ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6e série, tome 6, nº 2 (1997), p. 277-296

<http://www.numdam.org/item?id=AFST_1997_6_6_2_277_0>

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Annales de la Faculté des Sciences de Toulouse Vol. VI, n° 2, 1997

Interpolation by holomorphic functions in the unit ball with polynomial growth[*]

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RÉSUMÉ. — Nous nous occupons de deux problèmes concernant l'espace $A^{-\infty}$ des fonctions holomorphes à croissance polynomiale dans la boule unité de $\mathbb{C}^n$. D'un côté, nous donnons une condition suffisante pour qu'une suite dans la boule soit d'interpolation pour $A^{-\infty}$ et prouvons qu'une séparation faible est nécessaire pour ces suites. D'un autre côté, nous donnons des conditions sur une variété analytique X de dimension complexe $n-1$ pour que toute fonction de $A^{-\infty}(X)$ puisse être prolongée en une fonction de $A^{-\infty}$ dans la boule.

ABSTRACT. — We deal with two related problems for the space $A^{-\infty}$ of holomorphic functions with polynomial growth in the unit ball of $\mathbb{C}^n$. On the one hand we give a sufficient condition for a sequence in the ball to be $A^{-\infty}$-interpolating and prove that a weak separation in necessary for such sequences. On the other hand we give conditions on an analytic variety X of complex dimension $n-1$ so that every $A^{-\infty}(X)$ function can be extended to a $A^{-\infty}$ function in the ball.

0. Introduction

In this paper we want to study for the space of holomorphic functions with polynomial growth in $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ two classical problems: the interpolation on sequences of points and the extension of functions from $(n-1)$ dimensional complex analytic varieties.

(*) Reçu le 4 septembre 1995

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Partially supported by DGICYT grant PB92-0804-C02-02 and CIRIT grant 1996 SRG 00026

The space of holomorphic functions with polynomial growth, defined as

$$A^{-\infty} = \left\{ f \in H(\mathbb{B}_n) : \sup_{z\in\mathbb{B}_n} \frac{\log|f(z)|}{\log\left(\frac{e}{1-|z|}\right)} < +\infty \right\},$$

is the smallest algebra of holomorphic functions that contains the class H^∞ of bounded functions and is closed by differentiation. $A^{-\infty}$ can be viewed as the union of the spaces

$$A^{-p} = \left\{ f \in H(\mathbb{B}_n) : \|f\|_{A^{-p}} =: \sup_{z\in\mathbb{B}_n} (1-|z|)^p |f(z)| < +\infty \right\}, \quad p > 0,$$

as well as the union of the Bergman spaces

$$B^p = \left\{ f \in H(\mathbb{B}_n) : \int_{\mathbb{B}_n} |f(z)|^p \, dm(z) < +\infty \right\}, \quad p > 0.$$

A sequence of different points $\{a_j\} \subset \mathbb{B}_n$ is called $A^{-\infty}$-interpolating when every $A^{-\infty}$ function on $\{a_j\}_j$ can be extended to a $A^{-\infty}$ function in all $\mathbb{B}_n$. More precisely, $\{a_j\}_j$ is *$A^{-\infty}$-interpolating* if for any sequence $\{b_j\}_j$ in some space

$$\ell^{-p}(\{a_j\}) = \left\{ \{b_j\} : \|\{b_j\}\| := \sup_{j\in\mathbb{N}} (1-|a_j|)^p |b_j| < +\infty \right\}, \quad p > 0,$$

there exists $m > 0$ and $f \in A^{-m}$ such that $f(a_j) = b_j$ for all j. It is important to remark that the values p and m such that $\{b_j\} \in \ell^{-p}$ and $f \in A^{-m}$ need not be the same. The sequences for which m is required to coincide with p are called *A^{-p}-interpolating*, and will not be considered in this paper. A characterization of A^{-p}-interpolating sequences in case $n = 1$ can be found in [14] (see also [11] for some partial results for $n > 1$).

In dimension $n = 1$, the space $A^{-\infty}$ was deeply studied by Korenblum [10]. Using Korenblum's characterization of the $A^{-\infty}$ zero sets, Bruna and Pascuas gave a complete description of $A^{-\infty}$-interpolating sequences. In order to state this result denote by ϕ_a the automorphism of the disc commuting a and 0. A finite set F on the boundary of the unit disc is a *Carleson set* if

$$\mathcal{X}(F) := \frac{1}{2\pi} \int_0^{2\pi} \log\left(\frac{e}{d(e^{i\theta}, F)}\right) d\theta < +\infty,$$

where d denotes the Euclidian distance on $\partial\mathbb{B}_1$ normalized so that the distance between two diametrally opposite points is 1.

THEOREM [6, p. 455].— *Given $a \in (0,1)$, a necessary and sufficient condition for a sequence $\{a_j\}_j$ to be $A^{-\infty}$-interpolating is*

$$\exists\, c > 0 : \sum_{j:\phi_{a_j}(a_k)\in G_{F,a}} \log \frac{1}{|\phi_{a_j}(a_k)|} \le c\left(\mathcal{X}(F) + \log\left(\frac{e}{1-|a_k|}\right)\right) \quad \forall\, k \in \mathbb{N},$$

and for all Carleson sets F, where $G_{F,a} = \{0\} \cup \{z \in \mathbb{B}_1 : 1-|z| \ge a\, d(z/|z|\,, F)\}$.

This condition implies some separation between points of $\{a_j\}_j$ in terms of the *Gleason distance*

$$d_G(a,b) = \sup\left\{|f(b)| : f \in H^\infty(\mathbb{B}_n)\,,\ \|f\|_\infty \le 1\,,\ f(a) = 0\right\} = |\phi_a(b)|\,, \tag{1}$$

since by the properties of the automorphisms of the ball

$$1 - d_G^2(a,b) = 1 - |\phi_a(b)|^2 = \frac{(1-|a|^2)(1-|b|^2)}{|1-\overline{a}b|^2}\,. \tag{2}$$

The first part of the paper deals with the study of $A^{-\infty}$-interpolating sequences in case $n > 1$. We provide first a sufficient condition for a sequence to be $A^{-\infty}$-interpolating, which is formally the same that characterizes the radial $A^{-\infty}$-interpolating sequences in the unit disc [6, p. 458].

THEOREM 1.— *Let $\{a_j\}_j$ be a sequence in $\mathbb{B}_n$. If there exist $p > 0$ and $c > 0$ such that*

$$\prod_{j:j\neq k} d_G(a_j, a_k) = \prod_{j:j\neq k} |\phi_{a_j}(a_k)| \ge (1-|a_j|)^p\,, \quad \forall\, k \in \mathbb{N}\,,$$

then $\{a_j\}_j$ is $A^{-\infty}$-interpolating.

