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Levels of concentration between exponential and Gaussian [*]

FRANCK BARTHE [1]

RÉSUMÉ. — Soit μ une mesure de probabilité log-concave sur $\mathbb{R}^n$. Si μ vérifie pour $c > 1, d > 0, r \in [1,2]$ fixés et pour tout t positif, $\mu(\{x; |x| > d.t\}) \leqslant c\exp(-t^r)$, alors μ vérifie une famille d'inégalités de Sobolev. En conséquence, les mesures produit μ^k ont, indépendamment de k, une propriété de concentration sur le modèle de $\exp(-|t|^r)$.

ABSTRACT. — Let μ be a log-concave probability measure on $\mathbb{R}^n$. If for fixed $c > 1, d > 0, r \in [1,2]$ and all positive t, the measure μ satisfies $\mu(\{x; |x| > d.t\}) \leqslant c\exp(-t^r)$, then μ satisfies a family of Sobolev inequalities. Consequently, the product measures μ^k have independently of k a concentration property on the model of $\exp(-|t|^r)$.

We are interested in proving dimension free concentration inequalities for product measures. For this purpose, the famous Poincaré and log-Sobolev inequalities are very efficient. Recall that a Borel probability measure μ on the Euclidean space $(\mathbb{R}^n, |\cdot|)$ satisfies a Poincaré inequality with constant C if

$$\int f^2\, d\mu - \left(\int f\, d\mu\right)^2 \leqslant C \int |\nabla f|^2\, d\mu$$

holds for every smooth $f : \mathbb{R}^n \to \mathbb{R}$. It is said to verify a log-Sobolev

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inequality with constant D whenever

$$\int f^2 \log f^2 \, d\mu - \Big(\int f^2 \, d\mu\Big) \log \Big(\int f^2 \, d\mu\Big) \leqslant D \int |\nabla f|^2 \, d\mu$$

holds for f as before. These inequalities have the tensorisation property: if they hold for μ, then they automatically hold for all the product measures μ^k, $k \geqslant 1$. Moreover, the Poincaré inequality implies a concentration inequality of exponential type, whereas the log-Sobolev inequality implies Gaussian concentration (see e.g. [17] for details and precise references). Bobkov [7] recently studied these inequalities for log-concave probability measures (this reduces to absolutely continuous probabilities, with log-concave densities by [9]). Using the Prékopa-Leindler inequality, he proved that any log-concave probability satisfies a Poincaré inequality. Under the additional condition $\int \exp(\epsilon |x|^2) \, d\mu(x) < \infty$ for a positive ϵ, the measure μ satisfies a log-Sobolev inequality (this result was proved first by Wang [21], but Bobkov's approach is somewhat simpler). As a corollary, one gets the following: a log-concave probability μ on $\mathbb{R}^n$ such that

$$\mu(\{x; |x| > t\}) \leqslant c e^{-(t/d)^2},$$

holds for fixed $c > 1, d > 0$ and all positive t, satisfies an infinite dimensional concentration inequality, which can be stated on functions as follows. For any integer $k \geqslant 1$, and any $F : \mathbb{R}^{nk} \to \mathbb{R}$ with Lipschitz constant $\|F\|_{\mathrm{Lip}} \leqslant 1$, one has

$$\mu^k \left(\left\{ x;\ F(x) \geqslant \int F \, d\mu + r \right\} \right) \leqslant e^{-Cr^2}, \quad r > 0,$$

where $C > 0$ depends only on c, d. Equivalently, concentration can be expressed on sets; for any Borel set $A \subset \mathbb{R}^{nk}$ with $\mu^k(A) > 0$, one has

$$\mu^k\Big((A_r)^c\Big) \leqslant e^{-Cr^2 \mu^k(A)}, \quad r > 0,$$

where $A_r := \{x; d(x, A) \leqslant r\}$ is the Euclidean enlargement of A by a length r. Note that the hypothesis $\mu(\{x; |x| > t\}) \leqslant c e^{-(t/d)^2}$ is some very weak form of Gaussian concentration for μ itself. In some sense, for log-concave probability measures, Gaussian concentration for balls implies Gaussian concentration for the infinite product measure. Our aim here is to extend this result to other levels of concentration, between exponential and Gaussian. We follow Bobkov's approach, but we need some other ingredients. In particular, we shall use functional inequalities interpolating between Poincaré and log-Sobolev. Such inequalities were presented by Beckner for the Gaussian measure [4] and established for densities $c_r \exp(-|t|^r)$,

