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Asymptotic stability for a homogeneous singularly perturbed system of differential equations with unbounded delay(*)

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RÉSUMÉ. — On a obtenu des conditions suffisantes pour la stabilité équiasymptotique de la solution nulle d'un système d'équations différentielles avec une perturbation singulière et un retard non borné.

ABSTRACT. — Sufficient conditions for equiasymptotic stability of the null solution of a homogeneous singularly perturbed system of differential equations with unbounded delay are obtained.

1. Introduction

Singularly perturbed systems of differential equations are often used in the applications. In the recent decades the theory of singularly perturbed ordinary differential equations develops intensively. A principal problem in this theory is to find sufficient conditions under which certains properties of the solutions of the degenerate system (for $\mu = 0$) are preserved for sufficiently small values of the perturbing parameter μ.

In some mathematical models the history of the process described is taken into account. Thus naturally the necessity of investigation of singularly perturbed equations with retarded argument arises. The linear case was considered in the papers [2]-[3], [9], [12]-[14], [18], and the nonlinear case — in the papers [5], [9], but with constant delay of the argument. In [5],

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the existence of periodic solutions is discussed and in [9] — the asymptotic stability of the null solution. In the present paper sufficient conditions are found for equiasymptotic stability of the null solution of a homogeneous singularly perturbed system with unbounded delay. The results obtained are a generalization of some results of [9] and [18]. We shall note that the results in [9] are based on a linear approximation of the respective nonlinear system while in the present paper such an approach is not possible.

2. Preliminary notes

We shall denote the Euclidean norm by $|\cdot|$. If v is scalar function of a scalar argument t, denote by $\dot{v}$ its right derivative and by D^+v, D_+v, D^-v, D_-v its Dini derivatives, *i.e.*

$$\begin{aligned} D^+v(t) &= \limsup_{h\to 0^+}\big[v(t+h)-v(t)\big]h^{-1}, \\ D^-v(t) &= \limsup_{h\to 0^-}\big[v(t+h)-v(t)\big]h^{-1}, \\ D_+v(t) &= \liminf_{h\to 0^+}\big[v(t+h)-v(t)\big]h^{-1}, \\ D_-v(t) &= \liminf_{h\to 0^-}\big[v(t+h)-v(t)\big]h^{-1}. \end{aligned}$$

If x is a vector-valued function of a scalar argument, set $D^+x = \mathrm{col}(D^+x_1, \dots, D^+x_n)$ and analogously for D_+x, D^-x, D_-x. Introduce the notation

$$\begin{aligned} \mathcal{D}^+x &= \max\big\{|D^+x|, |D_+x|\big\}, \\ \mathcal{D}^-x &= \max\big\{|D^-x|, |D_-x|\big\}, \\ \mathcal{D}x &= \max\{\mathcal{D}^+x, \mathcal{D}^-x\}, \end{aligned}$$

$$x_t(s) = x(t+s), \quad I = (t_0, +\infty) \quad \text{for fixed } t_0 \in \mathbb{R}$$

and

$$G = \left\{ g \in C(I) : g(t) \le t, \lim_{t\to\infty} g(t) = +\infty \right\}.$$

Let Φ be the linear space of functions $\varphi : (-\infty, 0] \to \mathbb{R}^p$ and let E be a linear subspace of Φ provided with a seminorm $\|\cdot\|$. For $\tau \ge 0$, set $E_\tau = \big\{\varphi \in \Phi : \varphi \in C[-\tau, 0], \varphi_{-\tau} \in E\big\}$. The space E is said to be *admissible* if for any $\tau \geq 0$ and $\psi \in E_\tau$ the following conditions (H) are met:

H1 $\psi_t \in E$, for $t \in [-\tau\,,\,0]$;

H2 ψ_t is continuous in t with respect to $\|\cdot\|$ for $t \in [-\tau\,,\,0]$;

H3 $M_0^{-1}|\psi(0)| \leq \|\psi\| \leq K(\tau) \max_{s\in[-\tau\,,\,0]} |\psi(s)| + M(\tau)\|\psi_{-\tau}\|\,,$

where $M_0 > 0$ is a constant and K and M are continuous functions. An admissible space E has a *fading memory* if the functions K and M in H3 satisfy the condition H4

H4 $K(s) \equiv K_0 = \text{const}$, $M(s) \to 0$ as $s \to +\infty$.

Consider the initial value problem

$$\dot{x}(t) = F(t, x_t)\,, \quad x_\sigma = \varphi \in E\,, \tag{1}$$

where $t \geq \sigma \in I$, $F(t,\varphi)$ is a functional defined and continuous for $(t,\varphi) \in I \times E$ and $F(t,0) = 0$. The local existence of a solution of problem (1) is guaranteed by Theorem 2.1 [6].

DEFINITION 1.— *The null solution of (1) is said to be:*

(α_1) *stable in* $\mathbb{R}^p$ *if for* $\varepsilon > 0$, $\sigma \in I$ *there exists* $\delta = \delta(\sigma,\varepsilon) > 0$ *such that each solution* x *of the initial value problem (1),* $x_\sigma = \varphi$ *is defined for* $t \geq \sigma$, $\|\varphi\| < \delta$ *and satisfies the inequality*

$$|x(t)| < \varepsilon\,.$$

(α_2) *uniformly stable in* $\mathbb{R}^p$ *if in* (α_1) *the number* δ *does not depend on* σ.

(α_3) *equiasymptotically stable in* $\mathbb{R}^p$ *if it is stable and there exist functions* $\delta_0 = \delta_0(\sigma)$ *and* $T = T(\sigma,\varepsilon)$ *such that each solution* x *of (1),* $x_\sigma = \varphi$ *satisfies for* $t \geq T(\sigma,\varepsilon)$, $\|\varphi\| < \delta_0(\sigma)$, *the inequality*

$$|x(t)| < \epsilon\,.$$

(α_4) *uniformly asymptotically stable in* $\mathbb{R}^p$ *if* (α_2)-(α_3) *are valid and* $\delta_0(\sigma)$, $T(\sigma,\varepsilon) - \sigma$ *do not depend on* σ.

(α_5) *exponentially stable in* $\mathbb{R}^p$ *if there exist positive constants* α, M_1 *such that each solution* x *of (1),* $x_\sigma = \varphi$ *satisfies for* $t \geq \sigma \in I$ *the inequality*

$$|x(t)| \leq M_1\|\varphi\| \exp\bigl(-\alpha(t-\sigma)\bigr)\,.$$

If in the above definition we replace $|x(t)|$ and $\mathbb{R}^p$ respectively by $||x_t||$ and E, we obtain corresponding definitions of stability in E. If E is an admissible space with a fading memory, the notions of uniform asymptotic stability in E and $\mathbb{R}^p$ are equivalent (*cf.* Theorem 6.1 of [6]). If the null solution of (1) is stable, then it is necessary for the initial value problem (1), $x_0 = \varphi = 0$ to have a unique solution but for $\varphi \not\equiv 0$ it may have more than one solution.

