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MAURICE GAULTIER

MIKEL LEZAUN

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Spectral study of a self-adjoint operator on $L^2(\Omega)$ related with a Poincaré type constant[(*)]

MAURICE GAULTIER[(1)] and MIKEL LEZAUN[(2)]

RÉSUMÉ. — Soit Ω un ouvert borné et connexe de $\mathbb{R}^N$, $N \geq 2$, de frontière lipschitzienne. L'espace $H_0^1(\Omega)$ est muni de la norme du gradient. L'inégalité suivante a lieu pour les éléments de $L^2(\Omega)$.

$$|u|^2_{L^2(\Omega)} - \frac{1}{\operatorname{mes}(\Omega)} \left| \int_\Omega u(x)\,\mathrm{d}x \right|^2 \leq C(\Omega) \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|^2_{H^{-1}(\Omega)}$$

où $C(\Omega) > 0$ ne dépend que de Ω. À l'aide d'un opérateur autoadjoint sur $L^2(\Omega)$, on caractérise la meilleure constante dans l'inégalité précédente. Lorsque Ω est une boule de $\mathbb{R}^N$, $N \geq 2$, on fait l'analyse spectrale de cet opérateur et on montre que la meilleure valeur de la constante est N.

ABSTRACT. — Let Ω be a connected bounded open set in $\mathbb{R}^N$, $N \geq 2$, with lipschitzian boundary. $H_0^1(\Omega)$ is equipped with the gradient norm. The following inequality holds for the elements of $L^2(\Omega)$:

$$|u|^2_{L^2(\Omega)} - \frac{1}{\operatorname{mes}(\Omega)} \left| \int_\Omega u(x)\,\mathrm{d}x \right|^2 \leq C(\Omega) \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|^2_{H^{-1}(\Omega)}$$

where $C(\Omega) > 0$ depends only on Ω. This paper provides a characterization of the best constant in the previous inequality using a self-adjoint operator on $L^2(\Omega)$. When Ω is a ball in $\mathbb{R}^N$, $N \geq 2$, the spectral study of this operator is made and in this case, we obtain that the best constant is N.

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(1) Université de Bordeaux I et C.R.M. B, 351 cours de la Libération, F-33405 Talence Cedex (France)

(2) Departemento de Matemática Aplicada, Estadística e Investigagación Operativa, Facultad de Ciencias, Universidad del País Vasco, Apartado 644, Bilbao (Spain)

1. Introduction

Throughout this paper Ω is a connected bounded open set in $\mathbb{R}^N$, $N \geq 2$, and its boundary Γ is Lipschitz-continuous as [5]. The space $H_0^1(\Omega)$ will always be equipped with the gradient norm. Derivates of functions on Ω will be taken in the sense of distributions.

We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ normed by:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \left(\frac{\langle f, v \rangle}{\|v\|_{H_0^1(\Omega)}} \right),$$

where $\langle \cdot , \cdot \rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

The following inequality

$$|u|^2_{L^2(\Omega)} \leq C(\Omega) \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{H^{-1}(\Omega)}, \quad \forall\, u \in L^2(\Omega)\,, \int_\Omega u(x)\, \mathrm{d}x = 0\,, \quad (1)$$

or the equivalent inequality: $\forall\, u \in L^2(\Omega)$,

$$|u|^2_{L^2(\Omega)} - \frac{1}{\mathrm{mes}(\Omega)} \left| \int_\Omega u(x)\, \mathrm{d}x \right|^2 \leq C(\Omega) \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{H^{-1}(\Omega)} \quad (2)$$

where $C(\Omega) > 0$ depends only on Ω occurs in very many problems in the mechanics of continuous media [4]. Constant $C(\Omega)$ then occurs in the conditions for the uniqueness and sometimes for the existence of solutions. Knowledge of a value of $C(\Omega)$ is also important for the Numerical Analysis of these problems.

This paper provides a characterization of the best constant $P(\Omega)$ in inequalities (1)-(2) using a self-adjoint operator on $L^2(\Omega)$. Except if Ω is a ball in $\mathbb{R}^N$, the explicit value of this best constant $P(\Omega)$ is out of reach. In the particular case where Ω is a rectangle in 2-D, we obtain an approximate value of this best constant $P(\Omega)$.

The reader will recall that if $u \in H^1(\Omega)$, we have the following inequality, called Poincaré's inequality:

$$|u|^2_{L^2(\Omega)} - \frac{1}{\operatorname{mes}(\Omega)} \left| \int_\Omega u(x)\,\mathrm{d}x \right|^2 \le K(\Omega) \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^2_{L^2(\Omega)},$$

where $K(\Omega) > 0$ depends only on Ω. It is well known ([3]) that the best constant in this inequality, called Poincaré's constant, is the inverse of the smallest (positive) eigenvalue of the operator $-\Delta$ (in $L^2(\Omega)$) for the Neumann problem. The explicit value of this constant is in general out of reach.

This paper is organized as follows.

Section 2 introduces some function spaces and we draw attention to an important inequality due to J. Necas [6] for the elements of $L^2(\Omega)$ which plays an essential role in the proof of inequalities (1)-(2).

In Section 3, we propose a proof of coercivity inequality (2) and, in particular, as a consequence, we obtain the following well known result:

$$\text{if } f \in D'(\Omega) \text{ with } \operatorname{grad}(f) \in \left(H^{-1}(\Omega)\right)^N, \quad \text{then } f \in L^2(\Omega).$$

In Section 4, we prove that the best constant in inequalities (1)-(2) is the inverse of the smallest spectral value of the operator $T = -\operatorname{div} \circ (-\Delta)^{-1} \circ \operatorname{grad}$ on $L^2(\Omega)/\mathbb{R}$.