$r \in (1,2)$ in a recent work by Latała and Oleszkiewicz [15]. They still enjoy the tensorisation property (see [16] or the latter paper), and imply concentration but with rates $\exp(-t^r)$ for $r \in [1,2]$. We summarize these facts in the following

THEOREM 1 ([15]).— *Let $r \in [1,2]$ and $C > 0$. Let μ be a probability measure on $\mathbb{R}^n$. Assume that for any smooth $f : \mathbb{R}^n \to \mathbb{R}$ and any $p \in [1,2)$, one has*

$$\int f^2\, d\mu - \left(\int |f|^p\, d\mu\right)^{\frac{2}{p}} \leqslant C(2-p)^{2(1-\frac{1}{r})} \int |\nabla f|^2\, d\mu.$$

Then for any integer $k \geqslant 1$ and any $h : \mathbb{R}^{nk} \to \mathbb{R}$ with $\|h\|_{\mathrm{Lip}} \leqslant 1$, one has $\int |h|\, d\mu < \infty$ and

$$\mu^k\left(\left\{x;\ h(x) - \int h\, d\mu \geqslant t\sqrt{C}\right\}\right) \leqslant e^{-\frac{t^2}{3}}, \quad t \in [0,1],$$

$$\mu^k\left(\left\{x;\ h(x) - \int h\, d\mu \geqslant t\sqrt{C}\right\}\right) \leqslant e^{-\frac{t^r}{3}}, \quad t \geqslant 1.$$

Next, we need a systematic way to derive the latter inequalities from isoperimetric inequalities. Before exposing this, we recall that the isoperimetric function of a Borel probability measure ν is by definition

$$I_\nu(a) := \inf\{\nu^+(A); A \text{ Borel set with } \nu(A) = a\}, \quad a \in [0,1],$$

where $\nu^+(A) := \liminf_{\epsilon \to 0^+} (\nu(A_\epsilon) - \nu(A))/\epsilon$ is the boundary measure of A in the sense of ν.

1. A transfer principle

Lipschitz mappings are a convenient tool to prove concentration inequalities (see for example [13], [18]). Let μ and ν be Radon probability measures, say on Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ a Lipschitz map with Lipschitz constant L. Assume that T transports μ onto ν, meaning for every Borel $A \subset \mathbb{R}^n$, one has $\nu(A) = \mu(T^{-1}(A))$. Then ν concentrates at least as much as μ. More precisely, let $A \subset \mathbb{R}^n$, and denote $B = \mu(T^{-1}(A))$. The Lipschitz property of T ensures that for $h > 0$,

$$T(B_h) \subset A_{Lh}.$$

Thus $\nu(A) = \mu(B)$ and $\nu(A_{Lh}) \geqslant \mu(B_h)$. In particular, we obtain the following comparison of isoperimetric functions: $I_\nu \geqslant I_\mu / L$.

Further, any Sobolev type inequality satisfied by μ will transfer to ν with a change in the constants depending only on L. For example, assume that there exist $C > 0$, $a \in [0,1]$ and $p \in [1,2)$ such that for any locally Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$, one has

$$\int f^2\,d\mu - \left(\int |f|^p\,d\mu\right)^{\frac{2}{p}} \leqslant C(2-p)^a \int |\nabla f|^2\,d\mu.$$

Now let $g : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Applying the latter inequality to $f = g \circ T$, noticing that $|\nabla f| = |\nabla T| \cdot |(\nabla g) \circ T| \leqslant L|(\nabla g) \circ T|$, and using the fact that ν is the image of μ by T, we obtain

$$\int g^2\,d\nu - \left(\int |g|^p\,d\nu\right)^{\frac{2}{p}} \leqslant CL^2(2-p)^a \int |\nabla g|^2\,d\nu.$$

