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Pointwise estimates for the Poisson kernel on NA groups by the Ancona method(*)

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RÉSUMÉ. — Soit S un produit semi-direct d'un groupe homogène N et du groupe $A = \mathbb{R}^+$, N étant distingué et A opérant sur N par dilatations. On décrit la compactification de Martin pour une classe d'opérateurs sous-elliptiques et on donne des estimations supérieures et inférieures pour le noyau de Poisson sur N.

ABSTRACT. — Let S be a semi-direct product of a homogeneous group N and the group $A = \mathbb{R}^+$, acting on N by dilations. We describe the Martin compactification for a class of subelliptic operators on S and give sharp pointwise estimates for the corresponding Poisson kernel on N.

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1. Introduction

In this paper we investigate positive harmonic functions with respect to left-invariant operators on a class of solvable Lie groups S. The group S is a semi-direct product of a homogeneous group N and the group $A = \mathbb{R}^+$, acting on N by dilations $\{\delta_a\}_{a>0}$. We consider a left-invariant second order operator

$$L = X_1^2 + \cdots + X_m^2 + X_0 , \tag{1.1}$$

where $X_1, \ldots, X_m$ generate the Lie algebra s of S. The Poisson boundary for bounded L-harmonic functions has been fully described in [D], [DH1] (also in [R] for bounded functions F satisfying $F * \mu = F$ for a probability measure μ). Depending on X_0, it is trivial or it is the group N considered

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as an S-space $N = S/A$ together with a probability measure ν – the Poisson kernel. S acts on N by $sx = y\,\delta_a(x)$ where $s = ya$. ν is, in fact, a bounded, smooth function such that the formula

$$F(s) = \int_N f(sx)\nu(x)\,\mathrm{d}x \tag{1.2}$$

($\mathrm{d}x$ being the Haar measure on N), gives one-one correspondence between bounded L-harmonic functions F on S and L^∞ functions f on N. Moreover ν has a positive moment and is dominated at infinity by $\|x\|^{-\gamma}$ for some positive γ ([D], [DH1]).

In the present paper we go a step further and we give sharp pointwise estimates for the Poisson kernel ν. Our work is motivated on one hand by the results concerning Laplace–Beltrami operator on symmetric spaces ([Dy], [K], [G1]) and more generally on harmonic spaces [DR], and on the other, by the general approach to the Martin compactification for a class of operators on manifolds with negative curvature due to A. Ancona. By the result of E. Heintze [He] there is always a left-invariant Riemannian metric on S with the sectional curvature K satisfying $-a^2 < K < -b^2$ for some a, $b > 0$. Therefore by [A1] for an appropriate class of second order differential operators, the Martin boundary ∂M for the pair (S, L) is homeomorphic to the sphere at infinity S_∞ of S. This class incudes operators (1.1), whenever $\pi_A(X_0) \neq 0$, where $\pi_A(xa) = a$ is the canonical homomorphism S onto $A = S/N$. This is not made explicit in [A1], but easily derivable from the set of "axioms" formulated there.

What is of interest here is not so much the abstract description of ∂M as S_∞, but rather as the one point compactification of N and an identification of minimal positive functions for L in terms of ν. This is obtained by a careful examination of Ancona's boundary Harnack inequalities. For that purpose we have to understand them not only in terms of geodesics for a negative curvature metric but in a way which make explicit their connection with N and ν. This gives quite smoothly estimates for ν, identification of S_∞ as the one point compactification of N and minimal positive harmonic functions as translations and dilations of ν understood in a appropriate way. More precisely : the Martin boundary coincides with the set of minimal positive functions, which consists of the function $xa \to a^\alpha$, where α is a real number depending on L and of the functions $P_y(s) = P(y^{-1}s)$, $y \in N$, where $P(xa) = \nu(a^{-1}x^{-1}a)a^{-Q}$, $x \in N$, $a \in A$ and a^{-Q} is the determinant of the action $x \mapsto a^{-1}xa$.

In the proof we follow closely the "Φ-chain" method of Ancona [A1], but rather its abstract axiomatic formulation than the negative curvature one, and we do not define Φ-chains in terms of geodesic cones. In fact one can make use of any left-invariant Riemannian metric (not necessarily imposing negative sectional curvature) because all they are equivalent at ∞. This emphasises the role of the action of A on N and this action not curvature is important. This phenomenon is not surprising at all, because on one hand the choice of the group A is not canonical (we can always choose the algebra $\mathbf{a}$ of A being orthogonal to $\mathbf{n}$ – the algebra of N) and on the other, modifying the underlying scalar product by a constant on $\mathbf{a}$ we obtain the negative curvature object [He].

The Ancona's approach was presented by him during wonderful Tempus lectures held in Wroclaw in May 1993. The author is grateful to A. Ancona for detailed discussion of his method and, in particular, for showing how it works in the case of an operator

$$L = \sum_{i,j=0}^{n} a_{ij}(xa)a^{d_i+d_j}\partial_i\partial_j + \sum_{i=0}^{n} b_i(xa)a^{d_i}\partial_i \tag{1.3}$$

on $\mathbb{R}^n \times \mathbb{R}^+$ with the metric $\mathrm{d}s^2 = \sum_{i=0}^{n} a^{-2d_i}\,\mathrm{d}x_i^2$, which is a model situation for us. The operators (1.3) however do not cover our situation when N is abelian because of some further assumptions made in [A1] and also pontwise estimates for minimal harmonic functions are not derived there.