In Section 5, we prove that the inverse of this best constant $P(\Omega)$ is the limit of a decreasing positive sequence which has for general term the smallest eigenvalue of the matrix corresponding to a positive definite quadratic form on a suitable finite dimensional euclidean space.

In Section 6, we consider two particular cases:

- Ω is a rectangle in 2-D. Using an appropriate basis of $L^2(\Omega)$, we obtain an approximate value of this constant $P(\Omega)$;
- Ω is a ball in $\mathbb{R}^N$ with $N \ge 2$. We make the complete spectral study of operator $-\operatorname{div} \circ (-\Delta)^{-1} \circ \operatorname{grad}$ and we obtain that $P(\Omega) = N$.

2. Preliminaries

Throughout this paper we suppose for simplicity that all functions are real.

We use the usual product topology on the product spaces. We denote

$$\partial_i = \frac{\partial}{\partial x_i}.$$

For $u = (u_1, \dots, u_N) \in \left(D'(\Omega)\right)^N$, we set $\Delta u = (\Delta u_1, \dots, \Delta u_N)$. For $f = D'(\Omega)$, we set $\operatorname{grad}(f) = (\partial_1 f, \dots, \partial_N f)$.

In $L^2(\Omega)$ the Hilbert norm and the scalar product are written $|\cdot|_2$ and $(\cdot, \cdot)_2$.

Let $M(\Omega)$ be the closed subspace of $L^2(\Omega)$ of functions of zero mean (orthogonal to constants):

$$M(\Omega) = \left\{ u \in L^2(\Omega) \;\middle|\; \int_\Omega u(x)\, \mathrm{d}x = 0 \right\},$$

$M(\Omega)$ is equipped with the norm induced by Hilbert space $L^2(\Omega)$.

The quotient space $L^2(\Omega)/\mathbb{R}$, equipped with the usual quotient norm, is isometrically isomorphic to $M(\Omega)$. This isomorphism maps each equivalence class to its element of minimal norm, which is also the unique element of mean zero in the class. By convention we write $L^2(\Omega)/\mathbb{R} \equiv M(\Omega)$.

We recall that $H_0^1(\Omega)$ is equipped with the gradient norm, denoted by $\|\cdot\|$ and $H^{-1}(\Omega)$ is equipped with the dual norm. $\left(H_0^1(\Omega)\right)^N$ is isomorphic to $\left(H^{-1}(\Omega)\right)^N$ and $-\Delta$ is this isometric isomorphism.

For simplicity, in the remainder of this paper, we shall write indiscriminately $\|\cdot\|$ for the norm on $H_0^1(\Omega)$ or on $\left(H_0^1(\Omega)\right)^N$ and $\|\cdot\|_{-1}$ (resp. $((\cdot, \cdot))_{-1}$) for the norm (resp. scalar product) on $H^{-1}(\Omega)$ or on $\left(H^{-1}(\Omega)\right)^N$.

We introduce the following closed subspaces of $\left(H_0^1(\Omega)\right)^N$:

$$V = \left\{ u \in \left(H_0^1(\Omega)\right)^N \;\middle|\; \operatorname{div}(u) = 0 \right\},$$

$$V^\perp : \text{ the subspace of } \left(H_0^1(\Omega)\right)^N \text{ orthogonal to } V.$$

V and $V^\perp$ are equipped with the norm induced by $\left(H_0^1(\Omega)\right)^N$ (for properties of V see eg. [7]).

The important inequality which follows is proved in [6].

There exist a positive constant $N(\Omega)$ which depends only on Ω such that:

$$|u|_2 \leq N(\Omega)\left(\|u\|_{-1} + \|\mathrm{grad}(u)\|_{-1}\right), \quad \text{for all } u \in L^2(\Omega). \tag{3}$$

3. Poincaré type inequality on $L^2(\Omega)$

PROPOSITION 1.— *There exist a constant $C(\Omega) \geq 1$, depending only on Ω, such that:*

$$|u|_2^2 \leq C(\Omega)\|\mathrm{grad}(u)\|_{-1}^2 + \frac{1}{\mathrm{mes}(\Omega)}\left(\int_\Omega u(x)\,\mathrm{d}x\right)^2, \quad \forall\, u \in L^2(\Omega).$$

Proof.— grad $\in L\big(L^2(\Omega), (H^{-1}(\Omega))^N\big)$ and the kernel of grad is $\mathbb{R}$ because Ω is connected. Consequently, grad is a linear continuous injective mapping from $L^2(\Omega)/\mathbb{R}$ into $\left(H^{-1}(\Omega)\right)^N$.

Now we are going to show, by contradiction, that grad is bicontinuous.

We suppose that grad^{-1} is not bounded at 0. Then there exists a sequence $\{\widehat{u}_p\}$ of $L^2(\Omega)/\mathbb{R}$ such that $|\widehat{u}_p|_{L^2(\Omega)/\mathbb{R}} = 1$ and

$$\lim_{p\to\infty} \|\mathrm{grad}(\widehat{u}_p)\|_{-1} = 0,$$

whence $|u_p^0|_2 = 1$ and

$$\lim_{p\to\infty} \left\|\mathrm{grad}(u_p^0)\right\|_{-1} = 0$$

where u_p^0 is the unique element of $\widehat{u}_p$ of minimal norm.

Taking into account that the injection from $L^2(\Omega)$ into $H^{-1}(\Omega)$ is compact, there exist a subsequence $\{u_{p_k}^0\}$ which converges in $H^{-1}(\Omega)$.

