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Invariant local Dirichlet forms on locally compact groups (*)

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RÉSUMÉ. — Soit G un groupe localement compact connexe localement connexe métrisable, typiquement de dimension infinie, i.e., qui n'est pas un groupe de Lie. Nous considérons les espaces de Dirichlet locaux dans $L^2(G)$ (par rapport à une mesure de Haar gauche ou droite) naturellement associés à un semigroupe de convolution gaussien $(\mu_t)_{t>0}$. Nous considérons la question de savoir si $(\mu_t)_{t>0}$ admet une densité continue. Nous montrons que tout groupe comme ci-dessus porte des semigroupes de convolution gaussiens admettant une densité continue. Nous estimons cette densité hors de la diagonale en fonction du comportement sur la diagonale et donnons quelques applications.

ABSTRACT. — Let G be a locally compact connected locally connected metric group, typically infinite dimensional, i.e., not a Lie group. We consider those local Dirichlet spaces in $L^2(G)$ (with respect to either the left- or the right-invariant Haar measure) whose Markov semigroup is naturally associated to a Gaussian convolution semigroup of measures $(\mu_t)_{t>0}$. We consider the questions of whether or not $(\mu_t)_{t>0}$ can have a continuous density and what type of short-time estimates of this density can be obtained. We prove that Gaussian semigroups having a continuous density do exist on any group G as above. Under some specific assumptions on the on-diagonal behavior of the density, we give off-diagonal estimates.

1. Introduction

Let G be a locally compact connected metric group. Unless explicitly stated, all the groups considered in this paper are of this type. We will equip G with its right or left Haar measure denoted respectively by ν_r and ν_l.

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The aim of this paper is to study certain short time properties of Gaussian convolution semigroups of measures on G when G is (typically) infinite dimensional. The case of compact connected groups is studied in detail in [6], [7]. Here, we deal mostly with non-compact groups. We will be concerned with Gaussian convolution semigroups that are associated with self-adjoint semigroups of operators on either $L^2(G, \nu_r)$ or $L^2(G, \nu_l)$. One of the main result of this paper is Theorem 5.1 which shows that, on any locally compact, connected, locally connected metrizable group, there are many Gaussian semigroups having a continuous density with respect to Haar measure.

1.1. Notation

Let ν_l (resp. ν_r) denote a fixed left-invariant (resp. right-invariant) Haar measure . Let $m : G \to (0, +\infty)$ be the modular function of G, i.e., the function such that

$$\nu_l(Vg) = m(g)\nu_l(V) \tag{1.1}$$

for all Borel subsets V of G. We also have

$$d\check{\nu}_l(x) = m(x^{-1})d\nu_l(x) \tag{1.2}$$

where $\check{\nu}$ is defined by $\check{\nu}(V) = \nu(V^{-1})$ for any measure ν. Recall that G is *unimodular* if $m \equiv 1$ and that any compact group is unimodular. We assume that the Haar measures ν_l, ν_r are chosen such that

$$d\nu_l(x) = m(x)d\nu_r(x). \tag{1.3}$$

This is equivalent to say $\check{\nu}_l = \nu_r$.

For clarity, let us recall the definitions of convolutions of two measures, two functions, and measures and functions; See [21, I, Chp. V]. For two finite complex Borel measures μ, ν, the convolution $\mu * \nu$ is a finite complex Borel measure defined by

$$\forall f \in \mathcal{C}_0(G), \quad \mu * \nu(f) = \int_{G\times G} f(xy)d\mu(x)d\nu(y).$$

For functions $f, g \in \mathcal{C}_0(G)$, the convolution $f * g$ is the function

$$f * g(x) = \int_G f(xy)g(y^{-1})d\nu_l(y) = \int_G f(y)g(y^{-1}x)d\nu_l(y).$$

For a function $f \in \mathcal{C}_0(G)$ and a finite complex Borel measure μ, we set

$$\mu * f(x) = \int_G f(y^{-1}x)d\mu(y), \quad f * \mu(x) = \int_G m(y)^{-1}f(xy^{-1})d\mu(y).$$

These definitions are consistent when the measure μ (resp. ν) has density f (resp. g) with respect to the left Haar measure ν_l.

Let $(\mu_t)_{t>0}$ be a weakly continuous convolution semigroup of probability measures on G. This means precisely that $(\mu_t)_{t>0}$ satisfies

(a) $\mu_t * \mu_s = \mu_{t+s}$, $t, s > 0$

(b) $\mu_t \to \delta_e$ weakly as $t \to 0$.

Recall that this semigroup is a *Gaussian* convolution semigroup if it also satisfies

(c) $t^{-1}\mu_t(V^c) \to 0$ as $t \to 0$ for any neighborhood V of the identity element $e \in G$.

A Gaussian measure is a measure μ that can be embedded in a Gaussian convolution semigroup $(\mu_t)_{t>0}$ so that $\mu_1 = \mu$.

A measure μ is symmetric if and only if $\check{\mu} = \mu$. By definition, a convolution semigroup $(\mu_t)_{t>0}$ is symmetric if each μ_t, $t > 0$, is symmetric.

Each Gaussian convolution semigroup $(\mu_t)_{t>0}$ defines a semigroup of contractions $(H_t)_{t>0}$ on $L^\infty(G, \nu_r) = L^\infty(G, \nu_l)$ given by

$$H_t f(x) = \int_G f(xy) d\mu_t(y). \tag{1.4}$$

Clearly, $(H_t)_{t>0}$ is Markov semigroup. Moreover, it commutes with left translations. If G is unimodular, H_t can be expressed in terms of convolution by

$$H_t f = f * \check{\mu}_t.$$

If G is non-unimodular, the formula reads

$$H_t f(x) = m(x) \left[\left(\frac{f}{m} \right) * \check{\mu}_t \right] (x).$$

The semigroup $(H_t)_{t>0}$ extends to $L^2(G, d\nu_r)$ and its adjoint is given by

$$H_t^* f(x) = \int f(xy) d\check{\mu}_t(y).$$

Thus H_t is self-adjoint if and only if μ_t is symmetric. It follows that each symmetric Gaussian convolution semigroup defines a regular strictly local

Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(G, \nu_r)$. Let $-L$ denote the $L^2(G, \nu_r)$ infinitesimal generator of $(H_t)_{t>0}$. We will also refer to $-L$ as the infinitesimal generator of the convolution semigroup $(\mu_t)_{t>0}$. For background on Dirichlet spaces, see [19]. For background on convolution semigroups of measures, see [22].

1.2. Overview of the paper

In Section 3.2 of this paper, we consider the Dirichlet spaces $(\mathcal{E}, \mathcal{F})$ on $L^2(G, \nu_r)$ associated to Gaussian convolution semigroups as above and we describe their infinitesimal generators in terms of "partial differential operators". When G is not unimodular, these Dirichlet spaces are not left-invariant because they are defined on $L^2(G, \nu_r)$ which is not left-invariant. Thus, in section 3.3, we consider and characterize strictly local, left-invariant Dirichlet spaces on $L^2(G, \nu_l)$. For Lie groups, these descriptions are consequences of celebrated theorems of Hunt and Beurling-Deny; See [22], [19]. The general case follows from the fact that any locally compact connected group is the projective limit of connected Lie groups. In this context, a useful tool is the notion of projective basis for the (infinite dimensional) Lie algebra $\mathcal{G}$ of G. See [13], [14] and Section 2 below.

In Section 4 we discuss the basic properties of the intrinsic (quasi-) distance associated to these Dirichlet spaces (see, e.g., [11], [15], [27]). For instance, we show that under a natural non-degeneracy condition, the subset of all the points that are at finite distance from the neutral element is dense and either has Haar measure zero or is actually equal to G. Following [6], we introduce another distance called the relaxed (quasi-)distance which plays an important role when the intrinsic distance is ∞ almost everywhere.

On Lie groups, the intrinsic distance introduced in Section 4 is useful to estimate the density of the associated Gaussian semigroup. On more general groups, it is not entirely clear that Gaussian semigroups having a continuous density exist. In Section 5, we state and prove Theorem 5.1 which asserts the existence of many Gaussian semigroups having a continuous density. In Section 6, we discuss Gaussian type estimates involving either the intrinsic or the relaxed distance.

1.3. Notation concerning the densities of Gaussian semigroups

Our main interest is to study certain analytic properties that a Gaussian convolution semigroup $(\mu_t)_{t>0}$ may have or not. To start we consider the following property.

(CK) For all $t > 0$, μ_t is absolutely continuous with respect to the Haar measure $d\nu_l$ and admits a continuous density $\mu_t(\cdot)$.