3. Main results

Let $\sigma \in I$. Consider for $t > \sigma$ the system

$$\begin{aligned} \dot{X}(t) &= H_1(t, X_t, \mu) + B_1(t,\mu)Y(t) \\ \mu\dot{Y}(t) &= H_2(t, X_t, \mu) + B_2(t,\mu)Y(t)\,, \end{aligned} \tag{3}$$

with initial conditions $X_\sigma = \varphi \in E$, $Y(\sigma) = y_0 \in \mathbb{R}^n$, where E is an admissible space, $\mu \in (0\,,\,\mu_0\,]$, $X(t) \in \mathbb{R}^p$, $Y(t) \in \mathbb{R}^n$, B_1 and B_2 are real matrices whose entries are continuous functions of $(t,\mu) \in I \times [\,0\,,\,\mu_0\,]$, H_1 and H_2 are real vectors whose components are continuous functionals defined for $(t,\varphi,\mu) \in I \times E \times [\,0\,,\,\mu_0\,]$ and homogeneous in φ, *i.e.*

$$H_i(t, \lambda\varphi, \mu) = \lambda H_i(t,\varphi,\mu) \quad \text{for } \lambda > 0\,,\ i = 1,\,2\,. \tag{4}$$

THEOREM 1.— *For each $(s,\mu) \in I \times [\,0\,,\,\mu_0\,]$ there exists a function $Q(\,\cdot\,,s,\mu)$ continuous and monotone increasing in the interval $[\,s\,,\,+\infty)$ and such that any solution (X,Y) of the initial value problem (3), $X_\sigma = \varphi$, $Y(\sigma) = y_0$ satisfies for $t \geq \sigma$ the inequality*

$$||X_t|| + |Y(t)| \leq Q(t,\sigma,\mu)\big(||\varphi|| + |y_0|\big)\,. \tag{5}$$

If, moreover, it is given that for any $\mu \in (0\,,\,\mu_0\,]$ the components of $B_i(t,\mu)$ and $H_i(t,\varphi,\mu)$ are bounded for $t \in I$, $||\varphi|| = 1$, $i = 1,\,2$, then $Q(t,\sigma,\mu)$ depends only on $t - \sigma$ and μ.

Proof.— Let $\mu \in (0\,,\,\mu_0\,]$ be fixed. From the continuity of H_i, B_i, $i = 1,\,2$ and from relation (4) it follows that there exists a continuous function $q(s,\mu)$ such that

$$\big|H_i(s,\varphi,\mu) + B_i(s,\mu)Y\big| \leq q(s,\mu)\big(||\varphi|| + |Y|\big)\,,\quad i = 1,\,2\,. \tag{6}$$

Taking into account that E is an admissible space, inequalities (3) imply the inequality

$$\|X_t\| + |Y(t)| \le$$
$$\le a(t-\sigma)\big(\|\varphi\| + |y_0|\big) + b(t-\sigma,\mu)\int_\sigma^t q(s,\mu)\,\big(\|X_s\| + |Y(s)|\big)\,\mathrm{d}s\,,$$

where

$$a(s) = \max_{\theta\in[\,0\,,\,s\,]}\big(M_0K(\theta) + M(\theta) + 1\big)$$
$$q(s,\mu) = \max_{\theta\in[\,0\,,\,s\,]} K(\theta) + \mu^{-1}$$

are continuous monotone increasing functions of s. Since $q(s,\mu)$, $|Y(s)|$ and $\|X_s\|$ are continuous functions of s, from Gronwall's inequality there follows inequality (5) with

$$Q(t,\sigma,\mu) = a(t-\sigma)\exp\left(b(t-\sigma,\mu)\int_\sigma^t q(s,\mu)\,\mathrm{d}s\right)\,.\ \square$$

From Theorem 1 it follows that the null solution of (3) is unique. From inequalities (5), (6) and from Theorem 2.2 of [17] it follows that the solutions of the initial value problem (3), $X_\sigma = \varphi \in E$, $Y(\sigma) = y_0$ are defined for all $t \ge \sigma$.

In case that $\det B_2(t,0) \ne 0$ for $t \in I$, the degenerate system corresponding to (3) (for $\mu = 0$) is

$$\begin{aligned} \dot{x}(t) &= \big[H_1 - B_1B_2^{-1}H_2\big](t,x_t,0) \\ y(t) &= \big[-B_2^{-1}H_2\big](t,x_t,0) \end{aligned} \tag{7}$$

with initial condition $x_\sigma = \varphi$.

We shall say that conditions (A) are met if the following conditions holds.

(A1) The components of H_i, B_i $(i = 1,\,2)$ are continuous for $(t,\varphi,\mu) \in I \times E \times [\,0\,,\,\mu_0\,]$, $B_2(t,0) \in C^1(I)$ and there exists a constant M_2 such that

$$|B_i(t,\mu)| \le M_2\,,\quad |D^+B_2(t,\mu)| \le M_2\,,$$
$$|H_i(t,\varphi,0) - H_i(t,\psi,0)| \le M_2\|\varphi-\psi\|\,,\quad i = 1,\,2\,.$$

(A2) There exist functions $g \in G$, $\rho \in C[0, \mu_0]$ such that $\rho(\mu) \to 0$ as $\mu \to 0$ and for $(t, \mu) \in I \times [0, \mu_0]$, $\varphi \in E_{t-g(t)}$, $i = 1, 2$ the following inequalities are valid

$$\left|B_i(t,\mu) - B_i(t,0)\right| \le \rho(\mu),$$
$$\left|H_i(t,\varphi,\mu) - H_i(t,\varphi,0)\right| \le \rho(\mu) \sup_{s\in[g(t)-t,0]} \|\varphi_s\|.$$

(A3) There exists a function $g \in G$ and a constant $M_3 > 0$ such that for $(t, \mu) \in I \times [0, \mu_0]$, $\varphi \in E_{t-g(t)}$ the following inequality is valid

$$\left|D^+ H_2(t,\varphi,\mu)\right| \le M_3 \sup_{s\in[g(t)-t,0]} \left(\|\varphi_s\| + \mathcal{D}\varphi(s)\right).$$

(A4) There exists a constant $\beta > 0$ such that all eigenvalues $\lambda_k(t)$, $k = 1, 2, \ldots, n$ of the matrix $B_2(t, 0)$ satisfy the inequality

$$\operatorname{Re}\lambda_k(t) \le -\beta \quad \text{for } t \in I.$$

(A5) The null solution of the degenerate system (7) is uniformly asymptotically stable in $\mathbb{R}^p$.

We shall note that in condition (A4) the matrix $B_2(t, 0)$ is nondegenerate since $\left|\det B_2(t,0)\right| \ge \beta^n > 0$.

The main result in the paper is the following Theorem.

THEOREM 2.— *Let E be an admissible space with fading memory and conditions (A) hold. Then the null solution of (3) is equiasymptotically stable in E. If, moreover, it is given that the functions $t-g(t)$ and $H_i(t,\varphi,\mu)$ are bounded for $(t,\mu) \in I \times [0, \mu_0]$, $\|\varphi\| = 1$, $i = 1, 2$ the null solution of (3) is uniformly asymptotically stable.*

The proof of Theorem 2 is given in section 4. It is based on a modification of Lyapunov's direct method summarized in the following lemmas.