It follows from (3) that there exist $u^0 \in L^2(\Omega)$ such that:

$$\lim_{k\to\infty} u_{p_k}^0 = u^0 \text{ in } L^2(\Omega) \quad \text{and} \quad \lim_{k\to\infty} \left\|\mathrm{grad}(u_{p_k}^0)\right\|_{-1} = 0.$$

This result implies that $\operatorname{grad}(u^0) = 0$, therefore

$$\lim_{k\to\infty} \widehat{u}_{p_k} = \widehat{0}$$

in contradiction with the definition of sequence $\{\widehat{u}_p\}$.

Consequently, there exist a positive constant $C(\Omega)$, depending only on Ω, such that:

$$|u|_2^2 \leq C(\Omega)\|\operatorname{grad}(u)\|_{-1}^2, \quad \text{for all } u \in M(\Omega) \equiv L^2(\Omega)/\mathbb{R}. \qquad (4)$$

On the other hand, for each $u \in L^2(\Omega)$, there exist an unique element $v \in M(\Omega)$ such that

$$u = v + \frac{1}{\operatorname{mes}(\Omega)} \int_\Omega u(x)\,\mathrm{d}x.$$

Hence

$$|u|_2^2 = |v|_2^2 + \frac{1}{\operatorname{mes}(\Omega)} \left(\int_\Omega u(x)\,\mathrm{d}x\right)^2.$$

Then, it follows from inequality (4),

$$|u|_2^2 \leq C(\Omega)\|\operatorname{grad}(u)\|_{-1}^2 + \frac{1}{\operatorname{mes}(\Omega)} \left(\int_\Omega u(x)\,\mathrm{d}x\right)^2, \quad \forall\, u \in L^2(\Omega). \quad (5)$$

As $|\operatorname{div}(\phi)|_2 \leq \|\phi\|$ for all $\phi \in \left(H_0^1(\Omega)\right)^N$ [7, p. 140],

$$\|\operatorname{grad}(u)\|_1 = \operatorname*{Sup}_{\|v\|\leq 1} \langle \operatorname{grad}(u), v\rangle = \operatorname*{Sup}_{\|v\|\leq 1} \left(u, \operatorname{div}(v)\right)_2 \leq |u|_2 |\operatorname{div}(v)|_2.$$

Therefore $\|\operatorname{grad}(u)\|_{-1} \leq |u|_2$. Then, it follows from (4) that $C(\Omega) \geq 1$.

COROLLARY 1.— $-\operatorname{div}\circ(-\Delta)^{-1}\circ\operatorname{grad}$ *is an isomorphism from* $M(\Omega)$ *onto* $M(\Omega)$.

Proof.— We identify $L^2(\Omega)$ with its dual. It follows from proposition 1 that grad is an isomorphism from $M(\Omega)$ into $\left(H^{-1}(\Omega)\right)^N$. Then its adjoint, $-$ div, is an isomorphism from $\left(H_0^1(\Omega)\right)^N/V$ onto $M(\Omega)$. By transposition, grad is an isomorphism from $M(\Omega)$ onto the dual space of $\left(H_0^1(\Omega)\right)^N/V$ that is onto V^0 (the annihilator of V). It is not difficult to see that $V^0 = -\Delta(V^\perp)$.

We deduce from corollary 1 the following well known result.

COROLLARY 2.— *If* $f \in D'(\Omega)$ *and* $\mathrm{grad}(f) \in \left(H^{-1}(\Omega)\right)^N$, *then* $u \in L^2(\Omega)$.

Proof.— Let $v \in \left(D(\Omega)\right)^N$ such that $\mathrm{div}(v) = 0$. Then we have:

$$\left\langle \mathrm{grad}(f)\,,\, v\right\rangle = -\left\langle f\,,\, \mathrm{div}(v)\right\rangle = 0\,.$$

Therefore $\mathrm{grad}(f) \in V^0$. Thus, there exist $g \in L^2(\Omega)/\mathbb{R}$ such that $\mathrm{grad}(f) = \mathrm{grad}(g)$. It follows that $f = g + C$ because Ω is connected.

NOTATION .— *In the remainder of this paper, the best value of constant* $C(\Omega)$ *in inequalities (4)-(5) is denoted by* $P(\Omega)$*:*

$$P(\Omega)^{-1} = \inf_{\substack{u \in M(\Omega) \\ u \neq 0}} \frac{\|\mathrm{grad}(u)\|_{-1}^2}{|u|_2^2}\,.$$

4. The operator related with the Poincaré type constant $P(\Omega)$

From corollary 1, the operator $T = -\,\mathrm{div}\circ(-\Delta)^{-1}\circ \mathrm{grad}$ is an isomorphism from $M(\Omega)$ onto $M(\Omega)$. Moreover, for all $u \in M(\Omega)$:

$$(Tu, u)_2 = \sum_{i=1}^{N} \left\langle \partial_i u\,,\, (-\Delta)^{-1} \circ \partial_i u\right\rangle = \|\mathrm{grad}(u)\|_{-1}^2\,.$$

Consequently,

$$P(\Omega)^{-1} = \inf_{\substack{u \in M(\Omega) \\ |u|_2 = 1}} (Tu, u)_2\,.$$

Important properties of this operator T are as follows.

THEOREM 1

1) T is a self-adjoint and coercive operator.

2) $P(\Omega)$ is the inverse of smallest spectral value of T.

3) $T - I$ is a harmonic mapping in $M(\Omega)$.

4) $\|T\| = 1$, 1 is a eigenvalue of T and his eigenspace is infinite dimensional.