We also would like to express our gratitude to Andrzej Hulanicki, Martine Babillot, Yves Guivarc'h and John Taylor.

2. NA groups and invariant operators

Let $S = NA$ be a semi-direct product of a nilpotent group N and $A = \mathbb{R}^+$ acting on N by dilations, i.e. there is a basis $Y_1, \ldots, Y_n$ of the Lie algebra $\mathbf{n}$ of N and positive numbers $1 = d_1 \leq d_2 \leq \ldots \leq d_n$ such that the mappings

$$\delta_a\left(\exp\sum_{j=1}^{n} y_j Y_j\right) = \exp\left(\sum_{j=1}^{n} y_j a^{d_j} Y_j\right), \quad a \in A, \tag{2.1}$$

are automorphisms of N. Therefore the group structure in S is given by

$$xax_1a_1 = x\,\delta_a(x_1)aa_1\,. \tag{2.2}$$

N together with the family $\{\delta_a\}_{a>0}$ is called a homogeneous group. $Q = d_1 + \cdots + d_n$ is the homogeneous dimension of N. A homogeneous norm on N is a function

$$N \ni x \longmapsto |x| \in \mathbb{R}^+ ,$$

which is C^∞ outside $x = e$, satisfies $|\delta_a x| = a|x|$ and $|x| = 0$ if and only if $x = e$. Let $\mathbf{b}_r(x)$ be a ball with respect to $|\cdot|$ i.e.:

$$\mathbf{b}_r(x) = \left\{ y \in N : |x^{-1}y| < r \right\} .$$

We will also use a left-invariant Riemannian distance τ on S denoting balls with respect to τ by $B_r(s)$, i.e.:

$$B_r(s) = \left\{ w \in S : \tau(s^{-1}w) < r \right\} ,$$

where $\tau(s) = \tau(s, e)$. If W is a subset of S then by definition

$$\tau(s, W) = \inf_{w \in W} \tau(s, w) .$$

On the algebra level (2.2) becomes

$$[H , Y_j] = d_j Y_j , \quad j = 1, \ldots, n ,$$

for a basis H of the Lie algebra $\mathbf{a}$ of A. The choice of A is by no means unique. For any linear complement $\mathbf{a}' = \mathrm{lin}\langle H \rangle$ of $\mathbf{n}$ in the Lie algebra $\mathbf{s}$ of S there is a basis $Y_1', \ldots, Y_n'$ of $\mathbf{n}$ such that [DH2] :

$$[H , Y_j'] = d_j Y_j'$$

after an appropriate normalization of H. Therefore, S is a semi-direct product

$$S = NA'$$

of N and $A' = \exp \mathbf{a}'$ with $axa^{-1} = \delta_a(x)$ given by (2.1). Any decomposition S of type (2.2) will be called admissible.

We consider a left-invariant second order operator

$$L = X_1^2 + \cdots + X_m^2 + X_0 ,$$

where $X_1, \dots, X_m$ generate the Lie algebra $\mathfrak{s}$ of S. A simple calculation [DH2] shows that there is an admissible decomposition of S such that

$$Lf(xa) = \left((a\partial_a)^2 + \alpha a\partial_a + \sum_{i,j=1}^{n} \alpha_{ij} a^{d_i+d_j} Y_i Y_j + \sum_{i=1}^{n} \alpha_i a^{d_i} Y_i \right) f(xa), \tag{2.3}$$

where $x \in N$, $a \in A$, α, α_j, α_{ij} are real numbers and $[\alpha_{ij}] \geq 0$. The fact that the operator L can be written in the form (2.3) will be used in Lemma 4.1.

For a left-invariant vector field X on S and a distribution F on S, XF is defined by

$$\langle XF, \phi \rangle = -\langle F, X\phi \rangle, \quad \phi \in C_c^\infty(S). \tag{2.4}$$

Therefore,

$$L^* = X_1^2 + \cdots + X_m^2 - X_0$$

is the adjoint operator to L i.e.:

$$\langle LF, \phi \rangle = \langle F, L^*\phi \rangle. \tag{2.5}$$

A function $F \in L^1_{\mathrm{loc}}(m_R)$ is identified with the distribution $F\,\mathrm{d}m_R$, i.e. for ϕ, $\psi \in C_c(S)$,

$$\langle \phi, \psi \rangle = \int_S \phi(x)\psi(x)\,\mathrm{d}m_R(x),$$

where $\mathrm{d}m_R$ is a right Haar measure on S.

Let $\mu_t = p_t\,\mathrm{d}m_R$ be the convolution semi-group of measures with the infinitesimal generator L and

$$K_1 = (1-L)^{-1} = \int_0^\infty e^{-t}\mu_t\,\mathrm{d}t.$$

Since the right random walk with the law K_1 is transient [C],

$$G = \int_0^\infty \mu_t\,\mathrm{d}t = \sum_{n\geq 1} K_1^{*n}$$

is a Radon measure. Moreover, G does not have an atom at e. The density of G with respect to the right Haar measure will be denoted also by G, i.e.:

$$G = \int_0^\infty p_t\,\mathrm{d}t. \tag{2.6}$$

G is the fundamental solution of L i.e.:

$$LG = -\delta_e$$

as distributions. By the hypoellipticity of L, G is a smooth function on $S \setminus \{e\}$. Finally

$$G_y(x) = G(x,y) = G(y^{-1}x) \tag{2.7}$$

is the Green function for L on S with respect to the right Haar measure. Since $\check{\mu}_t$ is the semi-group generated by L^*,

$$G^*(x) = G(x^{-1})\,\Delta(x^{-1})\,,$$

where $\Delta = \mathrm{d}m_R/\mathrm{d}m_L$ is the modular function and

$$G^*(x,y) = G(y,x)\,\Delta(x^{-1}y)\,. \tag{2.8}$$

In this convention $G^*(x,y) \neq G(y,x)$. For every real β let

$$L'F = a^{-\beta}(L+\lambda I)(a^{\beta}F)\,.$$

Then

$$L'F = LF + 2\beta_a \partial_a F + (\beta^2 + \alpha\beta + \lambda)F\,. \tag{2.9}$$

