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Semilinear wave equation on manifolds (*)

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Dedicated to M. Milla Miranda in the occasion of his 60th. anniversary.

1. Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\Gamma$. Let $\nu$ be the outward normal unit vector to $\Gamma$ and $T > 0$ a real number. We consider the cylinder $Q = \Omega \times ]0, T[$ with lateral boundary $\Sigma = \Gamma \times ]0, T[$.

We investigate existence and asymptotic behaviour of weak solution for the problem

\begin{equation}
\begin{aligned}
 w'' + \frac{\partial w}{\partial \nu} + F(w) + \beta(x)w' &= 0 \quad \text{on} \quad \Sigma, \\
 w(0) &= w_0, \quad w'(0) = w_1 \quad \text{on} \quad \Gamma,
\end{aligned}
\end{equation}

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where the prime means the derivative with respect to $t$, the normal derivative of $w$ and $F : \mathbb{R} \to \mathbb{R}$ is a function that satisfies
\begin{equation}
F \text{ continuous and } sF(s) \geq 0, \forall s \in \mathbb{R}.
\end{equation}

It is important to call the attention to the reader that the idea employed in this work comes from Lions [8], pg. 134. The main point consists in adding to (1.1) an elliptic equation in $\Omega$ to reduce the problem to a canonical model of Mathematical Physics, but in this case on a manifold which is the lateral boundary $\Sigma$ of the cylinder $Q$. A Similar type of problem, also motivated by Lions [8], can be seen in Cavalvanti and Domingos Cavalcanti [2].

The plan of this article is the following: In the section 2, we give notations, terminology and we treat the linear case associated to (1.1). In the section 3, we prove existence for weak solution when $F$ satisfies the condition (1.2), approximating $F$ by Lipschitz functions. In this Lipschitz case, we employ Picard’s successive approximations and then we apply the Strauss’ method [9]. Finally in the section 4, we obtain the asymptotic behaviour by the method of perturbation of energy as in Zuazua [10].

2. Notations, Assumptions and Results

Denote by $|\cdot|$, $(\cdot, \cdot)$ and $\|\cdot\|$, $(\cdot, \cdot)$ the inner product and norm, respectively, of $L^2(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

For
\begin{equation}
G(s) = \int_0^s F(\sigma)d\sigma
\end{equation}
we will denote a primitive of $F$.

We consider the following assumption on $\beta$ in (1.1):
\begin{equation}
\beta \in L^\infty(\Gamma) \text{ such that } \beta(x) \geq \beta_0 > 0, \text{ a.e. on } \Gamma.
\end{equation}

As was said in the introduction, for $\lambda > 0$, let us consider the problem
\begin{equation}
\begin{aligned}
-\Delta w + \lambda w &= 0 \quad \text{in } Q, \\
\varphi'' + \frac{\partial \varphi}{\partial \nu} + F(w) + \beta(x)w' &= 0 \quad \text{on } \Sigma, \\
w(0) &= w_0, \quad w'(0) = w_1 \quad \text{on } \Gamma.
\end{aligned}
\end{equation}

From elliptic theory, we know that for $\varphi \in H^{\frac{1}{2}}(\Gamma)$, the solution $\Phi$ of the boundary value problem
\begin{equation}
\begin{aligned}
-\Delta \Phi + \lambda \Phi &= 0 \quad \text{in } \Omega, \\
\Phi &= \varphi \quad \text{on } \Gamma,
\end{aligned}
\end{equation}
belongs to $H^1(\Omega, \Delta) = \{ u \in H^1(\Omega); \Delta u \in L^2(\Omega) \}$. By the trace theorem, it follows that $\frac{\partial \Phi}{\partial \nu} \in H^{-\frac{1}{2}}(\Gamma)$.