In [11, theorem 4] it is proved that every separated sequence $\{a_j\}_j$ (that is, $\delta > 0$ with $d_G(a_j, a_k) > \delta$ for every $a_j \neq a_k$) is $A^{-\infty}$-interpolating. The separation and the condition provided by this theorem are clearly of different nature.

Using (2) one can rewrite the condition above as

$$\prod_{j:j\neq k}\left(1-\frac{(1-|a_j|^2)(1-|a_k|^2)}{\left|1-\overline{a}_j a_k\right|^2}\right)\geq c^2(1-|a_j|^{2p}\,,\quad \forall\, k\in\mathbb{N}\,,$$

which obviously implies, for some constant $C>0$:

$$\sum_{j=1}^{\infty}1-\left|\phi_{a_j}(a_k)\right|^2=\sum_{j=1}^{\infty}\frac{(1-|a_j|^2)(1-|a_k|^2)}{\left|1-\overline{a}_j a_k\right|^2}\leq C\log\left(\frac{1}{1-|a_k|}\right)\,. \quad (3)$$

Theorem 1 and its proof are modelled after a result of Berndtsson on H^∞-interpolating sequences [4], and can be generalized to give sufficient conditions for the interpolation by holomorphic functions with certain controlled growth (see the end of section 1). Related problems for Bergman and Hardy spaces have been studied in [1] and [15].

We also prove the following necessary weak separation condition.

THEOREM 2.— *Let $\{a_j\}_j$ be $A^{-\infty}$-interpolating. There exist constants c, $m>0$ such that*

$$d_G(a_j,a_k)\geq c\max(1-|a_j|\,,\,1-|a_k|)^m\,.$$

A simple example shows that, in contrast to the A^{-p}-situation, one cannot expect any stronger separation. In view of this result, the following is natural.

THEOREM 3.— *Let $\{a_j\}_j$ and $\{b_k\}_k$ be $A^{-\infty}$-interpolating sequences such that for some m, $c>0$:*

$$d_G(a_j,b_k)\geq c\max(1-|a_j|\,,\,1-|b_k|)^m\,,\quad \forall\, j,\,k\in\mathbb{N}\,.$$

Then $\{a_j\}_j\cup\{b_k\}_k$ is also $A^{-\infty}$-interpolating.

The second part of the paper deals with another possible generalization to $n>1$ of the interpolation problem for sequences in the unit disc, namely the extension of holomorphic functions defined on analytic varieties of complex dimension $n-1$. Let X be an analytic variety of complex dimension $n-1$ in $\mathbb{B}_n$. Define

$$A^{-p}(X)=\left\{f\in H(X):\sup_{\zeta\in X}(1-|\zeta|^2)^p|f(\zeta)|<+\infty\right\}$$

and

$$A^{-\infty}(X) = \bigcup_{p>0} A^{-p}(X) .$$

THEOREM 4.— *Let u be a $A^{-\infty}$ function such that $|\partial u(z)| \geq c(1-|z|)^s$, for some c, $s > 0$, and let $X = \{z \in \mathbb{B}_n : u(z) = 0\}$. Then, for any $p > 0$ there exist $q > 0$ and a linear operator E from $A^{-p}(X)$ to $A^{-q}(\mathbb{B}_n)$ such that $Ef|_X = f$ for every $f \in A^{-p}(X)$. The value q depends only on s, p, n and the k for which $u \in A^{-k}$.*

With a stronger assumption on the smoothness of X at the boundary of $\mathbb{B}_n$ the extension can be performed within A^{-p}.

THEOREM 5.— *Let $u \in H(\mathbb{B}_n) \cap C^\infty(\overline{\mathbb{B}}_n)$ and consider $X = \{z \in \overline{\mathbb{B}} : u(z) = 0\}$. If $\partial u \neq 0$ in $X \cap \overline{\mathbb{B}}_n$ then there exists a linear operator E from $A^{-p}(X)$ to $A^{-p}(\mathbb{B}_n)$ such that $Ef|_X = f$.*

The proof of this last result is based on the extension scheme of Amar, which was used to prove an analogous extension result for bounded functions [2]. It must be pointed out that the conditions in theorem 5 allow to extend from $A^{-p}(X)$ to $A^{-p}(\mathbb{B}_n)$, independently of the transversality of X.

The paper is organized as follows. In section 1, we give the proof of theorem 1. Section 2 is devoted to prove theorems 2 and 3. The last section of the paper gives the proof of theorems 4 and 5.

1. Proof of theorem 1

Given a sequence $\{b_j\}_j$ in some $\ell^{-r}(\{a_j\})$, it will be enough to construct a family of holomorphic functions F_k such that:

(a) $F_k(a_j) = \delta_{jk}$;

(b) $\forall\, r > 0$, $\exists\, s$, $M > 0$:

$$\sum_{k=1}^{\infty} (1-|z|)^s (1-|a_j|)^{-r} |F_k(z)| \leq M , \quad \forall\, z \in \mathbb{B}_n ,$$

since from (a) it follows immediately that the function $F(z) := \sum_k b_k F_k(z)$ has value b_k in a_k, and (b) implies that $F \in A^{-\infty}$:

$$\begin{aligned}|F(z)| &\leq \sum_{k=1}^{\infty} |b_k|\,|F_k(z)| \leq \|\{b_j\}\|_{\ell^{-r}} \sum_{k=1}^{\infty} (1-|a_j|)^{-r} |F_k(z)| \\ &\leq \|\{b_j\}\|_{\ell^{-r}} (1-|z|)^{-s}.\end{aligned}$$

To prove (a) and (b) define

$$F_k(z) = \left(\frac{1-|a_k|^2}{1-\bar{a}_k z}\right)^4 w_1(a_k,z)\, w_2(a_k,z) \prod_{j:j\neq k} \frac{\phi_{a_j}(a_k)\cdot\phi_{a_j}(z)}{|\phi_{a_j}(a_k)|^2},$$

where $w_1(a_k,z)$ and $w_2(a_k,z)$ are weights holomorphic in z and with value 1 on the diagonal. Note that with this definition (a) is immediatelly satisfied. The hypothesis of the theorem yields, for any $k \in \mathbb{N}$, the estimate

$$|F(z)| \leq \left(\frac{1-|a_k|^2}{|1-\bar{a}_k z|}\right)^4 |w_1(a_k,z)|\,|w_2(a_k,z)| C (1-|a_j|)^{-2p}, \tag{4}$$

for some $C > 0$. The crucial point of the construction is this choice of the weights w_1 and w_2 so that F_k has the suitable growth. w_1 will be chosen later. As w_2 we take the weight used in [4]:

$$w_2(\zeta,z) = \exp\left\{-\sum_{m=1}^{\infty}\left(\frac{1+\bar{a}_m z}{1-\bar{a}_m z} - \frac{1+\bar{a}_m\zeta}{1-\bar{a}_m\zeta}\right)\frac{(1-|a_m|^2)(1-|\zeta|^2)}{1-|\bar{a}_m\zeta|^2}\right\},$$

which by (3) satisfies (see [4, p. 4]):

$$|w_2(a_k,z)| \leq (1-|a_j|)^{-2cp} \exp\left(-\sum_{m=1}^{\infty}\frac{1+|\bar{a}_m z|^2}{|1-\bar{a}_m z|^2}\,\frac{(1-|a_m|^2)(1-|a_k|^2)}{1-|\bar{a}_m a_k|^2}\right).$$

