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Minimal Realizations of Classical Simple Lie Algebras Through Deformations(*)

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RÉSUMÉ. — Soit $\mathfrak{g}$ une algèbre de Lie simple complexe et $\mathcal{O}_{\min}$ son orbite coadjointe (non triviale) de dimension minimale. En utilisant un star-produit sur un revêtement de $\mathcal{O}_{\min}$ (ou d'un ouvert dense de $\mathcal{O}_{\min}$), on donne une construction explicite des réalisations minimales de $\mathfrak{g}$.

ABSTRACT. — Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathcal{O}_{\min}$ be its minimal non trivial coadjoint orbit. Using a star product on a covering space of $\mathcal{O}_{\min}$ (or an open subset of $\mathcal{O}_{\min}$), we give an explicit construction of the so called minimal realizations of $\mathfrak{g}$.

1. Introduction

The notion of star product was introduced by M. Flato and C. Fronsdal ([5], [7]) to give a new formulation for quantization of classical systems. In such a formulation, quantization is a deformation of the associative and Poisson structure of the algebra of classical observables. Applications of this notion to the theory of unitary representations of Lie groups were developped by many authors ([2], [3] and [4]).

The purpose of this paper is to give a new application of star products for the construction of the minimal realizations of a complex simple Lie algebra $\mathfrak{g}$ from its non trivial minimal coadjoint orbit $\mathcal{O}_{\min}$. In particular, we construct by a very simple and natural way the minimal realizations given by A. Joseph ([8], [9]). Recall that a realization of $\mathfrak{g}$ is a faithful

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representation of $\mathfrak{g}$ as differential operators on polynomial functions and it is minimal if the number of variables is minimal. Let us now describe our construction.

We first define a parametrization of an open subset of the minimal orbit $\mathcal{O}_{\min}$ (Sect. 2). This subset is an orbit under a solvable subgroup of the group G associated to $\mathfrak{g}$. $\mathfrak{g}$ is then embedded faithfully in a Poisson algebra $\mathcal{P}$ (an algebra of polynomials, or an algebra of polynomials localized at one generator). We introduce the Moyal star product [5] on $\mathcal{P}$ (Sect. 3). If the type of $\mathfrak{g}$ is A_n, $n \geq 1$ (Sect. 4), the Moyal star product has the covariance property and we obtain a one complex parameter family of minimal realizations of $\mathfrak{g}$. If it is C_n, $n \geq 2$, we obtain the usual realization of the symplectic algebra by quadratic polynomials ([10], [11]; Sect. 5). Finally if $\mathfrak{g}$ of type D_n, $n \geq 4$ (Sect. 6) or B_n, $n \geq 3$ (Sect. 7), the Moyal star product is not covariant thus we replace it by an equivalent star product which is covariant. This new star product leads us to the minimal realizations given in [8].

2. Parametrization of the minimal orbit

In section 2 and 3, we recall some properties of minimal orbits and star products and we describe the methods we shall use for each type of simple classical Lie algebra in section 4, 5, 6 and 7.

Let $\mathfrak{g}$ be a simple complex Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a system of positive roots, we obtain the usual triangular decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus n^+ \oplus n^- .$$

If ψ is the highest root, we fix a weight vector X_ψ. Using the Killing form β of $\mathfrak{g}$, we identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. Let ξ_0 be the element of $\mathfrak{g}^*$ corresponding to X_ψ. Let G be the adjoint group of $\mathfrak{g}$. The unique non trivial minimal nilpotent orbit for the adjoint action of G is $\mathcal{O}_{\min} = G \cdot \xi_0$.

PROPOSITION 1

1) Let $\mathfrak{g}(X_\psi)$ be the centralizer of X_ψ in $\mathfrak{g}$. Let H_ψ be the covector associated to the highest root ψ. Then there exists a Heisenberg subalgebra $\mathcal{H}$ of $\mathfrak{g}$ with central element $X_{-\psi}$ such that:

$$\mathfrak{g} = \mathfrak{g}(X_\psi) \oplus \mathbb{C}H_\psi \oplus \mathcal{H} .$$

2) Let $\mathfrak{t} = \mathbb{C}H_\psi \oplus \mathcal{H}$ *and* R *be the analytic subgroup of* G *with Lie algebra* $\mathfrak{t}$*. Then* $\widetilde{\mathcal{O}} = R \cdot \xi_0$ *is an open set of* $\mathcal{O}_{\min}$*. The coadjoint orbits* $\mathcal{O}_{\min} = G \cdot \xi_0$ *and* $R \cdot \left(\xi_0\big|_{\mathfrak{t}}\right)$ *being equipped with their Kirillov's canonical 2-forms* Ω *and* ω*, the map* $\xi \mapsto \xi\big|_{\mathfrak{t}}$ *is a symplectomorphism from* $\widetilde{\mathcal{O}}$ *onto* $R \cdot \left(\xi_0\big|_{\mathfrak{t}}\right)$*.*

Proof

1) See the following sections 4-7. See also [9].

