ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

TARO ASUKE

A remark on the Bott class

Annales de la faculté des sciences de Toulouse 6e série, tome 10, nº 1 (2001), p. 5-21

<http://www.numdam.org/item?id=AFST_2001_6_10_1_5_0>

© Université Paul Sabatier, 2001, tous droits réservés.

L'accès aux archives de la revue « Annales de la faculté des sciences de Toulouse » (http://picard.ups-tlse.fr/~annales/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Annales de la Faculté des Sciences de Toulouse Vol. X, n° 1, 2001
pp. 5–21

A remark on the Bott class (*)

TARO ASUKE (1)

RÉSUMÉ. — Nous définissons un cocycle dans le complexe de Čech-de Rham qui représente la classe ξ_q, la partie imaginaire de la classe de Bott. En utilisant le cocycle nous donnons une explication de l'annulation de la classe ξ_q pour des feuilletages transversalement affines complexes. Nous montrons aussi que la classe ξ_q a une relation avec un cocycle du groupe de automorphismes de variétés complexes.

ABSTRACT. — We define a cocycle in the Čech-de Rham complex which represents the class ξ_q, the imaginary part of the Bott class. By using the cocycle we give a simple explanation of the vanishing of the class ξ_q for transversely complex affine foliations. We show also that the class ξ_q has a certain relationship with a group cocycle.

1. Introduction

In the theory of the characteristic classes of foliations, the Godbillon-Vey class plays a central role. When we consider transversely holomorphic foliations, the imaginary part ξ_q of the Bott class seems more fundamental. For example, the Godbillon-Vey class is decomposed into the product of the classes ξ_q and the first Chern class of the complex normal bundle of the foliation [1]. However, the geometric properties of the class ξ_q is not yet clear. In this paper, we represent the class ξ_q in terms of the Čech-de Rham

(*) Reçu le 27 novembre 2000, accepté le 8 juin 2001
(1) Supported by the JSPS Postdoctoral Fellowships for Research Abroad (1999-2000)
Department of Mathematics, Faculty of Sciences, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8526, Japan.
email: asuke@math.sci.hiroshima-u.ac.jp

cocycles by using a similar technique used by Mizutani [10]. It turns out that the class ξ_q can be represented only by the local holonomy maps of the foliation, in fact, it is a combination of cocycles which represent how the holonomy maps expand and rotate local transversal discs.

As a simple application, we show the vanishing of the class ξ_q in some particular cases, for instance the case where the foliation is transversely complex affine or where the foliation admits a transverse invariant volume form. This can be considered as an analogue of the well-known fact that some of real secondary classes such as the Godbillon-Vey class vanish in the above cases.

Finally we introduce a group cocycle which is a complexification of the Thurston cocycle, and show its relationship with the class ξ_q in the case of foliated bundle with fiber $\boldsymbol{C}^q$.

This article is prepared during the author's stay at École Normale Supérieure de Lyon. He would like to express his gratitude for their warm hospitality.

2. The Čech-de Rham complex

First of all we recall and fix notations and conventions concerning the Čech-de Rham complex. A detail can be found in [6] and we follow the definitions in it. Note that the convention of [10] is slightly different from ours.

We choose and fix a simple open covering $\{U_i\}$ of M, namely, each non-empty intersection of U_i's is contractible. We consider the linear space of q-forms on $U_{i_0}\cap\cdots\cap U_{i_p}$ and denote by $A^{p,q}$ the direct sum of all such spaces obtained by varying the indices $i_0,\cdots,i_p$. There are two operators d and δ, namely, the operator d is naturally induced from the exterior derivative of differential forms. On the other hand, for an element $c=\{c_{i_0i_1\cdots i_p}\}$ of $A^{p,q}$ we set

$$(\delta c)_{i_0i_1\cdots i_{p+1}}=\sum_{j=0}^{p+1}(-1)^j c_{i_0\cdots\widehat{i_j}\cdots i_{p+1}},$$

where the symbol '$\widehat{\ }$' means omission of the index. We define the product $c_1\cdot c_2$ of the elements $c_1\in A^{p,q}$ and $c_2\in A^{r,s}$ by the following formula, namely,

$$(c_1\cdot c_2)_{i_0i_1\cdots i_{p+r}}=(-1)^{qr}(c_1)_{i_0\cdots i_p}\wedge(c_2)_{i_p\cdots i_{p+r}}.$$

We set $D'=\delta$ and $D''=(-1)^p d$. They satisfy the following equations, namely,

$$\begin{aligned} D'(c_1 \cdot c_2) &= D'(c_1) \cdot c_2 + (-1)^{p+q} c_1 \cdot (D' c_2) \quad \text{and} \\ D''(c_1 \cdot c_2) &= D''(c_1) \cdot c_2 + (-1)^{p+q} c_1 \cdot (D'' c_2). \end{aligned}$$

Finally we set $D = D' + D''$ and $\check{C}^r = \bigoplus_{p+q=r} A^{p,q}$. It is well-known that the cohomology of the pair $(\check{C}, D)$ is identified with the both of the Čech and the de Rham cohomology.

NOTATION 2.1. — If $c \in A^{p,q}$, then we denote by $c_{i_0,\cdots,i_p}$ the q-form defined on $U_{i_0} \cap \cdots \cap U_{i_p}$ which is determined by c. Conversely, we denote a family of q-forms $\{c'_{i_0,\cdots,i_p}\}$ with $(p+1)$-indices by omitting the subindices, namely, we say $c' \in A^{p,q}$.

3. A Cocycle in the Čech-de Rham complex

Let M be a manifold and let $\mathcal{F}$ be a transversely holomorphic foliation of complex codimension q. Then we can find a foliation chart $\{U_i\}$ of $(M, \mathcal{F})$, namely,

1) Each U_i is homeomorphic to $V_i \times T_i$, where V_i is an open ball of $\boldsymbol{R}^p$ and T_i is an open ball of $\boldsymbol{C}^q$.

2) Let p_i denote the mapping from U_i to T_i induced by the projection from $V_i \times T_i$ to T_i, then the foliation $\mathcal{F}$ restricted to U_i is given by the plaques $p_i^{-1}(\{z\})$, $z \in T_i$.

3) The local holonomy mappings γ_{ij} induced from the transition mapping φ_{ij} from U_j to U_i are biholomorphic diffeomorphisms.

We denote by $z_i = (z_i^1, \cdots, z_i^q)$ the natural coordinate of T_i and denote by γ'_{ij} the matrix $\dfrac{\partial \gamma_{ij}}{\partial z_j}$. It might be impossible to obtain a simple covering which is simultaneously a foliation chart. In this case we choose a simple covering which is a refinement of some foliation chart. Even after taking the refinement, the mappings γ_{ij} still have a sense and it suffices for our purpose. But for simplicity we pretend as if the refinement were also a foliation chart.