LEMMA 1 ([18]).— *Let z_0, z_1 and δ be constants, $z_0 \le z_1 < \tau \le +\infty$, $v \in C([z_0, \tau))$ and $D^+v(t) \le 0$ for all values of $t \in [z_1, \tau)$ which satisfy the conditions $v(t) \ge \delta$ and $v(t) \ge v(s)$ for $s \in [z_0, t]$. Then for $t \in [z_1, \tau)$ the following inequality is valid*

$$v(t) \le \max\left\{\delta, \sup_{s\in[z_0,z_1]} v(s)\right\}.$$

Proof.— For $t \in [z_1, \tau)$ set

$$\overline{v}(t) = \max_{s \in [z_0, t]} v(s) \quad \text{and} \quad w(t) = \max\{\delta, \overline{v}(t)\}.$$

If at least one of the relations $\overline{v}(t) \neq v(t)$ or $\overline{v}(t) < \delta$ is valid, then from the continuity of the function v it follows that $w(t) = 0$. Let us suppose that $D^+ w(t) > 0$ for some $t \in [z_1, \tau)$. Then $\overline{v}(t) = v(t) \geq \delta$ and there exists a monotone decreasing sequence of numbers $h_n > 0$ tending to zero and

$$\overline{v}(t + h_n) > \overline{v}(t) + \frac{1}{2} D^+ w(t) h_n \quad \text{for } n \in \mathbb{N}.$$

Consequently, for each $n \in \mathbb{N}$ there exists $\lambda_n \in (0, h_n)$ such that

$$v(t + \lambda_n) = \overline{v}(t + h_n) > v(t) + \frac{1}{2} D^+ w(t) \lambda_n,$$

from which we obtain that $D^+ v(t) \geq (1/2)\, D^+ w(t) > 0$. On the other hand, from the relations $\overline{v}(t) = v(t) \geq \delta$ it follows, by condition, that $D^+ v(t) \leq 0$ which contradicts the inequality $D^+ v(t) > 0$. Hence $D^+ w(t) \leq 0$ for all $t \in [\sigma, \tau)$. From Theorem 2.1 of [18, Appendix I] it follows that

$$v(t) \leq w(t) \leq w(z_1) \quad \text{for } t \in [z_1, \tau). \ \square$$

LEMMA 2 ([18]).— *Let $v \in C([z, \tau))$, $z < \tau \leq +\infty$, $q > 0$, $\delta \in \mathbb{R}$ and $z + T < \tau$, where*

$$T = q^{-1} \max\{0, v(z) - \delta\}.$$

For each $t \in [z, \tau)$ for which $v(t) \geq \delta$ let the relation $D^+ v(t) \leq -q$ be valid. Then

$$v(t) \leq \delta \quad \textit{for } t \in [z + T, \tau).$$

Proof.— If $v(s) \leq \delta$ for some $s \in [z, z + T]$, then from Lemma 1 (for $z_0 = z_1 = s$) it follows that $v(t) \leq \delta$ for $t \in [s, \tau) \supset [z + T, \tau)$. Suppose that this is not true, *i.e.* $v(t) > \delta$ for each $t \in [z, z + T]$. Then for the function $w(t) = v(t) + q(t - z)$, we obtain that $w(z) = v(z)$ and $D^+ w(t) \leq 0$ for $t \in [z, z + T]$. From Lemma 1 it follows that $w(z + T) \leq w(z) = v(z)$ which implies

$$v(z) \geq w(z + T) = v(z + T) + qT > \delta + qT$$

in a contradiction with the definition of T. $\square$

LEMMA 3 ([18]).— *Let* $f \in C([0, +\infty))$, $f(t) > t$ *for* $t > 0$, $r \in C([z_0, +\infty))$, $r(t) \le t$ *for* $t \ge z_0$ *and* $r(t) \to +\infty$ *as* $t \to +\infty$. *Let* $c_0 > 0$ *be a constant and let* τ *and* q *be functions defined for* $\delta > 0$ *so that* $\tau(\delta) = \tau(\delta, c_0, z_0) \ge z_0$ *and* $q(\delta) = q(\delta, c_0, z_0) > 0$.

Then there exists a function Γ *such that* $\Gamma(\delta) = \Gamma(\delta, c_0, z_0) \ge z_0$ *for* $\delta > 0$ *and for any function* $v \in C([z_0, +\infty))$ *for which the following relations are valid:* $v(t) \le c_0$ *for* $t \ge z_0$ *and* $D^+v(t) \le -q(\delta)$ *for each value of* t *satisfying the conditions* $t \ge \tau(\delta)$, $v(t) \ge \delta$, $r(t) \ge z_0$ *and* $f(v(t)) \ge v(s)$ *for* $s \in [r(t), t]$, *the relation* $v(t) \le \delta$ *is valid for* $t \ge \Gamma(\delta)$.

If, moreover, for $t \ge z_0$ *we have* $r(t) \ge t - h$, $h = \text{const} > 0$, *and the numbers* $\tau(\delta, c_0, z_0) - z_0$ *and* $q(\delta, c_0, z_0)$ *do not depend on* z_0, *then the number* $\Gamma(\delta, c_0, z_0) - z_0$ *also does not depend on* z_0.

Proof.— Let $\delta \in (0, c_0)$. By means of the numbers $q = q(\delta, c_0, z_0)$ and $\tau = \tau(\delta, c_0, z_0)$ we shall define $\Gamma(\delta)$. By the properties of the function f there exists a positive number $a = a(\delta, c_0)$ such that

$$f(s) - s > a \quad \text{for } s \in [\delta, c_0].$$

There exists a positive interger $m = m(\delta, c_0)$ such that $\delta + am \ge c_0$. From the properties of the function r it follows that there exists a finite monotone increasing sequence of numbers $t_n = t_n(\delta, c_0, z_0)$ $(n = 0, 1, \ldots, m)$ such that $t_0 = \tau$ and $r(t) \ge t_{n-1}$ for $t \ge t_n - (a/q)$, $n = 1, 2, \ldots, m$. Set

$$\Gamma(\delta) = t_m = t_m(\delta, c_0, z_0).$$

Let $v \in C([z_0, +\infty))$, $v(t) \le c_0$ for $t \ge z_0$ and $D^+v(t) \le -q$ for each value of t satisfying the conditions $t \ge \tau$, $v(t) \ge \delta$, $r(t) \ge z_0$ and $f(v(t)) \ge v(s)$ for $s \in [r(t), t]$. We shall prove that $v(t) \le \delta$ for $t \ge t_m$. For this purpose it suffices to prove that for $n = 0, 1, \ldots, m$ the following relation is valid

$$v(t) \le \delta + (m-n)a \quad \text{for } t \ge t_n. \tag{P$_n$}$$

We shall prove the above assertion by induction with respect to n. From the relations $v(t) \le c_0 \le \delta + ma$ for $t \ge z_0$ and from $t_0 = \tau \ge z_0$ it follows that relation (P_n) is valid for $n = 0$. Suppose that relation (P_n) is valid for some $n < m$. Then, for $t \ge t_{n+1} - (a/q)$ and $v(t) \ge \delta + (m-n-1)a$, the inequality $D^+v(t) \le -q$ is valid since $t \ge r(t) \ge t_n \ge \tau$, $v(t) \ge \delta$ and

$$f(v(t)) > v(t) + a \ge \delta + (m-n-1)a \ge v(s) \quad \text{for } s \in [r(t), t].$$

From Lemma 2 it follows that $v(t) \leq \delta + a(m-n-1)$ for $t \geq t_{n+1} - (a/q) + T_1$, where

$$T_1 = \frac{1}{q} \max\{0, \, v(t_{n+1} - aq^{-1}) - \delta - (m-n-1)a\} .$$

The inequalities

$$t_{n+1} - \frac{a}{q} \geq g\left(t_{n+1} - \frac{a}{q}\right) \geq t_n$$

and

$$v\left(t_{n+1} - \frac{a}{q}\right) \leq \delta + (m-n)a$$

show that $T_1 \leq a/q$. Hence $v(t) \leq \delta + (m-n-1)a$ for $t \geq t_{n+1}$, *i.e.* relation (P_{n+1}) holds. The first part of Lemma 3 is proved.