Whenever $\lambda \leq \alpha^2/4$ we can find β such that $\beta^2 + \alpha\beta + \lambda = 0$ and so there is a Green function

$$G^{\lambda}(x,y) = \int_0^{\infty} e^{\lambda t} p_t(y^{-1}x)\,\mathrm{d}t$$

for $L + \lambda I$, $\lambda \leq \alpha^2/4$. Indeed, $G^{\lambda}(x,y) = a^{\beta}G'(x,y)$, where G' is the Green function for L'. If $\alpha \neq 0$ this means that there is a positive such λ i.e. the operator L is coercive in the Ancona sense [A1].

Existence of a Green function for $L + \lambda I$ for some positive λ is one of the main properties required by the method used here. Therefore we restrict our attention to the case $\alpha \neq 0$. Moreover, by (2.9),

$$a^{\alpha}L(a^{-\alpha}F) = LF - 2\alpha a \partial_a F\,,$$

which proves that in order to describe the Martin boundary for L we can assume $\alpha < 0$. In view of (2.9), having the Martin compactification in this case, one gets it immediately for the operators $L + cI$ with $0 < \alpha^2/4 - c$.

Let L be as in (2.3) with $\alpha < 0$. Then, the bounded L-harmonic functions ([D],[R]) are given by the formula

$$F(s) = \int_N f(sx)\nu(x)\,\mathrm{d}x\,, \quad f \in L^\infty(N)\,, \tag{2.10}$$

where $sx = yaxa^{-1}$, if $s = ya$, $y \in N$, $a \in A$ and ν is a positive, smooth, bounded function integrable with respect to the Haar measure $\mathrm{d}x$ on N, called the Poisson kernel [DH1]. ν is a weak$*$ limit of $\pi_N(\check{\mu}_t)$ if $t \to \infty$, where $\pi_N : S \to N$ is defined by $\pi_N(ya) = y$. We have

$$F(xa) = \int_N f(u)a^{-Q}\nu\big(a^{-1}(x^{-1}u)\big)\,\mathrm{d}u\,.$$

Therefore, for every $u \in N$ the function

$$P_u(xa) = P_e(u^{-1}xa) = a^{-Q}\nu\big(a^{-1}(x^{-1}u)\big) \tag{2.11}$$

is L-harmonic. Moreover, the fact that $\big\{a^{-Q}\nu\big(a^{-1}(x)\big)\big\}_{a>0}$ is an approximate identity in $L^1(N)$, when $a \to 0$ implies the following result.

PROPOSITION 2.1.— *For every $u \in N$,*

$$P_u(xa) = a^{-Q}\nu\big(a^{-1}(x^{-1}u)\big)$$

is a minimal function. If $u_1 \neq u_2$ then P_{u_1} and P_{u_2} are not proportional.

Proof.— In view of (2.11) it is enough to show that P_e is minimal. Let R be a positive L-harmonic function satisfying $R \leq P$. For every $f \in C_c^\infty(N)$, the function

$$H_f(s) = \int_N f(x)R(x^{-1}s)\,\mathrm{d}x \tag{2.12}$$

is harmonic and if $f = \phi * \psi$, ϕ, $\psi \in C_c(N)$, then

$$H_f(s) = \int_N \phi(x)H_\psi(x^{-1}s)\,\mathrm{d}x\,.$$

By [DH1] there is $h \in L^\infty(N)$ such that

$$H_\psi(s) = \int_N h(x)P_e(x^{-1}s)\,\mathrm{d}x$$

and $h \neq 0$ if $\psi \neq 0$. Therefore,

$$H_f(s) = \int_N \phi * h(x)P_e(x^{-1}s)\,\mathrm{d}x\,. \tag{2.13}$$

Since $\phi{*}h \in C_b(N)$ and $P_e(\,\cdot\, a) = a^{-Q}\nu\big(a^{-1}(\,\cdot\,)\big)$ is an approximate identity

$$\lim_{a\to 0} H_f(xa) = \phi * h(x)\,. \tag{2.14}$$

Let $\lambda = \liminf_{a\to 0}\int_N R(xa)\,\mathrm{d}x$. In view of (2.12) and (2.14), $\lambda > 0$. If $a_n \to 0$ is a sequence such that

$$\lambda_n = \int_N R(xa_n)\,\mathrm{d}x \longrightarrow \lambda_0\,,$$

then

$$\lim_{a_n\to 0} H_f(xa_n) = \lambda_0 f(x)\,.$$

Indeed, let U be an arbitrary neighbourhood of e in N, then

$$\begin{aligned}&\big|H_f(xa_n) - \lambda_n f(x)\big| \leq \\ &\qquad \leq \int_U \big|f(xy^{-1}) - f(x)\big| R(ya_n)\,\mathrm{d}y + 2\|f\|_{L^\infty}\int_{N\setminus U} R(ya_n)\,\mathrm{d}y\end{aligned}$$

and

$$\lim_{n\to\infty}\int_{N\setminus U} R(ya_n)\,\mathrm{d}y \leq \lim_{n\to\infty}\int_{N\setminus U} P_e(ya_n)\,\mathrm{d}y = 0\,.$$

Hence $\lambda_0 f = \phi * f$ and by (2.12), (2.13), $R = \lambda P_e$.

