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## A Variational Approach for a Semi-Linear Parabolic Equation with Measure Data (\*)

MONGI MABROUK <sup>(1)</sup>

**RÉSUMÉ.** — On étudie, d'un point de vue variationnel, un problème parabolique à donnée initiale mesure:

$$\begin{aligned} \partial_t u(x, t) - \Delta_x u(x, t) + \beta(x, u(x, t)) &\ni 0 \text{ dans } Q, \\ u(0) = \mu \text{ dans } \Omega, u = 0 \text{ sur } \partial\Omega \times ]0, T[ \end{aligned}$$

où  $\Omega$  est un ouvert borné régulier de  $\mathbb{R}^N$ ,  $N \geq 2$ ,  $Q = \Omega \times ]0, T[$ ,  $\beta(x, \cdot)$  un graphe maximal monotone de  $\mathbb{R}^2$  dépendant mesurablement de  $x$ ,  $0 \in \beta(x, 0)$  pp et  $|\beta(x, r)| \leq C(1 + |r|^\alpha)$ ,  $\alpha < \frac{1}{2} + \frac{2}{N}$  et  $\mu$  est une mesure de Radon bornée. On montre que la solution faible de ce problème est en fait aussi la solution d'un problème de minimisation convexe dans l'espace  $L^r(Q)$ ,  $r = 2 + \frac{4}{N}$ . Les démonstrations font appel essentiellement à des arguments d'épi-convergence.

**ABSTRACT.** — We study, from a variational point of view, a parabolic problem with measure initial data:

$$\begin{aligned} \partial_t u(x, t) - \Delta_x u(x, t) + \beta(x, u(x, t)) &\ni 0 \text{ in } Q, \\ u(0) = \mu \text{ in } \Omega, u = 0 \text{ on } \partial\Omega \times ]0, T[ \end{aligned}$$

where  $\Omega$  is a smooth open bounded set in  $\mathbb{R}^N$ ,  $N \geq 2$ ,  $Q = \Omega \times ]0, T[$ ,  $\beta(x, \cdot)$  is a maximal monotone graph of  $\mathbb{R}^2$  depending measurably on  $x$  and verifying:  $0 \in \beta(x, 0)$  a.e,  $|\beta(x, r)| \leq C(1 + |r|^\alpha)$ ,  $\alpha < \frac{1}{2} + \frac{2}{N}$  and  $\mu$  is a bounded Radon measure in  $\Omega$ . We prove that the weak solution of this problem is in fact the solution of a convex minimization problem in the space  $L^r(Q)$ ,  $r = 2 + \frac{4}{N}$ . The proofs rely essentially on epi-convergence arguments.

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## 1. Introduction

The aim of this paper is to present a new approach for solving semi-linear parabolic problems involving measure data. More precisely, we are interested in the following problems  $P_\beta$ :

$$P_\beta := \begin{cases} \partial_t u(x, t) + Lu(x, t) + \beta(x, u(x, t)) \ni f & \text{in } \Omega \times ]0, T[, T > 0 \\ u(0) = \mu & \text{in } \Omega, u = 0 & \text{in } ]0, T[ \times \partial\Omega \end{cases}$$

where  $\Omega$  is a bounded open set of  $\mathbb{R}^N$ ,  $N \geq 2$  with a smooth boundary.  $L$  is a linear second order elliptic operator in divergence form,  $\beta(x, \cdot)$  is a measurable multi-valued mapping with values maximal monotone graphs of  $\mathbb{R} \times \mathbb{R}$  with  $\beta(x, 0) \ni 0$  for a.e.  $x$  in  $\Omega$ ,  $\mu \in m_b(\Omega)$ ,  $f \in m_b(Q)$ .

In what follows, we shall explain our approach by considering typically the case  $f = 0$ ,  $L = -\Delta$ .

Among the previous works related to this problem, we mention essentially the two following ones:

H. Brezis and A. Friedman [7] have previously studied the case  $L = -\Delta$ ,  $\beta(x, r) = |r|^{p-1} r$ . They showed the existence and uniqueness of a weak solution  $u$  if  $p < 1 + 2/N$ , and the non existence if  $p \geq 1 + 2/N$ . Their proofs are direct and rely on a priori estimates for the solutions of some approximating problems and on related compactness arguments.

P. Baras and M. Pierre [4] extended [7] to more general elliptic operators  $L$ , and possibly unbounded  $\mu$ . In particular, they showed under the same condition  $p < 1 + 2/N$ , that the problem  $P_\beta$  possesses a unique weak solution  $u$  which belongs to  $L^s(0, T; W_0^{1,q}(\Omega))$  with  $1 \leq s, q < \frac{N}{N-1}$ .

In the present work, the problem  $P_\beta$  is considered from a different point of view. Our aim is to give a variational formulation of this problem, more precisely to ask if the solution satisfies some variational principle. To this end, we use essentially the epi-convergence method, which has been already successfully applied to semi-linear elliptic equations involving measure data in [2], [8]. The key idea is to start from a variational formulation of the regular case  $\mu \in H^2(\Omega) \cap H_0^1(\Omega)$ , using the Brezis-Ekeland principle [6]. Then we introduce a smooth sequence  $\mu_n \in D(\Omega)$  converging weakly to  $\mu$ , and study the limit behaviour of the associated minimization problems. If the growth of  $\beta(x, r)$  is less than  $C(1 + |r|^\alpha)$ ,  $\alpha < \frac{1}{2} + \frac{2}{N}$ , then the limiting process yields a variational solution  $\bar{u}$  which is the solution of two alternative formulations (Th. 3.4 and Th.3.6) each having its own interest. This variational solution is also the weak solution. The fact that

it is only defined under the more restrictive condition  $\alpha < \frac{1}{2} + \frac{2}{N}$  is in some sense compensated by the fact that it possesses the mentioned minimizing properties and that it enjoys more regularity. Indeed it belongs to  $L^s(0, T; W_0^{1,q}(\Omega)) + L^2(0, T; H_0^1(\Omega))$ ,  $1 \leq s, q < \frac{N}{N-1}$ .

The literature on parabolic equations, especially concerning existence, uniqueness and regularity results, has been enriched by many authors in the last ten years. We mention particularly the contributions of L. Boccardo & co-workers (see [5] and the bibliography herein). But these works do not cover ours. In fact they are merely concerned with existence results for nonlinear problems with right-hand side measure, and this do not contain the semi-linear case with a measure as initial datum. Besides this, we seek not only existence results, but more important for us was to ask if the solutions verify some variational principle. In this sense, the present work can be more thought of as a generalization of the Brezis-Ekeland principle to a non-hilbertian, nonreflexive case.

The general idea of working via a variational principle has been used before in the elliptic case involving measures by L. Orsina in [11].

The main results of this work have been announced in the short note [9]. The paper is then organized as follows:

*2. The smooth case:* The problem with data  $\mu = gdx, g \in H^2(\Omega) \cap H_0^1(\Omega)$  is shown to be equivalent to the evolution equation in the Hilbert space  $H = L^2(\Omega)$ :

$$u' + \partial\varphi_H(t, u) \ni 0, u(0) = \mu \quad (1)$$

where  $\varphi_H(t, \cdot)$  is the functional:

$$H \ni u \rightarrow \varphi_H(t, u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx \quad (2)$$

$$+ \int_{\Omega} [j_{\beta}(x, E_{\mu}(x, t) + u(x)) - j_{\beta}(x, E_{\mu}(x, t))] dx$$

$E_{\mu}$  being the weak solution of the linear problem ( $\beta = 0$ ) and  $j_{\beta}(x, \cdot)$  a primitive of  $\beta(x, \cdot)$  verifying  $j_{\beta}(x, \cdot) \geq 0$  and  $j_{\beta}(x, 0) = 0$  for a.e.  $x$  in  $\Omega$ .

This problem is then transformed, via the variational principle of Brezis-Ekeland [6], into a minimization problem over a convex set  $K_{\mu}$  of the space  $H_2 = L^2(0, T; H)$ .