Formally, we have by (2.3) that

$$0 = \int_{\Omega} \nabla \Phi \nabla \Psi dx + \lambda \int_{\Omega} \Phi \Psi dx - \int_{\Gamma} \frac{\partial \Phi}{\partial \nu} \Psi d\Gamma.$$ 

We take $\Psi \in H^1(\Omega, \Delta)$ and we define

$$a(\Phi, \Psi) = \int_{\Omega} \nabla \Phi \nabla \Psi dx + \lambda \int_{\Omega} \Phi \Psi dx \quad (2.4)$$

Thus, by (2.4)

$$a(\Phi, \Psi) = \langle \gamma_1 \Phi, \gamma_0 \Psi \rangle,$$

where $\gamma_0$ and $\gamma_1$ are the traces of order zero and one, respectively, and $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

We consider the scheme

$$\Phi \in H^{\frac{1}{2}}(\Gamma) \quad \xrightarrow{A} \quad \varphi \in H^1(\Omega, \Delta)$$

$$\frac{\partial \Phi}{\partial \nu} \in H^{-\frac{1}{2}}(\Gamma)$$

Thus

$$A = \gamma_1 \circ \gamma_0^{-1}: H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \quad A \varphi = \frac{\partial \Phi}{\partial \nu}.$$ 

Therefore $A$ is self-adjoint and $A \in \mathcal{L} \left( H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma) \right)$.

Moreover, we have

$$\langle A \varphi, \varphi \rangle = a(\Phi, \Phi) \quad (2.5)$$

and so by (2.4) we get

$$\langle A \varphi, \varphi \rangle = \int_{\Omega} |\nabla \Phi|^2 dx + \lambda \int_{\Omega} |\Phi|^2 dx \geq \min \{1, \lambda\} \|\Phi\|_{H^1(\Omega)}^2 \geq \alpha \|\gamma_0 \Phi\|^2 = \alpha \|\varphi\|^2,$$

proving that $A$ is positive.
We formulate now the problem on $E$. For this, we define

$$w(t)|_{\Gamma} = u(t) \quad \text{and} \quad \frac{\partial w(t)}{\partial \nu}|_{\Gamma} = Au(t).$$

In this way, the problem (1.2) is reduced to find a function $u : \Sigma \to \mathbb{R}$ such that

$$u'' + Au + F(u) + \beta(x)u' = 0 \quad \text{on} \quad \Sigma,$$
$$u(0) = u_0, \quad u'(0) = u_1 \quad \text{on} \quad \Gamma,$$

which will be investigated in the section 3.

Firstly we will state a result that guarantees the existence and uniqueness of solution for the linear problem associated the (1.1).

**Theorem 2.1.** Given $(u_0, u_1, f) \in H^{\frac{3}{2}}(\Gamma) \times L^2(\Gamma) \times L^2(\Sigma)$, there exists a unique function $u : \Sigma \to \mathbb{R}$ such that

$$u \in C^0\left(0, T; H^{\frac{3}{2}}(\Gamma)\right) \cap C^1\left(0, T; L^2(\Gamma)\right),$$

$$u'' + Au + \beta u' = f \quad \text{in} \quad L^2\left(0, T; H^{-\frac{1}{2}}(\Gamma)\right),$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad \text{on} \quad \Gamma.$$  \hspace{1cm} (2.6)

Moreover we have the energy inequality

$$\frac{1}{2} |u'(t)|^2 + \frac{\alpha}{2} \|u(t)\|^2 \leq \frac{1}{2} |u_1|^2 + \frac{\alpha}{2} \|u_0\|^2 + \int_0^T \langle f(s), u'(s) \rangle ds, \ a.e \ in \ [0, T].$$  \hspace{1cm} (2.7)

**Proof.** In the proof of this linear case, we employ the Faedo-Galerkin’s method. \hfill \square

3. Existence of Solution

The goal of this section is to obtain existence of solutions for the problem (1.1).