Let, in addition $r(t) \geq t - h$, $h = \text{const} > 0$ and the numbers $\tau(\delta, c_0, z_0) - z_0$ and $q = q(\delta, c_0, z_0)$ do not depend on z_0. Setting $t_n = t_{n-1} + h + (a/q)$ for $n = 1, 2, \ldots, m$, we obtain

$$\Gamma(\delta, c_0, z_0) - z_0 = t_m - z_0 = m\left(h + \frac{a}{q}\right) + \tau(\delta, c_0, z_0) - z_0 ,$$

which completes the proof. □

Let $U = U(t, \varphi)$ be a functional continuous for $(t, \varphi) \in I \times E$,

$$a(\|\varphi\|) \leq U(t, \varphi) \leq b(\|\varphi\|) ,$$

where a, b are strictly increasing functions and $a(0) = b(0) = 0$. Set $v(t) = U(t, x_t)$, where x is a solution of the initial value problem (1), $x_\sigma = \varphi$, $(\sigma, \varphi) \in I \times E$. If $D^+v(t)$ satisfies appropriate inequalities, then by means of Lemmas 1 and 3 one can prove stability and equiasymptotic stability of the null solution of (1). It is essential that in the estimation of $D^+v(t)$ the properties of the solution x in the interval $[\sigma, t]$ or $[r(t), t]$ can be used, where the length of the interval $[r(t), t]$ can be unbounded for $t \in I$.

Lemma 1 and Lemma 3 represent a natural development of some ideas of B. S. Razumikhin [15], N. N. Krasovskii [10], R. D. Driver [4] and B. I. Barnea [1]. The results in [15], [10] and [4] require the existence of an *a priori* estimate for $D^+v(t)$ for any $t \geq \sigma$, $\varphi \in E$, in the paper of V. Lakshmikantham and S. Leela [11] an estimate of $D^+v(t)$ for any $t \geq \sigma$, $\varphi \in F_{\sigma - r(\sigma)}$ is required, and in [1] — respectively for $t \geq r(t) = t - h \geq \sigma$, $h = \text{const} > 0$. Results similar to Lemma 1 and Lemma 3 were obtained by J. Kato [8] but under some additional constraints on $r(t)$.

We shall use the following properties of the Dini derivatives.

LEMMA 4.— *Let f, g_1 and g_2 be scalar functions. Then the following relations are valid (provided that the respective algebraic operations have sense):*

(r_1) $\limsup\big(g_1(s)+g_2(s)\big) \le \limsup g_1(s) + \limsup g_2(s)$,
$\limsup\big(g_1(s)+g_2(s)\big) \ge \limsup g_1(s) + \liminf g_2(s)$,
$\limsup\big|g_1(s)\cdot g_2(s)\big| \le \limsup\big|g_1(s)\big| \cdot \limsup\big|g_2(s)\big|$,
$\limsup\big(-f(s) = -\liminf f(s)$;

(r_2) *if* $D^+ f(s) \in \mathbb{R}$ *then* $f(t) \to f(s)$ *as* $t \to s$, $t > s$;

(r_3) $\limsup\limits_{h\to 0^+}\big|f(s+h) - f(s)\big|h^{-1} = \mathcal{D}^+ f(s)$;

(r_4) *if* $\mathcal{D}^+ g_i(s) \in \mathbb{R}$, $i = 1, 2$, *then*

$$\mathcal{D}^+(g_1 g_2)(s) \le \big|g_1(s)\big|\,\mathcal{D}^+ g_2(s) + \big|g_2(s)\big|\,\mathcal{D}^+ g_1(s)\,;$$

(r_5) $\limsup\limits_{h\to 0^+}\ \sup\limits_{s\in(0,h]} f(s) = \limsup\limits_{h\to 0^+} f(h)$;

(r_6) $\mathcal{D}^+\big[g_1\big(g_2(s)\big)\big] \le \mathcal{D} g_1\big(g_2(s)\big)\mathcal{D}^+ g_2(s)$;

(r_7) *if the functions* f, g_1 *and* g_2 *are continuous*, $g_1(s) \le g_2(s)$ *for* $s \in (a,b)$ *and* $t \in (a,b)$, *then*

$$\mathcal{D}^+\Big(\max_{s\in[\,g_1(t)\,,g_2(t)\,]} f(s)\Big) \le \max_{i=1,\,2} \mathcal{D}^+\big[f\big(g_i(t)\big)\big]\ ;$$

(r_8) *if the functions* f, g_1 *and* g_2 *are continuous and*

$$g_1(t) \le D^+ f(t) \le g_2(t) \quad \text{for } t \in (a,b)\,,$$

then $D_+ f(t)$, $D^- f(t)$, $D_- f(t) \in \big[g_1(t)\,,\,g_2(t)\big]$.

Proof.— Relations (r_1)-(r_2) are obvious, relation (r_3) follows from the equality

$$\limsup\big|f(t)\big| = \max\big\{\big|\limsup f(t)\big|\,,\ \big|\liminf f(t)\big|\big\}\ ,$$

and relation (r_4) follows from (r_1)-(r_3). Let $d(h) = \sup_{s\in[\,0\,,\,h\,]} f(s)$, $h_n > 0$ for $n \in \mathbb{N}$ and $h_n \to 0$ as $n \to +\infty$. For any $n \in \mathbb{N}$ there exists $s_n \in (0\,,\,h_n\,]$ such that $f(s_n) \ge d(h_n) - n^{-1}$. Consequently,

$$\limsup_{n\to+\infty} d(h_n) \le \limsup_{n\to+\infty} f(s_n) \le \limsup_{s\to 0^+} f(s)$$

which implies relation (r_5). Let $\mathcal{D}^+ g_2(s) < +\infty$, $\mathcal{D}^+ g_1(g_2(s)) < +\infty$ and $\epsilon > 0$. There exists $\delta > 0$ such that for $t \in (s, s+\delta$ the inequality

$$\left|\left(g_1(g_2(s)) - g_1(g_2(s))\right)(t-s)^{-1}\right| \le \left(\mathcal{D} g_1(g_2(s)) + \epsilon\right)\left(\mathcal{D}^+ g_2(s) + \epsilon\right)$$