Proof

(1) For all $(u, v) \in M(\Omega) \times M(\Omega)$ we have:

$$(Tu, v)_2 = \sum_{i=1}^{N} \left\langle \partial_i v \, , (-\Delta)^{-1} \circ \partial_i u \right\rangle = \big((\mathrm{grad}(u) \, , \, \mathrm{grad}(v))\big)_{-1} = (v, Tv)_2 \, ,$$

$$(Tu, u)_2 = \|\mathrm{grad}(u)\|_{-1}^2 \geq P(\Omega)^{-1} |u|_2^2 \, .$$

Then T is a self-adjoint and coercive operator and $P(\Omega)^{-1}$ is the best value of the coercivity constant.

(2) We denote by $\sigma(T)$ the spectrum of T. It follows from (1) [1] that the residual spectrum of T is empty, $\sigma(T)$ is closed and it lies in the closed interval $[\, m \, , \, M \,]$ on the real axis, where

$$m = \operatorname*{Inf}_{\lambda \in \sigma(T)} \lambda = \operatorname*{Inf}_{\substack{u \in M(\Omega) \\ |u|_2 = 1}} (Tu, u)_2 = P(\Omega)^{-1} \, ,$$

$$M = \operatorname*{Sup}_{\lambda \in \sigma(T)} \lambda = \operatorname*{Sup}_{\substack{u \in M(\Omega) \\ |u|_2 = 1}} (Tu, u)_2 = \|T\| \, .$$

So, $P(\Omega)^{-1} \in \sigma(T)$ and it is the smallest spectral value of T.

(3) For all $u \in M(\Omega)$ we have, in the sens of distributions on Ω:

$$\Delta \circ T(u) = -\sum_{i=1}^{N} \Delta \left(\partial_i \circ (-\Delta)^{-1} \circ \partial_i \right) (u) = \Delta u \, .$$

So, $T(u) - u$ is a harmonic distribution on Ω, thus it is a harmonic function on Ω, for all $u \in M(\Omega)$.

(4) Let $H(\Omega)$ be the closed subspace of $M(\Omega)$ of harmonic functions:

$$H(\Omega) = \left\{ u \in M(\Omega) \mid \Delta u = 0 \right\}$$

and we denote by $H(\Omega)^{\perp}$ the orthogonal complement of $H(\Omega)$ in $M(\Omega)$.

Let $v \in H(\Omega)^{\perp}$ be such that $v \neq 0$ and $w \in H(\Omega)$. It follows from (3) that

$$\left(T(v) - v\,,\, w\right)_2 = \left(v\,,\, T(w) - w\right)_2 = 0\,.$$

Then $T(v) - v \in H(\Omega)^{\perp} \cap H(\Omega)$, consequently $Tv = v$ and 1 is an eigenvalue of T.

Now let $\phi \in D(\Omega)$ be with $\Delta\phi \neq 0$, we have that $T(\Delta\phi) = \Delta\phi$. Then, the eigenspace corresponding to the eigenvalue 1 is infinite dimensional.

On the other hand, for all $v \in H(\Omega)^{\perp}$, $v \neq 0$, we have:

$$\left|v\right|_2^2 = \left(T(v), v\right)_2 \leq \|T\|\,\left|v\right|_2^2\,, \quad \text{from where } \|T\| \geq 1\,.$$

Moreover:

$$\|T\| = \sup_{\substack{u \in M(\Omega) \\ u \neq 0}} \frac{\left(T(u), u\right)_2}{\left|u\right|_2^2} = \sup_{\substack{u \in M(\Omega) \\ u \neq 0}} \frac{\left\|\operatorname{grad}(u)\right\|_{-1}^2}{\left|u\right|_2^2} \leq 1\,.$$

Consequently, $\|T\| = 1$. Finally [1],

$$\|T\| = \sup_{\lambda \in \sigma(T)} \lambda\,.$$

So, the eigenvalue 1 is the largest spectral value of T.

COROLLARY 3.— *If u is an eigenvector of T corresponding to an eigenvalue $\lambda \neq 1$, then u is a harmonic function on Ω.*

In the following section, we are going to give a method to approximate the constant $P(\Omega)$.

5. Approximation of the Poincaré type constant $P(\Omega)$

$M(\Omega)$ is separable. Let $\{\varepsilon_j \mid 0 < j < \infty\}$ be an orthonormal basis of $M(\Omega)$, we consider the sequence $\left\{M_K(\Omega)\right\}_{K>0}$ of finite dimensional subspaces of $M(\Omega)$ defined as follows: for all integer $K > 0$, $M_K(\Omega)$ is spanned by the family of vectors $\{\varepsilon_j \mid 0 < j \leq K\}$.

Then $M_K(\Omega) \subset M_{K+1}(\Omega)$, $\forall\ K > 0$, and for all $u \in M(\Omega)$, there exist a sequence $\left\{u_K\right\}_{K>0}$, $u_K \in M_K(\Omega)$, such that $\lim_{K\to\infty}\left|u - u_K\right|_2 = 0$.

For all integer $K > 0$, let us put

$$\alpha_K = \operatorname*{Inf}_{\substack{w \in M_K(\Omega) \\ w \neq 0}} \frac{\|\text{grad}(w)\|_{-1}^2}{|w|_2^2} = \operatorname*{Inf}_{\substack{w \in M_K(\Omega) \\ w \neq 0}} \frac{(T(w), w)_2}{|w|_2^2}.$$

Then $\alpha_K \geq P(\Omega)^{-1}$ and $\alpha_K \geq \alpha_{K+1}$, $\forall\, K > 0$. Thus, the sequence $\{\alpha_K\}_{K>0}$ is convergent and its limit α is such that $\alpha \geq P(\Omega)^{-1}$. On the other hand, let u be some element of $M(\Omega)$. Then, there exist a sequence $\{u_K\}_{K>0}$ with $u_K \in M_K(\Omega)$, such that if $K \to \infty$, $u_K \to u$ in $L^2(\Omega)$ and therefore $\text{grad}(u_K) \to \text{grad}(u)$ in $(H^{-1}(\Omega))^N$. Furthermore, we have $\|\text{grad}(u_K)\|_{-1}^2 \geq \alpha_K |u_K|_2^2$. Passing to limit when $K \to \infty$, we obtain $\|\text{grad}(u)\|_{-1}^2 \geq \alpha |u|_2^2$. This result implies that $\alpha \leq P(\Omega)^{-1}$ and consequently $P(\Omega)^{-1} = \alpha$.