*3. The general case:* We take a regular sequence  $\mu_n$  converging in the weak-star topology to  $\mu$ . For each  $n$ , we have thus a minimization problem  $V_n$  over a convex set  $K_n$ . We show that  $V_n$  possesses a unique solution  $\bar{v}_n$ ,

for which we derive uniform estimates for the norm  $\|\bar{v}_n\|_X$ . Therefore, we can extract a subsequence, still denoted by  $\bar{v}_n$ , such that  $\bar{v}_n \xrightarrow{X} \bar{v}$  weakly. By using epi-convergence techniques, we show that  $\bar{v}$  is the solution of two minimization problems, the first one over the space  $L^r(Q)$  and the second one over the space  $W = \{u \in V, u' \in V'\}$ . Moreover we prove that  $\bar{v}$  is the solution of the evolution equation in  $H^{-1}(\Omega)$ :

$$\bar{v}' + \partial\varphi_V(t, \bar{v}) \ni 0, \quad \bar{v}(0) = 0$$

where  $\varphi_V(t, \cdot) : H_0^1 \rightarrow \overline{\mathbb{R}}$  is the restriction of  $\varphi_H(t, \cdot)$  to  $H_0^1$ . We call  $\bar{u} = \bar{v} + E_\mu$  the variational solution of  $P_\beta$ , and prove that it is the weak solution too.

#### 4. Concluding remarks.

**Notations:**  $\Omega$  is an open bounded set of  $\mathbb{R}^N$ ,  $N \geq 2$ , of class  $C^{2+\eta}$ ,  $\eta > 0$ ,  $Q = \Omega \times ]0, T[$ .

If  $u : Q \rightarrow \mathbb{R}$ , then  $u(t) = u(t, \cdot)$  and  $u'(t) = \partial_t u(t, \cdot)$ .

If  $B$  is a Banach space,  $B'$  is its (strong) dual,  $B_s, B_w = B$  equipped with the strong ( resp. weak) topology.

For every vector-valued function  $u : [0, T] \rightarrow B$  with values in the Banach space  $B$ , we denote by  $u'$  its (strong) derivative.

$L^p(\Omega), H_0^m(\Omega), H^m(\Omega), H^{-m}(\Omega)$  are the usual Lebesgue or Sobolev spaces, and  $L_{Loc}^p(\Omega), H_{Loc}^m(\Omega)$  are the local ones. Especially, we denote  $L^2(\Omega)$  by  $H$ .

$C^0(\overline{\Omega})$  is the space of continuous functions on  $\overline{\Omega}$ , vanishing on  $\partial\Omega$ ,  $m_b(\Omega)$  is the space of bounded Radon measures in  $\Omega$ , that is the dual of  $C^0(\overline{\Omega})$ ,  $\langle \cdot, \cdot \rangle_\sigma$  being the duality  $(C^0(\overline{\Omega}), m_b(\Omega))$ .  $u_n \xrightarrow{\sigma} \mu$  if  $u_n$  converges weak-star to  $\mu$ .

$V_r = L^r(0, T; H_0^1(\Omega))$ ,  $V_r' = L^{r'}(0, T; H^{-1}(\Omega))$ ; for  $r = 2$  we denote them  $V$  and  $V'$ .

$X = \{u \in V : u' \in V_r', u(T) \in H\}$ ,  $W = \{u \in V : u' \in V'\}$ .

$H_p = L^p(0, T; H)$ ,  $p \geq 1$ .  $D(\Omega), D'(\Omega)$  are the usual Schwartz spaces.

If  $F : X \rightarrow \overline{\mathbb{R}}$ , then  $dom F = \{x \in X : F(x) < +\infty\}$  is the effective domain of  $F$ .

$\|\cdot\|_p$  is the  $L^p$  norm. More generally  $\|\cdot\|_B$  is a norm in the space  $B$ .

If  $F : H \rightarrow \overline{\mathbb{R}}$ , then  $F^*(x)$  is its Fenchel conjugate in the duality  $\langle H, H \rangle$ .

If  $F : V_r \rightarrow \overline{\mathbb{R}}$ ,  $r \neq 2$ , then  $F^\otimes$  is the conjugate in the duality  $\langle V_r, V_r' \rangle$ . For  $r = 2$  we use the notation  $F^\oplus$ .

$$\Pi = \{u \in L^r(Q), u' \in L^{r'}(Q), r' = \frac{r}{r-1}\}.$$

By  $\langle \cdot, \cdot \rangle$ , we shall denote different duality brackets.

If  $f$  and  $g$  are two functions on the same space  $X$ , their infimal convolution (inf-convolution) is the function  $h = f \nabla g$  defined by  $h(x) = f \nabla g(x) = \inf_{y \in X} [f(y) + g(x - y)]$ .

The characteristic function of the set  $D$ , denoted  $I_D$ , is used in the sense of Convex Analysis, i.e.  $I_D(x) = 0$  if  $x \in D$  and  $+\infty$  elsewhere. We shall often denote by the same letter  $C$  different positive constants, and we shall often omit the symbol  $\Omega$  if there is no possible confusion.

## 2. The smooth case

Let us first give the precise formulation of the problem  $P_\beta$ :

DEFINITION 2.1. — Let  $\mu \in m_b(\Omega)$  and  $u \in L^1_{loc}(0, T; W_0^{1,1}(\Omega))$ . Then  $u$  is a weak solution of problem  $P_\beta$  if and only if:

- (i) there exists a function  $h \in L^1_{loc}(Q)$  s.t.  $h(x, t) \in \beta(x, u(x, t))$  a.e. and  $u' - \Delta u + h = 0$  in  $D'(Q)$ .
- (ii)  $\text{ess lim}_{t \downarrow 0} \int_\Omega u(x, t) \theta(x) dx = \langle \mu, \theta \rangle_\sigma \quad \forall \theta \in C_0(\overline{\Omega})$ .

In the linear case ( $\beta = 0$ ), we recall the following result of [4] concerning  $L^1$  data:

LEMMA 2.1. ([4] Lemma 3.3). — Let  $f \in L^1(Q)$  and  $\mu \in L^1(Q)$ . There exists a unique solution of the problem

$$P_0 := \left\{ \begin{array}{l} u \in C([0, T]; L^1(\Omega)) \cap L^1(0, T; W_0^{1,1}(\Omega)) \\ u' - \Delta u = f \in D'(Q), \quad u(0) = \mu \end{array} \right.$$

Moreover:

(i)  $\|u\|_{L^\infty(0, T; L^1(\Omega))} + \|u\|_{L^s(0, T; W_0^{1, q}(\Omega))} \leq C(\|f\|_1 + \|\mu\|_1)$  with  $C = C(s, q)$  and  $s, q \geq 1, \frac{2}{s} + \frac{N}{q} > N + 1$

(ii) the mapping  $(\mu, f) \rightarrow u$  is increasing and compact from  $L^1(\Omega) \times L^1(Q)$  into  $L^p(\Omega)$  for  $1 \leq p < 1 + 2/N$ .