is valid. Passing to the limit as $t \to s$, $t > s$, and then $\epsilon \to 0$, relation (r_6) is obtained. Let the functions f, g_1 and g_2 be continuous, $g_1(s) \le g_2(s)$ for $s \in (a,b)$ and $t \in (a,b)$. Set

$$y(s) = \max_{\lambda \in [g_1(s), g_2(s)]} f(y).$$

For $y(t+h) > y(t)$ the following inequality holds

$$y(t+h) \le \max_{i=1,2} \left\{ \sup_{s \in (t, t+h]} f(g_i(s)) \right\}.$$

For $y(t) > \max\{f(g_1(t)), f(g_2(t))\}$ there exists $h_0 > 0$ such that $y(t+h) = y(t)$ for $h \in (0, h_0]$. Consequently, for given t and sufficiently small values of h the inequality

$$\begin{aligned} &\left|y(t+h) - y(t)\right| \le \\ &\quad \le \max_{i=1,2} \max\left\{ \sup_{s \in (t, t+h]} f(g_i(s)) - f(g_i(t)), f(g_i(t)) - f(g_i(t+h)) \right\} \end{aligned}$$

is valid, whence by means of relations (r_3) and (r_5) there follows relation (r_7). Let

$$g_1(t) \le D^+ f(t) \le g_2(t) \quad \text{for } t \in (a,b),$$

where the functions f, g_1 and g_2 are continuous. There exists a differentiable function G_1 defined for $t \in (a,b)$ so that $\mathrm{d}G_1(t)/\mathrm{d}t = g_1(t)$. The function $F_1 = f - G_1$ satisfies for $t \in (a,b)$ the inequality

$$D^+ F_1(t) \ge D^+ f(t) - g_1(t) \ge 0$$

in view of relation (r_1). From Corollary 2.4 of [16, Appendix I], it follows that the Dini derivatives $D_+ F_1(t)$, $D^- F_1(t)$ and $D_- F_1(t)$ are also nonnegative for $t \in (a,b)$. Hence $D_+ f(t) > g_1(t)$ since

$$0 \le D_+ F_1(t) \le D_+ f(t) + D^+(-G_1)(t) = D_+ f(t) - g_1(t).$$

The remaining inequalities in relation (r_8) are proved in the same way. □

In order to construct the Lyapunov functional needed in the proof of Theorem 2 (see section 4), we shall use Lemma 5 of [18] and a particular case of Theorem 3.1 in [17] given below as Lemma 5 and Lemma 6 accordingly.

LEMMA 5.— *Let $D(r)$ be a real $n \times n$ matrix whose coefficients are smooth functions of $r \in I$. Suppose that there exist positive constants β and q such that for any $r \in I$ the characteristic roots $\lambda_i(r)$, $i = 1, 2, \ldots, n$ of $D(r)$ satisfy the condition $\operatorname{Re}\lambda_i(r) \le -\beta$ and, moreover, $|D(r)| \le q$, $|\dot{D}(r)| \le q$.*

Then there exists a continuous function $W(r,\eta) : I \times \mathbb{R}^n \to \mathbb{R}^1$ possessing partial derivatives with respect to all arguments and satisfying the conditions

$$a_2|\eta|^2 \le W(r,\eta) \le b_2|\eta|^2 ,$$
$$|W_r(r,\eta)| \le b_2|\eta^2| , \quad |W_\eta(r,\eta)| \le b_2|\eta| ,$$
$$(W_\eta(r,\eta) , D(r)\eta) = -|\eta|^2 ,$$

where

$$W_r = \frac{\partial W}{\partial r} \quad and \quad W_\eta = \operatorname{col}\left(\frac{\partial W}{\partial \eta_1}, \frac{\partial W}{\partial \eta_2}, \ldots, \frac{\partial W}{\partial \eta_n}\right) ,$$

and a_2, b_2 are positive constants depending only on β and q. Moreover, its partial derivatives $\partial W/\partial \eta_i (r,\eta)$, $i = 1, 2, \ldots, n$, are linear with respect to $\eta \in \mathbb{R}^n$.

LEMMA 6.— *Let E be an admissible space with fading memory,*

$$|F(t,\varphi) - F(t,\psi)| \le L\|\varphi - \psi\| \quad for\ t \in I ,\ \varphi,\ \psi \in F ,$$

and let the null solution of (1) be exponentially stable in E.

Then there exists a continuous real-valued functional $V(t,\varphi)$ defined on $I \times E$, which satisfies the following conditions:

i) $\|\varphi\| \le V(t,\varphi) \le L_1\|\varphi\|$,

ii) $|V(t,\varphi) - V(t,\psi)| \le L_2\|\varphi - \psi\|$,

iii) $V'_{(1)}(t,\varphi) \le -qV(t,\varphi)$,

for $t \in I$, φ, $\psi \in E$, where L_1, L_2 and q are positive constants, and

$$V'_{(1)}(t,\varphi) = \limsup_{h \to 0^+} \left[V\big(t+h , x_{t+h}(t,\varphi)\big) - V(t,\varphi)\right] h^{-1} .$$

We shall note that the conditions of Theorem 2 are natural. The importance of conditions (A2) and (A4) is seen in the case $g(t) = t$ from the examples E1 , E2 and E4 in [7]. Moreover, in conditions (A2) and (A3) the requirement $g(t) \to +\infty$ as $t \to +\infty$ cannot be omitted as seen from the following example.

Example 1.— Let the function g be Lipschitz continuous and $g(t) \leq t$ for $t > 1$. Consider the system

$$\begin{aligned} \dot{X}(t) &= -X(t) + \mu \max_{s\in[g(t),t]} X(s) - Y(t) \\ \mu\dot{Y}(t) &= \mu \max_{s\in[g(t),t]} X(s) - Y(t) \end{aligned} \tag{8}$$

for $\mu = [0, 1]$, $t \in I = (1, +\infty)$ with initial conditions $\sigma \in I$, $X_\sigma = \varphi \in E = C_\gamma$, $Y(\sigma) = y_0 \in \mathbb{R}$, where $\gamma > 0$ and

$$C_\gamma = \{\varphi \in C(-\infty, 0] \mid \exists \lim|\varphi(s)|e^{\gamma s} \text{ as } s \to -\infty\}$$

The space C_γ provided with the norm $||\varphi|| = \sup_{s\leq 0}|\varphi(s)|e^{\gamma s}$ is admissible and with fading memory. The initial value problem for system (8) satisfies all assumptions of Theorem 1 except for the condition $g(t) \to +\infty$. Suppose that $g(t) \leq Q = \text{const}$ for $t \in I$. Then the null solution of (8) is not asymptotically stable since for $\delta > 0$, $\sigma > Q$, $\varphi(s) \equiv \delta$, $y_0 = \mu\delta$ the initial value problem (8), $X_\sigma = \varphi$, $Y(\sigma) = y_0$ has a solution (X, Y) given by $X(t) = \delta \exp(\sigma - t)$, $Y(t) = \mu\delta$ for $t \geq \sigma$.

We shall illustrate Theorem 2 by two further examples.