We choose smooth functions θ_{ij} on $U_i \cap U_j$ such that $\det \gamma'_{ij} = \left|\det \gamma'_{ij}\right| e^{2\pi\sqrt{-1}\theta_{ij}}$ and that $\theta_{ji} = -\theta_{ij}$. It is well-known that if we set $c_{ijk} = \theta_{ij} + \theta_{jk} + \theta_{ki}$ on $U_i \cap U_j \cap U_k$, then c_{ijk} are integers and the collection $\{c_{ijk}\}$ defines the first Chern class of the normal bundle $Q(\mathcal{F})$ as an element of the

Čech-de Rham complex, where $Q(\mathcal{F})$ is the complex vector bundle locally spanned by $\dfrac{\partial}{\partial z_i^l}$ mod $T\mathcal{F} \otimes \boldsymbol{C}$.

As we are concerned only with the first Chern forms and its transgression forms, namely, the differential forms which are obtained by taking the traces of connection and curvature matrices, it is convenient to work on the determinant bundle $\bigwedge^q Q(\mathcal{F})$ and $\bigwedge^q Q(\mathcal{F})^*$. Indeed, if ∇ is a connection on $Q(\mathcal{F})$ and ω_i is the connection form on U_i with some local trivialization, say, $e_1, \cdots, e_q$, then ∇ naturally induces a connection of $\bigwedge^q Q(\mathcal{F})$ and its connection form with respect to the local trivialization $e_1 \wedge \cdots \wedge e_q$ is equal to $\operatorname{tr} \omega_i$.

From this point of view, we choose a family of local trivialization $\{a_i dz_i\}$, where $dz_i = dz_i^1 \wedge dz_i^2 \wedge \cdots \wedge dz_i^q$, as follows. We fix a Hermitian metric H on $Q(\mathcal{F})$ and let H_i denotes the local matrix on U_i with respect to $\dfrac{\partial}{\partial z_i^1}, \cdots, \dfrac{\partial}{\partial z_i^q}$. Then $\det H_i$ is a positive real function and the induced metric on $\bigwedge^q Q(\mathcal{F})$ is locally given by $(\det H_i) dz_i \otimes d\bar{z}_i$. We set $a_i = \sqrt{\det H_i}$, then a_i is an $\boldsymbol{R}^+$-valued smooth function and

$$(a_i \circ \varphi_{ij}) \left|\det \gamma'_{ij}\right| = a_j. \tag{3.1}$$

We consider the collection $\log a = \{\log a_i\}$ as a cochain of the Čech-de Rham complex, then the equation (3.1) implies $(\delta \log a)_{ij} = \log \left|\det \gamma'_{ij}\right|$. Note that the right hand side is already defined on an open subset of T_j, and we consider it as a differential form on U_j by pulling it back by p_j.

We now introduce some cochains.

Definition 3.2. — We define cochains Λ and $d\Theta$ by the following formulae, namely,

$$\begin{aligned} (\Lambda)_{ij} &= \tfrac{1}{2\pi} \log \left|\det \gamma'_{ij}\right| = \tfrac{1}{2\pi} (\delta \log a)_{ij}, \\ (d\Theta)_{ij} &= d\theta_{ij}. \end{aligned}$$

Note that Λ is $D'(=\delta)$-closed while $d\Theta$ is D''-closed. As we will see soon, $d\Theta$ is also D'-closed.

Remark 3.3. — The cochain $2\delta(\log a) = 4\pi\Lambda$ is essentially equal to the cochain L which appeared in [10] to study the Godbillon-Vey class. In our case, the Godbillon-Vey class is represented by the cocycle $-2^{2q+1}\Lambda \cdot (d\Lambda)^{2q}$ when the foliation is of complex codimension q.

We consider the Hermitian metric H that we used to define a_i, and adopt $e_i = \dfrac{1}{a_i}\dfrac{\partial}{\partial z_i^1}\wedge\cdots\wedge\dfrac{\partial}{\partial z_i^q}$ as a local trivialization of $\bigwedge^q Q(\mathcal{F})$ on U_i, then e_i is of norm 1. We fix a complex Bott connection ∇_b and a Hermitian connection ∇_h with respect to H on $Q(\mathcal{F})$. The connections ∇_b and ∇_h naturally induce connections on $\bigwedge^q Q(\mathcal{F})$. By abuse of notation, we call them again a Bott connection and a Hermitian connection, and denote them by the same symbols ∇_b and ∇_h because we do not use the connections on $Q(\mathcal{F})$ any more. We denote respectively by α_i and β_i the connection forms of ∇_b and ∇_h (which are now on $\bigwedge^q Q(\mathcal{F})$) on U_i with respect to e_i. Since $\{\alpha_i\}$ and $\{\beta_i\}$ are connections, they satisfy the equation

$$\begin{aligned}\varphi_{ij}^*\omega_i &= \omega_j - \frac{d(\det\gamma'_{ij})}{\det\gamma'_{ij}} + \frac{d\left|\det\gamma'_{ij}\right|}{\left|\det\gamma'_{ij}\right|} \\ &= \omega_j - 2\pi\sqrt{-1}d\theta_{ij},\end{aligned} \tag{3.4}$$

where $\{\omega_i\}$ is either $\{\alpha_i\}$ or $\{\beta_i\}$.

On the other hand, $\alpha_i = -d(\log a_i)+\zeta_i$ for some 1-form ζ_i which involves only dz_i^l, $l=1,\ldots,q$, and $\beta_i+\overline{\beta_i}=0$. We put $\rho_i=\sqrt{-1}\zeta_i$ and consider the collection $\rho=\{\rho_i\}$ as an element of $A^{0,1}$. By rewriting the equation (3.4) applied to $\{\alpha_i\}$, we see that

$$\delta\rho = 2\pi\sqrt{-1}(d\Lambda+\sqrt{-1}d\Theta).$$

Thus all the cochains $\delta\rho$, $\delta\overline{\rho}$, $d\Lambda$ and $d\Theta$ are both D'-closed and D''-closed.

DEFINITION 3.5. — We define a cochain u_1 of $A^{0,1}$ by setting

$$u_1 = \frac{-1}{2\pi\sqrt{-1}}\left(\frac{1}{\sqrt{-1}}\rho - d\log a\right).$$

Note that $u_1 = \frac{-1}{2\pi\sqrt{-1}}\alpha$.

Since $\beta+\overline{\beta}=0$, the equation $\widetilde{u}_1(\nabla_b,\nabla_h) = u_1-\overline{u_1}$ holds on each U_i, where $\widetilde{u}_1(\nabla_b,\nabla_h)$ is the transgression form of the imaginary part of the first Chern class computed by using ∇_b and ∇_h [1]. Let $v_1(\nabla_b)$ denote the first Chern form of the complex normal bundle of the foliation calculated by using the complex Bott connection ∇_b. We set $\overline{v}_1(\nabla_b)=\overline{v_1(\nabla_b)}$.

Here we recall the definition of the class ξ_q [5,1].