Our aim is to emphasize another transfer principle, based on rearrangement of functions: Given a function f, one builds a function f^* having the same distribution, monotonicity properties, and nicer level sets (chosen among a one-parameter family of sets, ordered for the inclusion). If a Sobolev inequality holds for f^*, then it will be satisfied by f, provided the rearrangement decreases the energy: that is $\int |\nabla f^*|^2 \leqslant \int |\nabla f|^2$, or more generally $\int F(|\nabla f^*|) \leqslant \int F(|\nabla f|)$ for positive non-decreasing convex functions F. This principle is classical and can be found in several texts (see e.g. [19], [2]). However, it is often exposed in special cases which do not fit with our purposes and the proof of the decrease of the energy under rearrangement is sometimes complicated or based on artificial tricks. This is why we write here another proof; it is an extension and a simplification of an argument of Bakry and Ledoux [1] for the Gaussian measure. These authors use the formalism of conditional expectation, which yields limpid proofs of the decrease of energy (one should notice that conditional expectations appeared implicitly in previous proofs, as in [11].)

LEMMA 2.— *Let μ be measure on $\mathbb{R}$, with density φ. We assume that φ is bounded continuous and positive on an interval (c,d) (c,d may be infinite), and vanishes outside. Let $\Phi(t) = \mu((-\infty,t])$ be its distribution function, and assume that it is finite for any $t \in \mathbb{R}$. Let ν be an absolutely continuous Borel measure on $\mathbb{R}^n$, with $\nu(\mathbb{R}^n) = \mu(\mathbb{R})$. Assume that the isoperimetric function of ν is estimated from below*

$$I_\nu(t) \geqslant \frac{1}{L} J_\mu(t), \qquad t \in (0, \mu(\mathbb{R}))$$

where L is a positive real number and $J_\mu = \varphi \circ \Phi^{-1}$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and denote by ν_f its distribution with respect to ν. Set $N(r) := \nu_f((-\infty, r])$ and assume that it is finite for all r. Further, we assume that ν_f is supported on $[a,b] \subset \mathbb{R}$, is absolutely continuous with respect to Lebesgue's measure and has a continuous density N' on (a,b).

Then, there exists a non-decreasing function k, defined on $\mathbb{R}$, such that its distribution with respect to μ coincides with the one of f with respect to ν, and μ-a.e.

$$0 \leqslant k'(t) \leqslant L\theta(k(t)),$$

where θ is a regular version of the conditional expectation of $|\nabla f|$ with respect to the sigma-field generated by f.

Proof. — The relation $I_\nu \geqslant J_\mu/L$ yields by integration that for Borel sets $A \subset \mathbb{R}^n$, if $\nu(A) \neq 0$, then $\nu(A_h) \geqslant \Phi(\Phi^{-1}(\nu(A)) + h/L)$. Let us use this fact to derive useful properties of f. We set $K := \|f\|_{\mathrm{Lip}}$. Let $h > 0$. The Lipschitz property easily yields that

$$\{x : f(x) \leqslant r\}_h \subset \{x : f(x) \leqslant r + Kh\}.$$

Thus by the isoperimetric inequality for the measure ν

$$N(r + Kh) \geqslant \Phi(\Phi^{-1}(N(r)) + h/L)$$

holds whenever $N(r) \neq 0$. This inequality ensures that ν_f has a positive density $N'(r)$ on (a,b). In particular, the inverse function $N^{-1} : (0, \mu(\mathbb{R})) \to [a,b]$ is well-defined. By construction, the function $k = N^{-1} \circ \Phi : \mathbb{R} \to [a,b]$ has distribution ν_f with respect to μ.