2) Let us remark that Ω and ω are R-invariant. Now if X and Y are two elements of $\mathfrak{g}$, one can write $X = X_1 + X_2$, $Y = Y_1 + Y_2$, where X_1, Y_1 are in $\mathfrak{g}(X_\psi)$ and X_2, Y_2 in $\mathfrak{t}$. Then, since $[X_\psi , X_1] = [X_\psi , Y_1] = 0$, one has:

$$\Omega_{\xi_0}(X \cdot \xi_0 , Y \cdot \xi_0) = \langle \xi_0 , [X , Y] \rangle = \beta(X_\psi, [X , Y]) = \beta(X_\psi, [X_2 , Y_2])$$
$$= \left\langle \xi_0\big|_{\mathfrak{t}} , [X_2 , Y_2] \right\rangle = \omega_{\xi_0 |_{\mathfrak{t}}}(X_2, Y_2) .$$

This proves our proposition. □

Now we obtain a parametrization of $\widetilde{\mathcal{O}} \subset \mathcal{O}_{min}$ by using the following facts.

1) It is easy to parametrize the coadjoint orbit of $\xi_0\big|_{\mathfrak{t}}$, under the action of the solvable group R by co-ordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ where $q_n \neq 0$ (here $n = 1/2 \dim \mathcal{O}_{min}$). This is equivalent to define a symplectomorphism between $\mathbb{C}^{2n} \setminus \{q_n = 0\}$ (with the 2-form $\mathrm{d}p_1 \wedge \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}p_n \wedge \mathrm{d}q_n$) and $R \cdot \left(\xi_0\big|_{\mathfrak{t}}\right)$ (see [2], [4]). We deduce by Proposition 1 a symplectomorphism:
$$\pi : \mathbb{C}^{2n} \setminus \{q_n = 0\} \longrightarrow \widetilde{\mathcal{O}} .$$

2) For X in $\mathfrak{g}$ and x in $\mathbb{C}^{2n} \setminus \{q_n = 0\}$, put $\widetilde{X}(x) = \langle \pi(x) , X \rangle$. Parametrizing the orbit $R \cdot \left(\xi_0\big|_{\mathfrak{t}}\right)$ or giving the functions $\widetilde{X}$, X in $\mathfrak{t}$, are of course equivalent. Now the explicit computation of the functions $\widetilde{X}$ (X in $\mathfrak{g}$), which define π, will become very easy in sections 4-7 because $\mathfrak{g}$ will be an algebra of matrices and the elements of $\mathcal{O}_{\min}$ are matrices of rank one or two.

3. Star products and minimal realizations

DEFINITION 1.— *A star product on a Poisson manifold* $(M , \{\cdot , \cdot\})$*, defined on a subalgebra* N *of* $C^\infty(M)$ *stable by the Poisson bracket, is a* *bilinear mapping $\star$ from $N \times N$, into the space $N[[\nu]]$ of formal power series in the variable ν with coefficients in N:*

$$(f,g) \mapsto f \star g = \sum_{n \geq 0} \nu^n c_n(f,g)$$

where c_n are bidifferential operators on M such that $c_n(N \times N) \subset N$ and

$$c_0(f,g) = f \cdot g\,, \quad c_1(f,g) = \{f,g\}$$
$$c_n(f,g) = (-1)^n c_n(g,f)\,, \quad c_n(1,f) = 0 \quad \text{if } n > 0\,;$$

furthermore, the star product satisfies, when extended to $N[[\nu]]$ by bilinearity, the associativity condition:

$$f \star (g \star h) = (f \star g) \star h\,, \quad f \in N\,,\ g \in N\,,\ h \in N\,.$$

Remark 1.— Such a star product gives simultaneously:

1) a deformation of the associative structure (for the usual pointwise product) of N,
2) a deformation of the Poisson structure of N (called the $\star$ bracket) namely, for f and g in N,

$$[f,g]_\star = \frac{1}{2\nu}(f \star g - g \star f) = \sum_{r \geq 1} \nu^{2r} c_{2r+1}(f,g)\,.$$

DEFINITION 2.— *Two star products $\star_1$ and $\star_2$ on M are equivalent if there exists a formal power series $T = \mathrm{Id} + \sum_{r \geq 1} \nu^r T_r$ where T_r, $r \geq 1$, are differential operators such that $T_r(N) \subset N$ and*

$$T(f \star_1 g) = T(f) \star_2 T(g)$$

for each f, g in N.