DEFINITION 3.6. — We denote by $\xi_q(\mathcal{F})$ the cohomology class represented by the differential form $\xi_q(\nabla_b,\nabla_h)$ which is defined by the formula

$$\xi_q(\nabla_b,\nabla_h) = \sqrt{-1}\,\widetilde{u}_1(\nabla_b,\nabla_h)(v_1(\nabla_b)^q+v_1(\nabla_b)^{q-1}\overline{v}_1(\nabla_b)+\cdots+\overline{v}_1(\nabla_b)^q).$$

It is known that $\xi_q(\mathcal{F})$ does not depend on the choice of connections. It is also known that when the Bott class is well-defined, its imaginary part multiplied by -2 and the class $\xi_q(\mathcal{F})$ agree as cohomology classes [1].

From now on, we consider only differential forms and omit the symbols like ∇_b or ∇_h, for example, we write $v_1(\nabla_b)$ simply by v_1.

The following formulae are for later use. We consider p-forms as cocycles in $A^{0,p}$.

LEMMA 3.7. — *We have the following formulae, namely,*

$$\begin{aligned}&D'u_1 = -d\Theta,\ D''u_1 = v_1 = \frac{1}{2\pi}d\rho,\\ &D'(u_1\cdot\overline{u_1}) = (d\Theta)\cdot(u_1-\overline{u_1}), \text{and}\\ &D''(u_1\cdot\overline{u_1}) = -(u_1\cdot\overline{v_1}-\overline{u_1}\cdot v_1).\end{aligned}$$

Proof. — Recall that α_i denotes the local connection form of the Bott connection ∇_b. The equation $(v_1)_i = \frac{-1}{2\pi\sqrt{-1}}d\alpha_i = \frac{1}{2\pi}d\rho_i$ is equivalent to the second formula. On the other hand, we have the following equations as cochains, namely,

$$\begin{aligned}D'u_1 &= \frac{-1}{2\pi\sqrt{-1}}\left(\frac{1}{\sqrt{-1}}\delta\rho - d\log|\det\gamma'|\right)\\ &= \frac{-1}{2\pi\sqrt{-1}}\left(2\pi\sqrt{-1}d\theta + d\log|\det\gamma'| - d\log|\det\gamma'|\right)\\ &= -d\Theta,\end{aligned}$$

where $\log|\det\gamma'|$ is a cochain in $A^{1,0}$ given by $(\log|\det\gamma'|)_{ij} = \log\left|\det\gamma'_{ij}\right|$.

Finally, we have the following relation:

$$D'(u_1\cdot\overline{u_1}) = (\delta u_1)\cdot\overline{u_1} - u_1\cdot(\delta\overline{u_1}) = (-d\Theta)\cdot\overline{u_1} - u_1\cdot(-d\Theta).$$

From the equation $((u_1)_j-(u_1)_i)\wedge d\theta_{ij} = (\delta u_1)_{ij}\wedge d\theta_{ij} = -d\theta_{ij}\wedge d\theta_{ij} = 0$, it follows that

$$\begin{aligned}(u_1\cdot(-d\Theta))_{ij} &= -(u_1)_i\wedge(-d\theta_{ij}) = (u_1)_j\wedge d\theta_{ij}\\ &= -d\theta_{ij}\wedge(u_1)_j = -((d\Theta)\cdot u_1)_{ij}.\end{aligned}$$

The last formula can be obtained in a similar way. □

The main result in this section is the following.

THEOREM 3.8. — *The cochain*

$$\Xi_q = -\sum_{i=0}^{q} 2\mathrm{Re}\left\{(-\sqrt{-1})^q\left(d\Lambda+\sqrt{-1}d\Theta\right)^i\cdot\Lambda\cdot(\sqrt{-1}d\Theta)^{q-i}\right\}$$

of $A^{q+1,q}$ is D-closed and in fact represents the class $\xi_q(\mathcal{F})$. In particular, the cohomology class determined by Ξ_q is independent of the choices we made.

For example, in the case of complex codimension one, Ξ_1 is given by the formula

$$\Xi_1 = -2\Lambda \cdot (d\Theta) - 2(d\Theta) \cdot \Lambda.$$

Proof. — We have the following equation as cocycles:

$$\begin{aligned}
\xi_q &= \sqrt{-1}\,\widetilde{u}_1(v_1^q + v_1^{q-1}\overline{v_1} + \cdots + \overline{v_1}^q) \\
&= \sqrt{-1}(u_1 - \overline{u_1}) \cdot (v_1^q + v_1^{q-1}\overline{v_1} + \cdots + \overline{v_1}^q) \\
&= \sqrt{-1}(u_1 \cdot v_1^q - \overline{u_1} \cdot \overline{v_1}^q) + \sqrt{-1}(u_1 \cdot \overline{v_1} - \overline{u_1} \cdot v_1) \cdot V(q-1),
\end{aligned}$$

where $V(m)$ is the element of $A^{0,2m}$ defined by the formula

$$V(m) = \begin{cases} v_1^m + v_1^{m-1}\overline{v_1} + \cdots + \overline{v_1}^m & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ 0 & \text{if } m < 0. \end{cases}$$

We set

$$\begin{aligned}
c_{m,2q-m} = \;& \sum_{i=0}^{m-1} 2\mathrm{Re}\left\{(-1)^i \left(\frac{1}{2\pi}\delta\rho\right)^i \cdot \Lambda \cdot (d\Theta)^{m-i-1} \cdot u_1 \cdot v_1^{q-m}\right\} \\
& -\sqrt{-1}(d\Theta)^m \cdot (u_1 \cdot \overline{u_1}) \cdot V(q-m-1) \\
& +(-1)^m 2\mathrm{Re}\left\{\left(\frac{1}{2\pi}\delta\rho\right)^m \cdot \left(\frac{1}{2\pi}\log a\right) \cdot v_1^{q-m}\right\},
\end{aligned}$$

where $m = 0, 1, \cdots, q$.