**Theorem 3.1.** Consider $F$ satisfying (1.2) and suppose

$$(u_0, u_1) \in H^{\frac{3}{2}}(\Gamma) \times L^2(\Gamma) \quad \text{and} \quad G(u_0) \in L^1(\Gamma).$$

Then there exists a function $u : \Sigma \to \mathbb{R}$ such that

$$u \in L^\infty\left(0, T; H^{\frac{3}{2}}(\Gamma)\right),$$

$$u' \in L^\infty\left(0, T; L^2(\Gamma)\right).$$  \hspace{1cm} (3.1)

\hspace{1cm} (3.2)
To prove Theorem 3.1, the following Lemma will be used:

**Lemma 3.1.** Assume that \((u_0, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)\) and suppose that the function \(F\) satisfies

\[
F : \mathbb{R} \to \mathbb{R} \text{ be Lipschitz function such that } sF(s) \geq 0, \forall s \in \mathbb{R}. \tag{3.5}
\]

Then there exists only one function \(u : \Sigma \to \mathbb{R}\) satisfying the conditions

\[
u \in L^\infty \left(0, T; H^{\frac{1}{2}}(\Gamma)\right), \tag{3.6}
\]

\[
u' \in L^\infty \left(0, T; L^2(\Gamma)\right), \tag{3.7}
\]

\[
u'' + Au + F(u) + \beta u' = 0 \quad \text{in} \quad L^2 \left(0, T; H^{-\frac{1}{2}}(\Gamma)\right), \tag{3.8}
\]

\[
u(0) = u_0, \quad \nu'(0) = u_1 \quad \text{on} \quad \Gamma. \tag{3.9}
\]

Furthermore

\[
\frac{1}{2} |\nu'(t)|^2 + \frac{\alpha}{2} \|\nu(t)\|^2 + \int_{\Gamma} G(u(x,t)) \, d\Gamma \leq \frac{1}{2} |u_1|^2 + \\
+ \frac{\alpha}{2} \|u_0\|^2 + \int_{\Gamma} G(u_0(x)) \, d\Gamma, \text{ a.e } [0, T]. \tag{3.10}
\]

**Proof of Lemma 3.1.** The proof will be done employing the Picard successive approximations method. Let us consider the sequence of successives approximations

\[
u_0, \nu_1, \nu_2, \ldots, \nu_n, \ldots \tag{3.11}
\]

defined as the solutions of the linear problems

\[
\begin{align*}
\left| & u_n'' + Au_n + F(u_{n-1}) + \beta u_n' = 0 \quad \text{on} \quad \Sigma, \\
& u_n(0) = u_0, \quad u_n'(0) = u_1 \quad \text{on} \quad \Gamma.
\end{align*} \tag{3.12}
\]

Using that \(F\) is Lipschitz and from Theorem 2.1, one can prove, using induction, that (3.12) has a solution for each \(n \in \mathbb{N}\) with the regularity claimed in the Theorem 2.1. We will prove now that the sequence (3.11) converges to a function \(u : \Sigma \to \mathbb{R}\) in the conditions of the Lemma 3.1.
For this end, we define $v_n = u_n - u_{n-1}$ which is the unique solution of the problem

$$
\begin{align*}
\begin{cases}
  v''_n + Av_n + F(u_{n-1}) - F(u_{n-2}) + \beta v'_n = 0 & \text{on } \Sigma, \\
  v_n(0) = 0, & v'_n(0) = 0 & \text{on } \Gamma.
\end{cases}
\end{align*}
$$

(3.13)

By the energy inequality (2.9), we have

$$
\frac{1}{2} |v'_n(t)|^2 + \frac{\alpha}{2} \|v_n(t)\|^2 \leq -\int_0^t (F(u_{n-1}) - F(u_{n-2}), v'_n(s)) \, ds. 
$$

(3.14)

Set

$$
e_n(t) = \text{ess sup}_{s \in [0,t]} \left\{ \frac{1}{2} |v'_n(s)|^2 + \frac{\alpha}{2} \|v_n(s)\|^2 \right\}. 
$$