To make the notation more comfortable we assume that the sequence is enumerated so that $|a_k|$ increases. In [4, lemma 6], it also shown that if $|a_m| \geq |a_k|$ then

$$\frac{1+|\bar{a}_m z|^2}{1-|\bar{a}_m a_k|^2} \geq \frac{1}{8}\,\frac{1-|a_m|^2}{1-|\bar{a}_k z|^2}.$$

From this and the estimate above we easily get an analog of the lemma 7 in [4].

LEMMA 1.1.— *Let $h(t)=\min(1,t^{-2})$ defined for $t>0$. Then*

$$\left(\frac{1-|a_k|^2}{|1-\bar{a}_k z|}\right)^2 |w_2(a_k,z)| \le c_2(1-|a_j|)^{-2cp} h\left(\sum_{m\ge k}\left(\frac{1-|a_m|^2}{|1-\bar{a}_m z|}\right)^2\right).$$

From the lemma and (4) it follows now that:

$$\begin{aligned}|F_k(z)| \le C(1-|a_j|)^{-2p}|w_1(a_k,z)|\left(\frac{1-|a_k|^2}{|1-\bar{a}_k z|}\right)^2 \times \\ \times\, c_2(1-|a_j|)^{-2cp} h\left(\sum_{m\ge k}\left(\frac{1-|a_m|^2}{|1-\bar{a}_m z|}\right)^2\right),\end{aligned}$$

thus multiplying both sides by $(1-|z|)^s(1-|a_j|)^{-r}$ and defining $c'=2p(c+1)+r$ it follows that:

$$\begin{aligned}(1-|z|)^s(1-|a_j|)^{-r}|F_k(z)| \le C(1-|z|)^s(1-|a_j|)^{-c'}|w_1(a_k,z)| \times \\ \times \left(\frac{1-|a_k|^2}{|1-\bar{a}_k z|}\right)^2 h\left(\sum_{m\ge k}\left(\frac{1-|a_m|^2}{|1-\bar{a}_m z|}\right)^2\right).\end{aligned} \tag{5}$$

Let us now define $w_1(\zeta,z)$. Given a convex function φ, the weight

$$w(\zeta,z) =: \exp\left(-2\langle\partial\varphi(\zeta)\,,\,\zeta-z\rangle\right) = \exp\left(2\sum_{j=1}^n \frac{\partial\varphi}{\partial\zeta_j}(\zeta)(z_j-\zeta_j)\right)$$

is holomorphic in z and satisfies the estimate

$$|w(\zeta,z)| = \left|\exp\left(2\sum_{j=1}^n \frac{\partial\varphi}{\partial\zeta_j}(\zeta)(z_j-\zeta_j)\right)\right| \le e^{\varphi(z)}\,e^{-\varphi(\zeta)}.$$

We consider the convex function $\varphi(z)=c'\log(e/(1-|z|))$ and the associated weight $w_1(\zeta,z)$ according to this construction. Then $|w_1(a_k,z)| \le (1-|z|)^{-c'}(1-|a_j|)^{c'}$, and taking $s=c'$ we deduce from (5) that:

$$(1-|z|)^s|b_k|\,|F_k(z)| \le C\left(\frac{1-|a_k|^2}{|1-\bar{a}_k z|}\right)^2 h\left(\sum_{m\ge k}\left(\frac{1-|a_m|^2}{|1-\bar{a}_m z|}\right)^2\right).$$

LEMMA 1.2 [4, p. 6].— *Let $h(t)$ be a function defined in $(0,\infty)$ not increasing and positive, and let $\{c_k\}_k$ be a sequence of positive terms such that $\sum_k c_k < +\infty$. Then*

$$\sum_{k=1}^{\infty} c_k h\left(\sum_{m\geq k} c_m\right) \leq \int_0^{\infty} h(t)\,\mathrm{d}t\,.$$

The function $h(t) = \min(1, t^{-2})$ is not increasing and positive. On the other hand, denoting

$$c_k = \left(\frac{1-|a_k|^2}{|1-\bar{a}_k z|}\right)^2,$$

we have

$$\sum_k c_k \leq \frac{1}{(1-|z|)^2}\sum_k (1-|a_k|^2)^2\,,$$

and by (3) this sum is clearly finite. Finally, an application of lemma 1.2 yields (b), and the proof of the theorem is finished. □

Theorem 1 can be easily generalized to show that the conditon $\exists\; p,\, c > 0$:

$$\prod_{j:j\neq k} d_G(a_j, a_k) \geq c\,e^{-p\lambda(|a_k|)}\,, \quad \forall\; k \in \mathbb{N}\,,$$

is sufficient for a sequence $\{a_j\}_j$ to be A^λ-interpolating, where $\lambda : [\,0\,,\,1\,) \to \mathbb{R}^+$ is a positive increasing convex function and

$$A^\lambda = \left\{ f \in H(\mathbb{B}_n) : \sup_{z\in\mathbb{B}_n} \frac{\log|f(z)|}{\lambda(|z|)} < +\infty \right\}.$$

This is immediate after choosing the weight w_1 associated to the convex function $\varphi(z) = c'\lambda(|z|)$ and following the corresponding steps in the proof above.

2. Proofs of theorems 2 and 3

First we state the following lemma, which is a direct consequence of the open mapping theorem for (LF) spaces.

LEMMA 2.1 [12].— *If $\{a_j\}_j$ is an $A^{-\infty}$-interpolating sequence, then for every $p > 0$ there exist $m = m(p, \{a_j\}) \geq p$ and $C = C(p, \{a_j\})$ such that for any $\{b_j\}_j \in \ell^{-p}(p, \{a_j\})$ there is $f \in A^{-m}$ satisfying $f(a_j) = b_j$ for all j and $\|f\|_{A^{-m}} \leq C\|\{b_j\}\|_{\ell^{-p}}$.*

One can also prove the invariance under automorphisms of the $A^{-\infty}$-interpolating sequences. Although the image under any ϕ_z of an $A^{-\infty}$-interpolating sequence is again $A^{-\infty}$-interpolating, the associated constants of lemma 2.1 are not, in general, preserved ([6], [12]).

The proof of theorem 2 is an immediate consequence of the following lemma.