DEFINITION 3.— *Let M be the space $\mathbb{C}^{2n}$ (or an open subset of $\mathbb{C}^{2n}$) endowed with the Poisson bracket defined in the co-ordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ by*

$$P^1(f,g) = \{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\,, \quad f\,,\ g \in C^\infty(M)\,.$$

We define on $N = C^\infty(M)$ (or on a subalgebra of $C^\infty(M)$ stable by the Poisson bracket) the Moyal star product $\star_M$ by

$$f \star_M g = \sum_{r \geq 0} \nu^r \frac{1}{r!} P^r(f, g)$$

where P^r is the rth tensorial power of the Poisson bracket P^1 on $\mathbb{C}^{2n}$ [5].

Remark 2.— When restricted to the localized algebra

$$N = \mathbb{C}\left[p_1, \ldots, p_n, q_1, \ldots, q_n, \frac{1}{q_n}\right]$$

of the polynomial algebra $\mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n]$ at q_n, the series $f \star_M g$ is always a finite sum. Fix the value of the deformation parameter $\nu = 1/2$, then the Moyal star product gives an associative product on N. Now, the Weyl's transform W is the linear mapping from N into the algebra of differential operators on $\mathbb{C}^n$ endowed with the co-ordinates $q_1, \ldots, q_n$ defined by

$$W(p_i) = \frac{\partial}{\partial q_i}, \quad W(q_i) = q_i, \qquad 1 \leq i \leq n$$
$$W(f \star g) = W(f) \circ W(g), \qquad f, g \in N.$$

W is a faithful representation of the algebra $(N, \star)$.

With the notations of Section 2, we shall see that the functions $\widetilde{X}$, X in $\mathfrak{g}$, are elements of the algebra $N = \mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n, 1/q_n]$. Since $\pi : \mathbb{C}^{2n} \setminus \{q_n = 0\} \to \widetilde{\mathcal{O}}$ is a symplectomorphism, the mapping $\varphi_0 : X \mapsto \widetilde{X}$ is an injective morphism of Lie algebras from $\mathfrak{g}$ to N endowed with the Poisson bracket.

DEFINITION 4.— *A star product $\star$ defined on N is called covariant if:*

$$\widetilde{X} \star \widetilde{Y} - \widetilde{Y} \star \widetilde{X} = 2\nu\{\widetilde{X}, \widetilde{Y}\}, \quad X \in \mathfrak{g}, Y \in \mathfrak{g}.$$

When $\mathfrak{g}$ is of type A_n or C_n we shall see in Sections 4 and 5 that the Moyal star product on $\widetilde{\mathcal{O}}$ is covariant. The mapping $\varphi_0 : X \mapsto \widetilde{X}$ leads to an injective morphism of Lie algebras from $\mathfrak{g}$ to $N[[\nu]]$ endowed with the star bracket $[\cdot, \cdot]_{\star_M}$. Fixing the value of the deformation parameter one obtains a representation of $\mathfrak{g}$ by considering the mapping $X \mapsto W(\widetilde{X})$.

When $\mathfrak{g}$ is of type B_n or D_n, the Moyal star product is not covariant (sects 6 and 7). φ_0 is not a Lie algebra morphism from $\mathfrak{g}$ to $\left(N[[\nu]]\,,\,[\,\cdot\,,\,\cdot\,]_{\star_M}\right)$. But there exists a differential operator T_2 of N such that the mapping:

$$\varphi : X \mapsto \widetilde{X} + \nu^2 T_2(\widetilde{X})$$

is a Lie algebra morphism from $\mathfrak{g}$ to $\left(N[[\nu]]\,,\,[\,\cdot\,,\,\cdot\,]_{\star_M}\right)$.

Replace φ_0 by φ means, in terms of star products, to replace the Moyal star product by an equivalent star product which is covariant. Fix $\nu = 1/2$, one obtains a representation of $\mathfrak{g}$ (the minimal realization of $\mathfrak{g}$) by considering the mapping $X \mapsto W\big(\varphi(X)\big)$.

All these considerations lead us to introduce the notion of star minimal realization.

DEFINITION 5

1) A star minimal realization o; $\mathfrak{g}$ is a morphism

$$\varphi : X \mapsto \widetilde{X} + \sum_{r \geq 0} \nu^r \varphi_r(X) \quad \textit{from } \mathfrak{g} \textit{ into } \left(N\,,\,[\,\cdot\,,\,\cdot\,]_{\star_M}\right).$$

2) Two star minimal realizations φ_1 and φ_2 of $\mathfrak{g}$ are equivalent if there exists an equivalence morphism

$$A = \mathrm{Id} + \sum_{r \geq 0} \nu^r A_r \quad \textit{of } \left(N\,,\,[\,\cdot\,,\,\cdot\,]_{\star_M}\right)$$

such that $\varphi_2 = A \circ \varphi_1$.

4. Lie algebra of type A_n

In this section and the following, the notations corresponding to the classical simple Lie algebras are those of [12].

