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Annales de la faculté des sciences de Toulouse 6^e série, tome 10,
n° 4 (2001), p. 595-617

http://www.numdam.org/item?id=AFST_2001_6_10_4_595_0

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Asymptotic properties of the Dulac map near a hyperbolic saddle in dimension three^(*)

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RÉSUMÉ. — Dans cet article, nous étudions l'application de Dulac associée à un champ de vecteurs définie au voisinage d'une singularité de type col, autrement dit l'application de passage de coin d'une section transverse à la variété stable de dimension 2 à une section transverse à la variété instable de dimension 1. Nous montrons, qu' à une fonction (dite de Mourtada) multiplicative près, cette application s'écrit comme si le champ était localement linéaire.

ABSTRACT. — In this paper, we study the Dulac map associated to a vector field defined in a neighborhood of a hyperbolic equilibrium point of saddle type i.e the map from a section transverse to the 2-dimensional stable manifold to another section transverse to the unstable one. We give an asymptotic expression of this Dulac map. We show that, this asymptotic expression has the same expression, up to some multiplication by a function of Mourtada type, as in the case where the vector field is linear.

1. Introduction

In the study of the bifurcation of a heteroclinic or homoclinic orbit tending to a saddle point p of a vector field it is important to have a good control of the Dulac map near p , that is roughly: the map describing the transition of the trajectories along the saddle point. Throughout this paper we always restrict to “a sufficiently small neighbourhood” of the saddle point.

(*) Reçu le 10 juillet 2001, accepté le 12 mars 2002

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In the planar case this problem was furthermore encountered in the context of the Hilbert 16th problem and it is discussed by authors such as Dumortier and Roussarie [3][9], Il'yashenko and Yakovenko [4], Moussu, [8] [9] and especially Mourtada [7].

Let us mention some facts about the planar case. Let $p = (0, 0) \in \mathbf{R}^2$ be a saddle point of the vector field \mathcal{X} , that is: the eigenvalues λ_1 and λ_2 of the linear part $d\mathcal{X}(0, 0)$ satisfy $\lambda_2 < 0 < \lambda_1$. The number $r = -\lambda_2/\lambda_1$ is called the ratio of hyperbolicity. Using invariant manifolds we can assume, up to a C^∞ change of variables, that \mathcal{X} takes the expression

$$\mathcal{X}(\bar{x}, \bar{y}) = \bar{x}(\lambda_1 + f_1(\bar{x}, \bar{y}))\frac{\partial}{\partial \bar{x}} + \bar{y}(\lambda_2 + f_2(\bar{x}, \bar{y}))\frac{\partial}{\partial \bar{y}} \quad (1)$$

on some neighbourhood U of $(0, 0)$. Take $\delta > 0$ a small real number. Up to a linear rescaling of the form $(\bar{x}, \bar{y}) = \delta(x, y)$, it is no restriction to assume that the points $(1, 0)$ and $(0, 1)$ belong to U . For small $\varepsilon_0 > 0$ we consider a semi-transversal $\Sigma_1 =]0, \varepsilon_0[\times \{1\}$ to $x = 0$, parameterized by x ; similarly $\Sigma_2 = \{1\} \times]0, \varepsilon_0[$ is parameterized by y . The Dulac map $\Sigma_1 \rightarrow \Sigma_2$ sends a point $(x, 1)$ to $(1, y)$ where the trajectory of the flow associated to \mathcal{X} through $(x, 1)$ hits for the first time Σ_2 . Denote $y = D(x)$. In [7] it is shown that $D(x) = x^r(1 + g(x))$ where for all $n \in \mathbf{N}$: $\lim_{x \downarrow 0} x^n d^n g/dx^n(x) = 0$.

In dimension three the so called 'almost planar case' was studied by Roussarie and Rousseau [10]. More explicitly: suppose that the linearization of the vector field \mathcal{X} at the singular point $(0, 0, 0)$ has real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with the extra properties $0 < \lambda_1 = -\lambda_2 > -\lambda_3$. It is also assumed that $\lambda_3/\lambda_2 \notin \mathbf{Q}$. One chooses coordinates such that $y = z = 0$ be the unstable manifold corresponding to the eigenvalue λ_1 , and $x = 0$ is the stable manifold. Because $\lambda_3/\lambda_2 \notin \mathbf{Q}$ one may moreover assume that the $x = z = 0$ is invariant. The Dulac map D is then defined on the section $\Sigma_1 = \{y = 1\}$ to $\Sigma_2 = \{x = 1\}$ and maps a point (x_0, y_0) to (y_1, z_1) where $(1, y_1, z_1)$ is the first point on the trajectory of the flow associated to \mathcal{X} through $(x_0, y_0, 1)$ hitting Σ_2 . In order to describe D it is hence no restriction to divide \mathcal{X} by any appropriate strictly positive function. So the eigenvalues may be assumed to be $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -\mu$ for some positive $\mu \notin \mathbf{Q}$. Using a smooth normal form, containing only the resonant monomial $u = xy$ it is then possible to compute a very explicit form of the Dulac map, exhibiting properties similar to in the planar case. More in particular: $y_1 = x_0(1 + f(x_0))$, $z_1 = z_0 x_0^\mu(1 + g(x_0))$ where for all integer $j \geq 0$, the property $\lim_{x_0 \rightarrow 0} x_0^j d^j(f, g)/dx_0^j(x_0) = 0$ holds.

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In the more general case, where such an explicit form is not at hand, we investigate whether a similar asymptotic behavior can be obtained. Let us give an example showing what can happen.

Example 1. — Let $\mathcal{X}(x, y, z) = x\partial/\partial x + ((-2 - \varepsilon)y + z^2)\partial/\partial y - z\partial/\partial z$ where ε is a small parameter. The trajectory through $(x_0, y_0, 1)$ is easily calculated by quadrature and variation of constants; it hits the point $(1, y_1, z_1) = (1, y_1, x_0)$ where

$$y_1 = x_0^{2+\varepsilon} \left(y_0 + \frac{x_0^{-\varepsilon} - 1}{\varepsilon} \right)$$

if $\varepsilon \neq 0$; for $\varepsilon = 0$ we get

$$y_1 = x_0^2 \left(y_0 + (-\log x_0) \right)$$

Note that we encounter the maps introduced in [9], i.e.

$$\begin{aligned} \omega_\varepsilon(x_0) &= \frac{x_0^\varepsilon - 1}{\varepsilon} \text{ if } \varepsilon \neq 0 \text{ and} \\ \omega_0(x_0) &= \log x_0, \end{aligned} \tag{2}$$

so $y_1 = x_0^{2+\varepsilon}(y_0 - \omega_{-\varepsilon}(x_0))$. Observe that for $\varepsilon = 0$ there is a resonance $-2=0.1+0.(-2)+2.(-1)$ and a resonant monomial $z^2\partial/\partial y$.