(3.15)

Thus, since $F$ is Lipschitz, we have

$$
-\int_0^t (F(u_{n-1}) - F(u_{n-2}), v'_n(s)) \, ds \leq C \int_0^t |v_{n-1}(s)|^2 \, ds + \frac{1}{2} e_n(t). 
$$

(3.16)

We have also

$$
|v_{n-1}(s)|^2 \leq C e_{n-1}(s). 
$$

(3.17)

Combining (3.14)–(3.17), we get

$$
e_n(t) \leq C \int_0^t e_{n-1}(s) \, ds,
$$

and, by interation, we obtain, for $n = 1, 2, \ldots$, that

$$
e_n(t) \leq e_0 C_T \frac{(Ct)^n}{n!},
$$

hence, we conclude that the series $\sum_{n=1}^{\infty} e_n(t)$ is uniformly convergent on $[0,T]$. By the definition of $e_n(t)$, see (3.15), it follows that the series $\sum_{n=1}^{\infty} (u'_n - u'_{n-1})$ and $\sum_{n=1}^{\infty} (u_n - u_{n-1})$ are convergents in the norms of $L^\infty(0,T;L^2(\Gamma))$ and $L^\infty(0,T;H^{1/2}(\Gamma))$, respectively. Therefore, there exists $u : \Sigma \to \mathbb{R}$ such that

$$
u_n \to u \quad \text{strong in } L^\infty(0,T;H^{1/2}(\Gamma)),
$$

(3.18)

and

$$
u_n' \to u' \quad \text{strong in } L^\infty(0,T;L^2(\Gamma)).
$$

(3.19)
Since $F$ is Lipschitz, we have by (3.18) that

$$F(u_n) \to F(u) \text{ strong in } L^\infty(0,T;L^2(\Gamma)).$$

(3.20)

Then, by the convergences (3.18) – (3.20), we can pass to the limit in (3.12) and we obtain, by standard procedure, a unique function $u$ satisfying (3.6) – (3.10).

We will prove now the main result.

**Proof of Theorem 3.1.** — By Strauss [9], there exists a sequence of functions $(F_\nu)_{\nu \in \mathbb{N}}$, such that each $F_\nu : \mathbb{R} \to \mathbb{R}$ is Lipschitz and $(F_\nu)_{\nu \in \mathbb{N}}$ approximates $F$ uniformly on bounded sets of $\mathbb{R}$. Since the initial data $u_0$ is not necessarily bounded, we have to approximate $u_0$ by bounded functions of $H^{\frac{1}{2}}(\Gamma)$. We consider the functions $\xi_j : \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_j(s) = \begin{cases} 
- j, & \text{if } s < -j, \\
|s|, & \text{if } |s| \leq j, \\
j, & \text{if } s > j.
\end{cases}$$

Considering $\xi_j(u_0) = u_{0j}$, we have by Kinderlehrer and Stampacchia [5] that the sequence $(u_{0j})_{j \in \mathbb{N}} \subset H^{\frac{1}{2}}(\Gamma)$ is bounded a.e. in $\Gamma$ and

$$u_{0j} \to u_0 \text{ strong in } H^{\frac{1}{2}}(\Gamma).$$

(3.21)

Thus, for $(u_{0j}, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)$, the Lemma 3.1 says that there exists only one solution $u_{j\nu} : \Sigma \to \mathbb{R}$ satisfying (3.6) – (3.9) and the energy inequality

$$\frac{1}{2} |u_{j\nu}'(t)|^2 + \frac{\alpha}{2} \|u_{j\nu}(t)\|^2 + \int_{\Gamma} G_\nu(u_{j\nu}(x,t)) \, d\Gamma \leq \frac{1}{2} |u_1|^2 +$$

$$+ \frac{\alpha}{2} \|u_{0j}\|^2 + \int_{\Gamma} G_\nu(u_{0j}(x)) \, d\Gamma.$$  

(3.22)