Whenever (CK) is satisfied, we denote by

$$x \mapsto \mu_t(x)$$

the corresponding density so that

$$d\mu_t = \mu_t d\nu_l. \tag{1.5}$$

Of course, having a continuous density with respect to left or right Haar measures are equivalent properties. Assuming that (CK) holds, we denote by

$$x \mapsto \mu_t^r(x)$$

the density with respect to the right Haar measure ν_r. We then have

$$\mu_t^r(x) = m(x)\mu_t(x).$$

It is important to note that the fact that the measure μ_t is symmetric is equivalent to say that its (left) density $x \mapsto \mu_t(x)$ satisfies

$$\mu_t(x^{-1}) = m(x)\mu_t(x) = \mu_t^r(x). \tag{1.6}$$

The existence of Gaussian semigroups of measures satisfying (CK) or even having a density is not an obvious fact in the present generality. In this respect, let us recall the following facts (see [22, Chapter 6.4] and [8]): (a) Each symmetric Gaussian measure is supported by a connected closed subgroup; (b) A locally compact connected group that admits a Gaussian measure which is absolutely continuous w.r.t. the Haar measure must be locally connected and be a metric group (i.e., must have a countable basis for its topology). Theorem 5.1 (proved below in Section 5) shows that any locally compact, connected, locally connected metrizable group admits plenty of symmetric Gaussian semigroups having property (CK) or even stronger properties.

The following strengthening of property (CK) are of interest to us:

(CK*) For all $t > 0$, μ_t has a continuous density $\mu_t(\cdot)$ w.r.t. the Haar measure $d\nu_l$ and this density satisfies

$$\lim_{t\to 0} t \log(\mu_t(e)) = 0. \tag{$*$}$$

(CK#) For all $t > 0$, μ_t has a continuous density $\mu_t(\cdot)$ w.r.t. the Haar measure $d\nu_l$ and this density satisfies

$$\lim_{t\to 0} \sup_{x\in K} \mu_t(x) = 0 \tag{$\#$}$$

for any compact set $K \subset G$ such that $e \notin K$.

The property (#) relates to the off-diagonal behavior of $\mu_t(\cdot)$ and can somehow be interpreted as a very weak Gaussian upper bound. It plays an important role in potential theory. See [1], [6].

The next properties involve a fixed non-decreasing positive function ψ.

(CKψ) For all $t > 0$, μ_t has a continuous density $\mu_t(\cdot)$ and

$$\forall\, t \in (0,1), \quad \sup_{0<t<1} \left\{ \frac{\log(\mu_t(e))}{\psi(1/t)} \right\} < +\infty. \tag{ψ}$$

(CKψ*) For all $t > 0$, μ_t has a continuous density $\mu_t(\cdot)$ and

$$\forall\, t \in (0,1), \quad \lim_{t\to 0} \frac{\log(\mu_t(e))}{\psi(1/t)} = 0 \tag{$\psi*$}$$

(CKWψ) For all $t > 0$, μ_t has a continuous density $\mu_t(\cdot)$ and

$$\forall\, t \in (0,1), \quad \sup_{0<t<1} \left\{ \frac{\log\log(\mu_t(e))}{\psi(1/t)} \right\} < +\infty. \tag{$W\psi$}$$

The W in this notation refers to the fact that condition (CKWψ) is much weaker than (CKψ) itself. Of course (CKWψ) is the same as (CK$e^{c\psi}$) for some constant c, but we find this additional notation useful.

Let us describe some functions ψ that are of special interest in the present context.

- $\psi(t) = \psi_\lambda(t) = t^\lambda$, for some $\lambda \in (0,+\infty)$. Note that (CK*) is a shorthand for (CKψ_1*). Concerning (CKWψ), we will use only the case (CKWt^λ), $\lambda \in (0,\infty)$. In fact, if $\lambda > 1$, (CKWt^λ) is already too weak to be of much use with the techniques available at present (that is, in some sense, we do not know significant differences between Gaussian semigroups satisfying (CK) and (CKWt^λ), $\lambda > 1$).

- ψ is a regularly varying non-decreasing function of some index $0 \leqslant \lambda < +\infty$. The function

$$\psi(t) = t^\lambda [\log_1 t]^{\alpha_1} \dots [\log_k t]^{\alpha_k}$$

where $\log_1 t = \log(1+t)$, $\log_k t = \log(1+\log_{k-1} t)$, is a good example of a regularly varying function of index λ. Note that this includes the possibility that $\psi(t) = [\log_1 t]^{\alpha_1} \dots [\log_k t]^{\alpha_k}$, when $\lambda = 0$. In this case, ψ is a *slowly* varying function. We refer to [10] for a book treatment of regularly varying functions.

Remark. — Consider the semigroup $(H_t)_{t>0}$ associated to a convolution semigroup $(\mu_t)_{t>0}$ by (1.4). The transition function of this semigroup is given by

$$h(t,x,V) = \mu_t(x^{-1}V)$$

for all Borel sets $V \subset G$. Clearly this transition function admits a density with respect to $d\nu_r$ if and only if μ_t does and, if it exists, this density w.r.t. $d\nu_r$ is given by

$$h(t,x,y) = m(y)\mu_t(x^{-1}y) = m(x)\mu_t^r(x^{-1}y). \tag{1.7}$$

Recall that, in general, a semigroup $(H_t)_{t>0}$ defined on $L^1(M,\nu)$ is *ultracontractive* if H_t sends $L^1(M,\nu)$ continuously in $L^\infty(M,\nu)$, for each $t>0$. This is equivalent to say that, for each $t>0$, H_t admits a bounded kernel $h(t,x,y)$ with respect to ν. It is important to make the following two observations:

- The properties (CK), (CK$*$), (CK$\#$) introduced above for the convolution semigroup $(\mu_t)_{t>0}$ are equivalent to the properties (CK), (CK$*$), (CK$\#$) for the semigroup $(H_t)_{t>0}$ as defined in [3], [4], [6]. This is because these properties (in terms of $(H_t)_{t>0}$) are local properties. See [6].
- As m is a multiplicative function, it follows that $(H_t)_{t>0}$ cannot have a bounded kernel w.r.t. ν_r unless G is unimodular. In particular, $(H_t)_{t>0}$ at (1.4) cannot be ultracontractive on $L^1(G,\nu_r)$ unless G is unimodular. See (1.7).

In [6], properties similar to (CKψ) are introduced in the setting of symmetric Markov semigroups with $\mu_t(e)$ replaced by $\sup_x h(t,x,x)$. These properties are called (CKU…) in [6] instead of simply (CK…) here. The extra U used in [6] stresses the fact that the properties introduced there are quantitative versions of ultracontractivity. When G is unimodular (and only in this case), the properties introduced above and in [6] coincide when applied respectively to a symmetric Gaussian semigroup $(\mu_t)_{t>0}$ and to the semigroup $(H_t)_{t>0}$ defined at (1.4). For non-unimodular groups, we will see that the properties (CK…) for symmetric Gaussian semigroups of measures coincide with the properties (CKU…) from [6] for another Markov semigroup acting not on $L^2(G,\nu_r)$ but on $L^2(G,\nu_l)$.

2. Gaussian semigroups and the projective structure

2.1. The projective structure

Let G be a locally compact, connected metrizable group (hence, G has a countable basis for its topology). Then, there exists a decreasing sequence of compact normal subgroups K_α, $\alpha \in I$, such that $\bigcap_\alpha K_\alpha = \{e\}$ and $G/K_\alpha = G_\alpha$ is a connected Lie group. See [20], [22], [29]. Here, the set I can be taken to be either a singleton or the countable set $I = \mathbb{N}$. Of course, I can be taken to be a singleton if and only if G is a Lie group. Denote by π_α the canonical homomorphism $G \to G_\alpha$ and by $\pi_{\alpha,\beta}$ the canonical homomorphism $G_\beta \to G_\alpha$ for $\alpha \leqslant \beta$. Then the group G is the projective limit of the family

$$(G_\beta, \pi_{\alpha,\beta}), \quad \alpha, \beta \in I, \quad \alpha \leqslant \beta.$$

For readers unfamiliar with projective limits, let us emphasize that the maps $\pi_{\alpha,\beta}$ are crucial ingredients of the notion of projective limit. A given sequence of abstract groups $(G_\alpha)_I$ can lead to very different projective limits depending on the specific nature of the maps $\pi_{\alpha,\beta}$.