On the other hand not all resonances entail this 'bad' behavior:

Example 2. — Let $\mathcal{X}(x, y, z) = x\partial/\partial x + ((-2\beta+1)y+xz^2)\partial/\partial y - \beta z\partial/\partial z$ where $\beta > 0$. The Dulac map is easily calculated by hand: $(x_0, y_0, 1) \rightarrow (1, y_1, z_1) = (1, x_0^{2\beta-1}(y_0 + x_0(-\log x_0)), x_0^\beta)$. The map $g(x_0) = x_0(-\log x_0)$ satisfies

$\lim_{x_0 \rightarrow 0} x_0^j d^j g/dx_0(x_0) = 0$. On the other hand there is a resonant monomial $xz^2\partial/\partial y$ since $-2\beta + 1 = 1.1 + 0.(-2\beta + 1) + 2.(-\beta)$.

We are in position to present the main result of this paper. Let $\{\mathcal{X}_\gamma; \gamma \in \mathcal{U}\}$ be a smooth family of vector field defined in a neighborhood of $O = (0, 0, 0) \in \mathbf{R}^3$ a hyperbolic equilibrium point, $\mathcal{U} \subset \mathbf{R}^2$. Suppose that for all $\gamma \in \mathcal{U}$ the eigenvalues of $d\mathcal{X}_\gamma(O)$ satisfy:

$$-\alpha(\gamma) < -\beta(\gamma) < 0 < \lambda_1(\gamma).$$

Up to a time rescaling, we can assume that $\lambda_1(\gamma) \equiv 1$. We choose linear coordinates (x, y, z) such that

$$d\mathcal{X}_\gamma(0) = \text{diag}[1, -\alpha(\gamma), -\beta(\gamma)].$$

We assume that the map

$$\mathcal{U} \rightarrow \mathbf{R}^2, \gamma \mapsto (\alpha(\gamma), \beta(\gamma))$$

define a local diffeomorphism. From now on, we identify γ with the eigenvalues of $d\mathcal{X}_\gamma(0)$. This means that the eigenvalues $(\alpha, \beta) \in \mathcal{U}$ represent the parameters, \mathcal{U} , the parameter space is now a small neighbourhood of (α_0, β_0) . For simplicity, we shall write \mathcal{X} instead of \mathcal{X}_γ .

Let Σ_1 , (respectively Σ_2) be a 2-dimensional cross sections transverse to $\{x = y = 0\}$ (respectively transverse to $\{y = z = 0\}$). Σ_1 intersects the stable manifold of the equilibrium point along a curve that disconnect Σ_1 into two connected components S^- and S^+ . Both Σ_1 and Σ_2 are chosen close enough to the origin in such a way that the Dulac map

$$\Delta : S^+ \rightarrow \Sigma_2, (x_0, y_0, 1) \mapsto (1, \Delta_y(x_0, y_0), \Delta_z(x_0, y_0))$$

is well defined.

Let $\tilde{\varepsilon} > 0$. We say that a function $f : (0, \tilde{\varepsilon}) \times (-\tilde{\varepsilon}, \tilde{\varepsilon})$ is a *Mourtada type function* if f is smooth and if for all integer $k \geq 0$,

$$\lim_{x_0 \rightarrow 0} x_0^k \frac{\partial^k f}{\partial x_0^k}(x_0, y_0) = 0$$

uniformly in y_0 .

THEOREM 1. — *Under the above notations, the following properties hold. Suppose that there exists an integer $m_0 > 1$ and $(\alpha_0, \beta_0) \in \mathcal{U}$ such that*

$$\alpha_0 = m_0 \beta_0.$$

Then, there exists a C^∞ coordinates such that

$$\begin{aligned} \Delta_y(x_0, y_0) &= x_0^\alpha \left(y_0 + p_{m_0} \omega_\varepsilon(x_0) + f_y(x_0, y_0) \right) \\ \Delta_z(x_0, y_0) &= x_0^\beta \left(1 + f_z(x_0, y_0) \right) \end{aligned}$$

where ω_ε satisfies (2), f_y and f_z are Mourtada type functions at 0 and

$$\varepsilon = \alpha - m_0 \beta.$$

Suppose that for all $(\alpha, \beta) \in \mathcal{U}$, $\alpha/\beta \notin \mathbf{N}$, then the above property holds with $p_{m_0} \equiv 0$.

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Remark. — We shall see that

$$\lim_{x_0 \rightarrow 0} x_0^k \frac{\partial^k f_y}{\partial x_0^k}(x_0, y_0) = 0, \quad x_0^k \frac{\partial^k f_z}{\partial x_0^k}(x_0, y_0) = 0.$$

The fact that those convergences are uniform with respect to y_0 will be evident, we even will see that they are uniform in α and β . We therefore avoid to mention the (α, β) or y_0 dependence when it is not necessary.

Obviously, ϵ tends to 0 as (α, β) tends to (α_0, β_0) . We also remark that in the case where $\alpha/\beta \notin \mathbb{N}$ the expression of the Dulac map satisfies the following. We put

$$D_y = \Delta_y - x_0^\alpha y_0, \quad D_z = \Delta_z - x_0^\beta.$$

Both D_y and D_z are the coordinates of the the difference between the Dulac maps associated to \mathcal{X} and the Dulac map associated to its linear part. Then, for all integer $i > 0$,

$$\frac{\partial^i D_y}{\partial x_0^i}(x_0, y_0) \leq o(x_0^{\alpha-i})$$

and

$$\frac{\partial^i D_z}{\partial x_0^i}(x_0, y_0) \leq o(x_0^{\beta-i}).$$

This asymptotic expression holds in the case of dimension three, we claim however that this is also true in higher dimension if we assume that the dimension of the unstable manifold is 1. In a more general context, the arguments developed in this paper do not work.

This result is also important for the following reason. Suppose that we are studying the bifurcations that arise in the unfolding of some heteroclinic cycle. In many cases authors usually assume (at least in dimension bigger than 2) that there is no resonance so that one can linearize the vector field at least C^r (r bigger than 2, 3..etc..) and then the Dulac map is easily expressed. We certainly believe that this argument is worth applying if we need to work in thick topology like the C^1 , even the C^2 topology. However (see for instance [6]) we some time need to work in thinner topology. In this case, we cannot ignore the existence of resonance and therefore the Dulac map hasn't this nice expression anymore.

Acknowledgments. — The authors would like to acknowledge Freddy Dumortier and Jiazhong Yang for all the discussions and comments given during the elaboration of this paper. Their remarks permit us to improve our initial result.