We need an estimate for the term $\int_{\Gamma} G_\nu(u_{0j}(x)) \, d\Gamma$. Since $u_{0j}$ is bounded a.e. in $\Gamma$, $\forall j \in \mathbb{N}$, it follows that

$$F_\nu(u_{0j}) \to F(u_{0j}) \text{ uniform in } \Gamma.$$

So

$$\int_{\Gamma} G_\nu(u_{0j}(x)) \, d\Gamma \to \int_{\Gamma} G(u_{0j}(x)) \, d\Gamma \text{ uniform in } \mathbb{R}.$$  

(3.23)
From (3.21), there exists a subsequence of \((u_{0j})_{j \in \mathbb{N}}\), which will still be denoted by \((u_{0j})_{j \in \mathbb{N}}\), such that

\[ u_{0j} \to u_0 \text{ a.e. in } \Gamma. \]

Hence, by continuity of \(G\), we have that \(G(u_{0j}) \to G(u_0)\) a.e. in \(\Gamma\). We also have that \(G(u_{0j}) \leq G(u_0) \in L^1(\Gamma)\). Thus, by the Lebesgue’s dominated convergence theorem, we get

\[ G(u_{0j}) \to G(u_0) \text{ strong in } L^1(\Gamma). \]  \hspace{1cm} (3.24)

Then, by (3.23) and (3.24), we obtain that

\[ \int_{\Gamma} G_{\nu}(u_{0j}(x)) \, d\Gamma \leq C, \]  \hspace{1cm} (3.25)

where \(C\) is independent of \(j\) and \(\nu\). In this way, using (3.21) and (3.25), we have from (3.22) that

\[ |u'_{j,\nu}|^2 + \|u_{j,\nu}\|^2 + \int_{\Gamma} G(u_{j,\nu}(x, t)) \, d\Gamma \leq C, \]  \hspace{1cm} (3.26)

where \(C\) is independent of \(j, \nu\) and \(t\).

From (3.26), we obtain that

\[ (u_{j,\nu}) \text{ is bounded in } L^\infty(0, T; H^{1/2}(\Gamma)), \]  \hspace{1cm} (3.27)

\[ (u'_{j,\nu}) \text{ is bounded in } L^\infty(0, T; L^2(\Gamma)). \]  \hspace{1cm} (3.28)

We have that (3.27) and (3.28) are true for all pairs \((j, \nu) \in \mathbb{N}^2\), in particular, for \((i, i) \in \mathbb{N}^2\). Thus, there exists a subsequence of \((u_{ii})\), which we denote by \((u_i)\), and a function \(u : \Sigma \to \mathbb{R}\), such that

\[ u_i \to u \text{ weak star in } L^\infty(0, T; H^{1/2}(\Gamma)), \]  \hspace{1cm} (3.29)

\[ u'_i \to u' \text{ weak in } L^\infty(0, T; L^2(\Gamma)). \]  \hspace{1cm} (3.30)

We also have by (3.8) that

\[ u''_i + Au_i + F_i(u_i) + \beta u'_i = 0 \quad \text{in} \quad L^2(0, T; H^{-1/2}(\Gamma)). \]  \hspace{1cm} (3.31)

From (3.29), (3.30) and observing that the injection of \(H^1(\Sigma)\) in \(L^2(\Sigma)\) is compact, there exists a subsequence of \((u_i)\), which we still denote by \((u_i)\), such that

\[ u_i \to u \text{ a.e. in } \Sigma. \]
Since $F$ is continuous
\[ F(u_i) \rightarrow F(u) \text{ a.e. in } \Sigma. \]

Furthermore, since $u_i(x, t)$ is bounded in $\mathbb{R}$,
\[ F_i(u_i) - F(u_i) \rightarrow 0 \text{ a.e. in } \Sigma. \]

Therefore, we conclude
\[ F_i(u_i) \rightarrow F(u) \text{ a.e. in } \Sigma. \] (3.32)