LEMMA 2.2.— *Let $\{a_k\}_k$ be a sequence in $\mathbb{B}_n$. If there exist numbers m, $c > 0$ and functions $f_j \in A^{-m}$ such that $f_j(a_k) = \delta_{jk}$ and $\|f_j\|_{A^{-m}} \leq c$ then*

$$d_G(a_j, a_k) \geq \frac{\max(1 - |a_j|\,,\, 1 - |a_k|)^m}{2^{m+2}c} \quad \textit{for any } j \neq k\,.$$

Proof.— For $|\zeta| < 1/2$ one has $|f_j(\zeta)| \leq c(1 - |\zeta|)^{-m} \leq 2^m c$, and by the Cauchy inequalities, also $|\partial f_j(\zeta)| \leq 2^{m+2}c$ for $|\zeta| < 1/4$.

Assume first $a_j = 0$ and take a_k with $d_G(a_j, a_k) = |a_k| < r$, for some $r < 1/4$. Then $1 = |f_j(a_j) - f_j(a_k)| < 2^{m+2}cr$, so $|a_k|$ can only be strictly smaller than r when $r > 1/2^{m+2}c$.

Let now a_j be an arbitrary point. Consider then the function $\psi_j = f_j \circ \phi_{a_j}^{-1}$ and the sequence $b_k = \phi_{a_j}(a_k)$. Immediately $\psi_j(b_k) = f_j(a_k) = \delta_{jk}$ and

$$\begin{aligned}
\|\psi_j\|_{A^{-m}} &= \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^m \left|f_j(\phi_{a_j}^{-1}(z))\right| = \sup_{z \in \mathbb{B}_n} \left(1 - |\phi_{a_j}(z)|^2\right)^m |f_j(z)| \\
&= \sup_{z \in \mathbb{B}_n} \frac{(1 - |z|^2)^m (1 - |a_j|^2)^m}{|1 - z\bar{a}_j|^{2m}} |f_j(z)| \leq \\
&\leq \frac{\|f_j\|_{A^{-m}}}{(1 - |a_j|^2)^m} \leq c(1 - |a_j|^2)^{-m}\,.
\end{aligned}$$

Using the estimate for the case $a_j = 0$ and the invariance under automorphisms of d_G, it follows finally

$$d_G(a_j, a_k) = d_G(\phi_{a_j}(a_j)\,,\, \phi_{a_j}(a_k)) = |b_k| \geq \frac{(1 - |a_j|^2)^m}{2^{m+2}c}\,.$$

The same inequality is true interchanging a_j and a_k. □

Proof of theorem 2

For any $j \in \mathbb{N}$ consider the sequence $\{w_k\}_k = \{\delta_{jk}\}_k$. Applying lemma 2.1 we obtain the conditions required in lemma 2.2, and the proof is finished. □

Remark.— An example given by Bruna and Pascuas for $n = 1$ shows that the weak separation condition of theorem 2 is optimal. A sequence $\{a_j\}_j$ is said to be *δ-uniformly separated* when

$$\inf_k \prod_{j:j\neq k} d_G(a_j, a_k) \geq \delta\,.$$

LEMMA [6, p. 460].— *Let $\{a_j\}_j$ be a δ-uniformly separated sequence.*

(a) Any sequence $\{a'_j\}_j$ such that $d_G(a_j, a'_j) \leq \delta/4$ for any $j \in \mathbb{N}$ is $\delta/4$-uniformly separated.

(b) $\displaystyle\prod_{j\in\mathbb{N}} d_G(z, a_j) \simeq d_G\left(z\,,\,\{a_j\}_j\right) =: \inf_j d_G(z, a_j)\,.$

Consider a fixed value $m > 0$ and a δ-uniformly separated sequence $\{a_j\}_j$. Pick another sequence $\{a'_j\}_j$ such that

$$0 < d_G(a_j, a'_j) \leq \frac{1}{8}\delta \quad\text{and}\quad d_G(a_j, a'_j) \simeq \left(1 - |a_j|\right)^m.$$

From the lemma one deduces that the union $\{a_j\} \cup \{a'_j\}$ satisfies the sufficient condition given by theorem 1, and therefore it is $A^{-\infty}$-interpolating. On the other hand, by construction $d_G(a_j, a'_j) \simeq \left(1 - |a_j|\right)^m$. This has to be contrasted with the A^{-p} situation, where the interpolating sequences are separated ([14], [11]).

In the proof of theorem 2 we have seen that when $\{a_j\}_j$ is interpolating there exists $m > 0$ such that for any $j \in \mathbb{N}$ there is $f_j \in A^{-m}$ with $f_j(a_k) = \delta_{jk}$ and $\|f_j\|_{A^{-m}} \leq c$. The converse is also true and will be used later on in the proof of theorem 3.

LEMMA 2.3.— *A sequence $\{a_j\}_j$ is $A^{-\infty}$-interpolating if and only if there exist p, m, $c > 0$ such that for any $j \geq 1$ there is $f_j \in A^{-m}$ with*

$$f_j(a_k) = \delta_{jk} \quad \textit{and} \quad \|f_j\|_{A^{-m}} \leq c\left(1 - |a_j|\right)^{-p}.$$

Proof.— The direct implication is immediate. To prove the reverse one notice first that the hypothesis can be self-improved, in the following sense: for any value $p' \geq 0$ there exist $m' > 0$ and functions g_j in $A^{-m'}$ such that $g_j(a_k) = \delta_{jk}$ and $\|g_j\|_{A^{-m'}} \leq c(1-|a_j|)^{p'}$. This is easily proven by considering the functions

$$g_j(z) = \left(\frac{1-|a_j|^2}{1-\bar{a}_j z}\right)^{p+p'} f_j(z).$$

Thus taking $p' = 0$ and applying lemma 2.2, we see that there exists $q > 0$ with $d_G(a_j, a_k) \geq c(1-|a_j|)^q$, where c is a constant which depends on q. This means that the hyperbolic balls $B_j = B\big(a_j\,,\, c(1-|a_j|)^q\big)$ are disjoint, and therefore the sum of their volumes is finite, that is

$$m\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} m(B_j) \simeq c^{n+1} \sum_{j=1}^{\infty} (1-|a_j|)^{q(n+1)} < +\infty. \qquad (6)$$

In order to show that $\{a_j\}_j$ is $A^{-\infty}$-interpolating take now $\{b_j\}_j \in \ell^{-s}(\{a_k\})$. As observed, there exist $m' > 0$ and functions $g_j \in A^{-m}$ with $g_j(a_k) = \delta_{jk}$ and $\|g_j\|_{A^{-m'}} \leq c(1-|a_j|)^{s+q(n+1)}$. Thus the series $\sum_j b_j g_j$ converges normally in A^{-m}, and by (6) :

$$\sum_{j=1}^{\infty} |b_j|\, \|g_j\|_{A^{-m'}} \leq c \left\| \{b_j\}_j \right\|_{\ell^{-s}} \sum_{j=1}^{\infty} (1-|a_j|)^{q(n+1)} < +\infty.$$

Hence $g = \sum_j b_j g_j$ is in $A^{-m'}$, for the convergence in the norm $\|\cdot\|_{A^{-m}}$ implies the punctual convergence, and it trivially verifies $g(a_j) = b_j$ for any $j \geq 1$. □

Before the proof of theorem 3 we need another lemma.