2. Basic assumptions

Consider \mathcal{X} be a C^∞ vector field near $0 \in \mathbf{R}^3$ with $\mathcal{X}(0) = 0$. We suppose that the linearization $d\mathcal{X}(0)$ at 0 is a saddle with three real eigenvalues $1, -\beta, -\alpha$ where $0 < \beta < \alpha$. Using invariant manifolds we let x denote a coordinate on the unstable manifold, and let (y, z) be coordinates on the stable manifold, y being the coordinate on the strong stable manifold. In this chart the vector field takes the form

$$\begin{aligned} x(1 + h_x(x, y, z)) \frac{\partial}{\partial x} - (\alpha y + a(x, y, z)y + b(x, y, z)z) \frac{\partial}{\partial y} \\ - (\beta z + c(x, y, z)y + d(x, y, z)z) \frac{\partial}{\partial z} \end{aligned} \quad (3)$$

with $(a, b, c, d, h_x)(x, y, z) = O(\|(x, y, z)\|)$. All these functions are defined on a neighbourhood U of the origin; up to a δ -rescaling (as we mentioned in the introduction), we may, and do, assume that $(1, 0, 0)$ and $(0, 0, 1)$ belong to U and that for all integer r the non linear terms satisfy

$$\|a\|_{C^r} + \|b\|_{C^r} + \|c\|_{C^r} + \|d\|_{C^r} \leq \mathcal{O}(\delta).$$

Let $S^+ = (0, \tilde{\varepsilon}) \times (-\tilde{\varepsilon}, \tilde{\varepsilon}) \times \{1\}$ for small $\tilde{\varepsilon} > 0$ and $\Sigma_2 = \{1\} \times (-\tilde{\varepsilon}, \tilde{\varepsilon}) \times (-\tilde{\varepsilon}, \tilde{\varepsilon})$. It is our purpose to describe the Dulac map $D : \Sigma_1 \rightarrow \Sigma_2$ sending a point $(x_0, y_0, 1)$ to the point $(1, y_1, z_1)$ where the trajectory of \mathcal{X} through $(x_0, y_0, 1)$ hits Σ_2 for the first time. For (x_0, y_0) small enough this map is well defined by the Hartman-Grobman theorem. We may divide the vector field by a strictly positive function, so we may assume that $h_x \equiv 0$. We then proceed to the normal form procedure. We get:

$$\left\{ \begin{array}{l} \dot{x} = x \\ \dot{y} = -\alpha y + xyG_y(x, y, z) + p_{m_0}z^{m_0} \\ \quad + x^{n_1}z^{m_1}B_y(x, z) + H_y(x, y, z) \\ \dot{z} = -\beta z + xzG_z(x, y, z) + x^{n_2}y^{m_2}B_z(x, y) \\ \quad + H_z(x, y, z) \end{array} \right. \quad (4)$$

where G_y, B_y, G_z, B_z are smooth functions in x, y, z they satisfy

$$G_y(x, 0, 0) \equiv 0, G_z(x, 0, 0) \equiv 0 \quad (5)$$

Moreover, $p_{m_0} = 0$ if $\alpha \neq m_0\beta$, and the integers n_1, m_1, n_2, m_2 are such that

$$x^{n_1}z^{m_1} \frac{\partial}{\partial y}, x^{n_2}z^{m_2} \frac{\partial}{\partial z}$$

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are resonant terms of the initial system. This means that

$$-m_1\beta_0 + \alpha_0 = -n_1 < 0 \quad -m_2\alpha_0 + \beta_0 = -n_2 < 0.$$

Therefore, one can assume that for all $(\alpha, \beta) \in \mathcal{U}$ the following inequalities hold.

$$-m_1\beta + \alpha < -\frac{n_1}{2} \quad -m_2\alpha + \beta < -\frac{n_2}{2}. \quad (6)$$

Both H_y and H_z are the flat terms, this means that all their partial derivative vanish at $(0, 0)$. We apply Sternberg's theorem (see [12], [13]): since 0 is a hyperbolic equilibrium point, we can eliminate the flat terms by conjugacy with a C^∞ local change of coordinates. To be more precise, Sternberg's theorem holds for a single vector field but we claim that this also holds even if the vector field depends on some parameters and the conjugacy depends smoothly with respect to the parameter. See [5] for a detailed proof or also [11], [1], [2]. For instance Rychlik [11], in the appendix of his article, give a very nice proof of this result using the so called Moser's homotopy method.

We will therefore assume from now on that both functions H_y and H_z vanish and we get the following ordinary differential equation.

$$\begin{cases} \dot{x} = x \\ \dot{y} = -\alpha y + xyG_y(x, y, z) \\ \quad + x^{n_1}z^{m_1}B_y(x, z) + p_{m_0}z^{m_0} \\ \dot{z} = -\beta z + xzG_z(x, y, z) \\ \quad + x^{n_2}y^{m_2}B_z(x, y) \end{cases} \quad (7)$$

In fact, we will call xyG_y and xzG_z the "good terms", $x^{n_1}z^{m_1}B_y$ and $x^{n_2}z^{m_2}B_z$ will be called the "bad terms". These "bad terms" prevent $\{y = 0\}$ resp. $\{z = 0\}$ from being invariant under the flow. We remark that, in the case of the dimension 2, the bad terms are inexistent. The integers n_1, m_1 , (respec. n_2, m_2) are such that

$$x^{n_1}z^{m_1} \frac{\partial}{\partial y}, \text{ (respec. } x^{n_2}z^{m_2} \frac{\partial}{\partial z} \text{)}$$

are the lowest order resonant terms that involve terms in x or z in the $\partial/\partial y$ direction (respectively terms in x or y in the $\partial/\partial z$) direction. Up to a rescaling, we have that for all integers r , $\|G_y\|_{C^r}, \|G_z\|_{C^r}, \|B_y\|_{C^r}, \|B_z\|_{C^r}$ and $|p_n|$ are bounded by some $K\delta$ where $K > 0$. As suggested by the examples and in the theorem, we need to distinguish two cases: for all $(\alpha, \beta) \in \mathcal{U}$, $\alpha/\beta \notin \mathbb{N}$ or there exists $(\alpha_0, \beta_0) \in \mathcal{U}$ such that $\alpha_0/\beta_0 \in \mathbb{N}$. The paper is organized as follows. In the next section we give a series of technical

results we need later. We will then introduce the ring of smooth functions \mathcal{M} , and the ring of Mourtada type functions. In the last two sections, we will prove theorem 1, we will treat separately the case $\alpha/\beta \notin \mathbf{N}$ (meaning $p_{m_0} = 0$) and the case where there exists $(\alpha_0, \beta_0) \in U$ such that $\alpha_0/\beta_0 \in \mathbf{N}$ (with $p_{m_0} \neq 0$).