The group G admits a Lie algebra $\mathcal{G}$ which is the projective limit of the system formed by the Lie algebras $\mathcal{G}_\alpha$ and the maps $d\pi_{\alpha,\beta}$. For clarity and later use, let us recall one possible way to understand projective limits. The fact that $G = \lim \text{proj } G_\alpha$ implies that we can regard G as a closed subgroup of the product $\prod_I G_\alpha$. Namely, G consists of those sequences $(g_\alpha)_I$ such that, for all $\alpha \leqslant \beta$, $g_\alpha = \pi_{\alpha,\beta}(g_\beta)$. In particular, π_α is the restriction to G of the canonical coordinate mapping $\prod_I G_\beta \mapsto G_\alpha$. This makes it clear that G is determined by the sequence $(G_\beta, \pi_{\alpha,\beta})$, $\alpha, \beta \in I$, $\alpha \leqslant \beta$. The same remark applies to the Lie algebra of G. In particular, the exponential map $\exp : \mathcal{G} \mapsto G$ is the restriction to $\mathcal{G}$ of the product of the exponential maps $\exp_\alpha : \mathcal{G}_\alpha \mapsto G_\alpha$, $\alpha \in I$. The differential $d\pi_\alpha : \mathcal{G} \to \mathcal{G}_\alpha$ is defined similarly.

The following two definitions are important for our purpose.

DEFINITION 2.1. — *Let $(\mu_t)_{t>0}$ be a Gaussian convolution semigroup on $G = \lim \text{proj } G_\alpha$. For each $\alpha \in I$, define $(\mu_{\alpha,t})_{t>0}$ to be the convolution semigroup on G_α given by*

$$\mu_{\alpha,t}(V) = \mu_t(\pi_\alpha^{-1}(V)).$$

It is easy to see that $(\mu_{\alpha,t})_{t>0}$ is Gaussian.

DEFINITION 2.2. — *We say that a Gaussian convolution semigroup $(\mu_t)_{t>0}$ is non-degenerate if for each $\alpha \in I$, $\mu_{\alpha,t}$ is absolutely continuous with respect to the Haar measure on G_α for all $t > 0$.*

Remark. — If μ_t is absolutely continuous w.r.t Haar measure for all $t > 0$, then clearly $\mu_{\alpha,t}$ is absolutely continuous w.r.t the Haar measure on G_α

for all $t > 0$ and all $\alpha \in I$. By a result of Siebert [26] this implies that $(\mu_{\alpha,t})_{t>0}$ admits a smooth density. In general, the converse is not true at all: it may of course be the case that each $(\mu_{\alpha,t})_{t>0}$ admits a smooth density and $(\mu_t)_{t>0}$ is singular with respect to Haar measure. See e.g., [1].

We now introduce in the discussion the notion of projective basis. As the sequence (K_α) is decreasing, one can show that there exists a family $(X_i)_{i\in J}$ which is a projective basis for $\mathcal{G}$ with respect to $(K_\alpha)_I$. That is, for any $\alpha \in I$, there is a finite subset J_α of J such that $d\pi_\alpha(X_i)$, $i \in J_\alpha$, form a basis of the vector space $\mathcal{G}_\alpha$ and $d\pi_\alpha(X_i) = 0$ if $i \notin J_\alpha$. Note that, in particular,

$$J = \bigcup_{\alpha\in I} J_\alpha.$$

In what follows, we fix a sequence $(K_\alpha)_I$ and an associated Lie projective basis $(X_i)_J$. Let $\mathcal{C}_\alpha$ be the set of all functions f on G of the form $f = \phi\circ\pi_\alpha$ with $\phi \in \mathcal{C}_0^\infty(G_\alpha)$. In this context, the set $\mathcal{C}$ of Bruhat test functions is defined by

$$\mathcal{C} = \bigcup_{\alpha\in I} \mathcal{C}_\alpha. \tag{2.1}$$

It plays the role that the set of smooth cylindric functions plays in the case of infinite products. Observe that

$$X_i f(x) = d\phi(\pi_\alpha(x)) \circ d\pi_\alpha(X_i), \quad i \in J, \quad x \in G$$

is well defined for $f = \phi\circ\pi_\alpha \in \mathcal{C}_\alpha$. See Born, [13].

In [14], Born proves a general Levy-Khinchin formula which describes the infinitesimal generator of any given convolution semigroup on G in terms of a fixed projective basis of the projective Lie algebra of G. A crucial step is to observe that $\mathcal{C}$, the set of Bruhat test functions, is contained in the (L^2 or C_0) domain of the infinitesimal generator of any convolution semigroup. This easily follows from Hunt's celebrated description of convolution semigroups on Lie groups. See [23].

In the special case of Gaussian semigroups the Levy-Khinchin formula reads as follows.

THEOREM 2.3 ([14]).— *Let G be a locally compact, connected, group having a countable basis for its topology. Fix a projective basis $(X_i)_{i\in J}$ of the projective Lie algebra of G. There is a one to one correspondence between* *the set of all Gaussian semigroups $(\mu_t)_{t>0}$ on G and the set of all pairs (A,b) where $A=(a_{i,j})_{J\times J}$ is a symmetric non-negative real matrix indexed by J and a sequence $b=(b_i)_J$ of reals such that the infinitesimal generator $-L$ of $(\mu_t)_{t>0}$ is given by*

$$Lf=-\sum_{i,j\in J}a_{i,j}X_iX_jf+\sum_{i\in J}b_iX_if \quad \text{for all } f\in \mathcal{C}.$$

Here and in the sequel, we say that a symmetric matrix $A=(a_{i,j})_{J\times J}$ is non-negative (resp. positive) if $\sum_{i,j}a_{i,j}\xi_i\xi_j\geqslant 0$ (resp. >0) for all $\xi=(\xi_i)_J\neq 0$ with finitely many non-zero real coordinates.

Note that for any given $f=\phi\circ\pi_\alpha$, $\alpha\in I$, the above formula for L involves only finitely many non-zero terms and reads

$$Lf=-\sum_{i,j\in J_\alpha}a_{i,j}(d\pi_\alpha(X_i)d\pi_\alpha(X_j)\phi)\circ\pi_\alpha+\sum_{i\in J_\alpha}(d\pi_\alpha(X_i)\phi)\circ\pi_\alpha=(L_\alpha\phi)\circ\pi_\alpha$$

where $-L_\alpha$ is the infinitesimal generator of $(\mu_{\alpha,t})_{t>0}$ on G_α. As we shall see below, a Gaussian convolution semigroup $(\mu_t)_{t>0}$ associated to a pair (A,b) as above is symmetric if and only if $b=0$.

We want to point out a consequence of the above theorem that will be important later in this paper.

Corollary 2.4. — *Let G be a locally compact, connected, group having a countable basis for its topology. Let H be a closed normal totally disconnected subgroup of G. Then any Gaussian semigroup $(\mu_t)_{t>0}$ on G/H can be lifted uniquely to a Gaussian semigroup $(\overline{\mu}_t)_{t>0}$ on G so that the image of $(\overline{\mu}_t)_{t>0}$ by the canonical projection $G\to G/H$ is $(\mu_t)_{t>0}$.*

The proof follows immediately from Born's result because G and G/H have the same projective Lie algebra. More precisely, the Lie algebras of G and G/H can be identified through the map $d\pi$ where π is the canonical projection from G onto G/H. Indeed, G/H is the projective limit of the groups G_α/H_α where H_α is the projection of H on G_α, that is $H_\alpha=HK_\alpha/K_\alpha\cong H/(H\cap K_\alpha)$. By construction H_α is a closed subgroup of the Lie group G_α. By a theorem of Cartan, it follows that H_α is either discrete or a Lie group. By [21, 3.5;7.11], the fact that H is totally disconnected implies that H_α is not a Lie group. Hence H_α is discrete. This shows that the groups G_α and G_α/H_α have the same Lie algebra. The result follows.

2.2. Sums of squares

We now want to give an alternative description of the generators of Gaussian convolution semigroups. Let $(X_i)_J$ be a fixed projective basis of the Lie algebra $\mathcal{G}$ of G. Consider a sequence of vectors

$$Y_i = \sum_{j \in J} \tau_{i,j} X_j \in \mathcal{G}, \quad i = 0, 1, 2, \dots$$

and define formally a differential operator L by setting

$$L = -\sum_{1}^{\infty} Y_i^2 + Y_0.$$

One can easily check that this operator is well defined on $\mathcal{C}$ if and only if

$$\forall j \in J, \quad \sum_{i=1}^{\infty} |\tau_{i,j}|^2 < +\infty. \tag{2.2}$$

Assuming (2.2), $-L$ is clearly the infinitesimal generator of a Gaussian convolution semigroup. Conversely, the following lemma from [6] shows that any infinitesimal generator of a Gaussian convolution semigroup is of this form. As we shall see below, this convolution semigroup is symmetric if and only if $Y_0 = 0$.