Let $\varepsilon > 0$ and $r \in (a,b)$. Consider the function

$$\psi_\varepsilon(t) = \mathbf{1}_{(-\infty,r]}(t) + \frac{r + \varepsilon - t}{\varepsilon}\mathbf{1}_{(r,r+\varepsilon]}(t), \qquad t \in \mathbb{R}.$$

Set $g_\varepsilon = \psi_\varepsilon \circ f$. By the co-area formula ([12]) and the isoperimetric inequality, one has

$$\int |\nabla g_\varepsilon| \, d\nu \geqslant \int_0^1 I_\nu(\nu(\{g_\varepsilon > t\})) \, dt \geqslant \frac{1}{L} \int_0^1 J_\mu(\nu(\{g_\varepsilon > t\})) \, dt.$$

Now, we let ε to zero. For $t \in (0,1)$, one has

$$\{x : f(x) \leqslant r\} \subset \{x : g_\varepsilon(x) \leqslant t\} \subset \{x : f(x) \leqslant r + \varepsilon\}.$$

Thus $\lim_{\varepsilon \to 0} \nu(\{x : g_\varepsilon(x) \leqslant t\}) = \nu(\{x : f(x) \leqslant r\}) = N(r)$. Since J_μ is continuous on $(0, \mu(\mathbb{R}))$, we obtain that

$$\lim_{\varepsilon \to 0} \int_0^1 J_\mu(\nu(\{g_\varepsilon > t\})) = J_\mu(N(r)).$$

On the other hand,

$$\int |\nabla g_\varepsilon|\, d\nu = \int \mathbf{1}_{f\in[r,r+\varepsilon]} \frac{1}{\varepsilon} |\nabla f|\, d\nu = \frac{1}{\varepsilon}\int_r^{r+\varepsilon} \theta(s)\, d\nu_f(s)$$

tends to $N'(r)\theta(r)$ when ε tends to zero, for almost every $r \in (a,b)$. Eventually, we have proved

$$L\theta(r)N'(r) \geqslant J_\mu(N(r))$$

for $r \in (a,b) \setminus \mathcal{N}$, where $\mathcal{N} \subset (a,b)$ is Lebesgue negligible. Next, we differentiate the relation that defines k and get for $x \in (c,d)$

$$N'(k(x))k'(x) = \Phi'(x) = J_\mu(\Phi(x)) = J_\mu(N(k(x)).$$

Let $x \in (c,d)$ such that $k(x) \in (a,b) \setminus \mathcal{N}$. The latter inequality applied with $r = k(x)$ gives

$$L\theta(k(x))N'(k(x)) \geqslant J_\mu(N(k(x)) = N'(k(x))k'(x).$$

As previously discussed $N'(k(x))$ is positive, so we have proved that $k'(x) \leqslant L\theta(k(x))$ holds whenever $x \in (c,d)$ and $k(x) \in (a,b) \setminus \mathcal{N}$. This condition is true μ-a.s. because $\mu((c,d)^c) = 0$ and

$$\mu(\{x;\, k(x) \notin (a,b) \setminus \mathcal{N}\}) = \nu_f(((a,b) \setminus \mathcal{N})^c) \leqslant \nu_f((a,b)^c) + \nu_f(\mathcal{N}) = 0,$$

by our assumptions on ν_f. The proof is complete. □

Remarks. — This lemma holds in more general settings, for example if ν is a measure on a Riemannian manifold. Also note that the function J_μ is larger than the isoperimetric function of μ. Indeed the sets $(-\infty, t]$ have μ-measure $\Phi(t)$ and μ-boundary measure $\varphi(t)$. So this hypothesis $I\nu \geqslant J_\mu$ is stronger than $I\nu \geqslant I_\mu$; these assertions are equivalent when μ is an even log-concave probability measure on $\mathbb{R}$ ($I_\mu = J_\mu$ in this case, see [10], [6].)

Let us now illustrate the transfer principle. For $r \in [1,2]$, let μ_r be the probability measure on the real line defined by

$$d\mu_r(t) = c_r \exp(-|t|^r)\, dt, \qquad t \in \mathbb{R}.$$

Latała and Oleszkiewicz proved in [15] that for any $p \in [1,2)$ and any smooth $k : \mathbb{R} \to \mathbb{R}$, one has

$$\int k^2\, d\mu_r - \left(\int |k|^p\, d\mu_r\right)^{\frac{2}{p}} \leqslant C(2-p)^{2(1-\frac{1}{r})} \int |k'|^2\, d\mu_r, \tag{1}$$

where C is an absolute constant.