To obtain a variational formulation in the smooth case, let  $\mu \in H^2 \cap H_0^1$ ,  $E_\mu$  the corresponding solution of  $P_0$ , and define the following functions for a.e  $t$  in  $[0, T]$ :

$$j(t, \dots) := \begin{cases} \Omega \times \mathbb{R} \rightarrow \overline{\mathbb{R}} \\ (x, r) \rightarrow j(t, x, r) = j_\beta(x, E_\mu(t, x) + r) - j_\beta(x, E_\mu(t, x)) \end{cases}$$

$$J(t, \cdot) := \begin{cases} H \rightarrow \overline{\mathbb{R}} \\ u \rightarrow J(t, u) = \int_\Omega j(t, x, u(x)) dx \end{cases}$$

$$\varphi_H(t, \cdot) := \begin{cases} H \rightarrow \overline{\mathbb{R}} \\ \varphi_H(t, u) = \frac{1}{2} \int_\Omega |Du|^2 dx + J(t, u) \text{ if } u \in H_0^1, J(t, u) < +\infty \end{cases}$$

and the associated functional  $\Phi_H = \begin{cases} H_2 \rightarrow \overline{\mathbb{R}} \\ \Phi_H(u) = \int_0^T \varphi_H(t, u(t)) dt \end{cases}$

**THEOREM 2.1.** — *Let  $\beta$  and  $\mu$  satisfy the following conditions*

- (i)  $|\beta(x, r)| \leq C(1 + |r|^\alpha)$ ,  $\alpha < \frac{1}{2}(1 + \frac{2}{N})^2 = \alpha_*$
- (ii)  $\mu = gdx$  with  $g \in H^2(\Omega) \cap H_0^1(\Omega)$

*Then:*

- (i) *The differential equation in  $H$*

$$v' + \partial\varphi_H(t, v) \ni 0, v(0) = 0 \tag{3}$$

*possesses a unique strong solution  $\bar{v}$ .*

- (ii)  *$\bar{v}$  solves the convex minimization problem*

$$V_\mu : \text{Min}\{\psi(v), v \in K_\mu\} \tag{4}$$

*where:*

$$K_\mu = \{v \in C(0, T; H), v' \in H_2, \varphi_H(t, v(t)) \in L^1(0, T), \varphi_H^*(t, -v'(t)) \in L^1(0, T), v(0) = 0\}$$

$$\psi(v) = \Phi_H(v) + \Phi_H^*(-v') + \frac{1}{2}|v(T)|_H^2$$

- (iii) *The problem  $P_\beta$  for smooth data possesses a unique strong solution  $\bar{u}$  given by  $\bar{u} = E_\mu + \bar{v}$ . We call  $\bar{u}$  the variational solution of  $P_\beta$ .*

As a consequence of this theorem, solving  $P_\beta$  for smooth data is equivalent to solving the differential equation (3) or the minimization problem  $V_\mu$ .

*Proof.* — (i) We need only to estimate  $\varphi_H(t, v) - \varphi_H(s, v)$  for  $s, t \in [0, T]$  and  $v \in \text{dom}\varphi_H(t, \cdot) = H_0^1(\Omega)$ . We have by using the subdifferential inequality:

$$\begin{aligned} |\varphi_H(t, v) - \varphi_H(s, v)| &\leq \|E_\mu(t) - E_\mu(s)\|_2 \max \\ &\quad (\|\beta(\cdot, v(\cdot) + E_\mu(t))\|_2; \|\beta(\cdot, v(\cdot) + E_\mu(s))\|_2, \\ &\quad \|\beta(E_\mu(t))\|_2, \|\beta(E_\mu(s))\|_2) \\ &\leq C|t - s| \|\Delta g\|_2 (2 + \|v\|_{2\alpha}^\alpha + \|E_\mu(s)\|_{2\alpha}^\alpha) \end{aligned}$$

But  $\alpha < \alpha_*$  gives us  $\sup_t \|E_\mu(t)\|_{2\alpha}^\alpha < +\infty$ , and  $\|v\|_{2\alpha}^\alpha \leq C_1 \|Dv\|_2^\alpha \leq C(1 + \|Dv\|_2^\alpha)$ , hence there exists positive constants such that:

$$|\varphi_H(t, v) - \varphi_H(s, v)| \leq C|t - s|(\varphi_H(t, s)) + C_1\|v\|_H + C_2).$$

Then it suffices to apply the results of [3] (Th.1, p.54).

(ii) This is simply the Brézis-Ekeland principle [6] applied to the equation (3).

(iii) We know from [3]Th.1 that  $\bar{v} \in W^{1,2}(0, T; H) \cap L^\infty(0, T; H_0^1)$ . With the assumptions made on  $\mu$  and  $\beta$ , the subdifferential of  $\varphi_H(t, \cdot)$  is characterized by:

$$\text{dom}(\partial\varphi_H(t, \cdot)) = \{u \in H \text{ s.t. } -\Delta u + \beta(t)u \in H\}$$

$$\partial\varphi_H(t, u) = -\Delta u + \beta(t)u$$

where, for each  $t$ ,  $\beta(t)$  is the maximal monotone operator in  $H$  associated to the graph (a family of graphs indexed by  $t$ )

$$r \rightarrow \beta(x, E_\mu(t, x) + r)$$

More precisely, for  $u \in L^2(\Omega)$  we have  $(\beta(t)u)(x) = \beta(x, E_\mu(t, x) + u(x))$  for a.e.  $x \in \Omega$ . Thus  $\bar{v}$  verifies

$$\bar{v}'(t) - \Delta\bar{v}(t) + \beta(t)\bar{v}(t) \ni 0 \text{ for a.e. } t \in (0, T), \bar{v}(0) = 0$$

Consequently  $\bar{u}'(t) - \Delta\bar{u}(t) + \beta(t)\bar{u}(t) \ni 0$  a.e. and  $\bar{u}(0) = \mu$  which means that  $\bar{u}$  is a strong solution of  $P_\beta$ .  $\square$

### 3. The general case

We suppose from now on that the condition:

$$C : \alpha < \frac{1}{2} + \frac{2}{N}$$

is fulfilled.

### 3.1. Approximate problems

Let  $\mu \in m_b(\Omega)$ . There exists a sequence  $\mu_n \in D(\Omega)$ ,  $\sup_n \|\mu_n\|_{L^1} < +\infty$ ,  $\mu_n \xrightarrow{\sigma} \mu$ . For each  $n$ , let  $E_n$  be the corresponding solution obtained by lemma 2.1. Then  $\mu_n$  (actually a subsequence) converges weakly to  $\mu$  as  $t \downarrow 0$ ,  $E_n \rightarrow E$  in  $L^p(Q)$  for  $p < 1 + 2/N$ , and  $E$  is the weak solution of the linear problem  $P_0$ .

Set  $K_n = K_{\mu_n}$ . For every  $n \in \mathbb{N}$ , we consider the problem:

$$P_n := \partial_t v(t, x) - \Delta_x v(t, x) + \beta(x, v(x, t) + E_n(x, t)) \ni 0, v(0, x) = 0 \quad (5)$$

Using  $E_n$  and  $E$  in place of  $E_\mu$ , we define as before, the functions  $j_n(t, \cdot)$ ,  $\varphi_H^n(t, \cdot)$ ,  $\Phi_H^n, \psi_H^n$  and  $j, \varphi_H, \Phi_H, \psi_H$ . We can then reformulate  $P_n$  as an equivalent evolution equation in  $H$ :

$$P_n : v' + \partial \varphi_H^n(t, v) \ni 0, v(0) = 0 \quad (6)$$

By theorem 2.1, for each  $n$  there is a unique strong solution  $\bar{v}_n$  for which we shall derive several estimates.

LEMMA 3.1. — *The functions  $\Phi_H^n$  are equi-coercive on  $V = L^2(0, T; H_0^1(\Omega))$  and we have  $\sup_n \Phi_H^{n*}(0) < +\infty$ .*

*Proof.* — Let  $v \in V$ . Then:

$$\begin{aligned} \Phi_H^n(v) &= \int_0^T \|Dv(t)\|_{H_0^1}^2 dt \\ &\quad + \int_0^T \int_\Omega j(t, x, v(t, x)) dx dt \geq C_1 \|v\|_V^2 - \|v\|_V \cdot \|\beta(\cdot, E_n(\cdot))\|_{V'} \end{aligned}$$

But  $E_n(t) \in L^q(\Omega)$  for  $1 \leq q \leq +\infty$  and we have the classical upper bound:

$$\|E_n(t)\|_{L^q(\Omega)} \leq \frac{C}{t^{n/2(1-1/q)}} \|\mu_n\|_{L^1(\Omega)}$$