Now, we are going to specify the elements of sequence $\{\alpha_K\}_{K>0}$.

Let K be some positive integer. Each $u \in M_K(\Omega)$ has a unique decomposition $u = \sum_{j=1}^{K} \lambda_j \varepsilon_j$. Put $\chi = Gu = (\lambda_1, \lambda_2, \ldots, \lambda_K) \in \mathbb{R}^K$.

In what follows, $M_K(\Omega)$ is equipped with the norm induced by $L^2(\Omega)$ and $\mathbb{R}^K$ is equipped with the usual euclidean norm $\|\cdot\|_0$. Then G is an isometric isomorphism from $M_K(\Omega)$ onto $\mathbb{R}^K$.

On the other hand,

$$(Tu, u)_2 = (T \circ G^{-1}\chi\,,\, G^{-1}\chi)_2 \geq P(\Omega)^{-1} \|\chi\|_0^2\,.$$

Therefore

$$\chi \to Q_K(\chi) = (T \circ G^{-1}\chi\,,\, G^{-1}\chi)_2 = \sum_{i,j=1}^{K} \lambda_i \lambda_j (T\varepsilon_i, \varepsilon_j)_2 \tag{6}$$

is a positive definite quadratic form on $\mathbb{R}^K$ and furthermore:

$$\alpha_K = \operatorname*{Inf}_{\substack{\chi \in \mathbb{R}^K \\ \chi \neq 0}} \frac{Q_K(\chi)}{\|\chi\|_0^2}\,.$$

So, we have proved the following result.

PROPOSITION 2.— *α_K is the smallest eigenvalue (minimum of a Rayleigh quotient) of the matrix A_K of the quadratic form defined on $\mathbb{R}^K$ by formula (6) and $\alpha_K \to P(\Omega)^{-1}$, when $K \to \infty$.*

6. Two particular cases

6.1 The open is the rectangle $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < L\,,\ 0 < y < \ell\}$

We use the previous results with $N = 2$.

We must now calculate the elements $(T\varepsilon_i, \varepsilon_j)_2$ of the matrix A_K. For all $(m,p) \in \mathbb{N}^2$, we put

$$\omega_m = \frac{m\pi}{L}\,, \quad \delta_p = \frac{p\pi}{\ell}\,, \quad \alpha_{m,p} = \omega_m^2 + \delta_p^2$$

and introduce real positive numbers $c_{m,p}$ defined by

$$c_{0,p}^2 = c_{m,0}^2 = \frac{1}{2}\, c_{m,p}^2 = \frac{2}{L\ell}\,, \quad \text{for } m \geq 1 \text{ and } p \geq 1\,.$$

We choose the following orthonormal basis of $M(\Omega)$:

$$\big\{e_{m,p}(x,y) = c_{m,p}\cos(\omega_m x)\cos(\delta_p y) \mid (x,y) \in \Omega\,,\ (m,p) \in \mathbb{N}^2\,, \\ m + p \geq 1\big\}\,.$$

The elements of the matrix A_K are the following real numbers:

$$(Te_{m,p}, e_{j,q})_2 \quad \text{with } (m,p) \in \mathbb{N}^2\,,\ (j,q) \in \mathbb{N}^2\,, \\ 0 \leq m,\ p,\ j,\ q \leq K\,,\ m + p \geq 1 \text{ and } j + q \geq 1\,.$$

More precisely, $(Te_{m,p}, e_{j,q})_2$ is the element of the $\big(m(K+1)+p\big)$-th row and $\big(j(K+1)+q\big)$-th columm of the matrix A_K.

In order to give explicit values of $(Te_{m,p}, e_{j,q})_2$, it is convenient to introduce $a(r,s)$ for $(r,s) \in \mathbb{N}^2$ with $r \geq 1$ and $s \geq 1$, given by

$$a(r,0) = a(0,s) = 1\,, \quad a(r,s) = \sqrt{2}\,.$$

The calculation of the scalar products $(Te_{m,p}, e_{j,q})_2$ is long but not very difficult. We obtain:

(i) $(Te_{m,p}, e_{j,q})_2 = 0$ if $m \neq j$ and $p \neq q$;

(ii) $(Te_{m,p}, e_{m,q})_2 = -\dfrac{\sqrt{2}\, a(p,q)\, \omega_m^3 \big(1 + (-1)^{p+q}\big) B_{m,p}}{\ell \alpha_{m,p} \alpha_{m,q}}$ if $p \neq q$,

(iii) $\left(Te_{m,p}, e_{j,p}\right)_2 = -\dfrac{\sqrt{2}\, a(m,j)\, \delta_p^3 (1 + (-1)^{m+j}) \Lambda_{m,p}}{L \alpha_{m,p} \alpha_{j,p}}$ if $m \neq j$,

(iv) $\left(Te_{m,p}, e_{m,p}\right)_2 = 1 - a(m,p)^2 \left(\dfrac{2\omega_m^3 B_{m,p}}{\ell \alpha_{m,p}^2} - \dfrac{2\delta_p^3 \Lambda_{m,p}}{L \alpha_{m,p}^2} \right)$,

where

$$B_{0,p} = 0\,,\; B_{m,p} = \frac{\exp(\omega_m \ell) + \exp(-\omega_m \ell) - 2(-1)^p}{\exp(\omega_m \ell) - \exp(-\omega_m \ell)}\,,\; \forall\, m \geq 1\,,\; \forall\, p \geq 0$$

$$\Lambda_{m,0} = 0\,,\; \Lambda_{m,p} = \frac{\exp(\delta_p L) + \exp(-\delta_p L) - 2(-1)^m}{\exp(\delta_p L) - \exp(-\delta_p L)}\,,\; \forall\, m \geq 0\,,\; \forall\, p \geq 1\,.$$

Remark 1.— A_K and consequently $P(\Omega)^{-1}$, depend only on the ratio of the dimensions of the rectangle.