Assume that ν is an absolutely continuous probability measure on $\mathbb{R}^n$ such that

$$I_\nu \geqslant \frac{1}{L} I_{\mu_r} = \frac{1}{L} J_{\mu_r}.$$

Then, we can show that ν satisfies the same inequality as μ_r with the constant C replaced by CL^2. When the density of ν is regular enough (log-concave suffices), standard approximation arguments show that it suffices to prove the inequality for Lipschitz functions f with a distribution function ν_f satisfying the hypothesis of Lemma 2 (for example, one can use polygonal approximations of the graph of f and modify them slightly in order to avoid horizontal pieces). For such a function, the lemma provides a non-decreasing function k on $\mathbb{R}$ with distribution ν_f. It satisfies inequality (1), where the left-hand side is exactly

$$\int f^2 \, d\nu - \left(\int |f|^p \, d\nu \right)^{\frac{2}{p}},$$

whereas the right-hand side is less than or equal to

$$CL^2 (2-p)^a \int |\theta(k(t))|^2 \, d\mu_r(t).$$

To conclude, notice that the latter integral is

$$\begin{aligned} \int \theta(u)^2 \, d\nu_f(u) &= \int E\Big[|\nabla f| \, \big| f = u \Big]^2 \, d\nu_f(u) \\ &\leqslant \int E\Big[|\nabla f|^2 \, \big| f = u \Big] \, d\nu_f(u) = \int |\nabla f|^2 \, d\nu. \end{aligned}$$

More generally, this method allows to transfer inequalities of the form

$$A(\mu_f) \leqslant \int G\Big(H(f), |\nabla f| \Big) \, d\mu$$

where G is jointly convex, and non-decreasing in the second variable. This was used to transfer Bobkov's isoperimetric inequality in [3]. We should like to emphasize that several results in the literature may be understood in a simple way from the preceding remarks. See e.g. in [5], [8].

To conclude this section, we explain how the lemma recovers a spherical symmetrization. In [14], Ilias shows the following: if (M, d, v) is Riemannian manifold with geodesic distance and normalized volume such that I_M is larger than the isoperimetric function I_{S_n} of the Euclidean sphere with uniform probability σ_n, then the Sobolev inequalities valid on S_n are still true on M. Let p be a point of the sphere, and f a function on $\mathbb{R}$. Applying a Sobolev inequality on the sphere to the function $x \to f(d(x,p))$ yields a Sobolev inequality on the real line for the image measure μ of σ_n by the map $x \to f(d(x,p))$. With the notation of the lemma, the function J_μ coincides with I_{S_n} (because $d(p,.)$ is a good parametrization of the extremal sets on the sphere.) So by the lemma, the Sobolev inequality for μ will transfer to (M,d,v).

2. Log-concave probabilities

We state the main result of the paper.

THEOREM 3.— *Let $r \in [1,2]$. Let μ be a log-concave probability on $\mathbb{R}^n$ such that there exists $c > 1$ and $d > 0$ such that*

$$\mu(\{x \in \mathbb{R}^n;\ |x| > t\}) \leqslant c\, e^{-\left(\frac{t}{d}\right)^r}, \qquad t > 0.$$

Then there exits a constant $C(c,d)$ depending only on c and d such that for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and any $p \in [1,2)$ one has

$$\int f^2\, d\mu - \left(\int |f|^p\, d\mu\right)^{\frac{2}{p}} \leqslant C(c,d)(2-p)^{2(1-\frac{1}{r})} \int |\nabla f|^2\, d\mu. \qquad (2)$$

In particular, the infinite product measure μ^∞ will concentrate, up to constant, as much as the measure $e^{-|t|^r}/c_r\, dt$, $t \in \mathbb{R}$.

In addition to the transfer principle exposed in the previous section, we need the following lemmas.

LEMMA 4.— *Let r, c, d as before and let μ be a log-concave probability measure on $\mathbb{R}^n$ such that*

$$\mu(\{x \in \mathbb{R}^n;\ |x| > t\}) \leqslant c\, e^{-\left(\frac{t}{d}\right)^r}, \qquad t > 0.$$

Then, the isoperimetric function of μ is bounded from below:

$$I_\mu(s) \geqslant \frac{1}{2d(\alpha \log c + \beta)^{\frac{1}{r}}} \min(s,\, 1-s)\left(\log \min(s,\, 1-s)\right)^{1-\frac{1}{r}}, \quad s \in [0,1],$$

where α and β are universal constants.