Finaly, the fact that for every neighbourhood U of e

$$\lim_{a\to 0}\int_U P(xa)\,\mathrm{d}x = 1$$

implies that P_{u_1} and P_{u_2} are not proportional for $u_1 \neq u_2$.

3. Potential theory for L

The shief of L-harmonic functions satisfies the axioms of Brelot [B]. In the sequel we shall use the standard terminology of Brelot's potential theory. The basic properties of L proved in [B] are the following ones.

PROPERTY 3.1.— *(Maximum principle) If $LF \geq 0$ on a domain Ω, then F can not attain its maximum in Ω unless F is constant.*

PROPERTY 3.2.— *The family of open sets which are Dirichlet regular is a basis of the topology of S.*

PROPERTY 3.3.— *(Harnack inequality) For an open set Ω, a compact set $K \subset \Omega$, a point $x_0 \in \Omega$ and a left-invariant differential operator X there is a constant C such that*

$$\sup_{xK} \left| XF(y) \right| \leq CF(xx_0)\,,$$

for every $x \in S$ and every nonnegative function F satisfying $LF = 0$ in $x\Omega$.

COROLLARY 3.4.— *For every $r > 0$, there is a constant $C(r)$ such that if $F > 0$, $LF = 0$ in $B_r(x)$ then*

$$\left| F(x) - F(y) \right| \leq C(r)\tau(x^{-1}y)F(x)\,,$$

whenever $\tau(x^{-1}y) < r/2$.

In view of (2.9) and Theorem 5.2 in [B] we can conclude that there is a basis $\mathcal{R}$ of open subsets of S which are Dirichlet regular with respect to all the operators $L+\lambda I$, $\lambda \leq \alpha^2/4$ and if $\lambda \geq 0$ we have the following inequality between harmonic measures corresponding to $L + \lambda I$ and L

$$\mu_x^\lambda \geq \mu_x\,.$$

From now on we assume that $\lambda \geq 0$.

Let Ω be an open subset of S. We are going to recall some properties of $L + \lambda I$-superharmonic functions on Ω, which are required by Ancona's method. They are consequences of § 3.1-3.3. For details we refer the reader to Brelot's potential theory as presented in [A2], [B1], [B2], [H], and [HH].

Let $\mathcal{S}_+^\lambda(\Omega)$ be the set of positive functions superharmonic in Ω with respect to $L+\lambda I$, $\lambda \geq 0$ nonequal identically to ∞. If $f \in \mathcal{S}_+^\lambda(\Omega)$ then f is locally integrable and $(L+\lambda I)f \leq 0$ as distributions [HH]. Conversely if $(L+\lambda I)f \leq 0$ then modyfing f possibly on a set of measure zero we obtain a function belonging to $\mathcal{S}_+^\lambda(\Omega)$. L-superharmonic functions satisfy the following minimum principle.

THEOREM 3.5.— *Let $D \subset \Omega$ be a closed set, $f \in \mathcal{S}_+(\Omega)$, p be an L-potential on Ω, harmonic in $\Omega \setminus D$, continuous in $\overline{\Omega \setminus D}$ and such that $f \geq p$ on ∂D. Then $f \geq p$ in $\Omega \setminus D$.*

For every open subset Ω there is a Green function $G_\Omega^\lambda(x,y)$ with respect to Ω given by

$$G_\Omega^\lambda(x,y) = G^\lambda(x,y) - H^\lambda(x,y)\,,$$

where $H^\lambda(\,\cdot\,,y)$ is the greatest harmonic minorant of $G^\lambda(\,\cdot\,,y)\restriction_\Omega$. $G_\Omega^\lambda(\,\cdot\,,y)$ is the only L-potential on Ω such that

$$(L+\lambda I)G_\Omega^\lambda(\,\cdot\,,y) = -\delta_y \tag{3.1}$$

as distributions.

Let μ be a positive measure on Ω. Then the Green potential of μ is defined by

$$G_\Omega^\lambda\mu(x) = \int_\Omega G_\Omega^\lambda(x,y)\,\Delta(y^{-1})\,\mathrm{d}\mu(y)\,,$$

where $\Delta = \mathrm{d}m_R/\mathrm{d}m_L$. Δ^{-1} inside the above formula matches with (2.4) and it is quite convenient here. Of course, putting Δ^{-1} or not does not change anything essentially. For a nonnegative function f on Ω we write $G_\Omega^\lambda f$ instead of $G_\Omega^\lambda f(f\,\mathrm{d}m_R)$, which gives

$$G_\Omega^\lambda f(x) = \int_\Omega G_\Omega^\lambda(x,y)f(y)\,\Delta(y^{-1})\,\mathrm{d}m_R(y) = \int_\Omega G_\Omega^\lambda(x,y)f(y)\,\mathrm{d}m_L(y)\,.$$

If $G_\Omega^\lambda\mu$ is nonequal to ∞ identically, then in view of (3.1), $G_\Omega^\lambda\mu$ is the only potential satisfying

$$(L+\lambda I)G_\Omega^\lambda\mu = -\mu$$

as distributions.