Numerical results.— We have performed a few numerical tests. Let K be a positive integer. We have computed an approximate value of the smallest eigenvalue α_K of the matrix A_K by means of the power of Mises [2, pp. 226-227]. We stopped this calculatioin when the relative error was less than 10^{-9}. We have ascertained that sequence $\{\alpha_K\}_{K>0}$ converges quickly.

The above mentioned values of the constant $P(\Omega)^{-1}$ have been rounded up to the 3-th decimal place.

$$\begin{aligned} L = 1\,, &\quad \ell = 1 : P(\Omega)^{-1} = 0.226 \\ L = 2\,, &\quad \ell = 1 : P(\Omega)^{-1} = 0.151 \\ L = 4\,, &\quad \ell = 1 : P(\Omega)^{-1} = 0.047\,. \end{aligned}$$

6.2 Then open is $\Omega = \left\{(x_1, x_2, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} \mid x_1^2 + \cdots + x_{n+2}^2 < 1\right\}$ with $n \in \mathbb{N}$

In this case, we are able to give the spectrum of T. In that follows, the following proposition is essential.

PROPOSITION 3.— *Each harmonic homogeneous polynomial in Ω of degree $m \geq 1$ is an eigenvector of T corresponding to the eigenvalue $m/(2m+n)$.*

Proof.— Let $u(x_1, x_2, \ldots, x_{n+2})$ be a harmonic homogeneous polynomial in Ω of degree $m \geq 1$. We have

$$\sum_{i=1}^{n+2} x_i \partial_i u = mu .$$

Hence, for all $j = 1, 2, \ldots, n+2$,

$$\partial_j u + \sum_{i=1}^{n+2} x_i \frac{\partial^2 u}{\partial x_j \partial x_i} = m\, \partial_j u . \tag{7}$$

On the other hand:

$$\Delta\big((x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - 1)\partial_j u\big) = 2(n+2)\partial_j u + 4 \sum_{i=1}^{n+2} x_1 \frac{\partial^2 u}{\partial x_i \partial x_j} + \\ + x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - 1)\partial_j(\Delta u) ,$$

and from (7)

$$\begin{aligned} \Delta\big((x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - 1)\partial_j u\big) &= 2(n+2)\partial_j u + 4(m-1)\partial_j u \\ &= (2n + 4m)\partial_j u . \end{aligned} \tag{8}$$

Since the function $(x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - 1)\partial_j u \in D(\Omega)$, it follows from (8) that

$$(-\Delta)^{-1}(\partial_j u) = -\frac{1}{4m + 2n}(x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - 1)\partial_j u .$$

Therefore,

$$\begin{aligned} &- \operatorname{div} \circ (-\Delta)^{-1} \circ \operatorname{grad}(u) = \\ &\quad = \frac{1}{4m + 2n} \left(2 \sum_{j=1}^{n+2} x_j \partial_j u + (x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - 1)\Delta u \right) , \end{aligned}$$

$$Tu = \frac{1}{2m + n} \sum_{j=1}^{n+2} x_j \partial_j u = \frac{m}{2m + n} u .$$

We denote by $H_m(\Omega)$ the eigenspace of T corresponding to the eigenvalue $m/(2m + n)$ with $m \in \mathbb{N}^*$.

We recall that $H(\Omega)$ is the subspace of $M(\Omega)$ of harmonic functions and $H(\Omega)^\perp$ its orthogonal complement in $M(\Omega)$. We have the following result.

PROPOSITION 4 [3]. — *The family of harmonic homogeneous polynomials in Ω of degree $m \geq 1$ is free and total in $H(\Omega)$.*

PROPOSITION 5. — *The orthogonal complement $H(\Omega)^\perp$ of $H(\Omega)$ in $M(\Omega)$ is the eigenspace of T corresponding to the eigenvalue 1.*

Proof. — We denote by $H_{-n}(\Omega)$ the eigenspace of T corresponding to the eigenvalue 1. It is proved in theorem 1(4) that $H(\Omega)^\perp \subset H_{-n}(\Omega)$.

On the other hand, eigenvectors corresponding to different eigenvalues of T are orthogonal; from propositions 3 and 4 we have $H_{-n}(\Omega) \subset H(\Omega)^\perp$.

Then, $H_{-n}(\Omega) = H(\Omega)^\perp$, and consequently,

$$M(\Omega) = H_{-n}(\Omega) \oplus H(\Omega)\,.$$

COROLLARY 4. — *The only eigenvalues of T are 1 and $m/(2m+n)$ with $m \in \mathbb{N}^*$.*

T is a bounded self-adjoint operator on $M(\Omega)$. Its spectrum $\sigma(T)$ is partitioned into two disjoint sets: the point spectrum $\sigma_p(T)$ and the continuous spectrum $\sigma_c(T)$.

Now, we are going to specify two cases.

1) $n = 0,\ \Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$

In this case, the harmonic homogeneous polynomials of degree $m \geq 1$ are eigenvectors of T corresponding to the eigenvalue 1/2.