Example 2.— Let $g(t) = t/2$ and consider system (8). It satisfies all conditions of Theorem 2, hence its null solution is equiasymptotically stable for sufficiently small μ. We shall prove that it is not uniformly asymptotically stable for $\mu \neq 0$. Suppose that this is not true. Then there exist positive numbers δ, μ and a function $T(\epsilon)$ defined for $\epsilon > 0$ so that for $||\varphi|| + |y_0| < \delta$, $t \geq \sigma + T(\epsilon)$, $\sigma \in I$ the following inequalities are valid

$$|X(t)| < \epsilon, \quad |Y(t)| < \epsilon, \tag{9}$$

where (X, Y) is a solution of the respective initial value problem for system (8). Set $\epsilon = \mu\delta/2$, $\sigma = 1 + T(\epsilon)$, $\varphi(s) = \delta/2$ for $s \leq 0$, $y_0 = \mu\delta/2$.

A straightforward verification shows that the functions X, Y defined by the equalities

$$\begin{cases} X(t) = \dfrac{\delta}{2} & \text{for } t \le \sigma, \\ X(t) = \dfrac{\delta}{2}\exp(\sigma - t) & \text{for } t \in [\sigma, 2\sigma], \\ Y(t) = \dfrac{\mu\delta}{2} & \text{for } t < 2\sigma, \end{cases}$$

are a solution of the initial value problem (8), $X_\sigma = \varphi$, $Y(\sigma) = y_0$, for $t \le 2\sigma$. But the equalities $2\sigma - 1 = \sigma + T(\epsilon)$ and $Y(2\sigma - 1) = \mu\delta/2 = \epsilon$ contradict inequalities (9). Hence the null solution of system (8) for $g(t) = t/2$ is equiasymptotically stable for sufficiently small $\mu > 0$ but it is not uniformly asymptotically stable.

Example 3.— Consider for $t > 1$ the system

$$\dot{X}(t) = aX(t) + \tilde{a} \max_{s\in[t-h,t]} X(s) + p\left(\mu X^2(t) \max_{s\in[\sqrt{t},t]} X(s)\right)^{1/3} + bY(t)$$

$$\mu\dot{Y}(t) = cX(t) + \tilde{c} \max_{s\in[t-h,t]} X(s) + q\left(\mu X^2(t) \max_{s\in[\sqrt{t},t]} X(s)\right)^{1/3} + dY(t) \tag{10}$$

where the two equations are scalar with constant coefficients, $\mu > 0$, $h > 0$, $d < 0$, $(a + \tilde{a})d - (c + \tilde{c})b > 0$. We assume that $E = C_\gamma$, $\gamma > 0$.

The corresponding degenerate system has the form

$$\dot{x}(t) = (a - bcd^{-1})x(t) + (\tilde{a} - b\tilde{c}d^{-1}) \max_{s\in[t-h,t]} x(s). \tag{11}$$

By the well known method of Razumikhin it is proved that for $(a - \tilde{a})d - b(c - \tilde{c}) > 0$ the null solution of (11) is uniformly asymptotically stable in $\mathbb{R}$, *i.e.* condition (A5) of Theorem 2 is satisfied. By means of Lemma 4, it is immediately verified that the remaining conditions of Theorem 2 are also satisfied. Hence, for $(a - \tilde{a})d > b(c - \tilde{c})$ and for sufficiently small values of $\mu > 0$ the null solution of (10) is equiasymptotically stable, while in the case when $p = q = 0$ it is uniformly asymptotically stable. We note that by Theorem 3.1 in [19] the same result is valid if we replace the condition $(a - \tilde{a})d > b(c - \tilde{c})$ by the weaker requirement that at least one of the inequalities

$$h(ad - bc) > d \quad \text{or} \quad pe\big[h(\tilde{a} - b\tilde{c}d^{-1})\big] + \exp\big[h(a - bcd^{-1})\big] < 0$$

hold.

4. Proof of theorem 2

From conditions (A1) and (A4) it follows that the functional $[H_1 - B_1 B_2^{-1} H_2](t, \varphi, 0)$ is continuous for $(t, \varphi) \in I \times E$ and Lipschitz continuous in φ. Since E is an admissible space with fading memory, from condition (A5) and Theorem 6.1 in [6] it follows that the null solution of the degenerate system (7) is uniformly asymptotically stable in E. Then it is exponentially stable in E since the functional $[H_1 - B_1 B_2^{-1} H_2](t, \varphi, 0)$ is homogeneous in φ. (It suffices in the proof of Theorem 3.2 in [17] to replace the expression " linear operator " by " homogeneous operator ".) In view of Lemma 6, there exists a continuous functional $\overline{V}(t, \varphi)$ such that the functional $V(t, \varphi) = \left[\overline{V}(t, \varphi)\right]^2$ satisfies the inequalities

$$a_1 \|\varphi\|^2 \le V(t, \varphi) \le b_1 \|\varphi\|^2 ,$$

$$\left|V(t, \varphi) - V(t, \psi)\right| \le b_1 \left(\|\varphi\| + \|\psi\|\right) \|\varphi - \psi\| , \tag{12}$$

$$V'_{(7)}(t, \varphi) \le -a_1 \|\varphi\|^2 , \tag{13}$$

where a_1 and b_1 are positive constants.

From conditions (A1), (A4) and from Lemma 5, it follows that there exists $W \in C^1(I \times \mathbb{R}^n)$ such that

$$a_2 |\eta|^2 \le W(t, \eta) \le b_2 |\eta|^2 ,$$

$$\left|W_t(t, \eta)\right| \le b_2 |\eta|^2 , \quad \left|W_\eta(t, \eta)\right| \le b_2 |\eta| , \tag{14}$$

$$\left(W_\eta(t, \eta), B_2(t, 0)\eta\right) = -\eta^2 ,$$

where a_2, b_2 are positive constants depending only on M_2 and β, $W_t = \partial W / \partial t W_\eta = \mathrm{col}(\partial W / \partial \eta_1, \ldots, \partial W / \partial \eta_n)$ and $(\cdot, \cdot)$ is the scalar product in $\mathbb{R}^n$. Moreover, the partial derivatives $\partial W / \partial \eta_i$ are linear in $\eta \in \mathbb{R}^n$.