We claim that $D'c_{m,2q-m} + D''c_{m+1,2q-m-1} = 0$ for $0 \leqslant m \leqslant q-1$ and that $D''c_{0,2q} = \xi_q$. Admitting this claim, we see that ξ_q is cohomologous to $-D'c_{q,q}$, which is calculated as follows:

$$\begin{aligned}
-D'c_{q,q} &= -\sum_{i=0}^{q} 2\mathrm{Re}\left\{(-1)^i \left(\frac{1}{2\pi}\delta\rho\right)^i \cdot \Lambda \cdot (d\Theta)^{q-i}\right\} \\
&= -\sum_{i=0}^{q} 2\mathrm{Re}\left\{(-\sqrt{-1})^i \left(d\Lambda + \sqrt{-1}d\Theta\right)^i \cdot \Lambda \cdot (d\Theta)^{q-i}\right\}.
\end{aligned}$$

Thus it suffices to establish the equations in our claim. By Lemma 3.7, we have the following equations, namely,

$$\begin{aligned}
D'c_{m,2q-m} = \;& \sum_{i=0}^{m} 2\mathrm{Re}\left\{(-1)^i \left(\frac{1}{2\pi}\delta\rho\right)^i \cdot \Lambda \cdot (d\Theta)^{m-i} \cdot v_1^{q-m}\right\} \\
& -\sqrt{-1}(d\Theta)^{m+1} \cdot (u_1 - \overline{u_1}) \cdot V(q-m-1),
\end{aligned}$$

and

$$\begin{aligned}
D''c_{m,2q-m} = & \sum_{i=0}^{m-1} 2\mathrm{Re}\left\{(-1)^i\left(\frac{1}{2\pi}\delta\rho\right)^i\cdot(-d\Lambda)\cdot(d\Theta)^{m-i-1}\cdot u_1\cdot v_1^{q-m}\right\}\\
& -\sum_{i=0}^{m-1} 2\mathrm{Re}\left\{(-1)^i\left(\frac{1}{2\pi}\delta\rho\right)^i\cdot\Lambda\cdot(d\Theta)^{m-i-1}\cdot v_1^{q-m+1}\right\}\\
& +\sqrt{-1}(d\Theta)^m\cdot(u_1\cdot\overline{v_1}-\overline{u_1}\cdot v_1)\cdot V(q-m-1)\\
& +(-1)^m 2\mathrm{Re}\left\{\left(\frac{1}{2\pi}\delta\rho\right)^m\cdot\left(\frac{1}{2\pi}d\log a\right)\cdot v_1^{q-m}\right\}.
\end{aligned}$$

In particular,

$$\begin{aligned}
D''c_{0,2q} & = \sqrt{-1}(u_1\cdot\overline{v_1}-\overline{u_1}\cdot v_1)\cdot V(q-1)+2\mathrm{Re}\left\{\left(\frac{1}{2\pi}d\log a\right)\cdot v_1^q\right\}\\
& = \sqrt{-1}(u_1\cdot\overline{v_1}-\overline{u_1}\cdot v_1)\cdot V(q-1)+\frac{1}{2\pi}(d\log a)\cdot(v_1^q+\overline{v_1}^q),
\end{aligned}$$

which is equal to ξ_q because

$$\begin{aligned}
& \frac{1}{2\pi}(d\log a)\cdot(v_1^q+\overline{v_1}^q)\\
= & \frac{-1}{2\pi}\left(\frac{1}{\sqrt{-1}}\rho-d\log a\right)\cdot v_1^q+\frac{-1}{2\pi}\left(-\frac{1}{\sqrt{-1}}\overline{\rho}-d\log a\right)\cdot\overline{v_1}^q\\
= & \sqrt{-1}(u_1\cdot v_1^q-\overline{u_1}\cdot\overline{v_1}^q).
\end{aligned}$$

The last equation holds because ρ involves only dz_i's and hence $\rho\cdot v_1^q=\rho\cdot(2\pi d\rho)^q=0$.

On the other hand,

$$\begin{aligned}
& D'c_{m,2q-m}+D''c_{m+1,2q-m-1}\\
= & -\sqrt{-1}(d\Theta)^{m+1}\cdot\Big((u_1-\overline{u_1})\cdot V(q-m-1)-(u_1\cdot\overline{v_1}-\overline{u_1}\cdot v_1)\\
& \qquad\cdot V(q-m-2)\Big)\\
& +\sum_{i=0}^{m} 2\mathrm{Re}\left\{(-1)^{i+1}\left(\frac{1}{2\pi}\delta\rho\right)^i\cdot d\Lambda\cdot(d\Theta)^{m-i}\cdot u_1\cdot v_1^{q-m-1}\right\}\\
& +(-1)^{m+1}2\mathrm{Re}\left\{\left(\frac{1}{2\pi}\delta\rho\right)^{m+1}\cdot\left(\frac{1}{2\pi}d\log a\right)\cdot v_1^{q-m-1}\right\}.
\end{aligned}$$

Noticing that

$$\begin{aligned}
& (u_1-\overline{u_1})\cdot V(q-m-1)\\
= & (u_1\cdot v_1^{q-m-1}-\overline{u_1}\cdot\overline{v_1}^{q-m-1})+(u_1\cdot\overline{v_1}-\overline{u_1}\cdot v_1)\cdot V(q-m-2),
\end{aligned}$$

we see that

$$\begin{aligned}
&-\sqrt{-1}(d\Theta)^{m+1}\cdot\Big((u_1-\overline{u_1})\cdot V(q-m-1)-(u_1\cdot\overline{v_1}-\overline{u_1}\cdot v_1)\\
&\qquad\cdot V(q-m-2)\Big)\\
=\ &-\sqrt{-1}(d\Theta)^{m+1}\cdot(u_1\cdot v_1^{q-m-1}-\overline{u_1}\cdot\overline{v_1}^{q-m-1})\\
=\ &-2\mathrm{Re}\left\{\sqrt{-1}(d\Theta)^{m+1}\cdot u_1\cdot v_1^{q-m-1}\right\}.
\end{aligned}$$

Finally, we have the following equations:

$$\begin{aligned}
&(-1)^{i+1}2\mathrm{Re}\left\{\sqrt{-1}\left(\frac{1}{2\pi}\delta\rho\right)^{i}\cdot(d\Theta)^{m-i+1}\cdot u_1\cdot v_1^{q-m-1}\right\}\\
&+(-1)^{i+1}2\mathrm{Re}\left\{\left(\frac{1}{2\pi}\delta\rho\right)^{i}\cdot d\Lambda\cdot(d\Theta)^{m-i}\cdot u_1\cdot v_1^{q-m-1}\right\}\\
=\ &(-1)^{i+2}2\mathrm{Re}\left\{\sqrt{-1}\left(\frac{1}{2\pi}\delta\rho\right)^{i+1}\cdot(d\Theta)^{m-i}\cdot u_1\cdot v_1^{q-m-1}\right\}
\end{aligned}$$

and

$$\begin{aligned}
&(-1)^{m}2\mathrm{Re}\left\{\sqrt{-1}\left(\frac{1}{2\pi}\delta\rho\right)^{m+1}\cdot u_1\cdot v_1^{q-m-1}\right\}\\
=\ &(-1)^{m}2\mathrm{Re}\left\{\sqrt{-1}\left(\frac{1}{2\pi}\delta\rho\right)^{m+1}\cdot\left(\frac{1}{2\pi\sqrt{-1}}d\log a\right)\cdot v_1^{q-m-1}\right\}.
\end{aligned}$$

Combining these formulae, one can show that $D'c_{m,2q-m}+D''c_{m+1,2q-m-1}=0$ for $0\leqslant m\leqslant q-1$ as claimed. □

Remark 3.9. — The equation $d\Xi_q=0$ corresponds to the Bott vanishing theorem applied to $v_1^{q+1}(\nabla_b)-\overline{v}_1^{q+1}(\nabla_b)$. We also remark that the decomposition of the Godbillon-Vey class into the class $\xi_q(\mathcal{F})$ and the first Chern class given in [1] can be obtained again from Theorem 3.8 and Remark 3.3.