THEOREM 3. — *T has a pure point spectrum. 1/2 and 1 are the only eigenvalues of T.*

Proof. — It follows from corollary 4 that $\sigma_p(T) = \{1/2\,,\,1\}$.

Now, let $\alpha \in \mathbb{R}$ be different of 1/2 or 1, and $v \in M(\Omega)$. We have

$$v = v_{1/2} + v_0 \quad \text{with } v_{1/2} \in H_{1/2}(\Omega) \text{ and } v_0 \in H(\Omega)^\perp\,,$$

where $H_{1/2}(\Omega)$ is the eigenspace of T corresponding to the eigenvalue 1/2. We take

$$w = \frac{2}{1-2\alpha}\, v_{1/2} + \frac{2}{1-\alpha}\, v_0\,;$$

then $w \in M(\Omega)$, and we check that $(T - \alpha I)w = v$. Therefore, $\alpha \notin \sigma(T)$.

Remark 2.— The two eigenspaces of T are infinite dimensional.

2) $n \geq 1, \Omega = \{(x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} \mid x_1^2 + \cdots + x_{n+2}^2 < 1\}$

THEOREM 2.— $\sigma_p(T) = \{1, m/(2m+n) \mid m \in \mathbb{N}^*\}$ *and* $\sigma_c(T) = \{1/2\}$.

Proof.— From corollary 4,

$$\sigma_p(T) = \left\{1, \frac{m}{2m+n} \;\middle|\; m \in \mathbb{N}^*\right\}.$$

Now, let $\alpha \in \mathbb{R}$ be different of $1/2$, 1 or $m/(2m+n)$ with $m \in \mathbb{N}^*$, and $v \in M(\Omega)$. We have

$$v = \sum_{m=1}^{\infty} v_m + v_0 \quad \text{with } v_m \in H_m(\Omega) \text{ and } v_0 \in H(\Omega)^\perp .$$

We take

$$w = \sum_{m=1}^{\infty} \frac{2m+n}{m - \alpha(2m+n)}\, v_m + \frac{1}{1-\alpha}\, v_0 ;$$

then $w \in M(\Omega)$, and we check that $(T - \alpha I)w = v$. Therefore, $\alpha \notin \sigma(T)$.

Consequently, the limit $1/2$ of the eigenvalues is the only element of the continuous spectrum of the operator T.

COROLLARY 5.— *The eigenspace of T corresponding to the eigenvalue $m/(2m+n)$ is finite dimensional.*

Proof.— Let P_m be the vector subspace of $M(\Omega)$ spanned by the harmonic homogeneous polynomials of degree $m \geq 1$ ($\dim P_m < +\infty$). It follows from propositions 3 and 4 that $H(\Omega)$ is spanned by the family $\{P_m \mid m \geq 1\}$:

$$M(\Omega) = \bigotimes_{m \geq 1} P_m \oplus H(\Omega)^\perp .$$

On the other hand, suppose that there is an eigenvector u corresponding to the eigenvalue $\lambda_m = m/(2m+n)$, such that $u \notin P_m$. From corollary 3, $u \in H(\Omega)$. Since $u \perp H(\Omega)^\perp$, u can be written

$$u = \sum_{\substack{j \geq 1 \\ j \neq m}} u_j , \quad \text{where } u_j \in P_j .$$

Then

$$\lambda_m u = Tu = \sum_{\substack{j\geq 1\\ j\neq m}} \lambda_j u_j\,, \quad \text{with } \lambda_j = \frac{j}{2j+n}$$

and

$$\sum_{\substack{j\geq 1\\ j\neq m}} (\lambda_j - \lambda_m) u_j = 0\,.$$

But this is impossible because the family $\{u_j\}_{j\geq 1}$ is free.

Hence $u \in P_m$; thus P_m is the eigenspace of T corresponding to the eigenvalue $m/(2m+n)$ and the corollary is proved.

We recall that the eigenspace of T corresponding to 1 is infinite dimensional.

COROLLARY 6.— *If Ω is the ball $\Omega = \{(x_1, \ldots, x_N) \mid x_1^2, \ldots, x_N^2 < 1\}$ with $N \geq 2$, then $P(\Omega) = N$.*

Remark 3.— The spectrum of the operator T is independent of the radius of the ball in $\mathbb{R}^N$.

Remark 4.— Now we are going to give a family of eigenvectors of T which is total in $M(\Omega)$.

First we consider the orthogonal basis for $M(\Omega)$ formed by the eigenvectors of $-\Delta$ (in $L^2(\Omega)$) for the Neumann problem.

To write these functions, the appropriate polar co-ordinates are

$$\begin{aligned}
x_1 &= \rho\cos\theta_1\,,\\
x_2 &= \rho\sin\theta_1\cos\theta_2\,,\\
x_3 &= \rho\sin\theta_1\sin\theta_2\cos\theta_3\,,\\
&\ldots\\
x_{n+1} &= \rho\sin\theta_1\sin\theta_2\cdots\sin\theta_n\cos\varphi\,,\\
x_{n+2} &= \rho\sin\theta_1\sin\theta_2\cdots\sin\theta_n\sin\varphi\,;
\end{aligned}$$

with $0 \leq \varphi \leq 2\pi$, $0 \leq \theta_1, \theta_2, \ldots, \theta_n \leq \pi$ and $0 \leq \rho < 1$.

For simplicity, we put:

$$L^{\nu_{i+1}+(n-i)/2}_{\nu_i+(n-i)/2}(t) = P^{\nu_{i+1}+(n-i)/2}_{\nu_i+(n-i)/2}(t)\,, \quad t \in [0\,,\,1]\,,$$

if $n-i$ is a positive even integer and

$$L^{\nu_{i+1}+(n-i)/2}_{\nu_i+(n-i)/2}(t) = Q^{\nu_{i+1}+(n-i)/2}_{\nu_i+(n-i)/2}(t)\,, \quad t \in [0\,,\,1]\,,$$

if $n-i$ is a positive odd integer, where P^{η}_{μ} is the associate Legendre function of the first kind of order μ and degree η, and Q^{η}_{μ} is the associate Legendre function of the second kind of order μ and degree η [8].