From conditions (A1), (A2) and (A4), it follows that there exists constants $\mu_1 \in (0, \mu_0)$ and b_3 such that for $(t, \mu) \in I \times [0, \mu_1]$, $\varphi \in E_{t-g(t)}$, $\eta \in \mathbb{R}^n$ the following inequalities are valid

$$\left|B_2^{-1}(t, \mu)\right| \le b_3 , \quad \mathcal{D}^+ B_2^{-1}(t, \mu) \le b_3 , \quad \rho(\mu) \le 1 , \tag{15}$$

$$\left|[B_1 B_2^{-1} H_2](t,\varphi,\mu) - [B_1 B_2^{-1} H_2](t,\varphi,0)\right| \le b_3 \rho(\mu) \sup_{s\in[g(t)-t,0]} \|\varphi_s\|, \tag{16}$$

$$\left|B_2^{-1}(t,\mu) H_2(t,\varphi,\mu)\right| \le b_3 \sup_{s\in[g(t)-t,0]} \|\varphi_s\|, \tag{17}$$

$$\left(W_\eta(t,\eta),\, B_2(t,\mu)\eta\right) \le -\frac{1}{2}|\eta|^2 . \tag{18}$$

Set

$$L = \max\{1,\, M_2,\, M_3,\, K_0,\, b_1,\, b_2,\, b_3\}, \quad m = \min\{1,\, a_1,\, a_2\}.$$

Then the functional $U = U(t,\varphi,\eta) = V(t,\varphi) + W(t,\eta)$ satisfies for $(t,\varphi,\eta) \in I \times E \times \mathbb{R}^n$ the inequality

$$m^2 \left(\|\varphi\| + |\eta|\right)^2 \le U(t,\varphi,\eta) \le L^2 \left(\|\varphi\| + |\eta|\right)^2 . \tag{19}$$

There exists a function d defined for $s \in I$ so that $g(t) > s$ for $t > d(s)$. Let $\mu \in (0,\mu_1)$, $(\sigma,\varphi,y_0) \in I \times E \times \mathbb{R}^n$. Let (X,Y) be a solution of the initial value problem (3), $X_\sigma = \varphi$, $Y(\sigma) = y_0$. Set $z_0 = d(\sigma)$,

$$z_1 = d\left(d\big(d(\sigma)\big)\right) \quad \text{and} \quad r(t) = \inf_{s\in[g(t),t]} g(s) \text{ for } t > d(\sigma).$$

From Theorem 1, it follows that for $t \in [\sigma, z_1]$ the following inequality holds

$$\|X_t\| + \left|Y(t)\right| \le C_1 \left(\|\varphi\| + |y_0|\right), \tag{20}$$

where $C_1 = Q(z_1,\sigma,\mu)$. For $t \ge z_1$ we shall estimate from above the quantities $\|X_t\| + \left|Y(t)\right|$ by means of the functional U and Lemmas 1 and 3. Introduce the functions η and ν defined for $t \ge z_0$ by the equalities

$$\eta(t) = Y(t,\mu) + B_2^{-1}(t,\mu) H_2(t,X_t,\mu), \tag{21}$$

$$v(t) = v_1(t) + v_2(t) = U\left(t, X_t, \eta(t)\right), \tag{22}$$

where

$$v_1(t) = v_1(\sigma,\varphi,y_0,\mu)(t) = V(t,X_t)$$

and

$$v_2(t) = v_2(\sigma,\varphi,y_0,\mu)(t) = W\left(t,\eta(t)\right).$$

We shall estimate $v'(t)$ taking into account that the functions X and Y satisfy system (3) in the interval $[\sigma, t]$.

Let $t \geq z_1$ and let w be a solution of the initial value problem (7), $x_t = X_t$. Since the space E is admissible and $X_t \in E$, from (12), (13) and Lemma 4 there follow the inequalities

$$\begin{aligned} v'(t) &\leq \limsup_{h\to 0^+} \big[V(t+h\,,\,X_{t+h}) - V(t+h\,,\,w_{t+h})\big]h^{-1} \\ &\quad + \limsup_{h\to 0^+} \big[V(t+h\,,\,w_{t+h}) - V(t\,,\,w_t)\big]h^{-1} \\ &\leq 2\,LK_0\|X_t\| \limsup_{h\to 0^+} \sup_{s\in(0\,,\,h\,]} \frac{\big|X(t+s) - w(t+s)\big|}{s} - m\|X_t\|^2 \\ &\leq 2\,L^2\|X_t\|\,\big|\dot{X}(t) - \dot{w}(t)\big| - m\|X_t\|^2 \end{aligned}$$

since the functions X and w satisfy equations (3) and (7). In view of conditions (A1), (A2) and relations (21), (16), we obtain that

$$v_1'(t) \leq -m\|X_t\|^2 + 2\,L^3\|X_t\| \left(\big|\eta(t)\big| + 2\,\rho(\mu) \max_{s\in[\,g(t)\,,\,t\,]} \|X_s\|\right). \tag{23}$$

Set

$$S(t) = B_2^{-1}(t,\mu)H_2(t,X_t,\mu) \quad \text{and} \quad \xi(\lambda) = \eta(\lambda) - \lambda B_2(t,\mu)\eta(t)\mu^{-1}$$

for $\lambda \geq t$. From conditions (A1), (A2), (A3), from relations (3), (15), (21) and from Lemma 4, there follows the inequality

$$\mathcal{D}^+S(t) \leq M_4 \sup_{\lambda\in[\,r(t)\,,\,t\,]} \big(\|X_\lambda\| + \big|\eta(\lambda)\big|\big) < \infty\,, \tag{24}$$

where the number M_4 depends only on L, n and p. By equalities (3), (21) and Lemma 4, we have that

$$\begin{aligned} D^+\eta(t) &= \mu^{-1}B_2(t,\mu)\eta(t) + D^+S(t)\,, \\ D_+\eta(t) &= \mu^{-1}B_2(t,\mu)\eta(t) + D_+S(t)\,. \end{aligned}$$

Hence $\mathcal{D}^+\eta(t) < \infty$ and $\mathcal{D}^+\xi(t) = \mathcal{D}^+S(t)$. Then by means of inequalities (14), (18) and Lemma 4, it is obtained that

$$\begin{aligned} v_2'(t) &\leq \limsup_{h\to 0^+} \Big(\big[\eta(t+h) - \eta(t)\big]h^{-1}\,,\, W_\eta\big(t,\eta(t)\big)\Big) + W_t\big(t,\eta(t)\big) \\ &\leq \mu^{-1} \limsup_{h\to 0^+} \big(B_2(t,\mu)\eta(t)\,,\, W_\eta\big(t,\eta(t)\big)\big) + \\ &\quad + L\big|\eta(t)\big|^2 + \limsup_{h\to 0^+} \Big(\big[\xi(t+h) - \xi(t)\big]h^{-1}\,,\, W_\eta\big(t,\eta(t)\big)\Big) \\ &\leq \big(L - (2\mu)^{-1}\big)\big|\eta(t)\big|^2 + L\big|\eta(t)\big|\mathcal{D}^+S(t)\,. \end{aligned} \tag{25}$$

From relations (23)-(25) it follows that for $t \geq z_1$, $\mu \in (0, \mu_1)$ the following inequality is valid

$$v'(t) \leq -m\|X_t\|^2 - (2\mu)^{-1}\left|\eta(t)\right|^2 + \\ + M_5\left(\rho(\mu)\|X_t\| + \left|\eta(t)\right|\right) \max_{\lambda \in [r(t),\, t]} \left(\|X_s\| + \left|\eta(s)\right|\right), \tag{26}$$

where the number M_5 depends only on L, n and p. From relations (19), (26) it follows that there exists $\mu_2 \in (0, \mu_1)$ such that for $\mu \in (0, \mu_2)$ for any $t \geq z_1$ satisfying the inequalities $v(t) \geq a > 0$ and $v(s) \leq 2\,v(t)$ for all $s \in \left[\, r(t)\,,\, t\,\right]$, the following inequality is valid

$$v'(t) \leq -ma(2L)^{-2} < 0\,. \tag{27}$$

Let the number $\mu \in (0, \mu_2)$ be fixed. We shall prove that the null solution of system (3) is equiasymptotically stable. With each $\sigma \in I$, $\delta > 0$, we associate the set $P(\sigma, \delta)$ consisting of all functions v defined by equality (22) for $\|\varphi\| + |y_0| < \delta$. Let $\sigma \in I$ and $\epsilon > 0$. Set $\delta(\sigma, \epsilon) = \epsilon m(L+1)^{-2}(LC_1)^{-1}$, where $C_1 = Q(z_1, \sigma, \mu)$. From relations (19)-(22) and (17), it follows that for $v \in P\big(\sigma\,,\, \delta(\sigma, \epsilon)\big)$ and $t \in [\, z_0\,,\, z_1\,]$ the inequality $v(t) < m^2\epsilon^2(L+1)^{-2}$ is valid. Since $r(t) \geq z_0$ for $t \geq z_1$, from inequality (27) and from Lemma 1, it follows that $v(t) < m^2\epsilon^2(L+1)^{-2}$ for $t \geq z_0$.