Remark 3.10. — The Bott class $u_1v_1^q(\mathcal{F})$ is defined if the first Chern class of the complex normal bundle $Q(\mathcal{F})$ is trivial. We can show as in [10] that

$$u_1v_1^q(\mathcal{F})=(-\sqrt{-1})^q\left(\Lambda+\sqrt{-1}\Theta\right)\cdot\left(d\Lambda+\sqrt{-1}d\Theta\right)^q$$

as cohomology classes. In fact, the above equation holds as cocycles under a suitable choice of connections. By taking the imaginary part of the right hand side, we obtain a cocycle cohomologous to $-2\Xi_q$.

Theorem 3.8 says that the class ξ_q reflects how the local holonomy maps expand and rotate transverse discs. There are two particular cases where the cocycle Ξ_q vanishes. One is the case where $\Lambda = 0$ as a cochain in $A^{1,0}$. In this case the foliation in fact preserves a transverse volume form locally defined by $dz_i \wedge \overline{dz_i}$ on T_i. Notice that if moreover the foliation is of complex codimension one, then the foliation is Riemannian. Conversely if the foliation preserves a transverse volume form, then we can find a Hermitian metric so that $(\det H_i) dz_i \otimes d\bar{z}_i$ coincides with the given transverse volume form. With this choice, $\Lambda = 0$. Here we used the notation as in the beginning of this section.

The other case is that $d\Lambda = 0$ or $d\Theta = 0$ as cocycles in $A^{1,1}$. In this case the functions $\det \gamma'_{ij}$ take values in the circle or the real line, hence they are some constants. Hence the both of the cocycles $d\Lambda$ and $d\Theta$ are equal to zero, and thus $\Xi_q = 0$. Note that if in addition the codimension q is equal to one, then the mappings γ_{ij} are complex affine transformations, namely, mappings of the form $z \mapsto az + b$ where $a \in \boldsymbol{C}^*$ and $b \in \boldsymbol{C}$. A typical example is given by the flow on a T^2-bundle over S^1 obtained as the suspension of an automorphism of T^2 defined by an element of $\mathrm{SL}(2, \boldsymbol{Z})$. Conversely, if the foliation is transversely complex affine, then $\det \gamma'_{ij}$ are constants and therefore $\Xi_q = 0$ even if the codimension of the foliation is greater than one.

Thus we showed the following well-known fact (cf. [4,11,14]).

COROLLARY 3.11. — *The class ξ_q and the Godbillon-Vey class GV_{2q} vanish in the following cases:*

1) The foliation $\mathcal{F}$ admits a transverse volume form.

2) The foliation is transversely complex affine.

Not only the Godbillon-Vey class but some other real secondary classes are factored by ξ_q [1]. These classes also vanish in the above cases.

Corollary 3.11 is already well-known for specialists. To say about transversely affine case, it is a corollary of the following general result, which is also well-known for specialists. The proof we give here is indicated by Tsuboi [18].

THEOREM 3.12. — *The secondary classes of transversely real affine foliations determined by $H^*(\mathrm{WO}_q)$ are trivial. The secondary classes of transversely complex affine foliations determined by $H^*(\mathrm{WU}_q)$ are also trivial.*

Proof. — We show only the real case since the proof of the complex case is almost parallel. Suppose that we are given a transversely affine foliation $\mathcal{F}$ of a manifold M. Then we can classify the foliation $\mathcal{F}$ by a map $f : M \to \mathrm{B}\Gamma_{\mathrm{Aff}_q}$, where Aff_q denotes the group of affine transformations of $\boldsymbol{R}^q$. Since the elements of Aff_q are analytic and globally well-defined on $\boldsymbol{R}^q$, the space $\mathrm{B}\Gamma_{\mathrm{Aff}_q}$ is equivalent to BAff_q. We now consider the natural projection $\pi : \mathrm{Aff}_q \to \mathrm{GL}(q, \boldsymbol{R})$, then π induces a mapping p from BAff_q to $\mathrm{BGL}(q, \boldsymbol{R})$. On the other hand, a theorem of Suslin [13,17] shows that the projection π induces an isomorphism $H_*(\mathrm{Aff}_q; \boldsymbol{R}) \to H_*(\mathrm{GL}(q, \boldsymbol{R}); \boldsymbol{R})$. This implies that the mapping $p \circ f : M \to \mathrm{BGL}(q, \boldsymbol{R})$, slightly perturbed if necessary, classifies a foliation which is cobordant to $\mathcal{F}$. Noticing that the $\boldsymbol{R}^q$-bundle associated with $\mathrm{EGL}(q, \boldsymbol{R}) \to \mathrm{BGL}(q, \boldsymbol{R})$ has a natural zero section, say s, we see that every secondary class reduces to a leaf corresponding to s and hence vanishes. $\square$

Remark 3.13. — It is shown that in [16] that the space BAff_q is in fact contractible.

We now consider a foliated vector bundle E over M whose fiber is isomorphic to $\boldsymbol{C}^q$ and assume that E is trivial when restricted to each U_i. Thus we can choose each T_i as $\boldsymbol{C}^q$ and $U_i \cong V_i \times \boldsymbol{C}^q$ for some open set V_i of M.

Once the formula presented in Theorem 3.8 is established, we can apply the argument used to prove Theorem B of [10]. Namely, since the cocycle Ξ_q is already defined on an open subset of $\amalg T_i$, we can define an element Γ_q of $A^{2q+1,0}$ by the formula

$$(\Gamma_q)_{i_0,\cdots,i_{2q+1}} = \int_{\Delta_0^q} (\Xi_q)_{i_0,\cdots,i_{q+1}},$$

where Δ_0^q is a singular cubic q-simplex given as

$$\Delta_0^q(t_{q+1}, \cdots, t_{2q}) = t_{q+1}\gamma_{i_{q+1},i_{q+2}}(t_{q+2}\gamma_{i_{q+2},i_{q+3}}(\cdots(t_{2q}\gamma_{i_{2q},i_{2q+1}}(0))\cdots).$$

The following theorem is shown completely parallel to the real case as in [10].

THEOREM 3.14. — *Γ_q is a cocycle in $A^{2q+1,0}$ which is cohomologous to the cocycle Ξ_q.*

Like the Godbillon-Vey class, the cocycle Ξ_q defines an element of a group cohomology. We will explain it in the next section.