LEMMA 2.5.— *Let $A = (a_{i,j})$ be an infinite symmetric non-negative matrix. There exists an infinite matrix τ with lines $\tau_i = (\tau_{i,k})_{k=1}^{\infty} \in \mathbb{R}^{\infty}$ such that:*

(i) $\tau_{i,k} = 0$ if $k < i$, that is, τ is upper-triangular.

(ii) $\forall\, \xi \in \mathbb{R}^{(\infty)}$, $(A\xi, \xi) = \sum_1^{\infty} (\tau_i, \xi)^2$.

Moreover, the lines $\tau_i = (\tau_{i,k})_k$ form a basis in $\mathbb{R}^{\infty}$ if and only if A is positive. This is the case if and only if $\tau_{i,i} > 0$ for all i.

Note that (2.2) is automatically satisfied here because of the upper-triangular nature of τ.

The following two definitions will help clarify future discussions (note the special role played by Y_0).

DEFINITION 2.6.— *Let $Y = (Y_i)_0^{\infty}$ be a system of vectors in $\mathcal{G}$ satisfying* (2.2).

1. *We say that Y is a Hörmander system if for each α the Lie algebra generated by the vectors $d\pi_\alpha(Y_i)$, $i = 1, 2, \dots$ is $\mathcal{G}_\alpha$. In other words, Y_i, $i = 1, 2, \dots$, generates $\mathcal{G}$.*

2. *We say that Y is a Siebert system if for each α the linear span of $d\pi_\alpha(Y_i)$, $i = 1, 2, \dots$ and the brackets of length at least two of the vectors $d\pi_\alpha(Y_i)$, $i = 0, 1, 2, \dots$, is $\mathcal{G}_\alpha$.*

With this definition, we can state the following result.

PROPOSITION 2.7.— *Let Y be a system of vectors in $\mathcal{G}$ satisfying* (2.2). *Then the associated Gaussian convolution semigroup $(\mu_t)_{t>0}$ is non-degenerate if and only if Y is a Siebert system.*

This immediately follows from Siebert's theorem [26]. It follows that $(\mu_t)_{t>0}$ can well be non-degenerate even if the system $Y = (Y_i)$ is not a basis of the Lie algebra $\mathcal{G}$ since it suffices that Y generates $\mathcal{G}$ as a Lie algebra (in the appropriate sense). Note that if $Y_0 = 0$ (i.e., if $(\mu_t)_{t>0}$ is symmetric) or, more generally, if Y_0 is in the span of Y_i, $i = 1, 2, \dots$, then Siebert condition reduces to Hörmander condition. In this case it follows that the projections $(\mu_{\alpha,t})_{t>0}$ of the non-degenerated convolution semigroup $(\mu_t)_{t>0}$ have smooth *strictly positive* densities.

3. Dirichlet forms associated to Gaussian semigroups

3.1. Semigroup of operators that commutes with left translations

Each weakly continuous convolution semigroup $(\mu_t)_{t>0}$ defines a weakly continuous semigroup of contractions $(H_t)_{t>0}$ on $L^\infty(G)$ given by

$$H_t f(x) = \int_G f(xy) d\mu_t(y).$$

Here $L^\infty(G)$ stands for $L^\infty(G, \nu_l) = L^\infty(G, \nu_r)$. Clearly, $(H_t)_{t>0}$ is Markov semigroup of operators and commutes with left translations. It is easy to check that this semigroup of operators also acts as a semigroup of contractions on $L^1(G, \nu_r)$ since

$$\int |H_t f| d\nu_r \leqslant \int \int |f(xy)| d\mu_t(y) d\nu_r(x) \leqslant \|f\|_1.$$

Hence, it extends to all $L^p(G, \nu_r)$, $1 \leqslant p \leqslant +\infty$. Moreover, its adjoint on $L^2(G, \nu_r)$ is given by

$$H_t^* f(x) = \int f(xy) d\check{\mu}_t(y).$$

It follows that $(H_t)_{t>0}$ is self-adjoint if and only if $(\mu_t)_{t>0}$ is symmetric. Thus each symmetric weakly continuous convolution semigroup defines a Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(G,\nu_r)$ where the form $\mathcal{E}$ is given by

$$\mathcal{E}(f,g)=\lim_{t\to 0}\frac{1}{t}\langle f-H_tf,g\rangle_r,\quad f,g\in\mathcal{F}$$

where $\mathcal{F}$ is the set of all functions in $L^2(G,\nu_r)$ such that

$$\lim_{t\to 0}\frac{1}{t}\langle f-H_tf,f\rangle_r<\infty.$$

Here $\langle .,.\rangle_r$ is the scalar product in $L^2(G,\nu_r)$.

Recall that a Dirichlet space $(\mathcal{E},\mathcal{F})$ is strictly local if $\mathcal{E}(u,v)=0$ for any $u,v\in\mathcal{F}$ such that v is constant on a neighborhood of the support of u (see [19, ($\mathcal{E}$7), pg. 6]). From the definition it follows that the Dirichlet spaces associated with symmetric Gaussian semigroups are strictly local.

We now want to characterize all Dirichlet spaces on $L^2(G,\nu_r)$ and $L^2(G,\nu_l)$ whose associated semigroup commutes with left translations. We start with the following known result.

Proposition 3.1. — *Let $(H_t)_{t>0}$ be a semigroup of bounded operators acting on $L^\infty(G)$ such that H_t commutes with left translations. Then there exists a convolution semigroup $(\mu_t)_{t>0}$ of signed measures with finite total variation such that*

$$H_tf(x)=\int_G f(xy)d\mu_t(y).$$

We sketch the proof for completeness. The first step is to show that, for any continuous function f with compact support, H_tf admits a continuous version. For any non-negative continuous function ϕ with compact support and $\int\phi d\nu_l=1$, let $T_\phi\psi(x)=\int\phi(z)\psi(xz)d\nu_l(z)$ where ψ is an arbitrary function in $L^\infty(G)$. Note that $T_\phi\psi$ is continuous. Now, for any x,y in G, we have

$$|T_\phi(H_tf)(x)-T_\phi(H_tf)(y)|=|T_\phi(H_t[f_x-f_y])(e)|\leqslant C_t\|f_x-f_y\|_\infty.$$

As we assume that f is continuous with compact support, it follows that the family $\{T_\phi H_tf\}$ obtained by varying ϕ is equicontinuous and uniformly bounded. Taking a sequence ϕ_n such that $\phi_n\to\delta_e$, the sequence $T_{\phi_n}H_tf$ converges weakly to H_tf and we can extract a subsequence which converges uniformly to a continuous function which represents H_tf.

Thus $f \mapsto H_t f(x)$ is a continuous functional on the space of continuous functions vanishing at infinity. Hence, for each t, x, there exists a unique signed measure $\mu(t, x, dy)$ with finite total variation such that

$$H_t f(x) = \int_G f(y)\mu(t, x, dy).$$

As H_t commutes with left translations, we must have

$$\mu(t, ax, V) = \mu(t, x, a^{-1}V)$$

for any Borel set V and any $a \in G$. Setting $\mu_t(V) = \mu(t, e, V)$, we get the representation

$$H_t f(x) = \int_G f(xy)d\mu_t(y)$$

as desired.

THEOREM 3.2.— *Let $(\mathcal{E}, \mathcal{F})$ be Dirichlet space on $L^2(G, \nu_r)$ (resp. $L^2(G, \nu_l)$) such that the associated semigroup $(H_t)_{t>0}$ extended to $L^\infty(G)$ commutes with left translations and satisfies $H_t 1 = 1$. Then, there exists a weakly continuous semigroup of probability measures $(\mu_t)_{t>0}$ such that*

$$H_t f(x) = \int_G f(xy)d\mu_t(y).$$

Moreover, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet space on $L^2(G, \nu_r)$ (resp. $L^2(G, \nu_l)$) and $\mathcal{C}$ is a core. Finally, $(\mathcal{E}, \mathcal{F})$ is strictly local if and only if $(\mu_t)_{t>0}$ is Gaussian.

Sketch of the Proof.— The representation of H_t follows from the previous proposition. By an argument similar to [19, Lemma 1.4.2], one shows that $\mathcal{C} \cap \mathcal{F}$ is a core. In particular $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet space. To show that $\mathcal{C}$ is a core, it suffices to show that $\mathcal{C} \subset \mathcal{F}$. This follows from Hunt's celebrated theorem [23]. To see that strict locality of $(\mathcal{E}, \mathcal{F})$ is equivalent to the Gaussian character of $(\mu_t)_{t>0}$, one uses the Beurling-Deny formula [19, Theorem 3.2.1] and the Lévy-Khintchin formula of Hunt [23]. See also [18].