For $t \geq z_1$ from relations (21), (17), (19), (22) and $g(t) > z_0$, it follows that

$$\|X_t\| + \left|Y(t)\right| \leq (L+1)m^{-1} \sup_{s \in [g(t)\,,\, t]} \sqrt{v(s)}\,, \tag{28}$$

hence $\|X_t\| + \left|Y(t)\right| < \epsilon$. The last inequality is valid for $t \in [\, \sigma\,,\, z_1\,]$ as well, by virtue of inequalities (20) and $L \geq m$. Hence the null solution of system (3) is stable for $\mu \in (0, \mu_2)$. It remains to prove that it is equiattractive. Let $\sigma \in I$. For $\delta_0 = \delta\big(\sigma, (L+1)m^{-1}\big)$, $v \in P(\sigma, \delta_0)$, $t \geq z_0$, we have $v(t) \leq 1$. From lemma 3 (for $f(\lambda) = 2\lambda$, $c_0 = 1$, $\tau(a, 1, z_0) = d\big(d(z_0)\big)$, $q(a) = ma(2L)^{-2}$) and from inequality (27), it follows that there exists a function $\Gamma = \Gamma(a, 1, z_0) \geq z_0$ defined for $a > 0$ so that for $v \in P(\sigma, \delta_0)$, $t \geq \Gamma(a, 1, z_0)$, the inequality $v(t) < a$ is valid. Set

$$T(\sigma, \epsilon) = d\left(\Gamma\left(\epsilon^2 m^2 (L+1)^{-2}\right),\, 1\,,\, d(\sigma)\right)\,.$$

From inequality (28), it follows that $\|X_t\| + \left|Y(t)\right| < \epsilon$ for $t \geq T(\sigma, \epsilon)$, $\|\varphi\| + |y_0| < \delta_0(\sigma)$. Hence the null solution of system (3) is equiasymptotically stable for $\mu \in (0, \mu_2)$.

Moreover, let it be given that $t-g(t) \leq h$, $|H_i(t,\varphi,\mu)| \leq h$, $h = \text{const} > 0$ for $t \in I$, $||\varphi|| = 1$, $i = 1, 2$. Then $d(t) = t + h$, $r(t) \geq t - 2h$, and from Theorem 1 and lemma 3, it follows that the numbers $\delta(\sigma,\epsilon)$, $\delta_0(\sigma)$ and $T(\sigma,\epsilon) - \sigma$ do not depend on $\sigma \in I$. □

References

[1] BARNEA (B. I.) .— *A method and new results for stability and instability of autonomous functional differential equations*, SIAM J. Appl. Math. **17** (1969), pp. 681-697.

[2] COOKE (K. L.) .— *The condition of regular degeneration for singularly perturbed linear differential-difference equations*, J. Differential Equations **1** (1965), pp. 39-94.

[3] COOKE (K. L.) and MEYER (K. R.) .— *The condition of regular degeneration for singularly perturbed systems of linear differential-difference equations*, J. Math. Anal. Appl. **14** (1966), pp. 83-106.

[4] DRIVER (R. D.) .— *Existence and stability of solutions of a delay-differential system*, Arch. Rat Mech. Anal. **10** (1962), pp. 401-426.

[5] HALANAY (A.) .— *Singular perturbations of systems with retarded argument*, Rev. Math. Pures Appl. **7** (1962).

[6] HALE (J. K.) and KATO (J.) .— *Phase space for retarded differential equations with infinite delay*, Funk. Ekvac. **21** (1978), pp. 11-41.

[7] HOPPENSTEADT (F. C.) .— *Singular perturbations on the infinite interval*, Trans. Amer. Math. Soc. **123** (1966), pp. 521-525.

[8] KATO (J.) .— *Stability problem in functional differential equations with infinite delay*, Funk. Ekvac. **21** (1978), pp. 63-80.

[9] KLIMUSHEV (A. I.) .— *On asymptotic stability of systems with after-effect containing a small parameter as coefficient of the derivative*, Prikl. Mat. Meh. **26** (1962), pp. 52-61. Translated as J. Appl. Math.. Mech. **26** (1962), pp. 68-81.

[10] KRASOVSKII (N. N.) .— *Some problems in the theory of stability of motion*, Moscow, 1959. English translation by J. Brenner, Stanford, California, 1962.

[11] LAKSHMIKANTHAM (V.) and LEELA (S.) .— *A unified approach to stability theory for differential equations with infinite delay*, J. Int. Eqns **10** (1985), pp. 147-156.

[12] MAGALHAES (L. T.) .— *Convergence and boundary layers in singularly perturbed linear functional differential equations*, J. Diff. Eqns **54** (1984), pp. 295-309.

[13] MAGALHAES (L. T) .— *Exponential estimates for singularly perturbed functional differential equations*, J. Math. Anal. Appl. **103** (1984), pp. 443-460.

[14] MAGALHAES (L. T.) .— *The asymptotics of solutions of singularly perturbed functional differential equations : distributed and concentrated delays are different*, J. Math. Anal. Appl. **105** (1985), pp. 250-257.

[15] RAZUMIKHIN (B. S.) .— *Application of Lyapunov's method to problems in the stability of systems with a delay*, Automat. and Remote Control **21** (1960), pp. 515-520.

[16] ROUCHE (N.), HABETS (P.) and LALOY (M.) .— *Stability theory by Liapunov's direct method*, Springer Verlag, 1977.

[17] SAWANO (K.) .— *Exponentially asymptotic stability for functional differential equations with infinite retardations*, Tohoku Math. J. **31** (1979), pp. 363-382.

[18] VOULOV (H. D.) and BAINOV (D. D.) .— *Asymptotic stability of singularly perturbed linear system with unbounded delay*, Differential Equations : Qualitative Theory, Vol. I, II (Szeged, 1984), pp. 1097-1124. Colloq. Maht. Soc. Janos Bolyai, North Holland, **47** (1987).

[19] VOULOV (H. D) and BAINOV (D. D.) .— *On the asymptotic stability of differential equations with "maxima"*, Rend. Circ. Mat. Palermo, Serie II, Tomo XL (1991), pp. 385-420.