3. Some technical preliminaries

LEMMA 1. — *Let $a > 0$, be a real number m be an integer. Let*

$$f : (0, a) \rightarrow \mathbf{R}, \quad x \mapsto f(x),$$

be a smooth function. Assume that

$$\lim_{x \downarrow 0} f(x) = a_0,$$

and suppose that for all integers $0 < n \leq m$,

$$\lim_{x \downarrow 0} x^n \frac{\partial^n f}{\partial x^n}(x) = a_n,$$

exists in \mathbf{R} , then $a_n = 0$.

This lemma is proven in the appendix. We then get the following corollary.

COROLLARY 1. — *Let m be an integer. Let*

$$\hat{f} : \mathbf{R}^+ \rightarrow \mathbf{R}$$

where $\hat{f}(w) = f(e^{-w})$, and f as in the previous lemma. Assume that for all integers $0 \leq n \leq m$

$$\lim_{w \rightarrow +\infty} \frac{\partial^n \hat{f}}{\partial w^n}(w)$$

exists. Then

$$\lim_{x \downarrow 0} x^n \frac{\partial^n f}{\partial x^n}(x) = 0,$$

and

$$\lim_{x \downarrow 0} f(x) = \lim_{w \rightarrow +\infty} \hat{f}(w).$$

This result is useful for the following purpose. In order to show that a function g satisfies

$$\lim_{x_0 \downarrow 0} x_0^n \frac{\partial^n g}{\partial x_0^n}(x_0) = 0,$$

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for some integer $n > 0$, it will suffice to show that the function $\hat{g}(w) = g(e^{-w})$ is such that

$$\lim_{w \rightarrow \infty} \frac{\partial^n \hat{g}}{\partial w^n}(w)$$

just exists for the same integer n . Such a function will be called in the next section, a function with *convergent derivatives* or a *CD*-function.

4. Some rings of functions

On the cross section S^+ , x_0 and y_0 are the coordinates. It means that the point $(x_0, y_0, 1)$ corresponds to initial condition of the flow \mathcal{X}_t associated to \mathcal{X} . In this section, we give some estimation on the higher order derivative of this flow with respect to the initial condition x_0 . Instead of using the variable x_0 , we shall use the variable w were $x_0 = e^{-w}$ and we denote by

$$\left(x(w, t), y(w, t), z(w, t) \right) = \mathcal{X}_t(e^{-w}, y_0, 1).$$

We remark that $x(w, t) = e^{t-w}$ and that the coordinates of the Dulac map associated to \mathcal{X} are such that

$$\Delta_y(x_0, y_0) = y(w, w), \quad \Delta_z(x_0, y_0) = z(w, w).$$

We now introduce some new rings of functions. Let $h : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}$ be a smooth and bounded function. We say that h belongs to \mathcal{M} if there exists $\nu > 0$ and if for all non-negative integer n and m there exists a function $g_{m,n}$ such that

$$\begin{aligned} h(w, t) &= g_{0,0}(w, t)e^{-\nu t} \\ \frac{\partial^{n+m} g_{0,0}}{\partial w^m \partial t^n}(w, t) &= g_{m,n}(w, t), \end{aligned} \tag{8}$$

and such that $g_{m,n}$ is bounded on the triangle

$$\mathcal{T} = \{(w, t), \mid 0 \leq t \leq w, \}.$$

We also say that $g_{0,0}$ has *bounded partial derivatives* on \mathcal{T} . Due to the Leibniz rule, it is easy to see that the set \mathcal{M} is a ring for the multiplication law. Let $\bar{h} : \mathbf{R}^+ \rightarrow \mathbf{R}$ be a smooth function. We say that \bar{h} is a function with *convergent derivatives* or is a *CD*-function if for all integer n ,

$$\lim_{w \rightarrow +\infty} \bar{h}^{(n)}(w)$$

exists. The set of *CD*-functions is also a ring for the multiplication law.

Remark. — If we go back to the variable $x_0 = e^{-w}$ and if we use corollary 1, this means that $g(x_0) = \bar{h}(-\log x_0)$ satisfies

$$\lim_{x_0 \downarrow 0} x_0^n g^{(n)}(x_0) = 0$$

for all integer $n > 0$ and of course

$$\lim_{x_0 \downarrow 0} g(x_0)$$

exists. In other words, if \bar{h} is a *CD*-function then g is a Mourtada type function.

Let $\nu > 0$ be close to 0. Then there exists λ and μ such that for all $(\alpha, \beta) \in \mathcal{U}$,

$$0 < \frac{\alpha + n_1 - \nu}{m_1} < \lambda < \beta, \quad 0 < \frac{\beta + n_2 - \nu}{m_2} < \mu < \alpha,$$

and

$$m_1 \lambda - \mu > 0, \quad -m_0 \lambda + \mu < 0, \quad m_2 \mu - \lambda > 0.$$

LEMMA 2. — *There exists smooth functions $\bar{y}(w, t)$ and $\bar{z}(w, t)$ such that*

$$y(w, t) = e^{-\mu t} \bar{y}(w, t), \quad z(w, t) = e^{-\lambda t} \bar{z}(w, t),$$

and such that \bar{y} and \bar{z} have bounded partial derivatives on

$$\mathcal{T} = \{(w, t) \mid 0 \leq t \leq w\}$$

i.e for all non-negative integer integers n, m

$$\frac{\partial \bar{y}}{\partial t^n \partial w^m}(w, t) \quad \text{and} \quad \frac{\partial \bar{z}}{\partial t^n \partial w^m}(w, t)$$

are bounded on \mathcal{T} .

We give the proof of this lemma in the appendix. From lemma 2 we know that both y and z as functions depending on w and t belong to \mathcal{M} . Suppose that $f = f(w, t) = F\left(y(w, t), z(w, t)\right)$ where F is a smooth function, then f belongs to \mathcal{M} .

PROPOSITION 1. — *Let $f \in \mathcal{M}$. Then there exists $\nu > 0$ and a smooth function \bar{F} with bounded partial derivatives on \mathcal{T} such that*

$$\int_0^t x(w, s) f(w, s) ds = e^{t-w} e^{-\nu t} \bar{F}(w, t) \tag{9}$$

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Moreover the functions

$$w \mapsto f(w, w), \quad w \mapsto \mathcal{A}_f(w) = \int_0^w x(w, t) f(w, t) dt$$

are *CD-functions*.

