CAM VAN QUACH HONGLER
CLAUDE WEBER

The link of an extrovert divide

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CAM VAN QUACH HONGLER AND CLAUDE WEBER [1]

RÉSUMÉ. — Nous donnons une bonne projection pour l'entrelac associé à un divide de Norbert A'Campo, pour une classe de divides que nous appelons extravertis et qui généralise la notion de slalom divide.

ABSTRACT. — We give a good projection of the link associated to a Norbert A'Campo divide, for a class of divides which we call extrovert and which generalizes the notion of slalom divides.

1. Introduction

According to the definition put forward by Norbert A'Campo in [A'C 2], a **divide** P is the image of an immersion α of the disjoint union $\coprod_j I_j$ of a finite number k of copies of the unit interval I into the unit disc D^2 in $\mathbf{R}^2$ such that:

1. The immersion is generic, i.e. multiple points are double points and at such a point the two local branches are transversal.

2. One has $\alpha^{-1}(\partial D^2) = \coprod_j (\partial I)_j$ and the immersion is orthogonal to ∂D^2.

3. the image of α is connected.

A'Campo defines the link $L(P)$ in S^3 of a divide P as follows:

$$S^3 = \{(x, u) \mid x \in D^2 , \; u \in T_x D^2 \;\; such \; that \;\; |x|^2 + |u|^2 = 1\}$$

Then $L(P)$ is defined to be the subset of S^3 of those pairs (x, u) with $x \in P$ and $u \in T_x P$. Here $T_x D^2$ stands for the tangent space of D^2 at the point $x \in D^2$.

A'Campo proves that $L(P)$ fibers over the circle S^1 for an orientation which is well defined up to a global change of orientation for all the components of the link. In [A'C 3] he presents a method (the "easy and fast method") to draw an abstract surface which is a model for the fiber. Using this model, he expresses the monodromy as a product of Dehn twists around explicit cycles (the red, white and blue cycles). We briefly recall this construction.

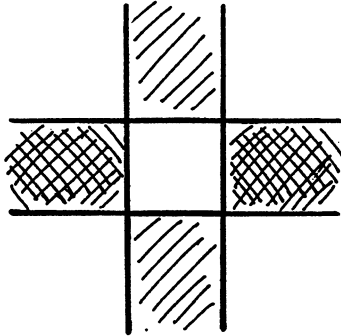1. At each double point of P one draws a piece of shaded surface as in figure 1.

Figure 1

2. One thickens each edge of P as shown in figure 2.
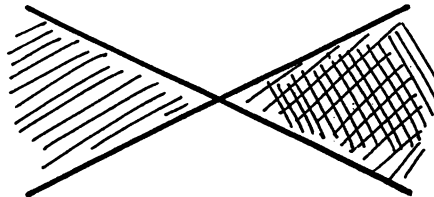

Figure 2

3. The pieces of shaded surfaces are glued together in an obvious way.

See [A'C 3] for more details.

The picture thus obtained must be understood as being the picture of an abstract surface made of twisted straps. It is easy to see that it is orientable, because the number of twisted straps along any closed curve is even.

In this article, we define a class of divides, which we call extrovert. They are a generalization of A'Campo's slalom divides [A'C 4]. We show that, for an extrovert divide, we can obtain a (so called fine) projection of the link $L(P)$ by choosing in an appropriate way the over and under crossings in the universe (in L. Kauffman sense) which is the boundary of the picture of the "easy and fast" surface. To prove that, we proceed in three steps.

First step. The shaded surface associated to the fine projection is a Seifert surface (because the "easy and fast" surface is orientable). We show that this surface is the fiber of a fibration onto the circle.

Second step. We determine the monodromy of this fibration.

Third step. We check that this monodromy coincides with A'Campo's monodromy.

Our proof uses ideas and results of M. Scharlemann-A. Thomson [S-T] and D. Gabai [Ga] which relate Conway moves, fibered links and Murasugi plumbings. See section 2.

