}(z)\}$$

and

$$\varphi_2(z) = \sup\{\varphi(y) \mid y \in \widehat{x_\alpha^* \widetilde{L}}(z)\} = \sup\{\varphi(y) \mid y \in x_\alpha^* \widetilde{L}(z)\}$$

for $z \in \mathbb{C}$.

Since φ_1 and φ_2 are plurisubharmonic on Δ and $\varphi_1 = \varphi_2$ on $\Delta \setminus \{z_0\}$ we have $\varphi_1(z_0) = \varphi_2(z_0)$. This is impossible because of the choice of φ. thus, $\operatorname{Re} x_\alpha^*(z) < \varepsilon_\alpha$ for all $z \in \widetilde{L}(z_0)$ and for all $\alpha \in I$. Hence $\widetilde{L}(z_0) \subset D$.

c) $\Rightarrow$ b) Obvious.

b) $\Rightarrow$ (1) By [1], it suffices to show that every holomorphic map $\beta : \mathbb{C} \to D$ is constant. By the hypothesis, β can be extended to an A.M.V. function $\widehat{\beta}$ on $\mathbb{C}P^1$. By the normality of $\mathbb{C}P^1$, it follows that $\widehat{\beta}$ is holomorphic on $\mathbb{C}P^1$ [2]. Since $\widehat{\beta} : \mathbb{C}P^1 \to D$ is holomorphic on the compact space $\mathbb{C}P^1$, it implies that $\widehat{\beta}$ and hence β is constant.

The theorem is proved. □

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