4. A Group Cocycle

Let F be a complex manifold, which is possibly non-compact, of complex dimension q. We assume that there is a holomorphic volume form of F, namely, we assume that there is a holomorphic q-form ω on F which is everywhere non-zero. We denote by G the group of biholomorphic automorphisms of F. The group G acts on the space $\Omega^*(F)$ of smooth differential forms on F by the formula

$$\omega \mapsto (f^{-1})^*\omega.$$

We denote the operator $(f^{-1})^*$ simply by f_*, then $(f \circ g)_* = f_* \circ g_*$.

We set

$$\mathcal{A}^{p,q} = \mathrm{Map}(G^p, \Omega^q(F))$$

and call it the space of (p,q)-cochains. We define mappings $\mathcal{D}' : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $\mathcal{D}'' : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$ by setting $\mathcal{D}' = \partial$ and $\mathcal{D}'' = (-1)^p d$, where d denotes the usual exterior differential and ∂ denotes the differential of group cochains, namely, the operator given by

$$\begin{aligned} & \partial\alpha(f_1, \cdots, f_r) \\ = \; & f_1 \cdot \alpha(f_2, \cdots, f_r) - \alpha(f_1 \circ f_2, f_3, \cdots, f_r) \\ & - \cdots + (-1)^{r-1}\alpha(f_1, \cdots, f_{r-2}, f_{r-1} \circ f_r) + (-1)^r \alpha(f_1, \cdots, f_{r-1}). \end{aligned}$$

The total complex $\mathcal{C}^n = \bigoplus_{p+q=n} \mathcal{A}^{p,q}$ equipped with the differential $\mathcal{D} = \mathcal{D}' + \mathcal{D}''$ defines the group cohomology $H^*(G; \Omega^*(F))$ of G with coefficients in $\Omega^*(F)$. Note that at the cochain level, the product $\alpha \cdot \beta$ of two elements $\alpha \in \mathcal{A}^{p,q}$ and $\beta \in \mathcal{A}^{r,s}$ is given by the formula

$$\alpha \cdot \beta(f_1, \cdots, f_{p+r}) = (-1)^{qr}\alpha(f_1, \cdots, f_p) \wedge (f_{1,\cdots,p})_*\beta(f_{p+1,\cdots,p+r}),$$

where $f_{1,\cdots,i} = f_1 \circ f_2 \circ \cdots \circ f_i$. The product $\alpha \cdot \beta$ is an element of $\mathcal{A}^{p+r,q+s}$. We have the equations

$$\begin{aligned} \mathcal{D}'(\alpha \cdot \beta) &= (\mathcal{D}'\alpha) \cdot \beta + (-1)^{p+q}\alpha \cdot (\mathcal{D}'\beta) \\ \mathcal{D}''(\alpha \cdot \beta) &= (\mathcal{D}''\alpha) \cdot \beta + (-1)^{p+q}\alpha \cdot (\mathcal{D}''\beta). \end{aligned}$$

We refer to [8] for more details of the group cohomology.

Definition 4.1. — For each element f of G, we define real valued functions $\lambda(f)$ and $d\theta(f)$ as follows, namely, first we define a holomorphic function Δ_f by the formula $\Delta_f\omega = f_*\omega$. Then we set

$$\begin{aligned} \lambda(f) &= \frac{1}{2\pi} \log |\Delta_f|, \\ d\theta(f) &= \frac{1}{2\pi} \mathrm{Im} \frac{d\Delta_f}{\Delta_f}. \end{aligned}$$

We consider λ as a $(1,0)$-cochain and $d\theta$ as a $(1,1)$-cochain, respectively, and define a $(q+1,q)$-cochain B^{q+1}_ω by the formula

$$B^{q+1}_\omega = -2\mathrm{Re}\sum_{i=0}^{q}\left\{(\sqrt{-1})^q(\sqrt{-1}d\theta)^{q-i}\cdot\lambda\cdot(d\lambda+\sqrt{-1}d\theta)^i\right\}.$$

For example, $B^2_\omega(f_1,f_2) = 2\lambda(f_1)((f_1)_*d\theta(f_2)) - 2d\theta(f_1)((f_1)_*\lambda(f_2))$.

We have the following lemma.

LEMMA 4.2. — *$d\lambda(f)+\sqrt{-1}d\theta(f) = \frac{1}{2\pi}\frac{d\Delta_f}{\Delta_f}$. Hence B^{q+1}_ω is $\mathcal{D}''$-closed, namely $B^{q+1}_\omega(f_1,\cdots,f_q)$ is a closed form.*

Proof. — The first claim is easy. The second claim is a direct consequence of the fact that $dB^{q+1}_\omega(f_1,\cdots,f_q)$ is a constant multiple of the real part of $(\sqrt{-1})^q\frac{d\Delta_{f_1}}{\Delta f_1}\wedge\frac{d\Delta_{f_{1,2}}}{\Delta f_{1,2}}\wedge\cdots\wedge\frac{d\Delta_{f_{1,2,\cdots,q+1}}}{\Delta_{f_{1,2,\cdots,q+1}}}$, which is equal to zero by the dimension reason. □

LEMMA 4.3. — *λ and $d\theta$ are $\mathcal{D}'$-closed, namely, the formulae*

$$\begin{aligned}\lambda(f\circ g) &= f_*\lambda(g)+\lambda(f)\\ d\theta(f\circ g) &= f_*d\theta(g)+d\theta(f)\end{aligned}$$

hold. Thus B^{q+1}_ω is also $\mathcal{D}'$-closed.

Proof. — This follows from the fact that $\Delta_{(f\circ g)} = (\Delta_g\circ f^{-1})\cdot\Delta_f$. □

Hence B^{q+1}_ω defines a cohomology class of $H^*(G;\Omega^*(F))$. Now we replace the volume form ω with another one, say, ω'. Then there is a non-vanishing holomorphic function α such that $\omega' = \alpha\omega$. We denote by λ' and $d\theta'$ the cocycles obtained by using ω', then we have the following. Recall that we denote by ∂ the differential of group cochains.

LEMMA 4.4. — *The equations*

$$\begin{aligned}\lambda'(f) &= \lambda(f)+\partial\left(\frac{1}{2\pi}\log|\alpha|\right)(f)\\ d\theta'(f) &= d\theta(f)+\partial\left(\frac{1}{2\pi}\mathrm{Im}\frac{d\alpha}{\alpha}\right)(f)\end{aligned}$$

hold, where the function $|\alpha|$ and the 1-form $\mathrm{Im}\dfrac{d\alpha}{\alpha}$ are viewed as a $(0,0)$-cochain and a $(0,1)$-cochain, respectively.