LEMMA 2.4.— *Let $\{a_j\}_j$ be $A^{-\infty}$-interpolating, and let $b \in \mathbb{B}_n$ such that $d_G(b, a_j) \geq \delta > 0$, for some $\delta > 0$ and for any $j \in \mathbb{N}$. Then there exist $p > 0$ and $f \in A^{-p}$ such that $f(b) = 1$, $f(a_j) = 0$ and $\|f\|_{A^{-p}} \leq c(\delta)(1-|b|)^{-p}$.*

Proof.— Suppose first that $b = 0$. By hypothesis $|a_j| \geq \delta$, so there exists a polydisc with radius $c_n\delta$ which does not contain any a_j. Separate the sequence $\{a_j\}_j$ in n subsequences $\{c_j^l\}_j$, $l = 1, \ldots, n$, such that the l-th coordinate is bounded below by $c_n\delta$, i.e. $|c_{j,l}^l| \geq c_n\delta$. Since

$$(1-|a_j|)\left|\frac{1}{c_{j,l}^l}\right| \leq \frac{1}{c_n\delta}, \quad \forall\, j \in \mathbb{N},$$

there exist $p > 0$ and $f_l \in A^{-p}$ with $f(a_j) = 1/c_{j,l}^l$. Without loss of generality we can assume that the value p is the same for every coordinate l. Besides, by the open mapping theorem there is a constant c such that $\|f_l\|_{A^{-p}} \leq c/c_n\delta$. Hence $g_l = 1 - z_l \cdot f_l$ belongs to A^{-p} and satisfies

$$g_l(0) = 1, \quad g_l(c_j^l) = 1 - c_{j,l}^l f_l(c_j^l) = 1 - c_{j,l}^l \frac{1}{c_{j,l}^l} = 0.$$

Define the product $g = g_1 \cdots g_n$. Trivially $g(0) = 1$ and $g(a_j) = 0$ for any j. Also $g \in A^{-pn}$ and $\|g\|_{A^{-pn}} \leq \|g_1\|_{A^{-p}} \cdots \|g_n\|_{A^{-p}} \leq c/\delta^n$.

Let now b be arbitrary. By the invariance under automorphisms of the Gleason distance $d_G(b, a_j) = d_G(\phi_b(b), \phi_b(a_j)) = d_G(0, \phi_b(a_j)) \geq \delta > 0$. Therefore, there exist $p > 0$ and $f \in A^{-p}$ such that

$$f(\phi_b(b)) = 1, \quad f(\phi_b(a_j)) = 0 \quad \text{and} \quad \|f\|_{A^{-p}} \leq c(\delta).$$

Defining $g = f \circ \phi_b$ we have $g(b) = 1$, $g(a_j) = 0$, and

$$\begin{aligned}\|g\|_{A^{-p}} &= \sup_{z\in\mathbb{B}_n} (1-|z|^2)^p |g(z)| = \sup_{z\in\mathbb{B}_n} \left(1 - |\phi_b(z)|^2\right)^p \left|g(\phi_b^{-1}(z))\right| \\ &= \sup_{z\in\mathbb{B}_n} \frac{(1-|z|^2)^p(1-|b|^2)^p}{|1-z\bar{b}|^{2p}} |f(z)| \leq \frac{c}{\delta^n(1-|b|^2)^p}. \;\square\end{aligned}$$

Proof of theorem 3

According to the hypothesis and lemma 2.4 for every b_k there exists f_k in A^{-p} with $f_k(b_k) = 1$, $f_k(a_j) = 0$ and $\|f\|_{A^{-p}} \leq C(c)(1-|b_k|)^{-(m+p)}$.

Since $\{b_k\}_k$ is $A^{-\infty}$-interpolating there exist functions g_k with $\|g_k\|_{A^{-p'}} \leq c$ and $g_k(b_j) = \delta_{jk}$. Then, the product function $f_k \cdot g_k$ is zero on $\{a_j\}_j$ and b_j for any $j \neq k$, and it is 1 on b_k:

$$(f_k \cdot g_k)(b_k) = 1, \quad (f_k \cdot g_k)(a_j) = 0, \quad \forall\, j \in \mathbb{N}$$

$$(f_k \cdot g_k)(b_j) = 0, \quad \forall\, j \neq k.$$

Furthermore

$$\|f_k g_k\|_{A^{-(p+p')}} \le \|f_k\|_{A^{-p}} \|g_k\|_{A^{-p'}} \le C(1-|b_k|)^{-(m+p)}.$$

In the same way we can construct functions $\widetilde{f}_k \cdot \widetilde{g}_k$ such that

$$(\widetilde{f}_k \cdot \widetilde{g}_k)(a_k) = 1, \quad (\widetilde{f}_k \cdot \widetilde{g}_k)(b_j) = 0, \quad \forall\, j \in \mathbb{N}$$
$$(\widetilde{f}_k \cdot \widetilde{g}_k)(a_j) = 0, \quad \forall\, j \neq k,$$

and

$$\|\widetilde{f}_k\, \widetilde{g}_k\|_{A^{-(p+p')}} \le C(1-|a_k|)^{-(m+p')}.$$

Denoting by m' the maximum of $m+p$ and $m+p'$ we get $\|f_k g_k\|_{A^{-(p+p')}} \le C(1-|a_k|)^{-m'}$, and the corresponding estimate for $\widetilde{f}_k \widetilde{g}_k$. Thus, by lemma 2.3, $\{a_j\}_j \cup \{b_k\}_k$ is $A^{-\infty}$-interpolating. □

3. Extension results. Proof of theorems 4 and 5

The proof of theorem 4 relies on the following extension result due to Berndtsson.

THEOREM 3.1 ([3, p. 405]).— *Let X be an analytic variety in the unit ball defined by a $C^1(\overline{\mathbb{B}}_n)$ function u. If $f \in H(X) \cap C^1(X)$, then*

$$F(z) = c_{n,p} \int_X f(\zeta) \left(\frac{1-|\zeta|^2}{1-\overline{\zeta} z} \right)^{N+n-1} \left[\partial\overline{\partial} \log \left(\frac{1}{1-|\zeta|^2} \right) \right]^{n-1} \wedge \nu$$

defines a holomorphic function such that $F\big|_X = f$.