In this section $\mathfrak{g} = \mathrm{sl}(n+1, \mathbb{C})$, $n \geq 1$, and $\mathfrak{h}$ is the abelian subalgebra of $\mathfrak{g}$ of diagonal matrices of $\mathfrak{g}$.

Let $\lambda_1, \ldots, \lambda_{n+1}$ be the linear forms defined on $\mathfrak{h}$ by

$$\lambda_i : \mathrm{Diag}(a_1, \ldots, a_n) \longmapsto a_i\,.$$

The system of roots is $\Delta = \{\lambda_i - \lambda_j \mid 1 \leq i \neq j \leq n+1\}$. Let $\Delta^+ = \{\lambda_i - \lambda_j \mid 1 \leq i < j \leq n+1\}$ be the set of positive roots. We denote by E_{ij} the usual matrix: the only non vanishing entries of E_{ij} has indices i and j and its value is 1. The eigensubspace corresponding to the root $\lambda_i - \lambda_j$ is $\mathfrak{g}_{\lambda_i - \lambda_j} = \mathbb{C}E_{ij}$. We identify $\mathfrak{g}$ with $\mathfrak{g}^\star$ by the trace form:

$$\langle X\,,\, Y\rangle = \mathrm{Tr}(XY)\,.$$

With the notations of the Section 2, one has $\psi = \lambda_1 - \lambda_{n+1}$, the orbit $\mathcal{O}_{\min}$ of $X_\psi = E_{1\,n+1}$ is the set of matrices of rank 1 and trace 0. We write $\mathfrak{g} = \mathfrak{g}(X_\psi) \oplus \mathbb{C}H_\psi \oplus \mathcal{H}$, where $H_\psi = E_{1\,1} - E_{n+1\,n+1}$ and $\mathcal{H}$ is the Heisenberg algebra generated by the $X_i = E_{n+1\,i+1}$, $Y_i = E_{i+1\,1}$, $1 \leq i \leq n-1$ and $Z = E_{n+1\,1}$.

Using the usual parametrization of coadjoint orbits of exponential group, we parametrize the orbit as follows:

$$\widetilde{X}_i = p_i q_n\,, \qquad \widetilde{Y}_i = q_i\,, \quad 1 \leq i \leq n-1 \qquad \text{and} \qquad \widetilde{Z} = q_n\,.$$

The commutation rules:

$$[\,H_\psi\,,\, X_i\,] = -X_i\,, \qquad [\,H_\psi\,,\, Y_i\,] = -Y_i\,, \quad 1 \leq i \leq n-1\,,$$

and

$$[\,H_\psi\,,\, Z\,] = -2Z$$

give $\widetilde{H}_\psi = -\sum_{i=1}^{n-1} p_i q_i - 2p_n q_n$.

By using the fact that the rank of the matrices in $\mathcal{O}_{\min}$ is 1, one obtains

$$\pi : (p_1\,, \ldots\,, p_n\,, q_1\,, \ldots\,, q_n) \longmapsto \xi = (\xi_{ij})$$

where

$$\xi_{k+1\,\ell+1} = p_k q_\ell\,, \quad 1 \leq k\,,\, \ell \leq n\,,$$

$$\xi_{1\,1} = -\sum_{i=1}^{n} p_i q_i\,, \qquad \xi_{1\,\ell+1} = q_\ell\,, \quad 1 \leq \ell \leq n$$

and

$$\xi_{k+1\,1} = -p_k \sum_{i=1}^{n} p_i q_i\,.$$

Now, on the algebra $N = \mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n, 1/q_n]$ we define a structure of $\mathfrak{g}$-module by

$$X \cdot f = \{\widetilde{X}, f\}, \quad X \in \mathfrak{g}, \; f \in N$$

and denote by $H^k(\mathfrak{g}, N)$ the corresponding k-th cohomological space.

PROPOSITION 2

1) The Moyal star product $\star_M$ defined on N is covariant.

2) The dimension of the space $H^1(\mathfrak{g}, N)$ is equal to 2. $H^1(\mathfrak{g}, N)$ is generated by the classes of φ_0 and φ_2, defined by

$$\begin{aligned}
&\varphi_0(E_{11} - E_{22}) = 1 \\
&\varphi_0(E_{ii} - E_{ii+1\,i+1} = 0\,, \quad 1 < i < n+1 \\
&\varphi_0(E_{k+1\,1}) = p_k\,, \quad 0 \le k \le n \\
&\varphi_0(E_{ij}) = 0\,, \quad 1 < j\,,\; i \neq j\,.
\end{aligned}$$

and

$$\varphi_1(X) = \frac{1}{q_n} \frac{\partial}{\partial p_n} \widetilde{X}\,, \quad X \in \mathfrak{g}\,.$$

3) Each minimal realization of $\mathfrak{g}$ in $\left(N\,,\, [\,\cdot\,,\cdot\,]_\star\right)$ is equivalent to a minimal realization

$$X \longmapsto \widetilde{X} + \left(\sum_{n\geq 1} a_n \nu^n\right) \varphi_0(X)$$

where a_n, $n \geq 1$, are complex numbers.