Taking duality between (3.31) and $u_i$ we obtain
\[
\int_0^T (F_i(u_i), u_i(t)) \, dt = \int_0^T |u_i'(t)|^2 \, dt - \alpha \int_0^T \|u_i(t)\|^2 \, dt - \alpha \int_0^T \|u_i(t)\|^2 \, dt - \\
- (u_i'(T), u_i(T)) + (u_1, u_0j) - \int_0^T (\beta u_i'(t), u_i(t)) \, dt.
\] (3.33)

Using (2.1), (3.6) and (3.7), we have by (3.33) that
\[
\int_0^T (F_i(u_i), u_i(t)) \, dt \leq C,
\] where $C$ is independent of $i$.

Thus, from (3.32) and (3.34), it follows by Strauss' theorem, see Strauss [9], that
\[ F_i(u_i) \rightarrow F(u) \text{ strongly in } L^1(\Sigma). \] (3.35)

By (3.29), (3.30) and (3.35) it is permissible to pass to the limit in (3.31) obtaining a function $u : \Sigma \rightarrow \mathbb{R}$ satisfying (3.1) – (3.4). \qed

### 4. Asymptotic Behaviour

In this section we study the exponential decay for the energy $E(t)$ associated to the weak solution $u$ given by the Theorem 3.1. This energy is given by
\[
E(t) = \frac{1}{2} |u'(t)|^2 + \alpha \|u(t)\|^2 + \int_{\Gamma} G(u(x,t))d\Gamma, \ t \geq 0.
\] (4.1)

We consider the followings additional hypothesis:
\[ 0 \leq G(s) \leq sF(s), \ \forall s \in \mathbb{R} \] (4.2)

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THEOREM 4.1. — Let $F$ satisfying (1.2) and (4.2). Then the energy (4.1) satisfies

$$E(t) \leq 4E(0)e^{-\frac{\varepsilon}{2}t}, \quad (4.3)$$

where $\varepsilon$ is a positive constant.

Proof. — For an arbitrary $\varepsilon > 0$, we define the perturbed energy

$$E_{\nu\varepsilon}(t) = E_\nu(t) + \varepsilon \eta(t) \quad (4.4)$$

where $E_\nu(t)$ is the energy similar to (4.1) associated to the solution obtained in the Lemma 3.1 and

$$\eta(t) = (u_\nu(t), u'_\nu(t)).$$

Note that

$$|\eta(t)| \leq C_2 E_\nu(t),$$

where $C_2 = \max \left\{ C_1, \frac{1}{\alpha} \right\}$, and $C_1$ is the immersion constant of $H^{\frac{1}{2}}(\Gamma)$ into $L^2(\Gamma)$.

Then,

$$|E_{\nu\varepsilon}(t) - E_\nu(t)| \leq \varepsilon C_2 E_\nu(t),$$

or

$$(1 - \varepsilon C_2) E_\nu(t) \leq E_{\nu\varepsilon}(t) \leq (1 + \varepsilon C_2) E_\nu(t).$$

Taking $0 < \varepsilon \leq \frac{1}{2C_2}$, we get

$$\frac{E_\nu(t)}{2} \leq E_{\nu\varepsilon}(t) \leq 2E_\nu(t), \forall t \geq 0. \quad (4.5)$$