The main result of this article is contained in theorem 4.1.

2. Checkboards, Conway moves and Hopf bands

Let D be an oriented link diagram in $\mathbf{R}^2$, i.e. a regular projection of an oriented link L in $\mathbf{R}^3$. We write $p : \mathbf{R}^3 \to \mathbf{R}^2$ for the projection. We shall always assume that D is connected.

We color the regions of $\mathbf{R}^2$ defined by D in such a way that the non compact region is unshaded (we say shaded-unshaded instead of black-white).

Suppose that the union S of the shaded regions is the projection of a Seifert surface Σ for L.

We define signs for S at a crossing point of D in the following way. A **left-hand twisted strap is positive**, while a right-hand twisted strap is negative. See figure 3.
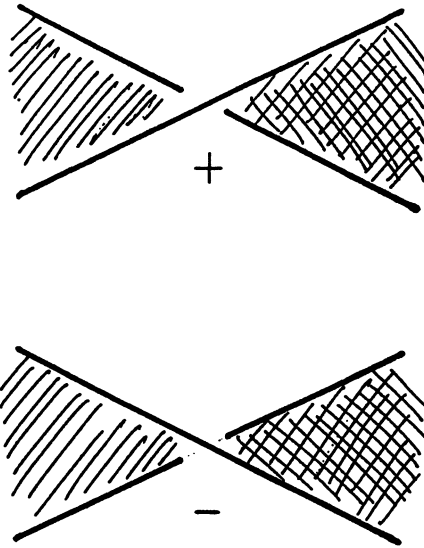
Figure 3

Remark. — Orient (arbitrarily) the shaded surface near the crossing point, and look at the orientation induced locally on the boundary. Then the crossing point is positive in Conway sense (for the exchange relation) if and only if S is a positive twisted strap.

Let R be a compact unshaded region determined by D. As D is connected, the interior of R is an open disc. To simplify matters, assume that R is a closed disc. Let C be a simple closed curve on S which is essentially ∂R pushed inside S. To make short, we shall say that C is roughly the boundary of R.

Let Γ be the simple closed curve on Σ such that $C = p(\Gamma)$. Orient C arbitrarily.

LEMMA 2.1. — *Assume that the signs of S at crossing points of D alternate when one proceeds along C. Then Γ is compressible in the complement of Σ.*

Proof. — The fact that the signs alternate implies that Γ slightly pushed off Σ bounds a disc which essentially projects by p onto R (the reader is advised to draw a picture). Now Γ is in general not compressible on both sides of Σ. The rule to detect the side off which Γ is compressible is the following. Orient C counterclockwise. The "good" side is the side of S which faces the observer before C passes through a positive twisted strap.

Definitions. — An **annulus** is a surface diffeomorphic to $S^1 \times I$. **A band** is an embedded annulus in $\mathbf{R}^3$ or in S^3. Orient a band arbitrarily. Look at the orientation induced on its boundary. Suppose that $\mathbf{R}^3$ is oriented. The band is said to be a **positive Hopf band** (resp. negative Hopf band) if the linking coefficient of the two boundary components is equal to $+1$ (resp. -1) and if its core is unknotted.

Let S be as above and let C be a simple closed curve on S. Let Γ be the simple closed curve on Σ such that $C = p(\Gamma)$. Let N be a tubular neighborhood of Γ on Σ. As Σ is orientable, N is a band. Then, for the right-hand orientation of $\mathbf{R}^3$, N is a positive Hopf band if and only if, along C, the signs of S at crossing points of D satisfy the obvious relation: (number of $+$) = (number of $-$) + 2.