4) The family of linear mappings $W_\alpha : X \mapsto W\big(\widetilde{X} + \alpha\varphi_0(X)\big)$, $\alpha \in \mathbb{C}$, is a family of representations of $\mathfrak{g}$. It coincides with the family of minimal realizations given in [8].

Proof

1) is obtained by a direct computation.

2) Let $\varphi : \mathfrak{g} \to N$ be a 1-cocycle, i.e.,

$$\varphi([X\,, Y]) - \{\widetilde{X}, \varphi(Y)\} - \{\varphi(X), \widetilde{Y}\} = 0$$

for X, Y in $\mathfrak{g}$. Modifying if necessary φ by a coboundary, i.e., by a mapping of the form $X \mapsto \{\widetilde{X}, f\}$, $f \in N$, we can suppose that $\varphi(X) = 0$ for all X in $\mathcal{H}$ and that $\varphi(E_{11} - E_{22})$ is a scalar. We deduce that φ is a linear combination of φ_0 and φ_1.

3) is deduced from 2). □

Remark 3.— The family of minimal realizations W_α of $\mathfrak{g}$ can also be found by the method given by N. Conze in [6].

5. Lie algebra of type C_n

In this section, $\mathfrak{g}$ is the Lie algebra $\mathrm{sp}(2n, \mathbb{C})$, $n \geq 2$, of matrices of type

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

where A is a $n \times n$ matrix, B and C are two symmetric $n \times n$ matrices. We denote by (A, B, C) such a matrix. With the previous notations:

$$\mathfrak{h} = \Big\{ \big(\mathrm{Diag}(a_1, \ldots, a_n), 0, \ldots, 0\big) \mid a_i \in \mathbb{C} \Big\}$$

$$\lambda_i : \big(\mathrm{Diag}(a_1, \ldots, a_n), 0, \ldots, 0\big) \mapsto a_i\,, \quad i = 1, 2, \ldots, n$$

$$\Delta^+ = \{\lambda_i - \lambda_j \mid 1 \leq i < j \leq n\} \cup \{\lambda_k + \lambda_\ell \mid 1 \leq k \leq \ell \leq n\}$$

$$\psi = 2\lambda_1\,, \quad X_\psi = (0, E_{11}, 0)\,, \quad H_\psi = (E_{11}, 0, 0)\,.$$

We identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by the form $\langle X\,, Y \rangle = (1/2)\,\mathrm{Tr}(XY)$. $\mathcal{O}_{\min}$ is the orbit of $\xi_0 = X_\psi$ and the elements of $\mathcal{O}_{\min}$ are matrices of $\mathrm{sp}(2n, \mathbb{C})$ with rank 1. Here $\mathcal{H}$ is the Heisenberg algebra generated by $X_i = (-E_{1\,i+1}, 0, 0)$, $Y_i = (0, 0, E_{1\,i+1} + E_{i+1\,1})$, $1 \leq i \leq n-1$, and $Z = (0, 0, 2E_{11})$ (then $[X_i\,, Y_i] = \delta_{ij} Z$).

We parametrize the orbit as follows:

$$\widetilde{X}_i = p_{n-i} q_n\,, \quad \widetilde{Y}_i = q_{n-i} q_n\,, \quad 1 \leq i \leq n-1$$

and

$$\widetilde{Z} = q_n^2\,.$$

As in Section 4, we deduce:

$$\widetilde{H}_\psi = -p_n q_n\,.$$

Here $\pi : \mathbb{C}^{2n} \setminus \{0\} \to \mathcal{O}_{\min}$ is a twofold covering of the whole orbit $\mathcal{O}_{\min}$ ([10], [11]), $\pi(p_1, \ldots, p_n, q_1, \ldots, q_n)$ is the matrix of rank 1 of $\mathfrak{g}$ whose the first line is

$$(-p_n q_n, -p_{n-1} q_n, \ldots, -p_1 q_n, q_n^2, q_{n-1} q_n, \ldots, q_1 q_n)$$

and the $n+1$-th column is

$$(q_n^2, q_{n-1} q_n, \ldots, q_1 q_n, p_n q_n, p_{n-1} q_n, \ldots, p_1 q_n).$$

The mapping $\varphi_0 : X \mapsto \widetilde{X}$ is the usual realization of $\mathfrak{g}$ as a Poisson algebra of quadratic polynomials.