In these polar co-ordinates this orthogonal basis for $M(\Omega)$ is the family of functions $\{\Phi^q_{\nu,k} \mid k \in \mathbb{N}\} \cup \{\Psi^q_{\nu,k} \mid k \in \mathbb{N}^*\}$ defined by

$$\begin{aligned}\Phi^q_{\nu,k} &= \rho^{-n/2} J_{\nu_1+n/2}(\lambda_{\nu_1+n/2,q}\rho) \times \\ &\times \left(\prod_{i=1}^{n} \left(L^{\nu_{i+1}+(n-i)/2}_{\nu_i+(n-i)/2}(\cos\theta_i)\left(\operatorname{cosec}\theta_i\right)^{(n-i)/2}\right)\right)\cos(k\varphi)\,,\end{aligned}$$

$$\begin{aligned}\Psi^q_{\nu,k} &= \rho^{-n/2} J_{\nu_1+n/2}(\lambda_{\nu_1+n/2,q}\rho) \times \\ &\times \left(\prod_{i=1}^{n} \left(L^{\nu_{i+1}+(n-i)/2}_{\nu_i+(n-i)/2}(\cos\theta_i)\left(\operatorname{cosec}\theta_i\right)^{(n-i)/2}\right)\right)\sin(k\varphi)\,.\end{aligned}$$

with $q \in \mathbb{N}^*$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}^n$, $k = \nu_{n+1} \leq \nu_n \leq \cdots \leq \nu_1$; where $J_{\nu_1+n/2}$ is the Bessel function of the first kind of order $\nu_1 + n/2$ [9], and $\lambda_{\nu_1+n/2,q}$ are the positive roots of equation

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^{-n/2} J_{\nu_1+n/2}(\lambda\rho)\right)\Big|_{\rho=1} = 0\,.$$

If we write the harmonic homogeneous polynomials of degree $\nu_1 \geq 1$ in these polar co-ordinates, we have

$$\begin{aligned}&u^{(1)}_{\nu,k}(\rho, \theta_1, \ldots, \theta_n, \varphi) = \\ &= \rho^{\nu_1}\left(\prod_{i=1}^{n}\left(L^{\nu_{i+1}+(n-i)/2}_{\nu_i+(n-i)/2}(\cos\theta_i)\left(\operatorname{cosec}\theta_i\right)^{(n-i)/2}\right)\right)\cos(k\varphi)\,, \quad k \in \mathbb{N}\,,\end{aligned}$$

$$u^{(2)}_{\nu,k}(\rho,\,\theta_1,\,\ldots,\,\theta_n,\,\varphi) = \\ = \rho^{\nu_1}\left(\prod_{i=1}^{n}\left(L^{\nu_{i+1}+(n-i)/2}_{\nu_i+(n-i)/2}\left(\cos\theta_i\right)\left(\operatorname{cosec}\theta_i\right)^{(n-i)/2}\right)\right)\sin(k\varphi)\,,\quad k\in\mathbb{N}^*$$

where $\nu = (\nu_1,\,\nu_2,\,\ldots,\,\nu_n) \in \mathbb{N}^n$, $k = \nu_{n+1} \le \nu_n \le \cdots \le \nu_1$. We say that this family of harmonic homogeneous polynomials is an orthogonal basis for $H(\Omega)$ and, as proved in proposition 3, they are eigenvectors of T corresponding to the eigenvalue $\nu_1/(2\nu_1+n)$.

Now we consider the functions $\Phi^q_{0,0}$, $\Phi^q_{\nu,k} - \gamma^q_{\nu,k}u^{(1)}_{\nu,k}$ and $\Psi^q_{\nu,k} - \delta^q_{\nu,k}u^{(2)}_{\nu,k}$, with $q\in\mathbb{N}^*$, $k\in\mathbb{N}^*$, $\nu = (\nu_1,\,\nu_2,\,\ldots,\,\nu_n)\in\mathbb{N}^n$, $k=\nu_{n+1}\le\nu_n\le\cdots\le\nu_1$; where

$$\gamma^q_{\nu,k} = \frac{(\Phi^q_{\nu,k}\,,\,u^{(1)}_{\nu,k})_2}{|u^{(1)}_{\nu,k}|^2_2} \quad\text{and}\quad \delta^q_{\nu,k} = \frac{(\Psi^q_{\nu,k}\,,\,u^{(2)}_{\nu,k})_2}{|u^{(2)}_{\nu,k}|^2_2}\,.$$

We obtain that these functions are eigenvectors of T corresponding to the eigenvalue 1.

Finally, we prove that the family of eigenvectors of T

$$\{u^{(1)}_{\nu,k}\,,\,u^{(2)}_{\nu,k}\,,\,\Phi^q_{0,0}\,,\,\Phi^q_{\nu,k}-\gamma^q_{\nu,k}u^{(1)}_{\nu,k}\,,\,\Psi^q_{\nu,k}-\delta^q_{\nu,k}u^{(2)}_{\nu,k}\}$$

with $q\in\mathbb{N}^*$, $k\in\mathbb{N}^*$, $\nu=(\nu_1,\,\nu_2,\,\ldots,\,\nu_n)\in\mathbb{N}^n$, $k=\nu_{n+1}\le\nu_n\le\cdots\le\nu_1$ is total in $M(\Omega)$.

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