As it is mentioned in Berndtsson's paper, the latter expression provides an extension of f whenever the integral converges. The measure ν is given by

$$\nu = g \wedge \overline{\partial} \left(\frac{1}{u} \right) = \frac{g \wedge \overline{\partial u}}{\|\partial u\|^2}\, \mathrm{d}\mu(\zeta),$$

where $\mathrm{d}\mu$ is the surface measure on X and the form $g = \sum_i g_j\, \mathrm{d}\zeta_j$ has coefficients

$$g_j(\zeta, z) = \int_0^1 \frac{\partial u}{\partial \zeta_j} (\zeta + t(z-\zeta))\, \mathrm{d}t.$$

Assuming $u \in A^{-k}$, it follows from the Cauchy inequalities that $\sup_{z\in\mathbb{B}_n}(1-|z|)^{k+1}|\partial u(z)| < +\infty$. Hence

$$\begin{aligned}
|g_j(\zeta,z)| &\leq \int_0^1 \left|\frac{\partial u}{\partial\zeta_j}(\zeta+t(z-\zeta))\right| \mathrm{d}t \leq \\
&\leq c\int_0^1 \frac{\mathrm{d}t}{(1-|\zeta+t(z-\zeta)|)^{k+1}} \leq \\
&\leq c\max\left(\frac{1}{1-|\zeta|}, \frac{1}{1-|z|}\right)^{k+1} \simeq \\
&\simeq \frac{(1-|\zeta|)^{k+1}+(1-|z|)^{k+1}}{(1-|\zeta|)^{k+1}(1-|z|)^{k+1}},
\end{aligned}$$

which implies

$$\frac{|g\wedge\overline{\partial u}|}{\|\partial u\|^2} \leq \frac{\|g\|}{\|\partial u\|} \preceq \frac{(1-|\zeta|)^{k+1}+(1-|z|)^{k+1}}{(1-|\zeta|)^{k+s+1}(1-|z|)^{k+1}}.$$

Take the extension given by theorem 3.1, with a constant N that will be fixed later. From the estimate above and the equality

$$\partial\overline{\partial}\log\left(\frac{1}{1-|\zeta|^2}\right) = \sum_{j,k=1}^{n} \frac{\delta_{jk}(1-|\zeta|^2)+\zeta_j\overline{\zeta}_k}{(1-|\zeta|^2)^2}\,\mathrm{d}\zeta_k\wedge\mathrm{d}\overline{\zeta}_j\,, \quad .$$

it follows that:

$$\begin{aligned}
&|F(z)| \leq \\
&\leq c\int_X |f(\zeta)|\left(\frac{1-|\zeta|^2}{|1-\overline{\zeta}z|}\right)^{N+n-1} \frac{(1-|\zeta|)^{k+1}+(1-|z|)^{k+1}}{(1-|\zeta|)^{2n+k+s-1}(1-|z|)^{k+1}}\,\mathrm{d}\mu(\zeta) \preceq \\
&\preceq \int_X \frac{(1-|\zeta|)^{N-n-p-s+1}}{|1-\overline{\zeta}z|^{N+n-1}(1-|z|)^{k+1}}\,\mathrm{d}\mu(\zeta) + \\
&\qquad + \int_X \frac{(1-|\zeta|)^{N-n-p-k-s}}{|1-\overline{\zeta}z|^{N+n-1}}\,\mathrm{d}\mu(\zeta) + \,.
\end{aligned}$$

With N so big enough and $q = 2n+p+s+\varepsilon-1$, for some $\varepsilon > 0$, we get finally

$$(1-|z|)^{k+1+q}|F(z)| \preceq \int_X (1-|\zeta|)^{1+\varepsilon}\,\mathrm{d}\mu(\zeta)\,.$$

For any variety defined by a $A^{-\infty}$ function this last integral is finite (see [8] or [7]), so the proof is finished. □

The proof of theorem 5, modelled after the result of Amar on extension of bounded holomorphic functions, is more technical.

LEMMA 3.2 [2, p. 28].— *Let $X = \{z \in \mathbb{B}_n : u(z) = 0\}$ be as in theorem 5, $f \in H(X)$ and let H be a distribution in $\mathbb{B}_n$ such that $\overline{\partial} H = f \overline{\partial}(1/u)$. Then $S = u \cdot H$ is a holomorphic extension of f in $\mathbb{B}_n$.*

In order to solve the $\overline{\partial}$-equation with datum $f\overline{\partial}(1/u)$ we consider the weighted solutions given by Berndtsson and Andersson [5], whose construction we now briefly recall.

Let $s = (s_1, \dots, s_n) : \overline{\mathbb{B}}_n \times \overline{\mathbb{B}}_n \to \mathbb{C}^n$ be the Cauchy–Leray section $s(\zeta, z) = \overline{\zeta}(1 - \zeta \cdot \overline{z}) - \overline{z}(1 - |\zeta|^2)$. Define $Q(\zeta) = \overline{\zeta}/\big(|\zeta|^2 - 1\big)$ and consider the 1-forms associated to Q and s:

$$Q = \sum_j Q_j(\mathrm{d}\zeta_j - \mathrm{d}z_j) \quad \text{and} \quad s = \sum_j s_j(\mathrm{d}\zeta_j - \mathrm{d}z_j)\,.$$

THEOREM 3.3 [5, p. 103].— *Define*

$$K^N = -c_n \left(\frac{1 - |\zeta|^2}{1 - \overline{\zeta} z}\right)^N \sum_{k=1}^{n} \gamma_k \left(\frac{1 - |\zeta|^2}{1 - \overline{\zeta} z}\right)^k \frac{s \wedge (\mathrm{d}Q)^k \wedge (\overline{\partial} s)^{n-1-k}}{\langle s\,,\, z - \zeta\rangle^{n-k}}\,,$$

where c_n and γ_k are certain positive constants depending respectively on n and k. Denote by $K^N_{p,q}$ the component of K^N with degrees (p, q) in z and $(n - p\,, n - q - 1)$ in ζ. Given $q \geq 1$ and a $\overline{\partial}$-closed (p, q)-form f with $\mathcal{C}^2$ coefficients, a solution of the equation $\overline{\partial} u = f$ is given by

$$u = c_{p,q,n} \int_{\mathbb{B}_n} f \wedge K^N_{p,q}\,.$$

Take $H = c \int_{\mathbb{B}_n} f\overline{\partial}(1/u) \wedge K^p_{0,0}$ as solution of $\overline{\partial} H = f\overline{\partial}(1/u)$. Observe that in Berndtsson and Andersson's theorem the forms are required to have $\mathcal{C}^2$ coefficients. This small difficulty is overcome just by taking the usual regularization.