The Moyal star product defined on $N = \mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n]$ is covariant and the representation $X \mapsto W(\widetilde{X})$ of $\mathfrak{g}$ in the space $\mathbb{C}[q_1, \ldots, q_n]$ is well known [10]. Its restriction to even polynomials of $\mathbb{C}[q_1, \ldots, q_n]$ is an irreducible representation of $\mathfrak{g}$ with dominant weight

$$\lambda : \big(\mathrm{Diag}(a_1, \ldots, a_n), 0, \ldots, 0\big) \to -\frac{1}{2}(a_1 + \cdots + a_n) \quad [1].$$

Let us remark that $H^1(\mathfrak{g}, N)$ vanishes, by the first Whitehead lemma.

6. Lie algebra of type D_n

Here $\mathfrak{g}$ is the Lie algebra $\mathrm{so}(2n, \mathbb{C})$, $n \geq 4$, of complex matrices of type

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

where A, B and C are $n \times n$ matrices such that $B^t = -B$, $C^t = -C$. We denote by (A, B, C) such a matrix. With the notations of section 4 and 5, one has

$$\mathfrak{h} = \Big\{\big(\mathrm{Diag}(a_1, \ldots, a_n), 0, \ldots, 0\big) \mid a_i \in \mathbb{C}\Big\}$$

$$\lambda_i : (\mathrm{Diag}(a_1, \ldots, a_n), 0, \ldots, 0) \mapsto a_i, \quad i = 1, 2, \ldots, n$$

$$\Delta^+ = \{\lambda_i - \lambda_j \mid 1 \leq i < j \leq n\} \cup \{\lambda_k + \lambda_\ell \mid 1 \leq k < \ell \leq n\}$$

$$\psi = \lambda_1 + \lambda_2, \quad X_\psi = (0, E_{21} - E_{12}, 0), \quad H_\psi = (E_{11} + E_{22}, 0, 0).$$

We identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by the form $\langle X\,,Y\rangle = (1/2)\,\mathrm{Tr}(XY)$. $\mathcal{O}_{\min}$ is the orbit of $\xi_0 = X_\psi$ and the elements of $\mathcal{O}_{\min}$ are matrices of $\mathfrak{g}$ with rank 2. Here $\mathcal{H}$ is the Heisenberg algebra generated by

$$X_i = (-E_{i+2\,2}, 0, 0)\,, \quad Y_i = (0, 0, E_{1\,i+2} - E_{i+2\,1})\,, \quad 1 \le i \le n-2$$
$$X_i = (E_{i+2\,1}, 0, 0)\,, \quad Y_i = (0, 0, E_{2\,i+2} - E_{i+2\,2})\,, \quad n-1 \le i \le 2n-4$$
$$Z = (0, 0, E_{1\,2} - E_{2\,1})\,.$$

We parametrize the orbit as follows:

$$\widetilde{X}_i = p_i q_{2n-3}\,, \quad \widetilde{Y}_i = q_i q_{2n-3}\,, \quad 1 \le i \le 2n-4$$

and

$$\widetilde{Z} = q_{2n-3}^2\,, \quad \widetilde{H} = -p_{2n-3} q_{2n-3}\,.$$

From the commutation rules we deduce

$$\widetilde{H} = -\sum_{i=1}^{n-2} p_i q_i + \sum_{j=n-1}^{2n-4} p_j q_j \quad \text{if } H = (\mathrm{Diag}(1, -1, 0, \ldots, 0), 0, 0)$$

$$\widetilde{X}_1 = -\sum_{i=1}^{n-2} p_i q_{i+n-2} \quad \text{if } X_1 = (E_{1\,2}, 0, 0)$$

$$\widetilde{X}_2 = -\sum_{i=1}^{n-2} p_{i+n-2} q_i \quad \text{if } X_2 = (E_{2\,1}, 0, 0)\,.$$

$\pi : \mathbb{C}^{2n} \setminus \{q_{2n-3} = 0\} \to \widetilde{\mathcal{O}}$ is still a twofold covering of an open subset of $\mathcal{O}_{\min}$ and $\pi(p_1, \ldots, p_{2n-3}, q_1, \ldots, q_{2n-3})$ is the matrix (ξ_{ij}) of $\mathfrak{g}$ with rank 2 with first line

$$(\xi_{11}, \widetilde{X}_2, p_{n-1} q_{2n-3}, \ldots,$$
$$p_{2n-4} q_{2n-3}, 0, -q_{2n-3}^2, -q_1 q_{2n-3}, \ldots, -q_{n-2} q_{2n-3})$$

and second line

$$(\widetilde{X}_1, \xi_{22}, -p_1 q_{2n-3}, \ldots,$$
$$-p_{n-2} q_{2n-3}, q_{2n-3}^2, 0, -q_{n-1} q_{2n-3}, \ldots, -q_{2n-4} q_{2n-3})$$

where

$$\xi_{11} = \frac{1}{2}\left(-\sum_{i=1}^{n-2} p_i q_i + \sum_{i=n-1}^{2n-4} p_i q_i - p_{2n-3} q_{2n-3}\right)$$

$$\xi_{22} = \frac{1}{2}\left(\sum_{i=1}^{n-2} p_i q_i - \sum_{i=n-1}^{2n-4} p_i q_i - p_{2n-3} q_{2n-3}\right) .$$

Let $\star_M$ be the Moyal star product defined on

$$N = \mathbb{C}\left[p_1, \ldots, p_{2n-3}, q_1, \ldots, q_{2n-3}, \frac{1}{q_{2n-3}}\right] :$$

$$f \star_M g = \sum_{k \geq 0} \nu^k c_k(f, g), \quad f, g \in N .$$

An easy computation shows that $\star_M$ is not covariant.