Proof. — Denote by $g(w) = f(w, w)$. We then get

$$\frac{d^k g}{dw^k}(w) = \sum_{i=1}^k C_k^i \frac{\partial^k f}{\partial t^i \partial w^{k-i}}(w, w),$$

and since $f \in \mathcal{M}$ there exists $\nu > 0$ such that

$$\left| \frac{\partial^k f}{\partial t^i \partial w^{k-i}}(w, w) \right| \leq \mathcal{O}\left(e^{-\nu w}\right).$$

This implies that for all integer n

$$\lim_{w \rightarrow \infty} \frac{d^n g}{dw^n}(w) = 0.$$

Now, since $f \in \mathcal{M}$, there exists a function \bar{f} with bounded partial derivatives on \mathcal{T} such that

$$x(w, t) f(w, t) = e^{t-w} e^{-\nu t} \bar{f}(w, t)$$

Therefore

$$\begin{aligned} \int_0^t x(w, s) f(w, s) ds &= \int_0^t e^{s-w} e^{-\nu s} \bar{f}(w, s) ds & (10) \\ &= (e^{t-w} e^{-\nu t}) \int_0^t e^{(s-t)(1-\nu)} \bar{f}(w, s) ds \\ &= (e^{t-w} e^{-\nu t}) \bar{F}(w, t) \end{aligned}$$

where \bar{F} is a smooth function with bounded partial derivatives on \mathcal{T} . It therefore turns out that $\mathcal{A}_f(w) = e^{-\nu w} \bar{F}(w, w)$, this implies that for all non-negative integer n

$$\left| \frac{d^n \mathcal{A}_f}{dw^n}(w) \right| \leq k_n e^{-\nu w}.$$

for some $k_n > 0$ and therefore \mathcal{A}_f is a *CD-function*. This ends the proof of the proposition

The next two sections are devoted to prove theorem 1.

5. The case where $\alpha/\beta \notin \mathbb{N}$

In this case we have, $p_{m_0} = 0$ in (6). Put

$$g_y(w, t) = x(w, t)G_y\left(x(w, t), y(w, t), z(w, t)\right),$$

$$g_z(w, t) = x(w, t)G_z\left(x(w, t), y(w, t), z(w, t)\right).$$

From (5) it turns out that there exist some smooth functions G_{yy} , G_{yz} , G_{zy} and G_{zz} such that:

$$G_y(x, y, z) = yG_{yy}(x, y, z) + zG_{yz}(x, y, z),$$

$$G_z(x, y, z) = yG_{zy}(x, y, z) + zG_{zz}(x, y, z).$$

We now have

$$g_y = xyG_{yy} + xzG_{yz}, \quad g_z = xyG_{zy} + xzG_{zz}.$$

Since yG_{yy} , zG_{yz} , yG_{zy} and zG_{zz} belong to \mathcal{M} , with proposition 1 and (10) we know that there exists $0 < \nu \leq 1$, some smooth functions φ and ψ with bounded partial derivatives on \mathcal{T} such that

$$\exp \int_0^t g_y(w, u)du = \exp\left(e^{t-w} e^{-\nu t} \bar{\varphi}(w, t)\right) \tag{11}$$

$$\exp \int_0^t g_z(w, u)du = \exp\left(e^{t-w} e^{-\nu t} \bar{\psi}(w, t)\right) \tag{12}$$

Let φ_2 and ψ_2 be such that

$$\left. \begin{aligned} \exp\left(e^{t-w} e^{-\nu t} \bar{\varphi}(w, t)\right) &= 1 + e^{t-w} e^{-\nu t} \varphi_2(e^{t-w}), \\ \exp\left(e^{t-w} e^{-\nu t} \bar{\psi}(w, t)\right) &= 1 + e^{t-w} e^{-\nu t} \psi_2(e^{t-w}), \end{aligned} \right\} \tag{13}$$

Here both φ_2 and ψ_2 have bounded partial derivatives on \mathcal{T} . We now put:

$$A(w, t) = \alpha t - \int_0^t g_y(w, s)ds, \quad \text{and} \quad B(w, t) = \beta t - \int_0^t g_z(w, s)ds \tag{14}$$

and

$$\left\{ \begin{aligned} \dot{y} &= -y \frac{\partial A}{\partial t}(w, t) + x^{n_1} z^{m_1} B_y\left(x(w, t), z(w, t)\right) \\ \dot{z} &= -z \frac{\partial B}{\partial t}(w, t) + x^{n_2} y^{m_2} B_z\left(x(w, t), y(w, t)\right) \end{aligned} \right. \tag{15}$$

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We obtain the following implicit solutions

$$y(w, t) = \exp(-A(w, t)) \left(y_0 + \int_0^t Q_y(w, s) ds \right) \quad (16)$$

$$z(w, t) = \exp(-B(w, t)) \left(1 + \int_0^t Q_z(w, s) ds \right) \quad (17)$$

where

$$Q_y(w, s) = e^{n_1(s-w)} z^{m_1}(w, s) B_y \left(x(w, s), z(w, s) \right) \exp \left(A(w, s) \right)$$

$$Q_z(w, s) = e^{n_2(s-w)} y^{m_2}(w, s) B_z \left(x(w, s), y(w, s) \right) \exp \left(B(w, s) \right)$$

With (13) we get

$$\begin{aligned} \exp \left(-A(w, t) \right) &= e^{-\alpha t} \exp \int_0^t g_y(w, u) du \\ &= e^{-\alpha t} \left(1 + e^{t-w} e^{-\nu t} \varphi_2(e^{t-w}) \right) \end{aligned} \quad (18)$$

$$\begin{aligned} \exp \left(-B(w, t) \right) &= e^{-\beta t} \exp \int_0^t g_z(w, u) du \\ &= e^{-\beta t} \left(1 + e^{t-w} e^{-\nu t} \psi_2(e^{t-w}) \right) \end{aligned} \quad (19)$$

Furthermore, since

$$y(w, t) = e^{-\mu t} \bar{y}, \quad z(w, t) = e^{-\lambda t} \bar{z}$$

and since $-m_1\lambda + \alpha < -n_1 + \nu < 0$, $-m_2\mu + \beta < -n_2 + \nu < 0$, there exists functions \hat{Q}_y and \hat{Q}_z with bounded partial derivatives such that

$$Q_y(w, t) = e^{(-m_1\lambda + \alpha)t} e^{t-w} \hat{Q}_y(w, t) \quad (20)$$

$$Q_z(w, t) = e^{(-m_2\mu + \beta)t} e^{t-w} \hat{Q}_z(w, t) \quad (21)$$

With proposition 1, there exists smooth functions φ_3 and ψ_3 with bounded partial derivatives on \mathcal{T} such that

$$\int_0^t Q_y(w, s) ds = e^{t-w} e^{-\nu_1 t} \varphi_3(w, t) \quad (22)$$

$$\int_0^t Q_z(w, s) ds = e^{t-w} e^{-\nu_2 t} \psi_3(w, t) \quad (23)$$

where $\nu_1 = m_1\lambda - \alpha > 0$ and $\nu_2 = m_2\mu - \beta > 0$. From (18), (19), (16) (17), (13), (22) and (23) there exists $\nu_4 \leq \inf\{\nu, \nu_1, \nu_2\}$ and some smooth functions $v(w, t)$ and $u(w, t)$ with bounded partial derivatives such that

$$y(w, t) = e^{-\alpha t} \left(y_0 + e^{t-w} e^{-\nu_4 t} v(w, t) \right) \quad (24)$$

$$z(w, t) = e^{-\beta t} \left(1 + e^{t-w} e^{-\nu_4 t} u(w, t) \right) \quad (25)$$