Let  $q = \alpha \frac{2N}{N+2}$ , then  $\beta(\cdot, E_n(\cdot)) \in L^2(0, T; L^{\frac{2N}{N+2}}(\Omega))$  iff  $q < \frac{N\alpha}{N\alpha-1}$  that is iff  $\alpha < \frac{1}{2} + \frac{2}{N}$ . Consequently

$$\begin{aligned} \sup_n \|\beta(\cdot, E_n(\cdot))\|_{L^2(0, T; H^{-1})} \\ \leq C \sup_n \|\beta(\cdot, E_n(\cdot))\|_{L^2(0, T; L^{\frac{2N}{N+2}}(\Omega))} \leq K \sup_n \|\mu_n\|_{L^1(\Omega)} \end{aligned}$$

and  $\Phi_H^n(v) \geq C_1 \|v\|_V^2 - C_2 \|v\|_V$  hence the equi-coercivity. This implies that

$$\Phi_H^{n*}(0) = -\inf_H \Phi_H^n = -\inf_V \Phi_V^n$$

is bounded uniformly in  $n$  (because  $\text{dom} \Phi_H^n \subset V$ ).  $\square$

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LEMMA 3.2. — (i) *The functions  $\Phi_H^n$  are equi-continuous on  $V_r, r = 2(1 + 2/N)$ .*

(ii) *The conjugate functions  $\Phi_H^{n*}$  are equi-coercive on  $V_r'$  and equi-continuous on  $V'$ .*

*Proof.* — (i) For  $v \in \text{dom}(\Phi_H^n)$  we have by using the Young's inequality and the subdifferential inequality:

$$|j_n(t, x, v)| = |j_\beta(x, v(t, x) + E_n(t, x)) - j_\beta(x, E_n(t, x))| \quad (7)$$

$$\leq C \max(|v\beta(E_n)|, |v\beta(v + E_n)|) \quad (8)$$

$$\leq C(2 + \epsilon|v|^{\alpha+1} + C_\epsilon|E_n|^{\alpha p'}), \quad \forall \epsilon > 0, p' = \frac{p}{p-1} \quad (9)$$

But  $v \in D(\Phi_H^n) \Rightarrow v \in V \cap C(0, T; H) \subset L^r(Q), r = 2(1 + \frac{2}{N})$ . If we take  $p = r, p' = r'$ , we have  $\alpha + 1 < 1 + \frac{N+2}{N} < 2\frac{N+2}{N} = r$  from which we deduce the following estimates

$$\int_\Omega |j^n(t, x)| dx \leq C_1 + C_2 \|v(t)\|_r^r + C_3 \|E_n(t)\|_{\alpha r}^{\alpha r}$$

$$|\varphi_H^n(t, v(t))| \leq C_1 \|v(t)\|_{H_0^1}^r + C_2 \|E_n(t)\|_{\alpha r}^{\alpha r} + C_3$$

$$\Phi_H^n(v) \leq C_1 \int_0^T \|v(t)\|_{H_0^1}^r dt + C_2 \int_0^T \|E_n(t)\|_{\alpha r}^{\alpha r} dt + C_3 \quad (10)$$

$$\leq C_1 \int_0^T \|v(t)\|_{H_0^1}^r dt + C \quad (11)$$

(ii) Let  $r' = 1 + \frac{N}{N+4}$  be the conjugate exponent of  $r$ . We shall derive a two-sided estimate for the functions  $\Phi_H^{n*}$  which implies the two desired properties. From lemma 3.1 and (i) we have:

$$C \int_0^T \|v(t)\|_{H_0^1}^2 dt - C' \leq \Phi_H^n(v) \leq C \int_0^T \|v(t)\|_{H_0^1}^r dt + C_1 \quad \forall v \in \text{dom} \Phi_H^n.$$

By taking the conjugates in  $H_2$ , we obtain:

$$-C + C \int_0^T \|w(t)\|_{H^{-1}}^{r'} dt \leq \Phi_H^{n*} \leq C \int_0^T (1 + \|w(t)\|_{H^{-1}}^2) dt \quad \forall w \in H_2 \quad \square$$

THEOREM 3.1. — *We have the uniform estimates:*

$$\sup_n \{ \|\bar{v}_n\|_V, \|\bar{v}'_n\|_{V_r'}, \sup_{t \in [0, T]} \|\bar{v}_n(t)\|_H^2 \} < +\infty$$

*Proof.* — By lemmas 3.1 and 3.2, the null function  $v(t) = 0$  a.e. belongs to  $K_n$  and to  $\text{dom}(\psi_H^n)$  for every  $n \in \mathbb{N}$ . Then we have for  $\bar{v}_n$ :

$$\sup_n \psi_H^n(\bar{v}_n) \leq \sup_n \psi_H^n(0) < +\infty$$

which gives us easily the desired estimates.  $\square$

Let us now define the following space:

$$X = \{u \in V = L^2(0, T; H_0^1(\Omega)), u' \in V_r' = L^r(0, T; H^{-1}(\Omega)), u(T) \in H\}$$

Equipped with the norm

$$\|u\|_X = (\|u\|_V^2 + \|u'\|_{V_r'}^2 + \|u(T)\|_H^2)^{1/2},$$

$X$  is a reflexive and separable Banach space, and the embedding  $X \hookrightarrow H_2$  is compact. Thus the sequence  $\bar{v}_n$  is bounded in  $X$ . We can therefore extract a subsequence, still denoted  $\bar{v}_n$ , of solutions of the variational problems  $V_n$  which converges weakly in  $X$  and strongly in  $H_2$  to a function  $\bar{v} \in X$ . We can also extract a subsequence  $E_n$  s.t.  $E_n \rightarrow E$  in  $L^p(Q)$  for  $p < 1 + 2/N$ . The question is then: In what sense is  $u = \bar{v} + E$  a solution of the initial problem? To answer this question, we pass to the limit in the variational problems  $V_n$  and in the corresponding Euler equations  $P_n$ . For the variational convergence, which is our main motivation, we give two approaches:

1) The first one is to seek the epigraphical limit of the  $\psi_H^n$  for the topology of  $X$ . In doing so, we use only the a priori estimates obtained from the variational formulations  $V_n$ , and never the fact that  $\bar{v}_n$  is also solution of the partial differential equation  $P_n$ . In this respect, this approach can be called purely variational. The resulting minimization problem is identified over the space  $L^r(Q)$ .

2) The second approach uses much more the equivalence  $V_n \Leftrightarrow P_n$ , to obtain more estimates for the derivatives  $\bar{v}_n'$ . The determination of the epilimit is then easier and yields us a convex minimization problem over the space  $W$ , which is strictly contained in  $L^r(Q)$ .

### 3.2. First variational formulation

To begin, we shall improve slightly the estimates of theorem 2.

LEMMA 3.3. — *It holds*

$$\sup_n \{\|\bar{v}_n\|_{L^r(Q)}, \sup_{t \in [0, T]} \|\bar{v}_n(t)\|_H^2\} \leq C(\mu, T) < +\infty \quad (12)$$

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*Proof.* — For every  $t$  in  $(0, T)$  define  $H_n(t, \cdot)$  as the function:  
 $u \rightarrow H_n(t, u) = \int_0^t \varphi_H^n(\tau, u(\tau)) d\tau + \int_0^t \varphi_H^{n*}(\tau, -u'(\tau)) d\tau + \frac{1}{2} \|u(t)\|_H^2$  on the moving convex set  $K_n(t)$  obtained from  $K_n$  by setting  $t$  instead of  $T$ . Since

$$\frac{1}{2} \|v(t)\|_H^2 = \int_0^t \langle v(\tau), v'(\tau) \rangle_{H, H} d\tau$$

for  $v$  in  $K_n(t)$ ,  $H_n(t, u)$  is, for fixed  $u$ , a nonnegative monotone increasing function of  $t$  bounded by  $\Psi_H^n(u)$ , hence  $0 \leq H_n(t, \bar{v}_n) \leq \psi_H^n(\bar{v}_n) = 0$  on  $K_n(t)$ , that is  $\bar{v}_n$  is a minimizer of  $H_n(t, \cdot)$ . But then

$$\sup_n \sup_t H_n(t, 0) \leq \Psi_H^n(0) < +\infty,$$

from which follows in particular:

$$\sup_n \|\bar{v}_n\|_{L^\infty(0, T; H)} < +\infty$$

and by interpolation

$$\sup_n \|\bar{v}_n\|_{L^r(Q)} \leq C(\mu, T). \quad \square$$

Theorem 3.1 and lemma 3.3 show that  $\bar{v}_n$  should be sought in a ball of the space  $X \cap L^r(Q)$ . To this end, let  $B^a$  be this ball of radius  $a$ , and  $I_{B^a}$  its characteristic function. We then consider the following functions:

$$H_2 \ni u \longrightarrow \Phi^n(u) = \Phi_H^n(u) + I_{B^a}(u) \quad (13)$$

$$H_2 \ni u \longrightarrow G^n(u) = \Phi^n(u) + \Phi^{n*}(-u') + \frac{1}{2} \|u(T)\|_H^2 \quad (14)$$