LEMMA 1.— *Let*

$$T_2 = \left(3n - \frac{15}{2}\right) \frac{1}{q_{2n-3}^2} \frac{\partial^2}{\partial p_{2n-3}^2} .$$

Then $T_2(N) \subset N$. Let $\theta : \mathfrak{g} \to N$ defined by

$$\theta(X) = T_2(\widetilde{X}), \quad X \in \mathfrak{g} .$$

Then, for each X and Y in $\mathfrak{g}$,

$$\partial\theta(X,Y) = \theta([X, Y]) - \{\widetilde{X}, \theta(Y)\} - \{\theta(X), \widetilde{Y}\} = c_3(\widetilde{X}, \widetilde{Y}) .$$

Proof.— It is a direct consequence of the following results:

$$c_3(\xi_{2\,k+2}, \xi_{n+1\,k+2}) = -c_3(\xi_{1\,k+2}, \xi_{n+2\,k+2}) = \alpha \frac{1}{q_{2n-3}^2}$$

$$c_3(\xi_{2\,k+2}, \xi_{n+1\,2}) = \alpha \frac{q_k}{q_{2n-3}^3} , \quad c_3(\xi_{n+1\,k+2}, \xi_{n+1\,2}) = \alpha \frac{p_k}{q_{2n-3}^3}$$

$$c_3(\xi_{1\,k+2}, \xi_{n+1\,2}) = -\alpha \frac{q_{n-2+k}}{q_{2n-3}^3} , \quad c_3(\xi_{n+2\,k+2}, \xi_{n+1\,2}) = \alpha \frac{p_{n-2+k}}{q_{2n-3}^3}$$

where
$$\alpha = -\frac{3}{2}(n-2) + \frac{3}{4}$$
and $k = 1, 2, \ldots, n-2$. All others $c_3(\widetilde{X}, \widetilde{Y})$ (for X and Y in a Chevalley basis of $\mathfrak{g}$) vanish. □

PROPOSITION 3

1) The mapping $\Phi : X \mapsto \widetilde{X} + \nu^2\theta(X)$ *is a Lie algebra morphism from* $\mathfrak{g}$ *into* $\left(N[[\nu]]\,,\, [\,\cdot\,,\,\cdot\,]_{\star_M}\right)$ *(i.e.,* Φ *is a star minimal realization). If we fix* $\nu = 1/2$*, the mapping* $X \mapsto W\big(\Phi(X)\big)$ *is a minimal realization of* $\mathfrak{g}$*.*

2) Put $T = \mathrm{id} + \nu^2 T_2$*. Then the star product denoted by* $\star$ *and defined by*
$$T(f \star g) = T(f) \star_M T(g)\,, \quad f\,,\, g \in N$$
is covariant.

Proof.— Let $r \geq 0$ and X, Y be in $\mathfrak{g}$, then
$$c_{2r+1}\big(\theta(X), \theta(Y)\big) = c_{2r+3}\big(\theta(X), \widetilde{Y}\big) = c_{2r+5}(\widetilde{X}, \widetilde{Y}) = 0\,.$$
Now we deduce 1) from the previous lemma. 2) is a consequence of 1). □

PROPOSITION 4

1) $H^1(\mathfrak{g}, N)$ *is generated by the class of the mapping*
$$X \longmapsto \frac{1}{q_{2n-3}} \frac{\partial}{\partial p_{2n-3}} \widetilde{X}\,.$$

2) Each star minimal realization is equivalent to the star minimal realization Φ*.*