The kernel $K^p_{0,0}$ has the decomposition $K^p_{0,0} = B^p + C^p \wedge \overline{\partial}\rho$, where

$$B^p \simeq \left(\frac{1-|\zeta|^2}{1-\overline{\zeta}z}\right)^p \frac{b^{n-1}}{d^n} \left((\rho\overline{z}\cdot \mathrm{d}\zeta + b\overline{\zeta}\cdot \mathrm{d}\zeta)\wedge \left(\overline{\partial}\partial\rho\right)^{n-1}\right)$$

$$C^p \simeq \left(\frac{1-|\zeta|^2}{1-\overline{\zeta}z}\right)^p \frac{b^{n-1}}{d^n} \left(\left(\overline{\partial}\partial\rho\right)^{n-2}\wedge (\overline{z}\cdot \mathrm{d}\zeta \wedge \overline{\zeta}\cdot \mathrm{d}\zeta)\right)$$

and ρ, b and d are defined by

$$\rho(\zeta) = \left|\zeta\right|^2 - 1\,,\ \ b(\zeta,z) = 1-\overline{\zeta}z\,,\ \ d(\zeta,z) = \left|1-\zeta\overline{z}\right|^2 - \left(1-\left|\zeta\right|^2\right)\left(1-\left|z\right|^2\right).$$

We use the decomposition of $K^p_{0,0}$ to separate the extending function S in two terms having the required growth. Define

$$S_1(z) = u(z)\int_X B^p(z,\zeta)\wedge f(\zeta)\,\overline{\partial}\left(\frac{1}{u}\right)(\zeta)$$

and

$$S_2(z) = u(z)\int_X C^p(z,\zeta)\wedge \overline{\partial}\rho(\zeta)\wedge f(\zeta)\,\overline{\partial}\left(\frac{1}{u}\right).$$

In order to carry out the estimates for S_1 and S_2, we use the non-isotropic pseudo-distance $\varrho(\zeta,z) = \left|\overline{\zeta}\cdot(\zeta-z)\right| + \left|\overline{z}\cdot(\zeta-z)\right| + \left|\zeta-z\right|^2$, which verifies the well-known estimates

$$\left|\zeta-z\right|^2 \le \varrho(\zeta,z) \le \left|\zeta-z\right|, \quad \left|1-\overline{\zeta}z\right| \simeq 1-|z| + \varrho(\zeta,z)$$

$$d(\zeta,z) \simeq \varrho^2(\zeta,z) + (1-|\zeta|)\left|\zeta-z\right|^2 \simeq \left(\varrho(\zeta,z) + (1-|z|)^{1/2}\left|\zeta-z\right|\right)^2.$$

We also have the obvious equivalences

$$|\overline{z}\cdot \mathrm{d}\zeta\wedge\overline{\zeta}\cdot \mathrm{d}\zeta| \simeq |\zeta-z| \quad\text{and}\quad |\rho\overline{z}\cdot \mathrm{d}\zeta + b\overline{\zeta}\cdot \mathrm{d}\zeta| \simeq (1-|z|^2)|\zeta-z| + \varrho(\zeta,z).$$

Denote by $\mathrm{d}\mu(\zeta)$ the Lebesgue measure on X. Since $\overline{\partial}(1/u) = \left(\overline{\partial u}/\left\|\partial u\right\|^2\right)\mathrm{d}\mu$ and $\partial u \ne 0$ on $\overline{\mathbb{B}}_n \cap X$ then $\left|\overline{\partial}(1/u)\wedge\overline{\partial}\rho\right| \simeq |\partial u\wedge\partial\rho|$ and $\left|\overline{\partial}(1/u)\right| \simeq \mathrm{d}\mu$. So:

$$\begin{aligned}
&(1-|z|)^p\left|S_1(z)\right| \preceq \\
&\preceq \left|u(z)\right|\int_X \frac{(1-|\zeta|^2)^p(1-|z|^2)^p}{\left|1-\overline{\zeta}z\right|^p}\,\frac{\left|1-\overline{\zeta}z\right|^{n-1}}{d^n(\zeta,z)} \times \\
&\qquad \times \left((1-|z|)|\zeta-z| + \varrho(\zeta,z)\right)\left|f(\zeta)\right|\mathrm{d}\mu(\zeta) \\
&\preceq \left|u(z)\right|\int_X \frac{\left|1-\overline{\zeta}z\right|^{n-1}}{d^n(\zeta,z)}\left((1-|z|)|\zeta-z| + \varrho(\zeta,z)\right)\mathrm{d}\mu(\zeta).
\end{aligned}$$

In the same way

$$(1-|z|)^p|S_2(z)| \preceq |u(z)| \int_X \frac{\left|1-\bar{\zeta}z\right|^{n-1}}{d^n(\zeta,z)} |\zeta - z| \left|\partial u(\zeta) \wedge \partial\rho(\zeta)\right| \mathrm{d}\mu(\zeta) .$$

We only prove that these two terms are uniformly bounded in case $n=2$. The general case can be carried out without essential changes. First, from the estimates above it follows that

$$\begin{aligned}
&(1-|z|)^p|S_1(z)| \preceq \\
&\preceq |u(z)| \int_X \frac{(1-|z|+\varrho(\zeta,z))\Big((1-|z|)|\zeta-z|+\varrho(\zeta,z)\Big)}{\Big(\varrho(\zeta,z)+(1-|z|)^{1/2}|\zeta-z|\Big)^4} \mathrm{d}\mu(\zeta) \\
&\preceq |u(z)| \left\{ \int_X \frac{\mathrm{d}\mu(\zeta)}{\varrho^2(\zeta,z)} + (1-|z|) \int_X \frac{\mathrm{d}\mu(\zeta)}{\Big(\varrho(\zeta,z)+(1-|z|)^{1/2}|\zeta-z|\Big)^3} \right\} .
\end{aligned}$$

and defining $m(\zeta) = \left|\partial\rho(\zeta) \wedge \partial u(\zeta)\right|$, also

$$\begin{aligned}
&(1-|z|)^p|S_2(z)| \preceq \\
&\preceq |u(z)| \left\{ \int_X \frac{m(\zeta)\,\mathrm{d}\mu(\zeta)}{\varrho^{5/2}(\zeta,z)} + (1-|z|) \int_X \frac{|\zeta-z| m(\zeta)\,\mathrm{d}\mu(\zeta)}{\Big(\varrho(\zeta,z)+(1-|z|)^{1/2}|\zeta-z|\Big)^4} \right\} .
\end{aligned}$$