With proposition 1, the following functions

$$w \mapsto e^{-\nu_4 w} v(w, w), \text{ and } w \mapsto e^{-\nu_4 w} u(w, w),$$

are *CD*-functions. Therefore, coming back to the previous coordinates i.e. $w = -\log x_0$, we then get

$$\begin{cases} \Delta_y(x_0, y_0) = x_0^\alpha \left(y_0 + x_0^{\nu_4} v(-\log x_0, -\log x_0) \right) \\ \Delta_z(x_0, y_0) = x_0^\beta \left(1 + x_0^{\nu_4} u(-\log x_0, -\log x_0) \right) \end{cases} \quad (26)$$

Put

$$\begin{aligned} f_y(x_0, y_0) &= x_0^{\nu_4} v(-\log x_0, -\log x_0), \\ f_z(x_0, y_0) &= x_0^{\nu_4} u(-\log x_0, -\log x_0), \end{aligned}$$

we finally get

$$\begin{cases} \Delta_y(x_0, y_0) = x_0^\alpha \left(y_0 + f_y(x_0, y_0) \right) \\ \Delta_z(x_0, y_0) = x_0^\beta \left(1 + f_z(x_0, y_0) \right) \end{cases} \quad (27)$$

and as a consequence of corollary 1 we have that for all integer m ,

$$\lim_{x_0 \rightarrow 0^+} x_0^m \frac{\partial^m f_y}{\partial x_0^m}(x_0, y_0) = \lim_{x_0 \rightarrow 0^+} x_0^m \frac{\partial^m f_z}{\partial x_0^m}(x_0, y_0) = 0$$

We moreover remark that those convergence are uniform in y_0 , α and β .

6. The case $\alpha_0/\beta_0 \in \mathbb{N}$

Put $m_0 = \alpha_0/\beta_0$. In this case we have

$$\begin{cases} \dot{y} &= -y \frac{\partial A}{\partial t}(w, t) + x^{n_1} z^{m_1} B_y(x(w, t), z(w, t)) + p_{m_0} z^{m_0} \\ \dot{z} &= -z \frac{\partial B}{\partial t}(w, t) + x^{n_2} y^{m_2} B_z(x(w, t), y(w, t)) \end{cases} \quad (28)$$

We therefore have the following implicit solution

$$y(w, t) = \exp(-A(w, t)) \left(y_0 + \int_0^t H_y(w, s) ds \right) \quad (29)$$

$$z(w, t) = \exp(-B(w, t)) \left(1 + \int_0^t Q_z(w, s) ds \right) \quad (30)$$

where

$$\begin{aligned} H_y(w, s) &= Q_y(w, s) + p_{m_0} \exp(A(w, s)) z^{m_0} \\ &= e^{n_1(s-w)} z^{m_1}(w, s) B_y(x(w, s), z(w, s)) \exp(A(w, s)) \\ &\quad + p_{m_0} \exp(A(w, s)) z^{m_0} \\ Q_z(w, s) &= e^{n_2(s-w)} y^{m_2}(w, s) B_z(x(w, s), y(w, s)) \exp(B(w, s)) \end{aligned}$$

Following the same computation in the previous section we get

$$z(w, t) = e^{-\beta t} \left(1 + e^{w-t} e^{-\nu_4 t} u(w, t) \right) \quad (31)$$

$$\begin{aligned} \exp(A(w, t)) &= e^{\alpha t} \exp \int_0^t -g_y(w, u) du \\ &= e^{\alpha t} \left(1 + e^{t-w} e^{-\nu t} \hat{\varphi}_2(e^{t-w}) \right) \end{aligned} \quad (32)$$

where both $\hat{\varphi}_2$ and u have bounded partial derivatives. With (31) we have

$$H_y(w, t) = Q_y(w, t) + p_{m_0} e^{(\alpha - m_0 \beta)t} \left(1 + e^{t-w} e^{-\nu_5 t} \varphi_4(w, t) \right)$$

where $\nu_5 = \inf\{\nu, \nu_4\}$ and where φ_4 has bounded partial derivatives. By putting $\varepsilon = \alpha - m_0 \beta$, and assume that $\varepsilon \leq \nu_5$, we therefore have

$$H_y(w, t) = Q_y(w, t) + p_{m_0} e^{\varepsilon t} + p_{m_0} e^{t-w} e^{-\nu_5 t} \varphi_4(w, t).$$

where $\nu_6 = \nu_5 - \varepsilon > 0$. By integration we get

$$\begin{aligned} \int_0^t H_y(w, t) ds &= \int_0^t Q_y(w, s) ds \\ &+ \int_0^t p_{m_0} e^{\varepsilon s} + \int_0^t p_{m_0} e^{s-w} e^{-\nu_6 s} \varphi_4(w, s) ds. \end{aligned} \quad (33)$$

Now with (22) we get

$$\begin{aligned} \int_0^t H_y(w, t) ds &= e^{t-w} e^{-\nu_1 t} \varphi_3(w, t) \\ &+ p_{m_0} \frac{e^{\varepsilon t} - 1}{\varepsilon} + p_{m_0} e^{t-w} e^{-\nu_6 t} \int_0^t e^{(1-\nu_6)(s-t)} \varphi_4(w, s) ds \\ &= e^{t-w} e^{-\nu_6 t} \varphi_6(w, t) + p_{m_0} \frac{e^{\varepsilon t} - 1}{\varepsilon} \end{aligned} \quad (34)$$

where φ_6 has bounded partial derivatives. With (29), (18), (35) and (31), we finally get

$$\begin{aligned} y(w, t) &= e^{-\alpha t} \left(y_0 + p_{m_0} \frac{e^{\varepsilon t} - 1}{\varepsilon} + e^{t-w} e^{-\nu_6 t} \varphi_7(w, t) \right) \\ z(w, t) &= e^{-\beta t} \left(1 + e^{w-t} e^{-\nu_4 t} u(w, t) \right) \end{aligned} \quad (36)$$

where φ_7 has bounded partial derivatives. We finally get

$$\begin{cases} \Delta_y(x_0, y_0) = x_0^\alpha \left(y_0 + \omega_\varepsilon(x_0) + f_y(x_0, y_0) \right) \\ \Delta_z(x_0, y_0) = x_0^\beta \left(1 + f_z(x_0, y_0) \right) \end{cases} \quad (37)$$

and as a consequence of corollary 1 we have that for all integer m ,

$$\lim_{x_0 \rightarrow 0^+} x_0^m \frac{\partial^m f_y}{\partial x_0^m}(x_0, y_0) = \lim_{x_0 \rightarrow 0^+} x_0^m \frac{\partial^m f_z}{\partial x_0^m}(x_0, y_0) = 0.$$

This finally ends the proof of theorem 1.