Then

LEMMA 3.4. —  $\bar{v}_n$  is a minimizer of  $G^n$  on the convex set

$$C_n = \{v \in C(0, T; H), \Phi^n(v) < +\infty, \Phi^{n*}(-v') < +\infty, v(0) = 0\}$$

*Proof.* — First,  $\bar{v}_n \in B^a$  yields  $\Phi^n(\bar{v}_n) = \Phi_H^n(\bar{v}_n)$   
 Then, for every  $w \in H_2$ , we have

$$\begin{aligned} \Phi^{n*}(w) &= (\Phi_H^{n*} \nabla I_{B^a}^*)^{**}(w) \\ &\leq (\Phi_H^{n*} \nabla I_{B^a}^*)(w) \\ &= \inf_{x \in H_2} (\Phi_H^{n*}(w - x) + I_{B^a}^*(x)) \\ &\leq \Phi_H^{n*}(w) \end{aligned}$$

Indeed  $0 \in H_2$  and  $I_{B^a}^*(0) = 0$ ,  $I_{B^a}^*$  being a positively homogeneous function. Consequently, we have:  $\bar{v}_n \in \text{dom}(G^n)$  and

$$0 \leq G^n(\bar{v}_n) \leq \Phi_H^n(\bar{v}_n) + \Phi_H^{n*}(-\bar{v}'_n) + \frac{1}{2} \|\bar{v}_n(T)\|_H^2 = \psi_H^n(\bar{v}_n) = 0$$

which means exactly  $G^n(\bar{v}_n) = 0$  and  $\bar{v}_n$  minimizes  $G^n$ .  $\square$

LEMMA 3.5. — *We have*

$$\sup_n \|\bar{v}'_n\|_{L^{r'}(Q)} < +\infty \quad (15)$$

*Proof.* — Let  $v \in \text{dom}\Phi^n$ . Then  $\|v\|_V \leq a$  and

$$\begin{aligned} \Phi^n(v) &= \int_0^T \|Dv(t)\|_H^2 dt + \int_0^T J_n(t, u(t)) dt \leq \int_0^T |J_n(t, u(t))| dt + C \\ &\leq C \int_0^T \|v\|_r^r dt + C = C\|v\|_{L^r(Q)} + C \end{aligned}$$

which gives

$$\Phi^{n*}(w) \geq C\|w\|_{L^{r'}(Q)} - C.$$

By the preceding lemma,  $\bar{v}_n$  minimizes  $G^n$  on  $C_n$ ,  $0 \in C_n$  and  $\sup_n G^n(0) < +\infty$ , hence

$$\sup_n \|\bar{v}'_n\|_{L^{r'}(Q)} < +\infty \quad \square$$

*Remark.* — This estimate could be in fact obtained starting from the Euler equation

$$(\partial_t - \Delta)\bar{v}_n + \beta(E_n + \bar{v}_n) \in L^{r'}(Q)$$

by using classical regularity results ([10]) for the linear heat equation, since  $\beta(E_n + \bar{v}_n) \in L^{r'}(Q)$ .

The following result will be frequently used in the sequel:

PROPOSITION 3.1. — *Let  $(v_n)_{n \in \mathbb{N}}$  be a sequence in  $L^r(Q)$ . Then:*

$\{v_n \text{ converges to } v \text{ in } L^r(Q) \text{ weakly and in measure}\} \implies \{J^n(v_n) \rightarrow J(v)\}$

*Proof.* — Let  $v \in L^1_{loc}(Q)$ ,  $j^n(v) = j_\beta(x, v(t, x) + E_n(t, x)) - j_\beta(x, E_n(t, x))$  and let  $\epsilon > 0$  be given. Then, for every pair of conjugate exponents  $\bar{r}, \bar{r}'$  we have

$$|j^n(v)|^{1+\epsilon} \leq C_1 + C_2(|v|^{(1+\epsilon)\bar{r}} + |v|^{(1+\epsilon)(1+\alpha)} + |E_n(x, t)|^{\alpha\bar{r}'(1+\epsilon)}) \quad (16)$$

$$\leq C_1 + C_2[|v|^{(1+\epsilon)\bar{r}} + |E_n(x, t)|^{\alpha\bar{r}'(1+\epsilon)}] \quad (17)$$

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Take  $\bar{r} = \frac{r}{1+\epsilon}$ , and let  $\gamma(\epsilon) = \alpha\bar{r}'(1+\epsilon) = \alpha\frac{\bar{r}}{\bar{r}-1}(1+\epsilon) = \alpha\frac{r}{r-(1+\epsilon)}(1+\epsilon)$ . Then  $\gamma(\epsilon)$  is continuous at  $\epsilon = 0$  and  $\gamma(0) = \alpha r' < 1 + 2/N$ . For  $\epsilon$  sufficiently small, we have  $\gamma(\epsilon) < 1 + 2/N$ . With this choice of  $\epsilon$ , we have

$$\begin{aligned} \int_0^T \int_{\Omega} |j^n(v)|^{1+\epsilon} dxdt &\leq C_1|Q| + C_2 \int_0^T \int_{\Omega} |v|^r dxdt + \int_0^T \int_{\Omega} |E_n(t, x)|^{\alpha r'} dxdt \\ &\leq C_1 + C_2 \int_Q |v|^r dxdt \end{aligned}$$

If  $v_n$  converges weakly to  $v$  in  $L^r(Q)$ , we have consequently

$$\sup_n \int_Q |j^n(v_n)|^{1+\epsilon} dxdt < +\infty, \quad (18)$$

that is the sequence  $j^n(v_n)$  is equi-integrable on  $Q$ . Moreover,  $E_n \rightarrow E$  in  $L^p(Q)$  for  $p < 1 + 2/N$ , hence in measure. By dominated convergence, we obtain:

$$j^n(v_n) \rightarrow j(v) \text{ in } L^1(Q)$$

Thus  $J^n(v_n) \rightarrow J(v)$ . □

Lemmas 3.4 and 3.5 show that  $\bar{v}_n$  lies in fact in the Banach space  $\Pi$ .

Let then:

(i)  $\Phi_r^n$  be the restriction of  $\Phi^n$  to  $L^r(Q)$

$$(ii) \Psi^n = \left\{ \begin{array}{l} L^r(Q) \rightarrow ]-\infty, +\infty[ \\ u \rightarrow \Psi^n(u) = \Phi_r^n(u) + \Phi_r^{n\otimes}(-u') + \frac{1}{2} \|u(T)\|_H^2 \end{array} \right\}$$

where  $\Phi_r^{n\otimes}$  is the Young-Fenchel conjugate of  $\Phi_r^n$  in the duality,  $< L^r(Q), L^{r'}(Q) >$ .

(iii)  $K_r^n = \{v \in \Pi, \Phi_r^n(v) < +\infty, \Phi_r^{n\otimes}(-v') < +\infty, v(0) = 0\} \subset L^r(Q)$ .

Then we can prove the following theorem:

**THEOREM 3.2.** — (i)  $\Psi^n$  is coercive and lower-semi-continuous (l.s.c.) on  $L^r(Q)$ .

(ii)  $\bar{v}_n$  is the unique solution of the convex minimization problem

$$\tilde{V}_n := \text{Min}\{\Psi^n(v), v \in K_r^n\}$$

*Proof.* — (i) The coercivity is quite obvious, we prove only the lower semi-continuity.