Suppose now that C is a simple closed curve on S which is roughly the boundary of an unshaded region R. Let Q be a crossing point of D on C and suppose that the sign of S at Q is positive. Perform Conway moves on D at Q. Write D_+ for D and let D_- and D_0 be the link diagrams obtained from D by the moves. Let $S_+ = S$ and S_-, S_0 be the corresponding unions of shaded regions. Let $C_+ = C$ and let C_- be the corresponding curve on S_-. Without further comment, we write $\Sigma_+ = \Sigma$, Σ_- , Σ_0 and $L_+ = L$, L_-, L_0. Let Γ_- be the curve on Σ_- which projects onto C_-.

THEOREM 2.2. — *Suppose that the signs of S_- at crossing points of D_- alternate on C_-. Then Σ_+ is a Murasugi plumbing of Σ_0 with a positive Hopf band whose core is Γ_+.*

COROLLARY 2.3. — *Suppose moreover that Σ_0 is the fiber of a fibration of L_0 onto S^1. Then Σ_+ is the fiber of a fibration of L_+ onto S^1.*

Proof. — Recall that a Murasugi plumbing requires an embedded 2-sphere in $\mathbf{R}^3$. Then, one of the surfaces has to be in one hemisphere, while the other surface lies in the other hemisphere. The intersection of the two surfaces is a neighborhood of an arc in each surface. For more details, see [Ga].

In our case, let Δ be a closed 2-disc in $\mathbf{R}^3$ such that $\Delta \cap \Sigma_- = \Gamma_- = \partial\Delta$. Take a regular neighborhood W of Δ modulo its boundary Γ_-. Then W is a 3-ball and its boundary ∂W is the 2-sphere required for a Murasugi plumbing. Indeed, Σ_- lies by construction in $\mathbf{R}^3 \setminus Int(W)$ and one can get that $\Sigma_- \cap \partial W$ is a tubular neighborhood of Γ_- in Σ_-. Now $\Gamma_- \cap \Sigma_0$ is an arc and Σ_0 lies in $\mathbf{R}^3 \setminus Int(W)$. By construction, a neighborhood of Γ_+ in Σ_+ is a positive Hopf band and that band can be slightly pushed in W.

A similar argument can be found in [Ko], together with nice pictures.

Proof of the corollary. — This is an immediate consequence of [Ga] and also of [St].

Remark. — The above arguments are an easy version of ideas due to D. Gabai and M. Scharlemann-A. Thomson [S-T]. Our situation is so simple that one does not even need foliations.

We now turn our attention to monodromies. In order to have the signs (hopefully) correct, we make some definitions explicit.

Let Θ be the unit tangent vector field on S^1 pointing in the positive direction. Let $\pi : X \to S^1$ be a differentiable fibration and let $\widehat{\Theta}$ be a vector field on X such that $d\pi(\widehat{\Theta}) = \Theta$. Then "the" monodromy of π is, by definition, the first return map for $\widehat{\Theta}$. See [La] p.28.

On the other hand, let F be an oriented surface and let c be a simple closed curve on F. The action on the first homology of F, of a $\pm$ **Dehn twist** τ around c, is given by the formula
$$\tau(x) = x + (\pm)I(x,c)c$$

where $I(x,c)$ is the intersection number (it is here that the orientation of F comes in).

If the orientation of F is locally counterclockwise, a negative Dehn twist is a right-hand earthquake.

The following proposition is easy to prove but good to know.

PROPOSITION 2.4. — *Suppose that S^3 is oriented. Let Σ be a fiber surface in S^3 and let Γ be a simple closed curve on Σ. Suppose that the monodromy of Σ contains a Dehn twist τ around Γ as a factor (for the composition of diffeomorphisms). Then, the sign of τ does not depend on the orientation of Σ.*

Proof. — Suppose that the orientation of Σ is reversed. Then the oriented normal to Σ changes side. Hence the monodromy h is replaced by h^{-1} because, along a fiber, the side indicated by the vector field $\widehat{\Theta}$ must agree with the side indicated by the oriented normal. This causes a first change of sign on τ. But a second change comes from the fact that the intersection form I is changed to $-I$.