Proof. — By definition $\Delta'_f\omega' = f_*\omega'$. Hence $\Delta'_f = \dfrac{f_*\alpha}{\alpha}\Delta_f$. The lemma follows from this equation. $\square$

LEMMA 4.5. — *The group cohomology class defined by B_ω^{q+1} does not depend on the choice of the volume form ω.*

Proof. — We set $\mu = d\lambda + \sqrt{-1}d\theta$, $d\tau = \frac{1}{2\pi}\frac{d\alpha}{\alpha}$, $\rho = \frac{1}{2\pi}\log|\alpha|$ and $d\sigma = \frac{1}{2\pi}\mathrm{Im}\frac{d\alpha}{\alpha}$, then $d\tau = d\rho + \sqrt{-1}d\sigma$, $\lambda' - \lambda = \partial\rho$ and $d\theta' - d\theta = \partial(d\sigma)$. By using these cochains, we define cochains A_j, B_j and C_j respectively by the following formulae:

$$\begin{aligned}
A_j = {} & \left(\sqrt{-1}d\theta'\right)^j \cdot \lambda' \cdot (\mu + \partial(d\tau))^{q-j}, \\
B_j = {} & \left(\sqrt{-1}d\theta\right)^j \cdot \lambda \cdot \mu^{q-j}, \\
C_j = {} & -\sum_{k=j+1}^{q} \left(\sqrt{-1}d\theta\right)^j \cdot \lambda \cdot \mu^{q-k} \cdot d\tau \cdot (\mu + \partial(d\tau))^{k-j-1} \\
& + \left(\sqrt{-1}d\theta\right)^j \cdot \rho \cdot (\mu + \partial(d\tau))^{q-j} \\
& + \sum_{k=1}^{j} \left(\sqrt{-1}d\theta\right)^{j-k} \cdot \left(\sqrt{-1}d\sigma\right) \cdot \left(\sqrt{-1}d\theta'\right)^{k-1} \cdot \lambda' \cdot (\mu + \partial(d\tau))^{q-j}.
\end{aligned}$$

Note that $\mu + \partial(d\tau) = d\lambda' + \sqrt{-1}d\theta'$. We claim that $\mathcal{D}'C_j = -B_j + A_j$. Indeed, we rewrite $d\theta' = d\theta + \partial(d\sigma)$ and expand this part, then we see that these terms are differential of the last sum in C_j except for the term $\left(\sqrt{-1}d\theta\right)^j \cdot \lambda' \cdot (\mu + \partial(d\tau))^{q-j}$. We divide this remaining term into two, namely, $\left(\sqrt{-1}d\theta\right)^j \cdot \lambda \cdot (\mu + \partial(d\tau))^{q-j}$ and $\left(\sqrt{-1}d\theta\right)^j \cdot \partial\rho \cdot (\mu + \partial(d\tau))^{q-j}$. Then a similar argument shows that except for $\left(\sqrt{-1}d\theta\right)^j \cdot \lambda \cdot \mu^{q-j}$, the former is the differential of the first sum of C_j. Finally, the latter is the differential of the second term of C_j.

On the other hand,

$$\begin{aligned}
\mathcal{D}''C_j = {} & \sum_{k=j+1}^{q} \left(\sqrt{-1}d\theta\right)^j \cdot d\lambda \cdot \mu^{q-k} \cdot d\tau \cdot (\mu + \partial(d\tau))^{k-j-1} \\
& + \left(\sqrt{-1}d\theta\right)^j \cdot d\rho \cdot (\mu + \partial(d\tau))^{q-j} \\
& + \sum_{k=1}^{j} \left(\sqrt{-1}d\theta\right)^{j-k} \cdot \left(\sqrt{-1}d\sigma\right) \cdot \left(\sqrt{-1}d\theta'\right)^{k-1} \cdot d\lambda' \cdot (\mu + \partial(d\tau))^{q-j}.
\end{aligned}$$

In particular, we have the following equation:

$$\begin{aligned}\mathcal{D}''C_0 = &\sum_{k=1}^{q} d\lambda\cdot\mu^{q-k}\cdot d\tau\cdot(\mu+\partial(d\tau))^{k-1}+d\rho\cdot(\mu+\partial(d\tau))^q\\ = &-\left(\sum_{k=1}^{q}\left(\sqrt{-1}d\theta\right)\cdot\mu^{q-k}\cdot d\tau\cdot(\mu+\partial(d\tau))^{k-1}\right)\\ &\qquad -\left(\sqrt{-1}d\sigma\right)\cdot(\mu+\partial(d\tau))^q\,,\end{aligned}$$

because all the cochains $\mu = d\lambda + \sqrt{-1}d\theta$, $\partial\mu$, $d\tau$, $\partial(d\tau)$ are valued in holomorphic differential forms and thus their product vanishes if its degree as a differential form exceeds q. We now claim that the following equation holds, namely,

$$\begin{aligned}&\mathcal{D}''(C_0+C_1+\cdots+C_j)\\ = &-\sum_{k=j+1}^{q}\left(\sqrt{-1}d\theta\right)^{j+1}\cdot\mu^{q-k}\cdot d\tau\cdot(\mu+\partial(d\tau))^{k-j-1}\\ &-\sum_{k=0}^{j}\left(\sqrt{-1}d\theta\right)^{j-k}\cdot\left(\sqrt{-1}d\sigma\right)\cdot\left(\sqrt{-1}d\theta'\right)^{k}\cdot(\mu+\partial(d\tau))^{q-j}\,.\end{aligned}$$

The equation is valid if $j=0$. We calculate $\mathcal{D}''(C_0+\cdots+C_j)+\mathcal{D}''C_{j+1}$. First,

$$\begin{aligned}&\sum_{k=j+2}^{q}\left(\sqrt{-1}d\theta\right)^{j+1}\cdot d\lambda\cdot\mu^{q-k}\cdot d\tau\cdot(\mu+\partial(d\tau))^{k-j-2}\\ &-\sum_{k=j+1}^{q}\left(\sqrt{-1}d\theta\right)^{j+1}\cdot\mu^{q-k}\cdot d\tau\cdot(\mu+\partial(d\tau))^{k-j-1}\\ = &-\sum_{k=j+2}^{q}\left(\sqrt{-1}d\theta\right)^{j+2}\cdot\mu^{q-k}\cdot d\tau\cdot(\mu+\partial(d\tau))^{k-j-2}\\ &-\left(\sqrt{-1}d\theta\right)^{j+1}\cdot d\tau\cdot(\mu+\partial(d\tau))^{q-j-1}\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sum_{k=1}^{j+1}\left(\sqrt{-1}d\theta\right)^{j+1-k}\cdot\left(\sqrt{-1}d\sigma\right)\cdot\left(\sqrt{-1}d\theta'\right)^{k-1}\cdot d\lambda'\cdot(\mu+\partial(d\tau))^{q-j-1}\\ &-\sum_{k=0}^{j}\left(\sqrt{-1}d\theta\right)^{j-k}\cdot\left(\sqrt{-1}d\sigma\right)\cdot\left(\sqrt{-1}d\theta'\right)^{k}\cdot(\mu+\partial(d\tau))^{q-j}\\ = &-\sum_{k=1}^{j+1}\left(\sqrt{-1}d\theta\right)^{j+1-k}\cdot\left(\sqrt{-1}d\sigma\right)\cdot\left(\sqrt{-1}d\theta'\right)^{k}\cdot(\mu+\partial(d\tau))^{q-j-1}\end{aligned}$$