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7. Appendix

We first start this appendix by writing the following sub-lemma about vector fields defined in a vicinity of a hyperbolic equilibrium point for which its unstable manifold is one-dimensional.

Sub-lemma 1. — Let n be an integer and $\hat{\mathcal{X}}$ be a smooth vector field defined in a neighborhood of $0 \in \mathbf{R}^{n+1}$. Suppose that 0 is an hyperbolic equilibrium point, and

$$\text{spect}\left(d\hat{\mathcal{X}}(0)\right) = \{1, -\mu_1, -\mu_2, \dots, -\mu_n\},$$

where the μ_i 's are strictly positive. Up to some linear rescaling, \mathcal{X} is equivalent to the following ordinary differential equation.

$$\begin{aligned} \dot{x} &= x \\ \dot{w}_1 &= -\mu_1 w_1 + \bar{\delta} H_1(x, w_1, \dots, w_n) \\ \dot{w}_2 &= -\mu_2 w_2 + \bar{\delta} H_2(x, w_1, \dots, w_n) \\ \dots &= \dots \\ \dot{w}_n &= -\mu_n w_n + \bar{\delta} H_n(x, w_1, \dots, w_n) \end{aligned}$$

where $\bar{\delta} > 0$ is close to 0 and the functions H_1, \dots, H_n are smooth, bounded and they consist in the higher order terms. Without loss of generality, we can assume that equation is valid in the following neighbourhood

$$\{(x, w_1, \dots, w_n) \mid x^2 + \sum_{i=1}^n w_i^2 \leq 3\}.$$

Fix $\delta > 0$. We define the following sections S_1^+, \dots, S_n^+ and Σ

$$S_j^+ = \{(x, w_1, \dots, w_n) \mid 0 < x < \delta, w_j = 1, \sum_{i=1}^n w_i^2 < w_j^2 + 1/2\},$$

and

$$\Sigma_0 = \{(x, w_1, \dots, w_n) \mid x = 1, w_1^2 + \dots + w_n^2 < 1/2\}.$$

We parameterize by (x, P_j) a point in S_j^+ . Let $w > -\log \delta$, and $\hat{\mathcal{X}}_t$ be the flow associated to $\hat{\mathcal{X}}$. For all integer $j = 1 \dots n$, we denote by $T_j(w, P_j)$ the following piece of trajectory

$$T_j(w, P_j) = \{\hat{\mathcal{X}}_t(e^{-w}, P_j), 0 \leq t \leq w\},$$

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that connects the point $(e^{-w}, P_j) \in S_j^+$ to the point $\hat{\mathcal{X}}_w(e^{-w}, P_j) \in \Sigma_0$ and the following transition maps associated to $\hat{\mathcal{X}}$

$$\Delta_{j,\zeta} : S_j^+ \rightarrow \Sigma_0, (x, P_j) \mapsto (1, \Delta_{j,1,\zeta}(x, P_j), \dots, \Delta_{j,n,\zeta}(x, P_j)).$$

Recall that

$$\Delta_{j,\zeta}(x, P_j) = \hat{\mathcal{X}}_{\log \frac{1}{x}}(x, P_j).$$

Then for all integer i, j the following properties hold.

(i) For all point $P_j \in S_j^+$, the sets $T(w, P_j)$ and $\Delta_{j,k}$ are bounded over w ,

$$(ii) \quad \lim_{w \rightarrow +\infty} \Delta_{j,k}(e^{-w}, P_j) = 0, \quad \lim_{w \rightarrow +\infty} \hat{\mathcal{X}}_{w-\zeta}(x, P_j) = 0.$$

This sub-lemma is a direct consequence of the Hartman-Grobman theorem.

PROOF OF LEMMA 1. — We prove this lemma by contradiction. Suppose that there exists an integer $i > 1$ such that $a_j = 0$ if $j < i$, and $a_i \neq 0$. Without loss of generality we can assume that $a_i > 0$. Fix $0 < \nu < a_i$. Therefore there exists $b > 0$ such that for all $0 < x \leq b$,

$$a_i - \nu < x^i \frac{\partial^i f}{\partial x^i} < a_i + \nu.$$

This implies that for all $0 < u < b$,

$$\int_u^b \frac{a_i - \nu}{x^i} dx < \int_u^b \frac{\partial^i f}{\partial x^i}(x) dx < \int_u^b \frac{a_i + \nu}{x^i} dx.$$

We therefore get

$$\frac{a_i - \nu}{i-1} \left(\frac{1}{u^{i-1}} - \frac{1}{b^{i-1}} \right) < \frac{\partial^{i-1} f}{\partial x^{i-1}}(b) - \frac{\partial^{i-1} f}{\partial x^{i-1}}(u)$$

This implies that

$$\frac{a_i - \nu}{i-1} \left(1 - \frac{u^{i-1}}{b^{i-1}} \right) < u^{i-1} \frac{\partial^{i-1} f}{\partial x^{i-1}}(b) - u^{i-1} \frac{\partial^{i-1} f}{\partial x^{i-1}}(u)$$

We now let u tend to 0. We know that

$$\lim_{u \rightarrow 0} u^{i-1} \frac{\partial^{i-1} f}{\partial x^{i-1}}(u) = a_{i-1} = 0, \quad (38)$$

and anyway we have

$$\lim_{u \rightarrow 0} u^{i-1} \frac{\partial^{i-1} f}{\partial x^{i-1}}(b) = 0.$$

Also for u close enough to 0,

$$\frac{a_i - \nu}{i - 1} \left(1 - \frac{u^{i-1}}{b^{i-1}} \right) \geq \frac{a_i - \nu}{2(i - 1)}$$

This implies that

$$\frac{a_i - \nu}{2(i - 1)} \leq u^{i-1} \frac{\partial^{i-1} f}{\partial x^{i-1}}(b) - u^{i-1} \frac{\partial^{i-1} f}{\partial x^{i-1}}(u),$$

which contradicts (38). Now, if $i = 1$ we come to the following inequality

$$\int_u^b \frac{a_1 - \nu}{x} dx < \int_u^b \frac{\partial f}{\partial x}(x) dx < \int_u^b \frac{a_1 + \nu}{x} dx.$$

This implies that

$$(a_1 - \nu) (\log b - \log u) < f(b) - f(u),$$

and this will implies that f does not converge to 0 as x tends to 0. We then get another contradiction and this ends the proof of the lemma.