PROPOSITION 2.5. — *The monodromy of a positive Hopf band in S^3 is a negative Dehn twist.*

Proof. — The Milnor fiber of an ordinary quadratic singularity at the origin of $\mathbf{C}^2$ is a positive Hopf band in a little Milnor sphere, oriented as the boundary of a little Milnor ball, equipped with the complex orientation. The monodromy is explicitly described in [La] as a negative Dehn twist. See sections 4 and 5, especially formula (5.3.4) p.37, with $n = 1$.

ADDENDUM TO COROLLARY 2.3. — *Let h_0 be the monodromy of L_0 and let τ be a negative Dehn twist around C_+. Then the monodromy of L_+ is equal to:*

a) $h_0 \circ \tau$ *if the Hopf band is attached on the positive side of Σ_0 (positive with respect to a vector field like $\widehat{\Theta}$).*

b) $\tau \circ h_0$ *if the Hopf band is attached on the negative side of Σ_0.*

3. Shaded Seifert surfaces for signed divides

Let P be a divide. At a double point O of P the tangent space $T_O(P)$ is the union of the two one-dimensional subspaces of $T_O(D^2)$ which are the tangent spaces to the local branches of P passing through O. From the definition of the link $L(P)$ it then easy to deduce that the contribution of the double point $O \in P$ to $L(P)$ consists of four disjoint segments.

We decide to associate to O the 8-tangle described in figure 4. We call it **the red-cross tangle**.
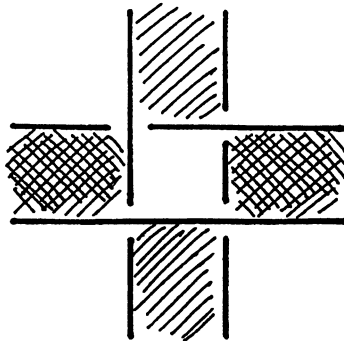

Figure 4

Next we want the shaded surface as shown in figure 4 to be locally the projection of a Seifert surface for the red-cross tangle (this is equivalent to some conditions on the local orientations of the strings, which need not to be made explicit).

Let R be the square unshaded region which is in the middle of the picture. Let C be the simple closed curve on that local Seifert surface which is roughly the boundary of R, as in section 2. Notice that the signs of the shaded surface along C are $+++-$.

Let P be a divide. Let us attribute signs $+$ or $-$ to P in the following way:

1. Each corner at a double point of P receives a sign, in such a way that one corner gets a $-$ and the other three a $+$.

2. Each edge receives a $+$ or a $-$.

A sign attibution will be written A and the pair (P, A) will be called a **signed divide**.

To (P, A) we associate a link diagram $D(P, A)$ in the following way:

1. For each double point O of P we draw a red-cross tangle, in such a way that the $-$ crossing falls into the corner that has received the $-$ sign. This can be achieved by just rotating Figure 4.

2. For each signed edge we draw a shaded twisted strap, according to the rule of figure 3. The shade goes along the edge.

By construction, the shaded surface, written $S(P, A)$, for the diagram $D(P, A)$ is the union of the local shaded surface for the red-cross tangles and of the shaded twisted straps associated to the signed edges. But for the over or under crossings on its boundary, the drawing of the surface $S(P, A)$ coincides with A'Campo's "easy and fast" surface. But the two objects are quite different. A'Campo's picture represents an abstract surface, while $S(P, A)$ represents a projection of a surface embedded in $\mathbf{R}^3$.

Let $L(P, A)$ be "the" link in $\mathbf{R}^3$ which projects onto $D(P, A)$ and let $\Sigma(P, A)$ be "the" orientable surface in $\mathbf{R}^3$ which projects onto $S(P, A)$. It is connected. Hence $L(P, A)$ has two possible orientations as boundary of $\Sigma(P, A)$, depending on which orientation we choose for $\Sigma(P, A)$.