3.2. Symmetric convolution semigroups

Let us now assume that $(\mu_t)_{t>0}$ is Gaussian and fix a projective basis $(X_i)_J$ associated to $(G_\alpha)_I$. By Theorem 2.3 above, the infinitesimal generator $-L$ of $(H_t)_{t>0}$ is given on $\mathcal{C}$ by

$$Lf = -\sum_{i,j \in J} a_{i,j} X_i X_j f + \sum_{i \in J} b_i X_i f \quad \text{for all } f \in \mathcal{C}$$

for some (A, b). As left-invariant vector fields are skew adjoint with respect to the right Haar measure ν_r and L must be self-adjoint, one easily see that b must vanish.

This leads to the following statement.

THEOREM 3.3.— *Let G be as in Section 2.1 and let $(X_i)_J$ be a fixed projective basis of the Lie algebra $\mathcal{G}$.*

(i) For any strictly local Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(G, \nu_r)$ such that $(H_t)_{t>0}$ commutes with left translations, there exists a unique symmetric non-negative matrix $A = (a_{i,j})_{i,j\in J}$ such that

$$\forall\, f \in \mathcal{C}, \quad \mathcal{E}(f, f) = \int_G \sum_{i,j\in J} a_{i,j} X_i f X_j f d\nu_r$$

Moreover, $\mathcal{C}$ is a core for $(\mathcal{E}, \mathcal{F})$.

(ii) Conversely, for any symmetric non-negative matrix $A = (a_{i,j})_{i,j\in J}$, the quadratic form

$$\mathcal{E}^r_A(f, f) = \int_G \sum_{i,j\in J} a_{i,j} X_i f X_j f d\nu_r, \quad f \in \mathcal{C}$$

is closable and its minimal closure defines a strictly local Dirichlet space $(\mathcal{E}^r_A, \mathcal{F}^r_A)$ on $L^2(G, \nu_r)$ such that the corresponding semigroup $(H_t)_{t>0}$ commutes with left translations.

(iii) For a Dirichlet space as above with associated matrix A, the infinitesimal generator $-L^r_A$ is given on $\mathcal{C}$ by

$$L^r_A f = -\sum_{i,j\in J} a_{i,j} X_i X_j f.$$

Using Lemma 2.5 we obtain another representation of these Dirichlet spaces.

THEOREM 3.4.— *Let $(\mathcal{E}, \mathcal{F})$ be a strictly local Dirichlet space on $L^2(G, \nu_r)$ whose semigroup $(H_t)_{t>0}$ commutes with left translations. Then there exists a system $Y = (Y_i)_{\mathcal{J}} \subset \mathcal{G}$ which is projective and can be obtained from the given basis $(X_i)_{\in J}$ by an upper-triangular matrix $\tau = (\tau_{i,j})_{i,j\in J}$ and such that:*

(i) *The domain $\mathcal{F}$ of $\mathcal{E}$ coincides with the Sobolev space*

$$W_2^1(Y,\nu_r) = \left\{ f \in L^2(G,\nu_r) : \forall\, i \in \mathcal{J},\ \ Y_i f \in L^2(G,\nu_r) \right.$$

$$\left. \text{and } \int_G \sum_{\mathcal{J}} |Y_i f|^2 d\nu_r < \infty \right\}$$

and

$$\forall\, f \in \mathcal{F},\ \ \mathcal{E}(f,f) = \int_G \sum_{\mathcal{J}} |Y_i f|^2 d\nu_r;$$

(ii) *The infinitesimal generator $-L$ associated to $(\mathcal{E},\mathcal{F})$ is given on $\mathcal{C}$ by*

$$Lf = -\sum_{\mathcal{J}} Y_i^2 f.$$

(iii) *The Gaussian convolution semigroup $(\mu_t)_{t>0}$ associated to $(H_t)_{t>0}$ by Theorem 3.2 is non-degenerate if and only if Y is a Hörmander system (see definition 2.6).*

Remarks. — 1. The system $Y = (Y_i)_{\mathcal{J}}$ is obtained from $X = (X_i)_I$ by $Y = \tau X$ where τ is given by Lemma 2.5, with the convention that any zero vector that might appear is disregarded. This explains why we use a different set $\mathcal{J}$ to index the Y_i's.

2. The system Y forms a basis of $\mathcal{G}$ if and only if the matrix A is positive. In the abelian case (see [6]), this is also equivalent to the fact that each $\mu_{\alpha,t}$ has a smooth density. This is no longer true in the non-abelian case. Let us emphasize and explain this important difference between the abelian and non-abelian case. Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet space as in Theorem 3.4. By Proposition 2.7 $(\mu_t)_{t>0}$ is non-degenerated if and only if the infinitesimal generator $-L_\alpha$ of $(\mu_{\alpha,t})_{t>0}$ is a sum of squares of vector fields having the Hörmander property. Actually, in the notation of Theorem 3.4,

$$L_\alpha f = -\sum_{\mathcal{J}} [d\pi_\alpha(Y_i)]^2 f.$$

Hence, $(\mu_t)_{t>0}$ is non-degenerate if and only if

$$\forall\, \alpha \in I,\quad \mathrm{Lie}\{d\pi_\alpha(Y_i), i \in \mathcal{J}\} = \mathcal{G}_\alpha.$$

This is equivalent to say that $\mathrm{Lie}\{Y\} = \mathcal{G}$, that is, Y is a Hörmander system. This proves the last assertion of Theorem 3.4.

3.3. Left-invariant Dirichlet spaces on G

We now define left-invariant Dirichlet spaces on G. See [16]. First, observe that left translations act as isometries on $L^2(G, d\nu_l)$.

DEFINITION 3.5.— *A Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(G, \nu_l)$ is left-invariant if for all $x \in G$ and for all $u \in \mathcal{F}$ the function $y \mapsto u_x(y) = u(xy)$ belongs to $\mathcal{F}$ and $\mathcal{E}(u_x, u_x) = \mathcal{E}(u, u)$.*

The notion of right-invariant Dirichlet space on $L^2(G, d\nu_r)$ is defined similarly and, if G is unimodular, a Dirichlet space is bi-invariant if it is both left and right invariant. Note that the Markov semigroup $(H_t)_{t>0}$ associated with a left-invariant Dirichlet space commutes with left translations. Hence, Theorem 3.2 applies to left-invariant Dirichlet spaces.

If G is unimodular, Theorems 3.3 and 3.4 describe all left-invariant strictly local Dirichlet spaces on G. If G is not unimodular, these theorems do not treat directly left-invariant Dirichlet spaces. The aim of this section is to describe all left-invariant Dirichlet spaces when G is not unimodular. First we show that the modular function is a smooth cylindric function that is, in some sense, depends only on finitely many coordinates.

LEMMA 3.6.— *The modular function m is a smooth cylindric function and, for any $X \in \mathcal{G}$, it satisfies*

$$Xm = \lambda_X m$$

where $\lambda_X = Xm(e)$ is a constant.

Proof.— First, one easily check this result for any Lie group using the fact that m is multiplicative. Second, by a well-known theorem of Malčev and Iwasawa, any locally compact connected group G is locally the direct product of a local Lie group E and a compact group K. That is, there exists a neighborhood V of e in G and a neighborhood U of e_E in E such that $V = U \times K$. Let $(K_\alpha)_I$ be a descending sequence of compact normal subgroups of G such that G/K_α is a Lie group G_α. By excluding finitely many α's, we can assume without loss of generality that the K_α's are in fact of the form $\{e_E\} \times \tilde{K}_\alpha$ with $\tilde{K}_\alpha \subset K$. It then follows that G_α is locally isomorphic to $V_\alpha = U \times (K/\tilde{K}_\alpha)$. Let m_α be the modular function on G_α. We claim that

$$m_\beta = m_\alpha \circ \pi_{\alpha,\beta}$$

for all $\beta \geqslant \alpha$. This claim easily follows from the following four facts: 1) the modular function can be computed locally in any neighborhood of the neutral element, 2) any (left-invariant, say) Haar measure on G_β projects to a (left-invariant) Haar measure on G_α. 3) the Haar measure in V_α is the product of the Haar measure of E in U and the Haar measure of $(K/\tilde{K}_\alpha)$, 4) compact groups are unimodular.

From the claim it follows that the modular function m is given by

$$m = m_\alpha \circ \pi_\alpha$$

for any α large enough. That is, m is a cylindric function. The rest of Lemma 3.6 follows from its finite dimensional version.