Let  $u$  in  $L^r(Q)$  and  $u_n$  a sequence such that  $u_n$  converges to  $u$  in  $L^r(Q)$  and  $\sup_n \Psi^n(u_n) < +\infty$ . But then (after extracting a subsequence if necessary)  $u_n \rightarrow u$  in  $X \cap L^r(Q)$  weak, in  $L^2(Q)$  strong and in measure. Moreover  $u'_n \rightarrow u'$  in  $L^{r'}$  weak, hence  $u_n$  converges to  $u$  in  $\Pi$  weak. From the continuity of the trace map,  $u_n(T) \rightarrow u(T)$  in  $H$  weakly and  $\liminf_n |u_n(T)|_H^2 \geq |u(T)|_H^2$ . The conclusion follows by invoking the lower-semi-continuity of  $\Phi_r^n$  and  $\Phi_r^{n^{\otimes}}$  for the  $L^r$  and  $L^{r'}$  topologies respectively.

(ii) We remark first that  $\bar{v}_n$  belongs to  $K_r^n$  and that  $G^n(\bar{v}_n) = 0$ . For  $u$  in  $\Pi$  we use the identity:

$$\frac{1}{2} \|u(T)\|_H^2 = \int_0^T \langle u(t), u'(t) \rangle_{L^r(\Omega), L^{r'}(\Omega)} dt = \langle u, u' \rangle_{L^r(Q), L^{r'}(Q)},$$

which permits us to write

$$\Psi^n(\bar{v}_n) = \Phi_r^n(\bar{v}_n) + \Phi_r^{n^{\otimes}}(-\bar{v}'_n) + \langle \bar{v}_n, -\bar{v}'_n \rangle_{L^r(Q), L^{r'}(Q)},$$

hence

$$0 \leq \Psi^n(\bar{v}_n) = (\bar{v}_n) + \Phi_r^{n^{\otimes}}(-\bar{v}'_n) + \frac{1}{2} \|\bar{v}_n(T)\|_H^2 \leq \Phi^n(\bar{v}_n) + \Phi^{n^*}(-\bar{v}'_n) + \frac{1}{2} \|\bar{v}_n(T)\|_H^2 = 0.$$

The uniqueness follows from the strong convexity .  $\square$

Thus, we have proved the equivalence of the three problems

$$P_n \Leftrightarrow V_n \Leftrightarrow \tilde{V}_n$$

We shall now pass to the limit on the last problem by studying the epi-convergence of the functionals  $\Psi^n$ . First, we recall some basic features concerning epigraphical convergence in Banach spaces (see [1] for details):

**DEFINITION 3.1.** — *Let  $(X, \tau)$  be a Banach space equipped with its strong topology  $\tau$  and let  $F^n, F : X \rightarrow ]-\infty, +\infty]$  be a sequence of  $\tau$ -l.s.c proper functions. Then  $F$  is the  $\tau$ -epi-limit of the sequence  $F^n$  at  $x \in X$  iff:*

(i)  $\forall x_n \xrightarrow{\tau} x$ , we have:  $\liminf_n F^n(x_n) \geq F(x)$ .

(ii) *There exists a sequence  $x_n \xrightarrow{\tau} x$  s.t.  $\limsup_n F^n(x_n) \leq F(x)$ . If this takes place at every point  $x \in X$ , we say that  $F$  is the  $\tau$ -epi-limit of the sequence  $F^n$ .*

Another, closely related notion of epigraphical convergence is the *Mosco-convergence*, which is obtained from the preceding one if we use the weak

convergence in the first sentence. Hence it implies  $\tau$  - *epiconvergence*. We recall also that epi-convergence implies the convergence of the minima and that the Mosco-convergence is bi-continuous with respect to the Young-Fenchel transformation.

We are going first to precise a little more the domains of our fonctions, more precisely we shall show that they are independent of  $n$ .

PROPOSITION 3.2. — *It holds*

- i)  $dom\Phi_r^n = dom\Phi_r$
- ii)  $dom\Phi_r^{n\otimes} = dom\Phi_r^\otimes$
- iii)  $dom\Psi^n = dom\Psi$ .

*Proof.* — We start from the majorizations:

$$\begin{aligned} |\Phi_H^n(v) - \Phi_H(v)| &= \left| \int_0^T \int_\Omega (j_n(t, v(t, x)) - j(t, v(t, x))) dx dt \right| \\ &\leq C_1 \int_0^T \|v(t)\|_r^r dt + C_2 \int_0^T \|E_n(t)\|_{\alpha r'}^{\alpha r'} dt + C_3 \\ &\leq C_1 \|v\|_{L^r(Q)} + C \end{aligned}$$

Hence  $L^r(Q) \cap dom\Phi_H^n = L^r(Q) \cap dom\Phi_H$ . But since  $V \cap L^r(Q) \subset dom\Phi_H^n \subset V$ , we obtain:

$$L^r(Q) \cap dom\Phi_H^n = L^r(Q) \cap dom\Phi_H = V \cap L^r(Q)$$

and consequently  $dom\Phi_r^n = dom\Phi_r \subset B^a$ .

For  $\Phi_r^{n\otimes}$ , we remark that if  $w \in dom\Phi_r^{n\otimes}$ , we have

$$\begin{aligned} \Phi_r^{n\otimes}(w) &= \sup\{[\langle u, w \rangle_{L^r, L^{r'}} - \Phi_r^n(u)], u \in dom\Phi_r^n\} \\ &\leq \sup\{[\langle u, w \rangle_{L^r, L^{r'}} - \Phi_r(u)], u \in dom\Phi_r\} \\ &\quad + \sup\{|\Phi_r^n(u) - \Phi_r(u)|, u \in dom\Phi_r\} \\ &\leq \Phi_r^\otimes(w) + C \end{aligned}$$

By interchanging  $\Phi_r^{n\otimes}$  and  $\Phi_r^\otimes$ , we obtain the desired result, from which  $dom\Psi_n = dom\Psi$  follows obviously.  $\square$

LEMMA 3.6. — *We have:  $\Phi_r^n$  Mosco-converges to  $\Phi_r$  in  $L^r(Q)$ .*

*Proof.* — (i) Let  $u \in L^r(Q)$  and  $u_n$  converging weakly to  $u$  in  $L^r(Q)$ .

We can assume that  $\Phi_r^n(u_n)$  is bounded, otherwise there is nothing to prove. Then, there exists a subsequence, still denoted  $u_n$ , such that  $\|u_n\|_X$  is bounded,  $u_n$  converges to  $u$  in  $X$  weakly, in  $L^2(Q)$  strongly, almost everywhere and  $u$  belongs to  $B^a$ , from what we deduce easily that  $\liminf \Phi_r^n(u_n) \geq \Phi_r(u)$ .

(ii) Let  $u \in L^r(Q)$ . Then

- i) If  $u \in \text{dom}\Phi_r$ , we have  $u \in \text{dom}\Phi_r^n$  and  $J_n(u) \rightarrow J(u)$  by proposition 1, so that  $\Phi_r^n(u) \rightarrow u$ .
- ii) If not, we have  $+\infty = \Phi_r^n(u) \rightarrow \Phi_r(u) = +\infty$ . Hence the constant sequence  $u_n = u$  works too.  $\square$

COROLLARY 3.1. —  $\Phi_r^{n^\otimes}$  Mosco-converges to  $\Phi_r^\otimes$  in  $L^{r'}(Q)$ .

THEOREM 3.3. —  $\Psi^n$  Mosco-converges to  $\Psi$  in  $L^r(Q)$ .

*Proof.* — (i) Let  $u \in L^r(Q)$  and  $u_n$  converging weakly to  $u$  in  $L^r(Q)$ . We may suppose that  $\Psi^n(u_n)$  is bounded, which gives:

$$\sup_n \|u_n\|_{X \cap L^r(Q)} < +\infty, \sup_n \|u'_n\|_{L^{r'}} < +\infty, \sup_n \|u_n(T)\|_H^2 < +\infty$$

Then (at least for a subsequence):

- i)  $u_n \rightarrow u$  in  $X$  weak and in  $L^r(Q)$  weak.
- ii)  $u_n \rightarrow u$  in  $L^2(Q)$  strong and a.e.
- iii)  $u'_n \rightarrow u'$  in  $L^{r'}(Q)$  weak

This implies that  $u \in \text{dom}\Psi$ ,  $J_n(u) \rightarrow J(u)$ ,  $\liminf_n \Phi_r^n(u_n) \geq \Phi_r(u)$ ,  $\liminf_n \Phi_r^{n^\otimes}(-u'_n) \geq \Phi_r^\otimes(-u')$  and finally  $\liminf_n \Psi^n(u_n) \geq \Psi(u)$ .