Moreover, for every $f \in \mathcal{S}_+^\lambda(\Omega)$, $f \geq 0$, $f \neq \infty$ idendically, there is a nonnegative Radon measure μ and a harmonic function h_f such that

$$f = G_\Omega^\lambda\mu + h_f\,.$$

We have also the following resolvent equation on Ω:

$$G_\Omega^\lambda = G_\Omega^\eta + (\lambda - \eta) G_\Omega^\lambda G_\Omega^\eta \,, \quad 0 \le \eta < \lambda \,,$$

where

$$G_\Omega^\lambda G_\Omega^\eta(x,z) = \int G_\Omega^\lambda(x,y) G_\Omega^\eta(y,z)\, \mathrm{d}m_L(y)$$

is the G_Ω^λ-potential of $G_\Omega^\eta(\,\cdot\,,z)$.

For the Ancona method some uniform estimates for the Green function on all balls of a given radius are crucial. In our case it is more appropriate to use, instead of balls, the family of neighbourhoods $\{xV\}_{x\in S}$, where V is a given set in $\mathcal{R}$. Assume that $B_r(e) \subset V$. Since L is left-invariant

$$G_{xV}^\lambda(y,z) = G_V^\lambda(x^{-1}y\,,\, x^{-1}z)$$

and so, there is a constant $c = c(L,r,V)$ such that for every x, every y, $z \in xV$ and every $0 \le \lambda \le \alpha^2/4$,

$$G_{xV}^\lambda(y,z) \ge c \qquad \text{if } y,\ z \in B_{r/2}(x)\,, \tag{3.2}$$

$$G_{xV}^\lambda(y,z) \le c^{-1} \quad \text{if } \tau(y,z) \ge r/4\,. \tag{3.3}$$

4. Submultiplicative property of the Green function

In this section we are going to formulate the main estimate of the Green function due to A. Ancona [A1], which is true also in the case of subelliptic operators provided we can solve the Dirichlet problem for L and $L + \lambda I$ in arbitrary large sets.

LEMMA 4.1.— *For every $R > 0$ there is an open set $\Omega \in \mathcal{R}$ such that $B_R(e) \subset \Omega$.*

Proof.— Let L be as in (2.3). Changing coordinates $xa = x\, e^t$ we obtain the operator

$$L = \sum_{i,j=1}^n \alpha_{ij}\, e^{(d_i+d_j)t} Y_i Y_j + \sum_{i=1}^n \alpha_i\, e^{d_i t} Y_i + \partial_t^2 + \alpha \partial_t$$

defined on $N \times R$ with $[\alpha_{ij}]$ being positive semidefinite. We put

$$\Omega = \left\{ (x,t) : \sum x_i^2 + \left(t + \frac{R}{2}\right)^2 < R^2 \,,\ t \geq 0 \right\}$$

$$\cup \left\{ (x,t) : \sum x_i^2 + \left(t - \frac{R}{2}\right)^2 < R^2 \,,\ t \leq 0 \right\}.$$

The normal direction to V, in the sense of Bony, is given by

$$\begin{aligned} &\sum x_i \partial_i + \left(t + \frac{R}{2}\right) \partial_t \,, \quad t \geq 0 \\ &\sum x_i \partial_i + \left(t - \frac{R}{2}\right) \partial \,, \quad t \leq 0 \,. \end{aligned} \tag{4.1}$$

Writing the second order part of L in partial derivatives we obtain

$$\sum \alpha_{ij}(x,t) \partial_i \partial_j + \partial_t^2 \,,$$

with $[\alpha_{ij}(x,t)]$ being positive semidefinite. Therefore, the corresponding quadratic form

$$\sum \alpha_{ij}(x,t) \xi_i \xi_j + \xi_0^2$$

is positive definite on normal vectors (4.1), which proves that the Dirichlet problem is solvable in Ω [B, Theorem 5.2].

Now we are going to write down a few inequalities between Green functions and harmonic measures for L and $L + \lambda I$, which lead to the main estimate (Theorem 4.6). For the rest of this section we fix $\lambda > 0$ such that G^λ exists. Let $V \in \mathcal{R}$ be a neighbourhood of e such that $B_r(e) \subset V \subset B_1(e)$. In view of Property 3.3 and (3.2), we have the following lemma.

LEMMA 4.2 [A1].— *Let $\Omega \in \mathcal{R}$ be such that $xV \subset \Omega$ and $G_\Omega(x,y)$, $G_\Omega^\lambda(x,y)$ the corresponding Green functions for L ans $L + \lambda I$ respectively. There is $0 < \delta = \delta(\lambda)$ such that for every $x \in S$ and every Ω,*

$$G_\Omega^\lambda(x,y) \geq (1+\delta) G_\Omega(x,y) \,,$$

whenever $y \in \Omega \setminus xB_{2r/3}$.

An inequality between Green functions G^λ_Ω and G_Ω implies the same inequality between harmonic measures, namely the following lemma.

LEMMA 4.3. — *Let $\Omega \in \mathcal{R}$ and*

$$G^\lambda_\Omega(x,y) \geq cG_\Omega(x,y)$$

for y outside a given neighbourhood U of x, $\overline{U} \subset \Omega$. Then

$$\mu^\lambda_x \geq c\mu_x\,, \tag{4.2}$$

where μ^λ_x, μ_x are harmonic measures on $\partial\Omega$ corresponding to $L+\lambda I$ and L respectively.