Let $(\mathcal{E}, \mathcal{F})$ be a left-invariant, strictly local Dirichlet space on $L^2(G, d\nu_l)$. Fix a projective basis $(X_i)_J$. By Theorem 2.3, the infinitesimal generator $-L$ is given by

$$Lf = -\sum_{i,j\in J} a_{i,j} X_i f X_j f + \sum_{i\in J} b_i X_i f, \quad f \in \mathcal{C}.$$

Let us compute the adjoint of L on cylindric functions. By Lemma 3.6, for each $i \in J$, the modular function m satisfies

$$X_i m = \lambda_i m \tag{3.3}$$

where $\lambda_i = [X_i m](e)$ is a constant. Moreover, only finitely many λ_i are non-zero. Hence, for any $\phi, \psi \in \mathcal{C}$, we have

$$\begin{aligned}
\int (L\phi)\psi d\nu_l &= \int (L\phi)\psi m d\nu_r \\
&= \int \phi \Bigg(-\sum_{i,j} a_{i,j} X_i X_j \psi - \sum_i (b_i + 2\sum_j a_{i,j}\lambda_j) X_i \psi \\
&\qquad -(\sum_i b_i\lambda_i + \sum_{i,j} a_{i,j}\lambda_i\lambda_j)\psi \Bigg) m d\nu_r \\
&= \int \phi \Bigg(-\sum_{i,j} a_{i,j} X_i X_j \psi - \sum_i (b_i + 2\sum_j a_{i,j}\lambda_j) X_i \psi \\
&\qquad -(\sum_i b_i\lambda_i + \sum_{i,j} a_{i,j}\lambda_i\lambda_j)\psi \Bigg) d\nu_l
\end{aligned}$$

As L must be self-adjoint, we have

$$-(b_i + 2\sum_i a_{i,j}\lambda_i) = b_i$$

and

$$\sum_i b_i\lambda_i + \sum_{i,j} a_{i,j}\lambda_i\lambda_j = 0.$$

In fact, the first equality implies

$$b_i = -\sum_j a_{i,j}\lambda_j \tag{3.4}$$

which implies the second condition.

Given that (3.4) must be satisfied, we see that the Dirichlet form $\mathcal{E}$ is given on cylindric functions by

$$\mathcal{E}(f,f) = \int fLf d\nu_l = \int \sum_{i,j} a_{i,j}X_i f X_j f d\nu_l.$$

The infinitesimal generator $-L$ is given by

$$Lf = -\sum_{i,j} a_{i,j}X_iX_jf - \sum_{i,j} a_{i,j}\lambda_i X_j f, \quad f \in \mathcal{C}.$$

These findings are recorded in the following theorem.

THEOREM 3.7.— *Let G be as in Section 2.1. Let $(X_i)_J$ be a fixed projective basis of $\mathcal{G}$.*

(i) For any strictly local left-invariant Dirichlet space $(\mathcal{E},\mathcal{F})$ in $L^2(G,\nu_l)$, there exists a unique symmetric non-negative matrix $A = (a_{i,j})_{i,j\in J}$ such that

$$\forall\, f \in \mathcal{C}, \quad \mathcal{E}(f,f) = \int_G \sum_{i,j\in J} a_{i,j}X_i f X_j f d\nu_l.$$

Moreover, $\mathcal{C}$ is a core for $(\mathcal{E},\mathcal{F})$.

(ii) Conversely, for any symmetric non-negative matrix $A = (a_{i,j})_{i,j\in J}$, the quadratic form

$$\mathcal{E}^l_A(f,f) = \int_G \sum_{i,j\in J} a_{i,j}X_i f X_j f d\nu_l, \quad f \in \mathcal{C}$$

is closable and its minimale closure defines a strictly local left-invariant Dirichlet space $(\mathcal{E}^l_A,\mathcal{F}^l_A)$ on $L^2(G,\nu_l)$.

(iii) *Given a symmetric non-negative matrix $A = (a_{i,j})_{i,j \in J}$, the infinitesimal generator $-L^l_A$ corresponding to the Dirichlet space $(\mathcal{E}^l_A, \mathcal{F}^l_A)$ is given on $\mathcal{C}$ by*

$$L^l_A = -\left(\sum_{i,j} a_{i,j}(X_i + \lambda_i) X_j f \right),$$

with $(\lambda_i)_J$ is given by (3.3).

Remark. — When G is not unimodular, the Gaussian semigroup associated to a left-invariant strictly local Dirichlet space is not symmetric. Let us also point out that, if A is a positive matrix and G is a Lie group, then the above Dirichlet space is, up to a multiplicative constant, the Dirichlet space of a left-invariant Riemannian structure on G. Thus, from a Riemannian geometry viewpoint, the Dirichlet spaces considered in Theorem 3.7 are more natural than those of Theorem 3.3 when G is not unimodular.

We can now use Lemma 2.5 to obtain a different representation of left-invariant Dirichlet spaces on $L^2(G, \nu_l)$.

THEOREM 3.8. — *Let $(\mathcal{E}, \mathcal{F})$ be a strictly local left-invariant Dirichlet space on $L^2(G, \nu_l)$. Then there exists a system $Y = (Y_i)_{\mathcal{J}} \subset \mathcal{G}$ which is projective and can be obtained from the given basis $(X_i)_{\in J}$ by an upper-triangular matrix $\tau = (\tau_{i,j})_{i,j \in J}$ and such that:*

(i) *The domain $\mathcal{F}$ of $\mathcal{E}$ coincides with the Sobolev space*

$$W^1_2(Y, \nu_l) = \left\{ f \in L^2(G, \nu_l) : \forall\, i \in \mathcal{J},\ \ Y_i f \in L^2(G, \nu_l) \right.$$
$$\left. \text{and } \int_G \sum_{\mathcal{J}} |Y_i f|^2 d\nu_l < \infty \right\}$$

and

$$\forall\, f \in \mathcal{F},\ \ \mathcal{E}(f, f) = \int_G \sum_{\mathcal{J}} |Y_i f|^2 d\nu_l;$$

(ii) *The infinitesimal generator $-L$ associated with $(\mathcal{E}, \mathcal{F})$ is given on $\mathcal{C}$ by*

$$Lf = \sum_{\mathcal{J}} Y_i^* Y_i f.$$

Here Y_i^* is the adjoint of Y_i on $L^2(G, d\nu_l)$ which is given by

$$Y_i^* = -(Y_i + \lambda_i), \quad \lambda_i = \frac{Y_i m}{m}.$$

(iii) The Gaussian semigroup associated with the Markov semigroup $(e^{-tL})_{t>0}$ by Theorem 3.2 is non-degenerate if and only if Y is a Hörmander system.

Only part (iii) needs to be proved. It follows directly from Proposition 2.7 and the fact that

$$L = -\sum_1^\infty Y_i^2 + Y_0$$

where $Y_0 = -\sum \lambda_i Y_i$ belongs to the linear span of Y.

3.4. The two Dirichlet spaces associated to a symmetric matrix

Let G be the projective limit of a sequence $(G_\alpha)_I$ of Lie groups, with associated projective basis $(X_i)_J$. Given a symmetric non-negative matrix A, Theorems 3.3, 3.7 yield two Dirichlet spaces and two Gaussian convolution semigroups. These objects are really distinct when G is not unimodular, i.e., when $m \not\equiv 1$. Lemma 3.9 below shows how these objects are related.

Let $(\mathcal{E}_A^r, \mathcal{F}_A^r)$ (resp. $(\mathcal{E}_A^l, \mathcal{F}_A^l)$) be the Dirichlet space on $L^2(G, \nu_r)$ (resp. $L^2(G, \nu_l)$) given by Theorem 3.3 (resp. 3.7). Namely, for $f \in \mathcal{C}$, we have

$$\begin{aligned} \mathcal{E}_A^r(f,f) &= \sum_{i,j} \int_G a_{i,j} X_i f X_j f d\nu_r \\ \mathcal{E}_A^l(f,f) &= \sum_{i,j} \int_G a_{i,j} X_i f X_j f d\nu_l. \end{aligned}$$

Let $(\mu_t^A)_{t>0}$ be the symmetric Gaussian semigroup associated to $(\mathcal{E}_A^r, \mathcal{F}_A^r)$ and let

$$L_A^r = -\sum_{i,j} a_{i,j} X_i X_j$$

be (minus) its infinitesimal generator and let

$$H_t^A f(x) = e^{-tL_A^r} f(x) = \int f(xy) d\mu_t^A(y)$$

be the associated self-adjoint semigroup of operators on $L^2(G, \nu_r)$.

Let $(\rho_t^A)_{t>0}$ be the (non-symmetric) Gaussian semigroup associated to $(\mathcal{E}_A^l, \mathcal{F}_A^l)$, let

$$L_A^l = -\sum_{i,j} a_{i,j}(X_i + \lambda_i)X_j$$

be (minus) its infinitesimal generator and let

$$Q_t^A f(x) = e^{-tL_A^l} f(x) = \int f(xy)d\rho_t^A(y)$$

be the associated self-adjoint semigroup of operators on $L^2(G, \nu_l)$.