(ii) Let  $u \in L^r(Q)$ .

a) Suppose first:  $u \in \text{dom}\Psi$ .

Let  $(u_n), n \in \mathbb{N}$ , be the sequence defined by  $u_n = t_n u$ , where  $t_n$  denotes a sequence of real numbers  $0 < t_n < 1, t_n \rightarrow 1$ . Then  $u_n \in \text{dom}\Psi = \text{dom}\Psi^n, u_n \rightarrow u$  strongly in  $L^r(Q)$  and

$$\Psi^n(u_n) = \Phi_r^n(t_n u) + \Phi_r^{n^\otimes}(-t_n u') + \frac{1}{2} t_n^2 \|u(T)\|_H^2.$$

But

$$\Phi_r^n(u_n) = t_n^2 \int_0^T \|Du(t)\|_H^2 dt + J_n(t_n u) \tag{19}$$

$$< \int_0^T \|Du(t)\|_H^2 dt + t_n J_n(u) + (1 - t_n) J_n(0) \tag{20}$$

$$\Phi_r^{n^\otimes}(-t_n u') \leq t_n \Phi_r^{n^\otimes}(-u') + (1 - t_n) \Phi_r^{n^\otimes}(0) \tag{21}$$

$$< \Phi_r^{n^\otimes}(-u') + (1 - t_n) C \tag{22}$$

which yields:

$$\limsup_n \Phi_r^n(u_n) \leq \int_0^T \|Du(t)\|_H^2 dt + \limsup_n J_n(u) \quad (23)$$

$$= \int_0^T \|Du(t)\|_H^2 dt + J(u) + I^a(u) \quad (24)$$

$$= \Phi_r(u) \quad (25)$$

By Proposition 2.9 in [1], we know that if  $F$  is the epi-limit of the sequence  $F_n$  in some Banach space  $X$  then:  $\inf_X F \geq \limsup_n (\inf_X F^n)$ . If we apply this to the sequence  $F^n(\cdot) = \langle -u', \cdot \rangle_{L^r, L^{r'}} - \Phi_r^n(\cdot)$ , with  $X = L^{r'}(Q)$ , we obtain:

$$\begin{aligned} \inf_{v \in L^r(Q)} \{ \langle -u', v \rangle_{L^r(Q), L^{r'}(Q)} - \Phi_r(v) \} \\ \geq \limsup_n \inf_{L^r(Q)} \{ \langle -u', v \rangle_{L^r(Q), L^{r'}(Q)} - \Phi_r^n(v) \} \end{aligned}$$

hence

$$\liminf_n \Phi_r^{n^\otimes}(-u') \geq \Phi_r^\otimes(-u') \geq \limsup_n \Phi_r^{n^\otimes}(-u')$$

and

$$\limsup_n \Phi_r^{n^\otimes}(-u'_n) \leq \limsup_n \Phi_r^{n^\otimes}(-u') = \Phi_r^\otimes(-u').$$

b) If  $u \in L^r(Q) \setminus \text{dom} \Psi$ , then  $u \in L^r(Q) \setminus \text{dom} \Psi^n$  and the constant sequence  $u_n = u$  works.  $\square$

We can then state our first variational formulation:

**THEOREM 3.4.** — (i) *The sequence  $\bar{u}_n = E_n + \bar{v}_n$  converges in  $L^p(Q)$ ,  $p < 1/2 + 2/N$ , to  $\bar{u} = \bar{v} + E$  the unique solution of the variational problem:*

$$\begin{aligned} V_\mu := \inf \{ \Phi_r(u - E) + \Phi_r^\otimes(-u' + E') \\ + \frac{1}{2} \|(u - E)(T)\|_H^2, u = v + E, v \in \Pi, v(0) = 0 \} \end{aligned}$$

(ii)  $\bar{u}$  is also the weak solution of the problem  $P_\beta$  and

$$\bar{u} \in L^s(0, T; W_0^{1,q}(\Omega)) \text{ with } 1 \leq s, q < \frac{N}{N-1}.$$

*Preuve.* — (i) That  $\bar{v}$  is the solution of the variational problem is a straightforward consequence of the preceding theorem.

(ii) Consider the sequence  $\bar{u}_n$ . It satisfies the Euler equation

$$\partial_t \bar{u}_n - \Delta \bar{u}_n \in \beta(x, \bar{u}_n) \text{ a.e. in } Q, \quad \bar{u}_n(0) = \mu_n \text{ in } \Omega, \quad \bar{u}_n = 0 \text{ on } \partial\Omega \times ]0, T[ \text{ a.e.}$$

Let  $\bar{z}_n = (\partial_t - \Delta)\bar{u}_n$  then  $\bar{z}_n \in \beta(\cdot, \bar{u}_n)$  and  $|z_n| \leq C(1 + |\bar{u}_n|^\alpha) \Rightarrow \sup_n |z_n|_{L^r(Q)} < +\infty$ , whence the equi-integrability of the  $z_n$ . We can thus extract subsequences s.t:

$$\begin{aligned} \bar{z}_n &\rightharpoonup \bar{z} \text{ in } \sigma(L^1(Q), L^\infty(Q)) \\ \bar{u}_n &\rightarrow \bar{u} \text{ in } L^r(Q), \quad r < 1 + \frac{2}{N}, \text{ and in } D'(Q). \end{aligned}$$

In fact,  $u_n \rightarrow \bar{u}$  quasi-uniformly on  $Q$ . For every  $\epsilon > 0$ , there exists  $Q^\epsilon \subset Q$ ,  $meas(Q/Q^\epsilon) < \epsilon$  and a constant  $C(\epsilon)$  s.t.  $\sup_{Q^\epsilon} |\bar{u}_n(x, t)| \leq C(\epsilon)$ . Hence  $u_n \rightarrow \bar{u}$  in  $L^2(Q^\epsilon)$  strongly and  $\bar{z}_n \rightarrow \bar{z}$  in  $L^2(Q^\epsilon)$  weakly.

As the Nemitskii operator associated with  $\beta$  on  $L^2(Q^\epsilon)$  is strong  $\times$  weak closed, we obtain that

$$\bar{z}(x, t) \in \beta(x, \bar{u}(x, t)) \text{ a.e. in } Q^\epsilon$$

Since this is true for every  $\epsilon > 0$ , it follows that

$$\bar{z}(x, t) \in \beta(x, \bar{u}(x, t)) \text{ a.e.}$$

But since  $\bar{z}_n = (\partial_t - \Delta)\bar{u}_n$ , we obtain, taking the limit in  $D'(Q)$ :

$$\bar{z} = (\partial_t - \Delta)\bar{u}$$

$$\text{Thus } \begin{cases} (\partial_t - \Delta)\bar{u}(x, t) \in \beta(x, \bar{u}(x, t)) \text{ a.e. } (x, t) \in Q \\ \bar{u}(x, t) = 0 \text{ on } \partial\Omega \times ]0, T[ \text{ a.e.} \\ \text{ess - lim}_{t \downarrow 0} \bar{u}(t) = \text{ess - lim}_{t \downarrow 0} \bar{v}(t) + \text{ess - lim}_{t \downarrow 0} E(t) = 0 + \mu = \mu \end{cases}$$

The regularity of  $\bar{u}$  is immediate.  $\square$

### 3.3. Second variational formulation

We explain now very briefly the second approach.