PROOF OF COROLLARY 1. — We first remark that

$$\frac{\partial \hat{f}}{\partial w}(w) = -e^{-w} \frac{\partial f}{\partial x}(e^{-w}).$$

Therefore, suppose that

$$\lim_{w \rightarrow \infty} \frac{\partial \hat{f}}{\partial w}(w) = a_1,$$

we then have

$$\lim_{u \rightarrow 0} u \frac{\partial f}{\partial x}(u) = a_1,$$

and by using lemma 1, we have that $a_1 = 0$. Suppose now that there exists an integer $n \leq m$ such that for all integer $1 \leq i \leq n - 1$

$$\lim_{w \rightarrow \infty} \frac{\partial^i \hat{f}}{\partial w^i}(w) = a_i = 0$$

Now, one can easily show (by induction on n) that

$$\frac{\partial^n \hat{f}}{\partial w^n}(w) = \sum_{i=1}^{n-1} A_{i,n} e^{-iw} \frac{\partial^i f}{\partial x^i}(e^{-w}) + A_{n,n} e^{-nw} \frac{\partial^n f}{\partial x^n}(e^{-w}),$$

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where

$$A_{1,n} = (-1)^n, \quad A_{n,n} = (-1)^n$$

and for all integer $1 < i \leq j < n$, we have

$$A_{i,j+1} = -A_{i-1,j} - iA_{i,j}.$$

Now when w tends to ∞ we have

$$\lim_{w \rightarrow +\infty} \frac{\partial^n \hat{f}}{\partial w^n}(w) = \sum_{i=1}^{n-1} A_{i,n} \lim_{x \rightarrow 0^+} x^i \frac{\partial^i f}{\partial x^i}(x) + A_{n,n} \lim_{x \rightarrow 0^+} x^n \frac{\partial^n f}{\partial x^n}(x).$$

Since $a_i = 0$ for all integer $i \leq n-1$ we therefore have

$$\lim_{w \rightarrow +\infty} \frac{\partial^n \hat{f}}{\partial w^n}(w) = A_{n,n} \lim_{x \rightarrow 0^+} x^n \frac{\partial^n f}{\partial x^n}(x) = 0.$$

Since $A_{n,n} \neq 0$, the proof of the corollary is therefore finished.

PROOF OF LEMMA 2. — We put:

$$Y(w, t) = \bar{y}e^{-\mu t} Z = \bar{z}e^{-\lambda t}.$$

Recall that

$$\begin{cases} \dot{x} = & x \\ \dot{y} = & -\alpha y & + & p_{m_0} z^{m_0} \\ & + & G_y(x, y, z)xy & + & x^{n_1} z^{m_1} B_y(x, z) \\ \dot{z} = & -\beta z & + & G_z(x, y, z)xz \\ & + & x^{n_2} y^{m_2} B_z(x, y) \end{cases} \quad (39)$$

With (39) we get

$$\begin{cases} \dot{\bar{y}}e^{-\mu t} - \mu \bar{y}e^{-\mu t} = & -\alpha \bar{y}e^{-\mu t} + p_{m_0} \bar{z}^{m_0} e^{-m_0 \lambda t} \\ & + e^{-\mu t} x \bar{y} G_y(x, y, z) + x^{n_1} e^{-m_1 \lambda t} \bar{z}^{m_1} B_y(x, z) \\ \dot{\bar{z}}e^{-\lambda t} - \lambda \bar{z}e^{-\lambda t} = & -\beta e^{-\lambda t} \bar{z} + e^{-\lambda t} x \bar{z} G_z(x, y, z) \\ & + x^{n_2} \bar{y}^{m_2} e^{-m_2 \mu t} B_z(x, y) \end{cases} \quad (40)$$

By putting

$$u = e^{(\lambda - m_2 \mu)t}, \quad v = e^{(\mu - m_0 \lambda)t}, \quad \xi = e^{(\mu - m_1 \lambda)t},$$

we obtain the following equation

$$\left\{ \begin{array}{l} \dot{x} = x \\ \dot{y} = -\alpha y + p_{m_0} z^{m_0} \\ \quad + G_y(x, y, z)xy + x^{n_1} z^{m_1} B_y(x, z) \\ \dot{z} = -\beta z + G_z(x, y, z)xz \\ \quad + x^{n_2} y^{m_2} B_z(x, y) \\ \dot{\bar{y}} = -(\alpha - \mu)\bar{y} + p_{m_0} \bar{z}^{m_0} v \\ \quad + x\bar{y}G_y(x, y, z) + x^{n_1} \xi \bar{z}^{m_1} B_y(X, Z) \\ \dot{\bar{z}} = -(\beta - \lambda)\bar{z} + x\bar{z}G_z(x, y, z) \\ \quad + x^{n_2} \bar{y}^{m_2} u B_z(x, y) \\ \dot{u} = (\lambda - m_2 \mu)u \\ \dot{v} = (\mu - m_0 \lambda)v \\ \dot{\xi} = (\mu - m_1 \lambda)\xi \end{array} \right. \quad (41)$$

Now up to some linear rescaling we can assume that the conditions in sublemma 1 are satisfied, then both \bar{y} and \bar{z} are bounded on the triangle \mathcal{T} . Take $m > 0$ and $n \geq 0$. We put

$$\bar{y}_{m,n} = \frac{\partial^{m+n} \bar{y}}{\partial t^n \partial w^m}, \quad \bar{z}_{m,n} = \frac{\partial^{m+n} \bar{z}}{\partial t^n \partial w^m}$$

and

$$\mathcal{W}_{m,n} = (\bar{y}, \bar{z}, \dots, \bar{y}_{i,j}, \bar{z}_{i,j}, \dots, \bar{y}_{m,n}, \bar{z}_{m,n}).$$

From (41) by successive derivations we get

$$\left\{ \begin{array}{l} \dot{x} = x \\ \dot{y} = -\alpha y + p_{m_0} z^{m_0} \\ \quad + G_y(x, y, z)xy + x^{n_1} z^{m_1} B_y(x, z) \\ \dot{z} = -\beta z + G_z(x, y, z)xz \\ \quad + x^{n_2} y^{m_2} B_z(x, y) \\ \dot{\bar{y}}_{m,n} = -(\alpha - \mu)\bar{y}_{m,n} + H_{y,m,n}(v, \xi, x, y, z, \mathcal{W}_{m,n}) \\ \dot{\bar{z}}_{m,n} = -(\beta - \lambda)\bar{z}_{m,n} + H_{z,y,z}(u, x, y, z, \mathcal{W}_{m,n}) \\ \dot{u} = (\lambda - m_2\mu)u \\ \dot{v} = (\mu - m_0\lambda)v \\ \dot{\xi} = (\mu - m_1\lambda)\xi \end{array} \right. \quad (42)$$

where both $H_{y,m,n}$ and $H_{z,m,n}$ are smooth functions. Again up to some linear rescaling we can assume that the conditions in sub-lemma 1 are satisfied, and this shows that both $\bar{y}_{m,n}$ and $\bar{z}_{m,n}$ are bounded. This ends the proof of the lemma.