LEMMA 3.9.— *For all $f \in \mathcal{C}$, we have*

$$L_A^l f = m^{-1/2}(L_A^r + a)(m^{1/2} f)$$

where

$$a = \frac{1}{4}\sum_{i,j} a_{i,j}\lambda_i\lambda_j$$

with λ_i defined at (3.3). *In particular,*

$$Q_t^A = e^{-at} m^{-1/2} H_t m^{1/2}$$

and

$$\rho_t^A(dy) = e^{-ta} m^{1/2}(y)\mu_t^A(dy).$$

Proof. — We simply compute

$$\begin{aligned} L_A^r(m^{1/2}f) &= -\sum_{i,j} a_{i,j}X_iX_j(m^{1/2}f) \\ &= -m^{1/2}\left(\sum_{i,j} a_{i,j}X_iX_jf + \sum_{i,j} a_{i,j}\lambda_iX_jf + \frac{1}{4}\sum_{i,j} a_{i,j}\lambda_i\lambda_jf\right) \\ &= m^{1/2}(L_A^l - a)f. \end{aligned}$$

This prove the first identity. The other statements all follow from the first.

4. Intrinsic and relaxed distances

Let $(X_i)_J$ be a fixed projective basis of the Lie algebra of G. Fix a symmetric non-negative matrix $A = (a_{i,j})_{J\times J}$. Let also $Y = (Y_i)$, $Y_i = \sum_j \tau_{i,j}X_j$, $i \in \mathcal{J}$, be the system of left invariant vector fields given by Lemma 2.5 so that

$$\forall\, \xi \in \mathbb{R}^\infty, \quad (A\xi, \xi) = \sum_i (\sum_j \tau_{i,j}\xi_j)^2.$$

This notation will be used throughout this section.

4.1. The intrinsic distance

For $f, g \in \mathcal{C}$, set

$$\Gamma(f,g) = \Gamma_A(f,g) = \sum_{i,j} a_{i,j} X_i f X_j g$$

and note that Γ can also be computed as

$$\Gamma(f,g) = \sum_i Y_i f Y_i g.$$

Set

$$d(x,y) = d_A(x,y) = \sup\left\{f(x) - f(y) : f \in \mathcal{C},\quad \Gamma(f,f) \leqslant 1\right\}. \tag{4.1}$$

Note that, by definition, d is lower semi-continuous.

Consider now the two Dirichlet spaces $(\mathcal{E}^r_A, \mathcal{F}^r_A)$, $(\mathcal{E}^l_A, \mathcal{F}^l_A)$. They both have $\mathcal{C}$ as a core and, for $f \in \mathcal{C}$,

$$\mathcal{E}^r_A(f,f) = \int \Gamma_A(f,f) d\nu_r, \quad \mathcal{E}^l_A(f,f) = \int \Gamma_A(f,f) d\nu_l.$$

Let us recall that the intrinsic distance attached to the Dirichlet space $(\mathcal{E}^r_A, \mathcal{F}^r_A)$ is defined by

$$d^r_A(x,y) = \sup\left\{ f(x) - f(y) : f \in \mathcal{F}^r_{A,loc}, \text{ continuous}, \right.$$
$$\left. \text{with } \sum_i |Y_i f|^2 \leqslant 1, \;\; \nu_r \text{ a.e.} \right\}.$$

See e.g., [11], [15], [27]. A similar definition holds for d^l_A. As ν_r and ν_l are equivalent measures and $\mathcal{F}^r_{A,\mathrm{loc}}$, $\mathcal{F}^l_{A,\mathrm{loc}}$ are equal as sets, it is clear that $d^r_A = d^l_A$. Now, in the present context, for any compact $K \subset G$, it is easy to construct cylindric compactly supported functions that are equal to 1 on K and have arbitrarily small gradient (i.e., small Γ_A). Moreover, for any $f \in \mathcal{F}^r_{A,\mathrm{loc}}$, $\phi \in \mathcal{C}$, $\int \phi d\nu_r = 1$,

$$f_\phi(x) = \int f(yx)\phi(y) d\nu_r(y)$$

is a cylindric function such that

$$\sup_{x \in G} \Gamma(f_\phi, f_\phi)(x) \leqslant \sup_{x \in G} \Gamma(f,f)(x).$$

Using this regularization procedure and cut-off, it is easy to prove the following lemma.

LEMMA 4.1.— *For any symmetric non-negative matrix A, the distances d^r_A and d^l_A are both equal to the distance d_A defined at* (4.1).

We will call $d = d_A$ the intrinsic distance associated to A. Since d is left-invariant, we set $d(x) = d(e,x)$ so that $d(x,y) = d(x^{-1}y)$. Let also d_α be the intrinsic distance on G_α associated to $(\mathcal{E}^r_\alpha, \mathcal{F}^r_\alpha)$.

THEOREM 4.2.— *Referring to the notation introduced above,*

(i) $\forall x \in G$, $d(x) = \sup_{\alpha\in I} d_\alpha \circ \pi_\alpha(x)$. *In particular, the topology of G is finer or equal to the topology induced by d. They are equal if and only if d is continuous.*

(ii) If Y is a Hörmander system, i.e., $\mathrm{Lie}(Y) = \mathcal{G}$, *then the set*

$$D = \{x \in G : d(x) < \infty\}$$

is a dense Borel subgroup of G and $\nu(D) = 0$ or $D = G$.

(iii) If $D = G$ then d is bounded on each compact subset of G.

Proof.— Assertion (i) is clear from the definition (see (2.1)). The proof of (ii) is three steps.

Step 1.— As d is lower semicontinuous, D is a Borel set. The triangle inequality and the fact that $d(x) = d(x^{-1})$ show that D is a subgroup of G.

Step 2.— On a locally compact group, if U is a measurable subset with positive Haar measure then $U^{-1}U$ contains a non-empty open neighborhood of the neutral element. Hence, if $\nu_r(D) > 0$ then $D^{-1}D = D$ must contain a non-empty open neighborhood of the neutral element. Since any non-empty open neighborhood of the neutral element in a locally compact connected group G generates G, we must then have $D = G$.

Step 3.— To show that D is dense, it suffices to show that, for each α, $\pi_\alpha(\overline{D}) = G_\alpha$. In $\mathcal{G}$, consider the linear subspace

$$\mathcal{H}(Y) = \{Z = \sum_{\mathcal{J}} \theta_i Y_i ; \sum_{\mathcal{J}} |\theta_j|^2 < \infty\}$$

and set

$$H(Y) = \exp(\mathcal{H}(Y)) \subset G.$$

We now show that $H(Y) \subset D$. Let $\psi \in \mathcal{C}$ and set $\Psi(t) = \psi(\exp(tZ))$ where $Z \in \mathcal{H}(Y)$. Observe that

$$\begin{aligned}|\Psi'(t)| &= |Z\Psi(\exp(tZ))| \\ &= |\sum_1^\infty \theta_k[Y_k\psi](\exp(tZ))| \\ &\leqslant \|(\theta_k)\|_2 \left(\sum_1^\infty |[Y_k\psi](\exp(tZ))|^2\right)^{1/2} \\ &= \|(\theta_k)\|_2[\Gamma(\psi,\psi)(\exp(tZ))]^{1/2}.\end{aligned}$$

Observe that, since Y is obtained from X by an upper triangular matrix τ and $\psi \in \mathcal{C}$, it follows that the sums in the above computation are in fact finite sums. Now write

$$\begin{aligned}|\psi(x) - \psi(e)| &= |\Psi(1) - \Psi(0)| \leqslant \max_{(0,1)} |\Psi'| \\ &\leqslant \|(\theta_k)\|_2 \max_G \Gamma(\psi,\psi)^{1/2}.\end{aligned}$$

This shows that $d(x) \leqslant \|(\theta_k)\|_2 < \infty$. Note that $H(Y)$ is not a subgroup of G. Let

$$\mathcal{H}_\alpha(Y) = d\pi_\alpha(\mathcal{H}(Y)).$$

Then, clearly, $\mathcal{H}_\alpha(Y)$ is simply the linear span of $d\pi_\alpha(Y)$ in $\mathcal{G}_\alpha$ and thus the non-degeneracy hypothesis implies that $\mathcal{H}_\alpha(Y)$ generates $\mathcal{G}_\alpha$ as a Lie algebra. Moreover,

$$\pi_\alpha(H(Y)) = \pi_\alpha(\exp \mathcal{H}(Y)) = \exp_\alpha[d\pi_\alpha(\mathcal{H}(Y))] = \exp_\alpha(\mathcal{H}_\alpha(Y)).$$

It follows that the closed subgroup generated by $\pi_\alpha(H(Y))$ is G_α. As $\pi_\alpha(H(Y)) \subset \pi_\alpha(D)$, this proves that $\pi_\alpha(\overline{D}) = \overline{\pi_\alpha(D)} = G_\alpha$ which is the desired result.