This fact was proved for $r = 1,2$ in [7]. Our proof follows the same lines.

Proof.— Let A be a measurable subset of $\mathbb{R}^n$. Set $t = \min(\mu(A), 1 - \mu(A))$. Since μ is log-concave, inequality (2.4) in [7] is available; it provides the following lower bound on the boundary measure of A:

$$\mu^+(A) \geqslant \frac{t \log \frac{1}{t} + (1-t)\log \frac{1}{1-t} + \log \mu(RB_2^n)}{2R},$$

where $R > 0$ is arbitrary. So by hypothesis, and provided $c\,e^{-\left(\frac{R}{d}\right)^r} < 1$, one has

$$\mu^+(A) \geqslant \frac{t\log\frac{1}{t} + (1-t)\log\frac{1}{1-t} + \log\left(1 - c\,e^{-\left(\frac{R}{d}\right)^r}\right)}{2R}.$$

Choose $R = d\left(\gamma\log\frac{1}{t}\right)^{\frac{1}{r}}$, where $\gamma > 0$ will be specified later. The condition $c\,e^{-\left(\frac{R}{d}\right)^r} < 1$ is equivalent to $ct^\gamma < 1$. The possible values of t are $[0,1/2]$, so if $\gamma > \log c/\log 2$, the previous condition is satisfied and one has

$$\mu^+(A) \geqslant \frac{1}{2d\gamma^{\frac{1}{r}}}t\left(\log\frac{1}{t}\right)^{1-\frac{1}{r}} + \frac{v(t)}{R},$$

where $v(t) = (1-t)\log\frac{1}{1-t} + \log(1-ct^\gamma)$. When $\gamma \geqslant 1$, the function v is concave in $t \in [0,1/2]$. Clearly $v(0) = 0$, whereas $v(1/2)$ is non-negative if and only if $\gamma \geqslant (\log c + \log(2+\sqrt{2})/\log 2$. So, if one sets $\gamma = (\log c + \log(2+\sqrt{2})/\log 2$, all the previous requirements are satisfied and v is non-negative on the range of t. In this case, one obtains

$$\mu^+(A) \geqslant \frac{1}{2d\left(\frac{\log c + \log(2+\sqrt{2})}{\log 2}\right)^{\frac{1}{r}}}t\left(\log\frac{1}{t}\right)^{1-\frac{1}{r}}.$$

□

Next, we want an upper bound on the isoperimetric function I_{μ_r} of the probability measure on $\mathbb{R}$ with density $\varphi_r(t) = \exp(-|t|^r)/c_r$, with $c_r = \Gamma(1+1/r)$. Let $\Phi_r(t) = \int_{-\infty}^t \varphi_r$. Recall that $I_{\mu_r}(0) = I_{\mu_r}(1) = 0$ and for $t \in (0,1)$, $I_{\mu_r}(t) = \varphi_r(\Phi_r^{-1}(t))$. Notice also that $I_{\mu_r}(t) = I_{\mu_r}(1-t)$.

LEMMA 5.— *There exists a positive constant K such that for all $r \in [1,2]$ and all $t \in [0,1]$, one has*

$$I_{\mu_r}(t) \leqslant K\min(t,1-t)\left(\log\frac{1}{\min(t,1-t)}\right)^{1-\frac{1}{r}}.$$

Proof. — Since we consider only $r \in [1,2]$, we can make rough estimates. By symmetry of the latter quantities with respect to $1/2$, we may restrict our attention to $t \in (0,1/2]$. We make the change of variables $t = \Phi_r(x)$, $x \leqslant 0$, and rewrite the inequality we aim at as

$$\varphi_r(x) \leqslant K\Phi_r(x)\left(\log\frac{1}{\Phi_r(x)}\right)^{1-\frac{1}{r}}.$$

Equivalently, let us prove for arbitrary $y \geqslant 0$ that

$$e^{-y^r} \leqslant K \int_y^\infty e^{-t^r}\,dt \left(\log \frac{c_r}{\int_y^\infty e^{-t^r}\,dt}\right)^{1-\frac{1}{r}}. \tag{3}$$