Finally, as

$$\begin{aligned}
& \left(\sqrt{-1}d\theta\right)^{j+1} \cdot d\rho \cdot (\mu + \partial(d\tau))^{q-j-1} - \left(\sqrt{-1}d\theta\right)^{j+1} \cdot d\tau \cdot (\mu + \partial(d\tau))^{q-j-1} \\
= \quad & -\left(\sqrt{-1}d\theta\right)^{j+1} \cdot (\sqrt{-1}d\sigma) \cdot (\mu + \partial(d\tau))^{q-j-1},
\end{aligned}$$

our claim is verified. It follows that

$$\begin{aligned}
& \mathcal{D}''(C_0 + C_1 + \cdots + C_q) \\
= \quad & -\sum_{k=0}^{q} \left(\sqrt{-1}d\theta\right)^{q-k} \cdot \left(\sqrt{-1}d\sigma\right) \cdot \left(\sqrt{-1}d\theta'\right)^{k}.
\end{aligned}$$

Thus $2\mathrm{Re}\,(\sqrt{-1})^q \mathcal{D}''(C_0 + C_1 + \cdots + C_q) = 0$. Noticing that $-2\mathrm{Re}\,(\sqrt{-1})^q \sum_{j=0}^{q} A_j = B_{\omega'}^{q+1}$ and $-2\mathrm{Re}\,(\sqrt{-1})^q \sum_{j=0}^{q} B_j = B_{\omega}^{q+1}$, the proof is completed. □

DEFINITION 4.6. — We denote by β^{q+1} the cohomology class defined by B_ω^{q+1}.

By the definitions, the equations

$$\Lambda_{i_j,i_{j+1}} = \lambda((\gamma_{i_j,i_{j+1}})^{-1}) \text{ and } d\Theta_{i_j,i_{j+1}} = (d\theta)((\gamma_{i_j,i_{j+1}})^{-1})$$

hold, where the left hand sides are the differential forms defined in the previous section, and the right hand sides are the group cocycles evaluated by $\left(\gamma_{i_j,i_{j+1}}\right)^{-1}$. By putting $f_k = (\gamma_{i_{q+1-k},i_{q+2-k}})^{-1}$, $k = 1, \cdots, q+1$, we obtain the equation

$$\begin{aligned}
& (\Xi_q)_{i_0,\cdots,i_{q+1}} \\
= \quad & -2\mathrm{Re}(\sqrt{-1})^q \sum_{j=0}^{q} \Big\{ \gamma_{i_1,i_q}^* (\sqrt{-1}d\Theta_{i_0,i_1}) \cdots (\gamma_{i_j,i_q}^* \sqrt{-1}d\Theta_{i_{j-1},i_j}) \\
& \cdot \gamma_{i_j,i_q}^* \Lambda_{i_j,i_{j+1}} \cdot \gamma_{i_{j+1},i_{j+2}}^* (d\Lambda + \sqrt{-1}d\Theta)_{i_{j+1},i_{j+2}} \cdots (d\Lambda + \sqrt{-1}d\Theta)_{i_{q-1},i_q} \Big\} \\
= \quad & (-1)^{\frac{q(q-3)}{2}} B_{dz}^{q+1}(f_1, f_2, \cdots, f_q),
\end{aligned}$$

where dz denotes the standard complex volume form of $\boldsymbol{C}^q$. Thus we can reformulate Theorem 3.14 as follows.

THEOREM 4.7. — *In the case of foliated bundles with fiber $\boldsymbol{C}^q$, we can obtain the imaginary part of the Bott class by integrating the group cocycle $(-1)^{\frac{q(q-3)}{2}} B_{dz}^{q+1}$, where dz denotes the standard complex volume form of $\boldsymbol{C}^q$.*

Bibliography

[1] Asuke (T.). — *On the real secondary classes of transversely holomorphic foliations*, Annales de l'Institut Fourier **50**, No.3 (2000), 995–1017.

[2] Asuke (T.). — *On the real secondary classes of transversely holomorphic foliations II*, preprint (1999).

[3] Baum (P.) and Bott (R.). — *Singularities of Holomorphic Foliations*, Jour. Diff. Geom. **7** (1972), 279–342.

[4] Benson (C.) and Ellis (B.). — *Characteristic classes of transversely homogeneous foliations*, Trans. Amer. Soc. **289** (1985), 849–859.

[5] Bott (R.). — *On the Lefschetz Formula and Exotic Characteristic Classes*, Symposia Math. **10** (1972), 95–105.

[6] Bott (R.), Gilter (S.), James (I.M.). — *Lectures on Algebraic and Differential Topology*, Lecture Notes in Mathematics, No. 279, Springer-Verlag, 1972.

[7] Bott (R.), Haefliger (A.). — *On characteristic classes of* Γ*-foliations*, Bull. Amer. Math. Soc. **78** (1972), 1039–1044.

[8] Brown (K.S.). — *Cohomology of Groups*, Graduate Texts in Mathematics **87**, Springer-Verlag, 1994.

[9] Haefliger (A.). — *Homotopy and Integrability*, Manifolds–Amsterdam 1970 (Proc. Nuffic Summer School), Lecture Notes in Mathematics, No. 197, Springer-Verlag, 1971, 133-163.

[10] Mizutani (T.). — *The Godbillon-Vey cocycle of* $\mathrm{Diff}\,\mathbb{R}^n$, A fête of Topology, Academic Press, 1988, 49–62.

[11] Morita (S.). — *On characteristic classes of conformal and projective foliations*, J. Math. Soc. Japan **31** (1979), 693–718.

[12] Morita (S.). — *Discontinuous Invariants of Foliations*, Advanced studies in Pure Mathematics 5, Kinokuniya Book Store, 1985, 169–193.

[13] Nesterenko (Y.P.) and Suslin (A.A.). — *Homology of the full linear group over a local ring, and Milnor's* K*-Theory*, Math. USSR Izvestiya **34** (1990), 121–145.

[14] Nishikawa (S.) and Sato (H.). — *On characteristic classes of riemannian, conformal and projective foliations*, J. Math. Soc. Japan **28** (1976), 223–241.

[15] Pittie (H.V.). — *Characteristic classes of foliations*, Research Notes in Mathematics; 10, Pitman Publishing, 1976.

[16] Rozhe (K.). — *Homology of affine groups, and classifying spaces for piecewise-linear bundles* (Russian), Funktsional. Anal. i Prilozhen **13**, no. 4 (1979), 47–52.

[17] Suslin (A.A.). — *Homology of* GL_n*, characteristic classes and Milnor* K*-theory*, Algebraic K-theory, number theory, geometry and analysis (Bielefeld 1982), Lecture Notes in Mathematics 1046, Springer-Verlag, 357–375.

[18] Tsuboi (T.). — Private communication.