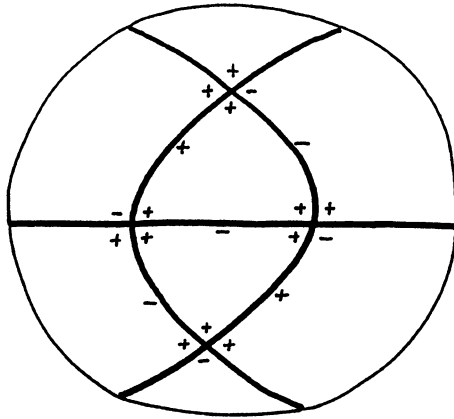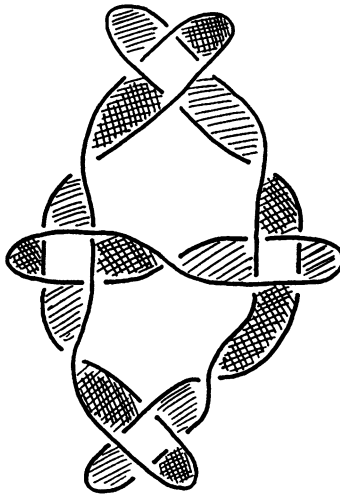Figure 5 exhibits a signed divide and figure 6 shows $S(P, A)$.

Figure 5

Figure 6

4. Extrovert divides

DEFINITION. — We say that a divide P is **extrovert** if it satisfies the following two conditions.

i) For each double point $O \in P$ at least one corner belongs to an exterior region.

ii) For each interior region, there is at least one edge which is also adjacent to an exterior region.

A slalom divide in the sense of [A'C 4] is extrovert, because slalom divides satisfy the more demanding conditions:

i') For each double point $O \in P$ exactly two corners belong to exterior regions.

ii') Each edge is adjacent to an exterior region.

Figure 5 exhibits a divide which is extrovert without being slalom.

One can prove that extrovert divides are indeed "partages" in the sense of [A'C 1], i.e. they satisfy condition Par3. The converse is false.

DEFINITION. — Let P be an extrovert divide. We shall say that a sign attribution A to P is fine if it satisfies the following two conditions:

a) At a double point $O \in P$ the $-$ sign falls into a corner which belongs to an exterior region.

b) Let R be an interior region. All edges of R but one receive a $-$ sign, the exception being an edge which is adjacent to an exterior region.

The corresponding diagram $D(P, A)$ will be called a **fine link projection** (or diagram). Figure 6 shows one.

THEOREM 4.1. — *Let P be an extrovert divide and let $D(P, A)$ be a fine link projection. Then:*

1) *$D(P, A)$ is a projection of A'Campo's link $L(P)$.*

2) *The shaded surface $S(P, A)$ is the projection of a fiber Seifert surface for $L(P)$.*

Proof. — Let R be a compact unshaded region of $\mathbf{R}^2$ determined by $D(P, A)$. This region can be of one of two types:

Type α: R is the "square" which sits in the middle of a red-cross tangle.

Type β: R is (essentially) an interior region of the divide P.

Let C be a simple closed curve on $S(P, A)$ which is roughly the boundary of R and let Γ be a simple closed curve on $\Sigma(P, A)$ which projects onto C. The following two observations are crucial. Indeed, they are the goal of the notion of fine sign attribution.

1) A tubular neighborhood of Γ on $\Sigma(P, A)$ is a positive Hopf band.

2) Suppose R is of type β. Perform a Conway move at the $+$ crossing which corresponds to the unique $+$ edge which is on the boundary of R. Then C_- is sign alternating.

Now perform a zero-Conway move at each $+$ crossing of $D(P, A)$ which corresponds to a $+$ edge. Write D_0 for the link projection which is obtained, and similarily S_0, Σ_0 etc.