Integration by parts easily yields, for $y \geqslant 0$

$$\int_y^\infty e^{-t^r}\,dt = \frac{e^{-y^r}}{ry^{r-1}} - \frac{r-1}{r}\int_y^\infty \frac{e^{-t^r}}{t^r}\,dt.$$

From this relation, we deduce upper and lower estimates of Φ_r. For $y \geqslant 0$,

$$\int_y^\infty e^{-t^r}\,dt \leqslant \frac{e^{-y^r}}{ry^{r-1}},$$

and for $y \geqslant 1$, one has

$$\int_y^\infty e^{-t^r}\,dt \geqslant \frac{e^{-y^r}}{ry^{r-1}} - \frac{r-1}{r}\int_y^\infty e^{-t^r}\,dt,$$

therefore

$$\int_y^\infty e^{-t^r}\,dt \geqslant \frac{1}{2-\frac{1}{r}}\frac{e^{-y^r}}{ry^{r-1}} \geqslant \frac{e^{-y^r}}{3y^{r-1}},$$

where we have used $r \leqslant 2$. Combining these estimates, we obtain for $y \geqslant 1$

$$\begin{aligned}\int_y^\infty e^{-t^r}dt \left(\log \frac{c_r}{\int_y^\infty e^{-t^r}dt}\right)^{1-\frac{1}{r}} &\geqslant \frac{e^{-y^r}}{3y^{r-1}}\left(\log\left(r\,c_r y^{r-1}e^{y^r}\right)\right)^{1-\frac{1}{r}} \\ &\geqslant \frac{e^{-y^r}}{3}\end{aligned}$$

where we have used $y \geqslant 1$ and $rc_r = r\Gamma(1+1/r) = \Gamma(1/r) \geqslant \Gamma(1) = 1$. The proof of (3) is complete for $y \geqslant 1$ and $K = 3$. The existence of a K such that (3) holds for $r \in [1,2]$ and $y \in [0,1]$ is easy by compactness arguments (an explicit value of K can be obtained by elementary estimates.) □

Proof of Theorem 3. — The hypothesis on μ and the previous two lemmas yield

$$I_\mu \geqslant \frac{I_{\mu_r}}{2Kd\left(\frac{\log c+\log(2+\sqrt{2})}{\log 2}\right)^{\frac{1}{r}}} \geqslant \frac{I_{\mu_r}}{2Kd\frac{\log c+\log(2+\sqrt{2})}{\log 2}}$$

because $r \geqslant 1$. By the transfer principle explained in the previous section, inequality (2) holds with

$$C(c,d) = \left(2Kd\frac{\log c + \log(2+\sqrt{2})}{\log 2}\right)^2 \leqslant K'd^2(1+\log c)^2.$$

Finally, Theorem 1 ensures, for $k \geqslant 1$ and $h : \mathbb{R}^{nk} \to \mathbb{R}$ with $\|h\|_{\mathrm{Lip}} \leqslant 1$,

$$\mu^k\left(\left\{x;\ \left|h(x) - \int h\,d\mu\right| \geqslant t\sqrt{C(c,d)}\right\}\right) \leqslant 2e^{-\frac{1}{3}\min(t^2,t^r)}, \quad t > 0.$$

□

Remark. — If one is not interested in Sobolev inequalities but only in concentration properties, one can follow another route, by combining the results of Section 2 with Talagrand's tensorisation Theorem 2.7.1 in [20]. The latter operates instead at the level of inf-convolution inequalities. For example Proposition 2.7.4 in [20] provides a precise concentration inequality for products of even log-concave densities on the real line. It ensures that when $1 \leqslant r \leqslant 2$ and $A \subset \mathbb{R}^n$ satisfies $\mu_r^n(A) \geqslant 1/2$ then for $t > 0$ one has

$$\mu_r^n(A + tB_r^n + t^{r/2}B_2^n) \geqslant 1 - ce^{-Kt^r},$$

where $c, K > 0$ are universal constants and $B_p^n = \{x \in \mathbb{R}^n;\ \sum_{i=1}^n |x_i|^p \leqslant 1\}$. This is somewhat stronger than the concentration statement in [15] since for $r \leqslant 2, t \geqslant 1$, one has $tB_r^n + t^{r/2}B_2^n \subset 2tB_2^n$.