Proof. — Let $\phi \in C(\partial\Omega)$ and Φ be the solution of the Dirichlet problem for L in Ω with the boundary value ϕ. Assume that $\phi \geq 0$. Then $\Phi \geq 0$. Let $\psi \in C^\infty(\Omega) \cap C(\overline{\Omega})$, $\psi \geq 0$, $\psi = 0$ in $\overline{U}$, $\psi = 1$ on $\partial\Omega$ and $\Phi' = \eta\Phi$. Then

$$\Phi^\lambda = G^\lambda\big((L+\lambda I)\Phi'\big) + \Phi'$$

is the solution of the Dirichlet problem for $L+\lambda I$ in Ω with the boundary value ϕ. Since $\Phi' \geq 0$ and $\Phi'(x) = 0$,

$$\Phi^\lambda(x) \geq G^\lambda(L\Phi')\,.$$

But

$$\begin{aligned} G^\lambda(L\Phi') = \int_\Omega G^\lambda(x,y)L\Phi'(y)\,\mathrm{d}m_L(y) \geq c\int_{\Omega\setminus\overline{U}} G(x,y)L\Phi'(y)\,\mathrm{d}m_L(y) = \\ = cG(L\Phi')(x) = c\big(G(L\Phi')(x) + \Phi'(x)\big) = c\Phi(x)\,, \end{aligned}$$

which gives (4.2).

COROLLARY 4.4. — *Let x and Ω be as in Lemma 4.2. Then*

$$\mu^\lambda_x \geq (1+\delta)\mu_x\,.$$

Lemma 4.2 and Corollary 4.4 imply, as in [A1], the following properties.

LEMMA 4.5 [A1]. — *There are η, $C > 0$ such that whenever $\overline{B_{R+1}(x)} \subset \Omega$ then*

$$G_\Omega(x, y) \le C\, e^{-\eta R} G_\Omega^\lambda(x, y)\,,$$

for $y \in \Omega \setminus B(x, R)$. In particular for every x, y

$$G(x, y) \le C'\, e^{-\eta\tau(x,y)} G^\lambda(x, y)\,.$$

COROLLARY 4.6. — *Let x and Ω be as in Lemma 4.5. Then*

$$\mu_x \le C\, \epsilon^{-\mu R} \mu_x^\lambda\,.$$

Let $\Phi : [0, \infty) \mapsto [c_0, \infty)$, $\Phi(0) = c_0$ be a positive, increasing function such that $\lim_{t\to\infty} \Phi(t) = \infty$. By a *$\Phi$-chain* we mean a sequence of open sets $V_1 \supset V_2 \supset \ldots \supset V_m$ together with a sequence of points $x_i \in \partial\overline{V_i}$, $i = 1, \ldots, m$, such that for every i and every $z \in \partial V_{i+1}$

$$\tau(z, \partial V_i) \ge \Phi\big(\tau(z, x_{i+1})\big)$$

and

$$\tau(x_{i+1}, x_i) \le c_0^{-1}\,.$$

Notice that $S \setminus \overline{V_i} \subset \ldots \subset S \setminus \overline{V_m}$ together with $x_1, \ldots, x_m$ is also a Φ-chain. A sequence of points $x_1, \ldots, x_m$ is called a Φ-chain if there exist open subsets $V_1, \ldots, V_m$ with $x_i \in \partial\overline{V_i}$ and the condition above.

Now we are ready to formulate the main estimate of the Green function on Φ-chains, which in view of the above lemmas and corollaries, follows as in [A1].

THEOREM 4.7. — *Let $x_1, \ldots, x_m$ be a Φ-chain. Then there is a constant depending only on L, ϕ and λ fixed above, such that for every $1 < k < m$*

$$c^{-1} G(x_k, x_1) G(x_m, x_k) \le G(x_m, x_1) \le c G(x_k, x_1) G(x_m, x_k)\,. \tag{4.3}$$

In view of (2.8), Theorem 4.7 is true for G^* with the same constant.

5. The Martin compactification and the asymptotics for the Poisson kernel

We fix a homogeneous norm $|\cdot|$ in N and let K be a ball in $|\cdot|$, i.e. $K = \mathbf{b}_r(e)$ for some $r > 0$. We consider

$$T^K = \{ya \mid y \in K\,,\ a < 1\}\,,$$

and, for $q \in A$,

$$qT^K = \{qs \mid s \in T^K\} = \{ya \mid y \in \delta_q K\,,\ a < q\}\,.$$

We have the following lemma.

LEMMA 5.1.— *There is a linear function* $\Phi = \Phi(K,q) : [0,\infty) \to (c_0,\infty)$, $\Phi(0) = c_0 > 0$, *such that*

$$\tau(s\,,\,\partial qT^K) \geq \Phi\big(\tau(s)\big) \quad \textit{for } s \in \partial T^K\,, \tag{5.1}$$

i.e. e, q together with T^K, qT^K is a Φ-chain.

Proof.— To prove Lemma 5.1 one can proceed as in [A1, Lemma 25]. But here, since τ is a subadditive function on S, the situation is simpler. There is a constant c such that [G2]

$$c^{-1}\left(\log\big(1+|x|\big) + |\log a| + 1\right) \leq \tau(xa) + 1 \leq$$
$$\leq c\left(\log\big(1+|x|\big) + |\log a| + 1\right)\,.$$

Now (5.1) follows easily.