Clearly Σ_0 is a connected sum of positive Hopf bands. The Hopf bands correspond to the red-cross tangles and they are connected together by a tree of (negative) twisted straps. Figure 7 illustrates this construction performed on the signed divide of figure 5. The surface Σ_0 is a fiber for the link L_0 and its monodromy is the product of negative Dehn twists around the cores of the Hopf bands (they are A'Campo's white cycles).
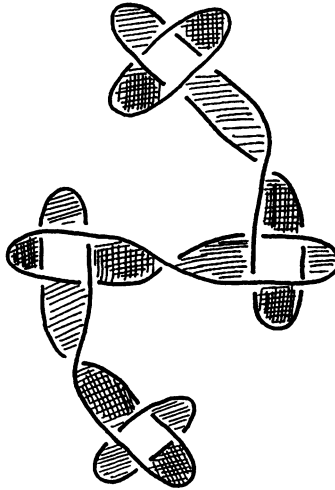

Figure 7

To go on, we need to choose a sign function (encore !) for the interior regions of D^2 determined by the divide P, as in [A'C 2]. We thus obtain a $\pm$ sign for each compact unshaded region R of type β. Let C be roughly the boundary of an R of type β and let Γ project onto C. As expected, Γ is a red cycle in A'Campo's sense if R is a $+$ region, while it is a blue cycle if R is a $-$ region.

On $\Sigma(P, A)$ orient (via $S(P, A)$ and the projection $p : \Sigma \to S$) the red and blue cycles counterclockwise and the white cycles clockwise. One can check that one can provide $\Sigma(P, A)$ with an orientation such that all intersection numbers $I(red, blue)$, $I(red, white)$ and $I(white, blue)$ are $\geqslant 0$

as they should be. Let us orient $\mathbf{R}^3$ with the right-hand convention. Let Γ_+ be a red cycle. It is not hard to see by drawing a picture that Γ_- is compressible on the positive side of Σ_-. On the other hand, if Γ_+ is a blue cycle, then Γ_- is compressible on the negative side of Σ_-.

We now make several uses of theorem 2.2 and corollary 2.3. First, we consider the positive side of the connected sum of (positive) Hopf bands obtained above. We "add" the red cycles (and the corresponding bands) one after the other. The order in which this is done is irrelevant, because any two red cycles are disjoint. Then we do a similar operation with the blue cycles on the negative side of the surface just obtained.

The addendum to corollary 2.3 shows that the monodromy of $L(P, A)$ is exactly A'Campo's monodromy: (product of negative Dehn twists around the blue cycles) ∘ (product of negative Dehn twists around the white cycles) ∘ (product of negative Dehn twists around the red cycles).

This completes the proof, because monodromies considered in the mapping class group of diffeomorphisms which are the identity on the boundary of the fiber determine the isotopy class of fibered links in S^3. However, the referee has aptly pointed out that some caution is needed here. The key point in the argument is that monodromies have to be the identity on the boundary of the fiber and that isotopies have to be fixed on the boundary. The link in S^3 can then be recovered from such monodromies by performing the open book construction, as in [W]. As a consequence, one needs to verify that all constructions and monodromies considered in this article take place in the setting of open book decompositions, rather than just fibered links and isotopy classes of monodromies with no restriction on the boundary. Now, Lamotke's determination of the monodromy of isolated complex singularities does just that ("isolated" is crucial here). See especially the definitions in section 2. Gabai's proof for Murasugi sums fits with open book decompositions. See theorem 4. Finally, one can check that monodromies and isotopies considered by A'Campo are the identity on the boundary.

Remark. — The proof shows that $D(P, A)$ is a projection of $L(P)$ if $\mathbf{R}^3$ is oriented with the right-hand rule.

A final remark. — Suppose that one reverses the sign function for the regions determined by the divide. Then the orientation of Σ is reversed. The orientation of each component of the link is reversed. The monodromy h is replaced by h^{-1}. The sign of the Dehn twists remain the same (Prop. 2.4). The red and blue cycles exchange their color.

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