Proof.— It is similar to the proof of Proposition 2. □

7. Lie algebra of type B_n

Here $\mathfrak{g}$ is the Lie algebra $\mathrm{so}(2n+1, \mathbb{C})$, $n \geq 3$, of complex matrices of type
$$\begin{pmatrix} 0 & a & b \\ -b^t & A & B \\ -a_t & C & -A^t \end{pmatrix}$$

where A, B, C are $n \times n$ matrices, a and b are $1 \times n$ matrices and $B^t = -B$, $C^t = -C$. We denote by $(a, b\,; A, B, C)$ such a matrix. One has,

$$\mathfrak{h} = \left\{ (0,0\,; \mathrm{Diag}(a_1, \ldots, a_n), 0, \ldots, 0) \mid a_i \in \mathbb{C} \right\}$$

$$\lambda_i : (0,0\,; \mathrm{Diag}(a_1, \ldots, a_n), 0, \ldots, 0) \mapsto a_i\,, \quad i = 1, 2, \ldots, n$$

$$\begin{aligned} \Delta^+ &= \{\lambda_i - \lambda_j \mid 1 \le i < j \le n\} \cup \{\lambda_k + \lambda_\ell \mid 1 \le k < \ell \le n\} \\ &\quad \cup \{\lambda_r \mid 1 \le r \le n\} \end{aligned}$$

$$\psi = \lambda_1 + \lambda_2\,, \quad X_\psi = (0,0\,; 0, E_{21} - E_{12}, 0)\,, \quad H_\psi = (0,0\,; E_{11} + E_{22}, 0, 0)\,.$$

We identify g^* and $\mathfrak{g}$ by the form $\langle X\,, Y \rangle = (1/2)\,\mathrm{Tr}(XY)$. $\mathcal{O}_{\min}$ is the orbit of $\xi_0 = X_\psi$. Here $\mathcal{H}$ is the Heisenberg algebra generated by

$$\begin{cases} X_i = (0,0\,; -E_{i+2\,2}, 0, 0) \\ Y_i = (0,0\,; 0, 0, E_{1\,i+2} - E_{i+2\,1})\,, \end{cases} \qquad 1 \le i \le n-2$$

$$\begin{cases} X_i = (0,0\,; E_{i+2\,1}, 0, 0) \\ Y_i = (0,0\,; 0, 0, E_{2\,i+2} - E_{i+2\,2})\,, \end{cases} \qquad n-1 \le i \le 2n-4$$

$$\begin{aligned} X_{2n-3} &= \big(0, (0, 1, 0, \ldots, 0)\,; 0, 0, 0\big) \\ Y_{2n-3} &= \big(0, (1, 0, \ldots, 0), 0, 0, 0\big) \\ Z &= (0,0\,; 0, 0, E_{12} - E_{21})\,. \end{aligned}$$

Put $\widetilde{X}_i = p_i q_{2n-2}$, $\widetilde{Y}_i = q_i q_{2n-2}$, $1 \le i \le 2n-3$, and $\widetilde{Z} = q_{2n-2}^2$. As in Section 6, we find that $\pi : \mathbb{C}^{2n} \setminus \{q_{2n-3} = 0\} \to \widetilde{\mathcal{O}}$. is the 2-covering given by: $\pi(p_1, \ldots, p_{2n-3}, q_1, \ldots, q_{2n-3})$ is the matrix (ξ_{ij}) of $\mathfrak{g}$ with rank 2 and second line:

$$\begin{aligned} \big(q_{2n-3} q_{2n-2}, \xi_{22}, \xi_{23}, p_{n-1} q_{2n-2}, \ldots, \\ p_{2n-4} q_{2n-2}, 0, -q_{2n-2}^2, -q_1 q_{2n-2}, \ldots, -q_{n-2} q_{2n-2}\big) \end{aligned}$$

and third line

$$\begin{aligned} \big(p_{2n-3} q_{2n-2}, \xi_{32}, \xi_{33}, -p_1 q_{2n-2}, \ldots, \\ -p_{n-2} q_{2n-2}, q_{2n-2}^2, 0, -q_{n-1} q_{2n-2}, \ldots, -q_{2n-4} q_{2n-2}\big) \end{aligned}$$

where

$$\xi_{22} = \frac{1}{2}\left(-\sum_{i=1}^{n-2} p_i q_i + \sum_{i=n-1}^{2n-4} p_i q_i - p_{2n-3}q_{2n-3} - p_{2n-2}q_{2n-2}\right)$$

$$\xi_{33} = \frac{1}{2}\left(\sum_{i=1}^{n-2} p_i q_i - \sum_{i=n-1}^{2n-4} p_i q_i + p_{2n-3}q_{2n-3} - p_{2n-2}q_{2n-2}\right)$$

$$\xi_{23} = \frac{1}{2}\, q_{2n-3}^2 - \sum_{i=1}^{n-2} p_{i+n-2} q_i$$

$$\xi_{32} = -\frac{1}{2}\, p_{2n-3}^2 - \sum_{i=1}^{n-2} p_i q_{i+n-2}\,.$$

Now we introduce the Moyal star product on the algebra

$$N = \mathbb{C}\left[p_1, \ldots, p_{2n-2}, q_1, \ldots, q_{2n-2}, \frac{1}{q_{2n-2}}\right]$$

and the operator

$$T_2 = 3(n-2)\frac{1}{q_{2n-2}^2}\frac{\partial^2}{\partial p_{2n-2}^2}\,;$$

we obtain the same types of results of Proposition 3 and Proposition 4. □

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