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Bibliography

[1] BAKRY (D.) and LEDOUX (M.). — Lévy-Gromov isoperimetric inequality for an infinite dimensional diffusion generator. *Invent. Math.*, 123:259–281, 1996.

[2] BANDLE (C.). — *Isoperimetric inequalities and applications.* Number 7 in Monographs and Studies in Math. Pitman, 1980.

[3] BARTHE (F.) and MAUREY (B.). — Somes remarks on isoperimetry of Gaussian type. *Ann. Inst. H. Poincaré, Probabilités et statistiques*, 36(4):419–434, 2000.

[4] BECKNER (W.). — A generalized Poincaré inequality for Gaussian measures. *Proc. Amer. Math. Soc.*, 105:397–400, 1989.

[5] BLOWER (G.). — The Gaussian isoperimetric inequality and transportation. *Preprint*, 1999.

[6] BOBKOV (S. G.). — Extremal properties of half-spaces for log-concave distributions. *Ann. Probab.*, 24(1):35–48, 1996.

[7] BOBKOV (S. G.). — Isoperimetric and analytic inequalities for log-concave probability measures. *Ann. Probab.*, 27(4):1903–1921, 1999.

[8] BOBKOV (S. G.) and HOUDRÉ (C.). — Isoperimetric constants for product probability measures. *Ann. Probab.*, 25(1):184–205, 1997.

[9] BORELL (C.). — Convex measures on locally convex spaces. *Ark. Math.*, 12:239–252, 1974.

[10] BORELL (C.). — Convex functions in *d*-space. *Period. Math. Hungar.*, 6:111–136, 1975.

[11] EHRHARD (A.). — Inégalités isopérimétriques et intégrales de Dirichlet gaussiennes. *Ann. Sci. Éc. Norm. Sup., 4^e série*, 17:317–332, 1984.

[12] FEDERER (H.). — *Geometric Measure Theory.* Springer-Verlag, New York, 1969.

[13] IBRAGIMOV (I. A.), SUDAKOV (V. N.) and TSIREL'SON (B. S.). — Norms of Gaussian sample functions. In *Proc. of the third Japan-USSR Symposium on Probability Theory*, number 550 in LMN, pages 20–41. Springer, 1976.

[14] ILIAS (S.). — Constantes explicites dans les inégalités de Sobolev sur les variétés riemanniennes compactes. *Ann. Inst. Fourier, Grenoble*, 33(2):151–165, 1983.

[15] LATAŁA (R.) and OLESZKIEWICZ (K.). — Between Sobolev and Poincaré. In *Geometric aspects of functional analysis*, volume 1745 of *Lecture Notes in Math.*, pages 147-168. Springer, Berlin, 2000.

[16] LEDOUX (M.). — On Talagrand's deviation inequalities for product measures. *ESAIM Prob. & Stat.*, 1:63–87, 1996.

[17] LEDOUX (M.). — Concentration of measure and logarithmic Sobolev inequalities. In *Séminaire de Probabilités, XXXIII*, number 1709 in *Lecture Notes in Math.*, pages 120–216. Springer, Berlin, 1999.

[18] PISIER (G.). — Probabilistic methods in the geometry of Banach spaces. In *Probability and Analysis, Varenna (Italy) 1985*, volume 1206 of *Lecture Notes in Math.*, pages 167–241. Springer-Verlag, 1986.

[19] POLYA (G.) and SZEGÖ (G.). — *Isoperimetric inequalities in mathematical physics.* Princeton University Press, Princeton, 1951.

[20] TALAGRAND (M.). — Concentration of measure and isoerimetric inequalities in product spaces. *Publications Mathématiques de l'I.H.E.S.*, 81:73–205, 1995.

[21] WANG (F.-Y.). — Logarithmic Sobolev inequalities on noncompact Riemannian manifolds. *Proba. Theory Relat. Fields*, 109:417–424, 1997.