Fix z in $\mathbb{B}_n$ and define $\tau(\zeta,\eta) = \varrho(\zeta,\eta) + \big(1-|z|\big)^{1/2}|\zeta-\eta|$. τ is a pseudo-distance, since it is a linear combination of two pseudo-distances. Consider also a point $\zeta_0 \in X$ such that $\varrho(z,X) = \varrho(z,\zeta_0) =: d$. Since $m(\zeta) = m(\zeta_0) + \mathrm{O}\big(|\zeta-\zeta_0|\big)$ we get, respectively:

$$(1-|z|)^p|S_1(z)| \preceq |u(z)| \left\{ \int_X \frac{\mathrm{d}\mu(\zeta)}{\varrho^2(\zeta,z)} + (1-|z|) \int_X \frac{\mathrm{d}\mu(\zeta)}{\tau^3(\zeta,z)} \right\}$$

and

$$\begin{aligned}
&(1-|z|)^p|S_2(z)| \preceq \\
&\preceq |u(z)| \left\{ \int_X \frac{m(\zeta)}{\varrho^{5/2}(\zeta,z)} \mathrm{d}\mu(\zeta) + (1-|z|) \int_X \frac{|\zeta-z| m(\zeta)}{\tau^4(\zeta,z)} \mathrm{d}\mu(\zeta) \right\} \\
&\preceq |u(z)| \left\{ \int_X \frac{m(\zeta)}{\varrho^{5/2}(\zeta,z)} \mathrm{d}\mu(\zeta) + \right. \\
&\qquad \left. + (1-|z|)^{1/2} \max\Big((1-|z|)^{1/2}, m(\zeta_0)\Big) \int_X \frac{\mathrm{d}\mu(\zeta)}{\tau^3(\zeta,z)} \right\} .
\end{aligned}$$

LEMMA 3.3 [2, p. 33].— *Let $z \in \mathbb{B}_n$, $\zeta_0 \in X$ and $B(\zeta_0, t)$ be the non isotropic ball of center ζ_0 and radius t. Then*

$$\left|u(z)\right| \preceq \varrho(z,\zeta_0) + |z-\zeta_0|\left|\partial\rho(\zeta_0)\wedge\partial u(\zeta_0)\right|.$$

LEMMA 3.4.— *Let $d = \varrho(z,\zeta_0)$ and $\tau_0 = \tau(z,\zeta_0)$. Then*

(a) $\displaystyle\int_X \frac{\mathrm{d}\mu(\zeta)}{\varrho^2(\zeta,z)} \preceq \frac{1}{m(\zeta_0)^2}\max\left(1\,,\,\log\left(\frac{m(\zeta_0)^2}{d}\right)\right)$;

(b) $\displaystyle\int_X \frac{m(\zeta)}{\varrho^{5/2}(\zeta,z)}\mathrm{d}\mu(\zeta) \preceq \frac{1}{\max(d\,,\,d^{1/2}m(\zeta_0)}$;

(c) $\displaystyle\int_X \frac{\mathrm{d}\mu(\zeta)}{\tau^3(\zeta,z)} \preceq \frac{1}{\max(1-|z|\,,\,m(\zeta_0)^2)}$.

These two lemmas yield

$$\begin{aligned}
&(1-|z|)^p\left|S_1(z)\right| \preceq\\
&\quad\preceq \frac{d+|z-\zeta_0|m(\zeta_0)}{m(\zeta_0)^2}\max\left(1\,,\,\log\left(\frac{m(\zeta)^2}{d}\right)\right)+\\
&\qquad+\frac{\left(d+|z-\zeta_0|\,m(\zeta_0)\right)(1-|z|)}{\tau_0\max(1-|z|\,,\,m(\zeta_0)^2)}\\
&\quad\preceq \frac{d}{m(\zeta_0)^2}\max\left(1\,,\,\log\left(\frac{m(\zeta)^2}{d}\right)\right)+\frac{d^{1/2}}{m(\zeta_0)}\max\left(1\,,\,\log\left(\frac{m(\zeta)^2}{d}\right)\right)+\\
&\qquad+\frac{d(1-|z|)}{\tau_0\max(1-|z|\,,\,m(\zeta_0)^2)}+\frac{|z-\zeta_0|\,m(\zeta_0)(1-|z|)}{\tau_0\max(1-|z|\,,\,m(\zeta_0)^2)}.
\end{aligned}$$

It is clear that the first two summands are uniformly bounded, since $m(\zeta_0) > c$, for some $c > 0$. From $d < \tau_0$ we see that the third one is also bounded. To estimate the last term it is enough to notice that by definition of $\tau(\zeta, z)$ one has $|\zeta_0 - z|(1-|z|)^{1/2} < \tau_0$.

In the same way, from the lemmas, we get $(1-|z|)^p\left|S_2(z)\right| \preceq 1$. □

It remains only to prove lemma 3.3.

Proof of lemma 3.3.— (a) and (b) are proved in [2, p. 36]. To prove (c) denote $a = 1-|z|$ and write $\varrho(\zeta, z)$ in non-isotropic real orthonormal coordinates $t = (t_1, t_2, t')$ centered at ζ_0, where t_1, t_2 indicate the two directions in the complex normal and t' the $2n-2$ directions in the complex tangent. Then $\tau(\zeta,\zeta_0) \simeq |t_1| + |t_2| + |t'|^2 + \sqrt{a}\,|t|$, and for the points in $\{\zeta \in \mathbb{B}_n : \tau(\zeta,\zeta_0) < r\}$ the estimates

$$|t_1| < \min\left(r, \frac{r}{\sqrt{a}}, \frac{r}{m(\zeta_0)^2}\right), \quad |t_2| < \min\left(r, \frac{r}{\sqrt{a}}\right)$$

$$|t'| < \min\left(\sqrt{r}, \frac{r}{\sqrt{a}}\right)$$

hold (see [2, p. 31]). Hence

$$\mu\{\zeta \in \mathbb{B}_n : \tau(\zeta,\zeta_0) < r\} \preceq \min\left(\sqrt{r}, \frac{r}{\sqrt{a}}\right)^{2n-4} \min\left(\sqrt{r}, \frac{r}{\sqrt{a}}, \frac{r}{m(\zeta_0)^2}\right)^2.$$

Define the dyadic balls $B_k = B(z, 2^k\tau_0)$, which for some positive constant c satisfy the inclusions $B_k \subset B'_k = B(z,\, c2^k\tau_0)$ [2, p. 30]. Then, from this estimate above and the equivalence $\tau(\zeta,z) \simeq \tau(\zeta,\zeta_0) + \tau_0$, it follows that

$$\int_X \frac{\mathrm{d}\mu(\zeta)}{\tau^3(\zeta,z)} \simeq \sum_k \frac{\mu(B'_k)}{(2^k\tau_0)^3} \preceq \sum_k \frac{1}{(2^k\tau_0)^3} \min\left(\sqrt{2^k\tau_0}, \frac{2^k\tau_0}{\sqrt{a}}, \frac{2^k\tau_0}{m(\zeta_0)}\right)^2.$$

A straightforward calculation, after considering the cases $m(\zeta_0)^2 > a$ and $m(\zeta_0)^2 < a$ yields finally the stated estimate. □

Acknowledgments

I would like to thank Joaquim Bruna for his enlightening suggestions and his generous help.

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