Since left translations are isometries, for every $s \in S$, $q \in A$,

$$s,\, sq,\, \ldots,\, sq^n \quad \text{together with} \quad sT^K,\, sqT^K,\, \ldots,\, sq^nT$$

is a Φ-chain with the above Φ (independent of s and n). For $s = xa \in S$ we write $a(s) = a$.

LEMMA 5.2.— *Let $q < 1$. Every $y \in \partial T^K$ and every $z \in \partial q^4 T^K$ can be joined by a Φ-chain passing trough y, q^2 and z for some Φ, which depends only on K and q.*

Proof.— We proceed as in [A1, Lemma (26)]. Let k be the largest integer such that

$$q < a(yq^{-k}) \leq 1$$

and ℓ the largest integer such that

$$q^5 < a(zq^{-\ell}) \leq q^4 .$$

In view of Lemma 5.1, for U sufficiently small,

$$yq^{-k}, \ldots, yq^{-1}, y \quad \text{with} \quad yq^{-k}T^U, \ldots, yq^{-1}T^U, yT^U$$

is a $\Phi = \Phi(U, q)$-chain contained in $S \setminus qT$. Analogously, for U sufficiently small

$$zq^{-\ell}, \ldots, zq^{-1}, z \quad \text{with} \quad zq^{-\ell}T^U, \ldots, zq^{-1}T^U, zT^U$$

is a $\Phi = \Phi(U, q)$-chain contained in q^3T^K. Now

$$zT^U, \ldots, zq^{-\ell}T^U, q^2T^K, S \setminus \overline{yq^{-k}T^U}, \ldots, S \setminus \overline{yT^U}$$

together with the distinguished points is a Φ-chain for an appropriate $\Phi = \Phi(K, U, q)$.

Now using the left translations we obtain the following corollary.

COROLLARY 5.3.— *There is $\Phi = \Phi(K, q)$ such that for every s, every $z \in \partial sT^K$ and every $y \in \partial sq^4T^K$ can be joined by a Φ-chain passing through z, sq^2 and y.*

Given $s = xb \in S$, $K = \mathbf{b}_r(e)$ and $q < 1$ we are going to consider the following family of neighbourhoods of x:

$$sT^K \supset sq^4T^K \supset \ldots \supset sq^{4n}T^K \supset \ldots \tag{5.2}$$

Every $z \in \partial sq^{4n}T^K$ and every $y \in \partial sq^{4(n+1)}T^K$ can be joined by a Φ-chain going through $u_n = sq^{4n+2}$, where Φ is the function in Corollary 5.3, i.e. is independent of s and n. The same property is satisfied by the family

$$S \setminus \overline{bT^K} \supset S \setminus \overline{bq^{-4}T^K} \supset \ldots \supset S \setminus \overline{bq^{-4n}T^K} \supset \ldots, \quad b \in A, \tag{5.3}$$

and the points $u_n = bq^{-4n-2}$, $n = 0, 1, \ldots$, which is a basis of neighbourhoods of the point ∞ in the one point compactification of $N \times [0, \infty)$. In what follows $V_1 \supset V_2 \supset \ldots \supset V_N \supset \ldots$ will denote any of the families

(5.2) or (5.3). Φ-chains (5.2) and (5.3) describe all possible limits of the Martin kernel, i.e. we have the following theorem, which in our context can be proved as it is proved in [A1].

THEOREM 5.4 [A1, Theorem 7].— *There is a minimal point ζ on the L-Martin boundary ∂S of S such that a sequence $\{s_j\} \subset S$ converges to ζ if and only if for each $n \geq 1$, $s_j \in V_n$ for j large enough.*

Theorem 5.4 implies, in fact, that the Martin compactification of (S, L) is the one point compactification of $N \times [0, \infty)$. The L-Martin boundary is $N \cup \{\infty\}$. Moreover we have the following boundary Harnack inequality.

THEOREM 5.5 [A1, proof of Theorem 7].— *Let f, g be positive L-harmonic functions on S such that for every n there is $z \in S$ with the property*

$$\sup_{y \notin V_n} \frac{f(y)}{G(y,z)} < \infty . \tag{5.4}$$

Then there is a constant independent of f, g and n such that

$$\frac{f(y)}{f(u_n)} \leq c \frac{g(y)}{g(u_n)} \quad \textit{for } y \notin V_n . \tag{5.5}$$

Let $K(\,\cdot\,,\zeta)$ be the limit of the Martin kernel

$$K(\,\cdot\,, s_n) = \frac{G(\,\cdot\,, s_n)}{G(e, s_n)}$$

when $s_n \to \zeta \in \partial S$. Assumption (5.4) is satisfied by $f(x) = K(x,\zeta)$. In view of Theorem 4.7 and Corollary 5.3 there is c such that for every $n \in \mathbb{N}$,

$$c^{-1} G(u_n, z) G(y, u_n) \leq G(y,z) \leq c G(u_n, z) G(y, u_n) , \tag{5.6}$$

whenever $y \in \partial V_n$, $z \in \partial V_{n+1}$. Applying Minimum principle 3.5 (for L and L^*) we obtain

$$c^{-1} G(y, u_n) \leq \frac{G(y,z)}{G(u_n, z)} \leq c G(y, u_n) \tag{5.7}$$

for $y \notin V_n$, $z \in \overline{V_{n+1}}$. Therefore,

$$K(y,\zeta) \leq c^2 \frac{G(y, u_n)}{G(e, u_n)} \quad \text{for } y \notin V_n \tag{5.8}$$

($n > 1$, to have e outside $\overline{V_n}$), which shows that $K(\,\cdot\,\zeta)$ satisfies (5.4).