$C(\mu, T)$  being the constant of Lemma 4, let  $a = C(\mu, T) + 1$ ; let  $B_H$  (resp.  $B_\infty$ ) be the closed ball of radius  $a$  of  $H$  (resp. of  $L^\infty(0, T; H)$ ), and  $I_{B_H}$  (resp.  $I_{B_\infty}$ ) its characteristic function. We consider the functions  $\varphi^n = \varphi_H^n + I_{B_H}$  and the associated functionals:

$$\begin{aligned} u \in H_2 &\rightarrow \Phi^n(u) = \int_0^T [\varphi_H^n(t, u(t)) + I_{B_H}(u(t))] dt = \Phi_H^n(u) + I_{B_\infty}(u) \\ u \in H_2 &\rightarrow \Phi(u) = \int_0^T [\varphi_H(t, u(t)) + I_{B_H}(u(t))] dt = \Phi_H(u) + I_{B_\infty}(u) \end{aligned}$$

As in the previous paragraph, we have the following result:

PROPOSITION 3.3. — (i)  $\Phi^n$  (resp.  $\Phi^{n^*}$ ) are equi-coercive on  $V$  (resp.  $V'$ )

(ii) For each  $n$ , the evolution equation  $\bar{P}_n : \{u'(t) + \partial\varphi^n(t, u(t)) \ni 0, u_n(0) = 0\}$  possesses a unique strong solution  $v_n$ , which is also the unique solution of the minimization problem:

$$\bar{V}_n := \text{Min}\{G^n(v), v \in K_n\}$$

where:

$$K_n = \{v \in H_2 : v' \in H_2, v(0) = 0, \varphi^n(t, u(t)) \in L^1(0, T), \varphi^{n*}(t, -u'(t)) \in L^1(0, T)\}$$

$$G^n : u \in H_2 \rightarrow G^n(u) = \Phi^n(u) + \Phi^{n*}(u) + \frac{1}{2}\|u(T)\|_H^2$$

(iii) The problems  $P_n$  and  $\bar{P}_n$  have the same solution  $v_n$  and the estimate:  $\sup_n \|v_n\|_W < +\infty$  holds.

The behaviour of the sequence  $\Phi^n$  is then given by the following lemma:

LEMMA 3.7. —  $\Phi^n \rightarrow \Phi$  in the Mosco sense on  $X$ .

*Proof.* — (i) Let  $u \in X$  and  $u_n \rightarrow u$  weakly in  $X$ . Let  $u_n$  be a subsequence s.t.  $\sup_n \Phi^n(u_n) < +\infty$ . Then:

$$\liminf_n \int_0^T \|Du_n\|_H^2 dt \geq \int_0^T \|Du(t)\|_H^2 dt$$

Besides this,  $u_n \xrightarrow{L^2(Q)} u$  hence in measure, and so

$$\liminf_n J_n(u_n) \geq J(u)$$

Finally

$$\sup_n \Phi^n(u_n) < +\infty \implies \sup_n I_{B_\infty}(u_n) < +\infty \implies \text{ess sup}_{[0, T]} \|u_n(t)\|_H < +\infty$$

and for a.e.  $t$  we have:  $u_n(t) \xrightarrow{H} w_t \in H$  weakly. By the continuity of the trace map  $X \rightarrow H^{-1}(\Omega)$ ,  $u'_n(t) \xrightarrow{H^{-1}} u'_t$  weakly in  $H^{-1}(\Omega)$  and hence  $w(t) = u'(t)$  a.e. Thus:

$$\liminf_n \|u_n(t)\|_H^2 \geq \|u(t)\|_H^2 \text{ a.e.,}$$

that is  $\|u(t)\|_H \leq a$  a.e.,  $u \in B_\infty$  and  $I_{B_\infty}(u) = 0$ .

In short we have

$$\liminf_n \Phi^n(u_n) \geq \Phi(u).$$

(ii) Let  $u \in X$ .

1) If  $I_{B_\infty}(u) = +\infty$ , then  $\phi(u) = +\infty, \phi^n(u) = +\infty$ . The sequence  $v_n = u$  converges strongly in  $X$  to  $u$  and  $\Phi^n(u_n) \rightarrow \Phi(u)$ .

2) If  $I_{B_\infty}(u) = 0$ , then  $\|u\|_{L^\infty(0,T;H)} \leq a$  and as  $u \in V$ , we have  $u \in L^r(Q)$ . Hence  $u \in L^r(Q) \cap \text{dom}\Phi_H$ . Then  $\Phi^n(u) = \Phi_H^n(u) + I_{B_\infty}(u) = \Phi_H^n(u)$ . But  $u \in L^r(Q)$  implies that  $J_n(u) \rightarrow J(u)$ , hence

$$\Phi_H^n(u) \rightarrow \Phi(u) \text{ and } \Phi^n(u) \rightarrow \Phi(u)$$

The constant sequence is still suitable.  $\square$

*Remark 3.2.* — If  $r' = 2$  we obtain that  $\Phi^n \rightarrow \Phi$  in the Mosco-sense on  $W = \{u \in V, u' \in V'\}$ .

Let us now define the following functions:

$F^n : u \in H_2 \rightarrow F^n(u) = \Phi^n(u) + \Phi^{n\oplus}(-u') + \frac{1}{2}\|u(T)\|_H^2$  if  $u \in W$ , and  $= +\infty$  otherwise

$F : u \in H_2 \rightarrow F(u) = \Phi(u) + \Phi^\oplus(-u') + \frac{1}{2}\|u(T)\|_H^2$  if  $u \in W$ , and  $= +\infty$  otherwise

Here  $\oplus$  is the conjugation in the duality  $\langle V, V' \rangle$ .

Then, using the same kind of arguments as in the proof of theorem 3.3, we obtain:

**THEOREM 3.5.** — *The sequence  $F^n$  converges in the Mosco-sense to  $F$  in  $W$ .*

We can then state our second variational formulation:

**THEOREM 3.6.** — (i) *The sequence  $\bar{v}_n$  of solutions of the variational problems  $(\bar{V}_n)$  converges weakly in  $W$  (and strongly in  $H_2$ ) to  $\bar{v}$ , unique solution of the variational problem:*

$$\bar{V}_\mu := \text{Min}\{F(v), v \in W, v(0) = 0\}$$

(ii)  $\bar{u} = \bar{v} + E_\mu$  is the weak solution of the problem  $P_\beta$  and

$$\bar{u} \in L^s(0, T; W_0^{1,q}(\Omega)) \text{ with } 1 \leq s, q < \frac{N}{N-1}.$$

(iii) Let  $\varphi_V : H_0^1 \rightarrow \bar{\mathbb{R}}$  be the restriction of  $\varphi$  to  $H_0^1(\Omega)$ . Then  $\bar{v}$  is the strong solution of the following evolution equation in  $H^{-1}(\Omega)$ :

$$u' + \partial\varphi_V(t, u(t)) \ni 0, u(0) = 0.$$

*Proof.* — (i) and (ii) are immediate consequences of the previous statements.

(iii) From the basic properties of epigraphical convergence, we assert that:

$$\inf_W F^n \xrightarrow{n} \inf_W F$$

thus  $\inf_W F = 0$ .

We remark that  $F$  may be written, for  $u \in \text{dom}F$ , as:

$$F(u) = \int_0^T [\varphi_V(t, u(t)) + \varphi_V^*(t, -u'(t)) + \langle u'(t), u(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}] dt$$

where  $\varphi_V^*$  is the conjugate of  $\varphi_V$  in the duality  $\langle H_0^1, H^{-1} \rangle$ . But then  $\inf_W F = 0 \Rightarrow F(\bar{v}) = 0$ , that is

$$\varphi_V(t, \bar{v}(t)) + \varphi_V^*(t, \bar{v}'(t)) + \langle \bar{v}'(t), \bar{v}(t) \rangle_{H^{-1}, H_0^1} = 0 \text{ for a.e. } t$$

or equivalently

$$\bar{v}'(t) + \partial\varphi_V(t, \bar{v}(t)) \ni 0. \quad \square$$

#### 4. Concluding Remarks

Many aspects of this work may be enlarged and deepened. Some extensions, as to other boundary conditions, are easy. On the other hand the extension to time dependent measures  $\mu(t)$  is more involved. Numerical aspects, asymptotic behaviour, periodic datum, abruptly changing data require a deeper investigation. Applications to mechanics of continua, for example to evolution problems in plasticity (quasi-static case) are possible.

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