Multiplying the equation in (3.8) for $u'_\nu$, using (2.1) and the fact of $A$ to be positive, we obtain

$$E'_\nu(t) \leq -\beta_0 |u'_\nu(t)|^2 \leq 0. \quad (4.6)$$

Differentiating the function $\eta(t)$ and using (3.8), (4.2) and the fact of $A$ to be positive comes that

$$\eta'(t) \leq \left(1 + \frac{\beta_1}{2\mu} \right) |u'_\nu(t)|^2 + \left(\frac{\beta_1\mu C_1}{2} - \alpha \right) \|u_\nu(t)\|^2 - \int_{\Gamma} G_\nu(u_\nu) d\Gamma, \quad (4.7)$$

where $\beta_1 = \|\beta\|_{L^\infty(\Gamma)}$ and $\mu > 0$ to be chosen.
It follows by (4.4), (4.6) and (4.7) that
\[
E_{\nu \epsilon} \left( t \right) \leq \left[ \epsilon \left( 1 + \frac{\beta_1}{2\mu} \right) - \beta_0 \right] |u_{\nu \epsilon}(t)|^2 - \epsilon \left( \alpha - \frac{\beta_1 \mu C_1}{2} \right) \|u_{\nu \epsilon}(t)\|^2 - \epsilon \int_\Gamma G_{\nu}(u_{\nu \epsilon}) d\Gamma.
\]
(4.8)

Taking \( \mu = \frac{\alpha}{\beta_1 C_1} \) and \( 0 < \epsilon \leq \frac{2\alpha \beta_0}{3\alpha + \beta_1^2 C_1} \) we get
\[
E_{\nu \epsilon} '(t) \leq -E_{\nu}(t).
\]
(4.9)

Choosing \( \epsilon \leq \min \left\{ \frac{1}{2C_2}, \frac{2\alpha \beta_0}{3\alpha + \beta_1^2 C_1} \right\} \) then (4.5) and (4.9) occur simultaneously, therefore
\[
E_{\nu \epsilon} '(t) + \frac{\epsilon}{2} E_{\nu \epsilon}(t) \leq 0,
\]
that is,
\[
E_{\nu}(t) \leq 4E_{\nu}(0)e^{-\frac{\epsilon}{2}t}.
\]
(4.10)

From (3.29), (3.30) and since \( G_{\nu} \) is continuous, we have
\[
G_{\nu}(u_{\nu}(\cdot, t)) - G_{\nu}(u(\cdot, t)) \to 0 \text{ a.e. in } \Gamma, \ \forall t \geq 0.
\]
(4.11)

But we know that \( F_{\nu} \to F \) uniformly on bounded sets of \( \mathbb{R} \). Then
\[
G_{\nu}(u(\cdot, t)) \to G(u(\cdot, t)) \text{ a.e. in } \Gamma, \ \forall t \geq 0.
\]
(4.12)

Thus, by (4.11) and (4.12)
\[
G_{\nu}(u_{\nu}(\cdot, t)) \to G(u(\cdot, t)) \text{ a.e. in } \Gamma, \ \forall t \geq 0.
\]
(4.13)

Moreover, we have, by (4.10), that
\[
\int_\Gamma G_{\nu}(u_{\nu}(x, t)) d\Gamma \leq 4E_{\nu}(0).
\]

Therefore, using (3.21), (3.23) and (3.24), we get
\[
\liminf_{\nu \to \infty} \int_\Gamma G_{\nu}(u_{\nu}(x, t)) d\Gamma \leq 4E(0).
\]
(4.14)
By (4.13), (4.14) and Fatou’s lemma, we have
\[
\int_{\Gamma} G(u(x,t)) \, d\Gamma \leq \liminf_{\nu \to \infty} \int_{\Omega} G_{\nu}(u_{\nu}(x,t)) \, d\Gamma.
\]

Hence, passing \( \liminf_{\nu \to \infty} \) in (4.10), we get (4.3).

\[ \square \]

Remark. — In the existence we can take \( \lambda = 0 \). For this end, we define in \( H^1(\Omega) \) the norm
\[
[v]^2 = \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Gamma} |\gamma_0 v|^2 \, d\Gamma, \quad \forall v \in H^1(\Omega),
\]

obtaining now the positivity of operator \( A + \zeta I \), for \( \zeta > 0 \) arbitrary, like in Lions [8]. For the asymptotic behaviour, we need the additional hypothesis \( \beta_0 > \zeta \).

**Bibliography**