Now we are going to identify the limits of the Martin kernel with the functions P_y defined in (2.11). As before for $s = xa \in S$ we write $a(s) = a$.

THEOREM 5.6.— *The Martin compactification of (S, L) is the point compactification of $N \times [0, \infty)$, the Martin boundary being the one point compactification of N. If $\alpha < 0$, $x_n \to x$ and $a_n \to 0$, then*

$$K(s\,,\, x_n a_n) \longrightarrow \frac{P_x(s)}{\nu(x)}\,,$$

and, if $|x_n| + a_n \to \infty$,

$$K(s\,,\, x_n a_n) \longrightarrow a(s)^{-\alpha}\,.$$

If $\alpha > 0$, the limits are

$$a(s)^{-\alpha} \frac{P_x(s)}{\nu(x)} = K(s, x) \quad \textit{and} \quad 1 = K(s, \infty)\,,$$

where ν and P_x are defined as above for $-\alpha$.

Proof.— To prove that $K(xa, \infty)$ does not depend on x, when $x \in N$ we consider

$$K(xs, a_n) = \frac{G(xs, a_n)}{G(e, a_n)} = \frac{G(s\,,\, a_n a_n^{-1} x^{-1} a_n)}{G(s, a_n)} \cdot \frac{G(s, a_n)}{G(e, a_n)}\,.$$

In view of Corollary 3.4,

$$\lim_{a_n \to \infty} \frac{G(s\,,\, a_n a_n^{-1} x^{-1} a_n)}{G(s, a_n)} = 1\,.$$

Therefore by Theorem 5.4,

$$K(xs, \infty) = K(s, \infty) \quad \text{for } x \in N\,,\ s \in S\,,$$

i.e. $K(s, \infty) = a(s)^{-\alpha}$. To identify $K(\,\cdot\,, e)$ we use the boundary Harnack inequality (5.5) with $f = K(\,\cdot\, e)$, $g = P_e$. More precisely, by (5.5),

$$K(xa, e) \leq cK(bq^2, e) \frac{P_e(xa)}{P_e(bq^2)}\,,$$

for $x \notin \mathbf{b}_b(e) = \delta_b \mathbf{b}_1(e)$ and all a. Therefore,

$$\int_{S\setminus \mathbf{b}_b(e)} K(xa,e)\,\mathrm{d}x \le c\frac{K(bq^2,e)}{P_e(bq^2)} \int_{S\setminus \mathbf{b}_b(e)} P_e(xa)\,\mathrm{d}x = $$
$$= c\frac{K(bq^2,e)}{P_e(bq^2)} \int_{S\setminus \mathbf{b}_{ba^{-1}}(e)} \nu(x)\,\mathrm{d}x \longrightarrow 0 \quad \text{when } a \to 0 \tag{5.9}$$

and so, $K(xa,e)$ is proportional to P_e, because P_y, for $y \neq e$ does not satisfy (5.9) for every b. The rest of the conclusion follows from the fact that $K(\,\cdot\,,x)$ is proportional to $K(x^{-1}\cdot\,,\,e)$.

Using boundary Harnack principle (5.5) we are able now to give precise pointwise estimates for P.

THEOREM 5.7.— *If α in (2.3) is negative then there is $c > 0$ such that*

$$c^{-1}\big(1+|x|\big)^{\alpha-Q} \le \nu(x) \le c\big(1+|x|\big)^{\alpha-Q}, \quad x \in N. \tag{5.10}$$

Proof.— Let f, g in (5.5) be P_e and $a(s)^{-\alpha}$ respectively, $n = 1$ and $V_1 = T^{\mathbf{b}_1(e)}$. Then there is a constant c such that

$$P_e(s) \le c a(s)^{-\alpha} \quad \text{for } s \notin V_1 .$$

In particular,

$$\nu\big(\delta_{a^{-1}}(x)\big) \le c a^{-\alpha+Q} \quad \text{for } |x| \ge 1 \text{ and } a \le 1 . \tag{5.11}$$

Now let $x = \delta_{|y|^{-1}} y$ and $a = |y|^{-1}$. Then by (5.11)

$$\nu(y) \le c|y|^{\alpha-Q} \quad \text{for } |y| \ge 1 .$$

Since ν is a continuous function [DH1], this implies the upper bound (5.10).

To obtain the lower bound we compare $f(s) = K(s,\infty)$ and $g = P_e$ inside $T^{\mathbf{b}_1(e)}$. By (5.5), there is c such that

$$a^{-\alpha} \le cP(s) \quad \text{for } s \in \overline{T^{\mathbf{b}_1(e)}} ,$$

i.e.:

$$a^{-\alpha} \le c a^{-Q} \nu\big(\delta_{a^{-1}}(x)\big)$$

for $|x| \leq 1$ and $a \leq 1$. In otherwords,

$$a^{-\alpha+Q} \leq c\nu\big(\delta_{a^{-1}}(x)\big)$$

for $|x| \leq 1$ and $a \leq 1$. As before, this together with the fact that ν is a positive continuous function, implies the lower bound (5.10).

Remark. — For

$$L = \big(a\partial_a\big)^2 + \alpha a\partial_a + a^2 \sum_{i,j}^{k} \alpha_{ij} Y_i Y_j$$

and $\alpha < 0$, the estimates (5.10) have been obtained in [S] by an other method.

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