The proof of (iii) is three steps.

Step 1.— As $D = G$ it follows that $\{x \in G_\alpha : d_\alpha(x) < +\infty\} = G_\alpha$. This in turn implies that Y is a Hörmander system and d_α is a continuous function, in fact, locally Hölder continuous with respect to any fixed Riemannian distance. See, e.g., [24], [28].

Step 2.— Consider the sets $V_k = \{x : d(x) \leqslant k\}$. As each of the sets $\{x : d_\alpha \circ \pi_\alpha(x) \leqslant k\}$ is a compact cylindric set, $V_k = \cap_\alpha\{x : d_\alpha(x) \leqslant k\}$ is compact.

Step 3. — As $V_k \uparrow G$, there exists a k_0 such that V_{k_0} has positive Haar measure. Then, $V_{k_0}V_{k_0}^{-1}$ contains an open neighborhood U of the identity element e and for any $z \in U$ we have $z = xy^{-1}$ with $x, y \in V_{k_0}$. Thus $\sup_{z\in U}\{d(z)\} \leqslant 2k_0$. As G is connected, U generates the whole group, that is, $G = \cup_1^\infty U^k$. As U is open, for any compact set $K \subset G$, there exists $k_1 < \infty$ such that $K \subset \cup_1^{k_1} U^k$. Thus $d(x) \leqslant 2k_0k_1$ for any $x \in K$. This finishes the proof of Theorem 4.2.

4.2. The relaxed distance

It was observed in [6] that another distance called "relaxed distance" is a useful substitute to the intrinsic distance in certain situation. In order to define the notion of relaxed distance we need to recall the notion of "good algebra".

DEFINITION 4.3. — *Let M be a locally compact separable metric space equipped with a positive Borel measure ν having full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular strictly local Dirichlet space on $L^2(M, \nu)$ with infinitesimal generator $(-L, \mathcal{D})$. An algebra $\mathcal{A}$ of compactly supported continuous functions is a good algebra for $(\mathcal{E}, \mathcal{F})$ if*

1. *$\mathcal{A} \subset \mathcal{D}$ and $\mathcal{A}$ is stable by L, i.e., $L\mathcal{A} \subset \mathcal{A}$;*
2. *$\mathcal{A}$ is a core for $(\mathcal{E}, \mathcal{F})$.*

Note that the existence of a good algebra is a strong assumption.. See [6]. If $(\mathcal{E}, \mathcal{F})$ is a regular strictly local Dirichlet space with infinitesimal generator $-L$ that admits a good algebra $\mathcal{A}$, we can set

$$\delta(x, y) = \sup\{f(x) - f(y) : f \in \mathcal{A}, \Gamma(f, f) \leqslant 1, |Lf| \leqslant 1\}.$$

The function $\delta : M \times M \to [0, +\infty]$ is called the relaxed (quasi) distance associated to $(\mathcal{E}, \mathcal{F}, \mathcal{A})$.

In the setting of the present paper there is a natural obvious choice for the good algebra $\mathcal{A}$. Indeed, by Theorems 3.4, 3.8, it is clear that the set $\mathcal{C}$ of all Bruhat test functions is a good algebra for the Dirichlet spaces $(\mathcal{E}^r_A, \mathcal{F}^r_A)$ and $(\mathcal{E}^l_A, \mathcal{F}^l_A)$. This leads to the definition of two (left and right), possibly distinct, relaxed distances δ^l_A, δ^r_A associated with a given matrix A because the infinitesimal generators $-L^l_A$ and $-L^r_A$ are different. We will only use the left version. Actually, it is easy to see that one always has $\delta^l_A \approx \delta^r_A$. This follows from the fact that L^l_A, L^r_A share the same Γ and are related by

$$L^l_A f = L^r_A f + \sum_{i,j} a_{i,j}\lambda_i X_j f$$

where

$$\left|\sum_{i,j} a_{i,j}\lambda_i X_j f\right|^2 \leqslant \left(\sum_{i,j} a_{i,j}\lambda_i\lambda_j\right)\Gamma(f,f).$$

DEFINITION 4.4.— *Given a symmetric, non-negative matrix A, we define the relaxed distance associated to A by setting*

$$\begin{aligned}\delta(x,y) = \delta_A(x,y) = \delta_A^l(x,y)\\ = \sup\{f(x)-f(y) : f \in \mathcal{C},\ \Gamma_A(f,f) \leqslant 1, |L_A^l f| \leqslant 1\}.\end{aligned}$$

Clearly, δ is lower semicontinuous and a result analogous to Theorem 4.2 holds true for the relaxed distance δ. Set $\delta(x) = \delta(e,x)$ and let δ_α denote the relaxed distance on G_α associated to the Dirichlet form $(\mathcal{E}_\alpha^l, \mathcal{F}_\alpha^l)$ with good algebra $\mathcal{C}_0^\infty(G_\alpha)$.

THEOREM 4.5.— *Referring to the definition above, we have*

(i) $\forall x \in G$, $\delta(x) = \sup_{\alpha\in I} \delta_\alpha \circ \pi_\alpha(x)$.

(ii) If Y is a Hörmander system, i.e., $\mathrm{Lie}(Y) = \mathcal{G}$, *then the set*

$$\Delta = \{x \in G : \delta(x) < \infty\}$$

is a dense Borel subgroup of G and $\nu(\Delta) = 0$ or $\Delta = G$.

(iii) If $\Delta = G$ then δ is bounded on each compact subset of G.

Note that $\delta_\alpha \leqslant d_\alpha$ always holds. By the triangular inequality, it follows that δ_α is continuous whenever d_α is continuous. It follows also that $D \subset \Delta$. These remarks and steps 1-2 of the proof of Theorem 4.2(ii) yields a proof of Theorem 4.5(ii). Step 3 need not be repeated here since $D \subset \Delta$. The proof of (iii) is similar to that of Theorem 4.2(iii).

The relaxed distance is difficult to compute explicitly. See [6] where it is computed for left-invariant Dirichlet form on the circle $\mathbb{T}$. In this case the relaxed and intrinsic distances may have completely different behaviors. In [6] we also noticed that the relaxed distance of a finite dimensional Euclidean space is equal to the Euclidean (i.e. intrinsic) distance. In general, we have the following result.

THEOREM 4.6. — *Let A be a symmetric non-negative matrix. Let d, δ be the corresponding intrinsic and relaxed distances on G and assume that $d(x)$ is finite for all $x \in G$. Then*

$$\delta \leqslant d \leqslant C(1+\delta)$$

where C depends on G and A.

Proof. — Let ϕ be a non-negative function in $\mathcal{C}$ such that $\int \phi d\nu_r = 1$ and $\phi \geqslant 1/2$ on a neighborhood V of the neutral element. As $d(x)$ is finite for each x, Theorem 4.2 shows that d is bounded on compact sets. Hence the formula

$$\rho(x) = \int d(y)\phi(yx)d\nu_r(y)$$

defines a smooth cylindric function on G. Moreover, ρ is strictly positive. Indeed, for any α, $d \geqslant d_\alpha \circ \pi_\alpha$ and d_α is the intrinsic distance associated with a Hörmander system of left-invariant vector fields on the Lie group G_α.

We claim that

$$\forall\, x \in G, \quad c(1+d(x)) \leqslant \rho(x) \leqslant C(1+d(x)). \tag{4.2}$$

To prove the upper bound, write $d(y) \leqslant d(x^{-1}) + d(yx) = d(x) + d(yx)$ and

$$\rho(x) \leqslant \int (d(x)+d(yx))\phi(yx)d\nu_r(y) = d(x) + \int d(z)\phi(z)d\nu_r(z) \leqslant d(x) + C_1$$

where $C_1 = \int d(z)\phi(z)d\nu_r(z)$.

To prove the lower bound, write

$$\rho(x) \geqslant \int (d(x)-d(yx))\phi(yx)d\nu_r(y) = d(x) - \int d(z)\phi(z)d\nu_r(z) = d(x) - C_1$$

with the same C_1 as before. Now, fix α and let $K = \{x : d_\alpha \circ \pi_\alpha(x) \leqslant 2C_1\}$. The set K is compact, and for $x \in K$, we have

$$\rho(x) \geqslant c' \geqslant c(1+d(x))$$

for some constant $c > 0$. Indeed, we already noticed that ρ is continuous and positive and d is bounded on compact sets. For $x \notin K$, we have $d(x) > d_\alpha \circ \pi_\alpha(x) \geqslant 2C_1$. As $\rho(x) \geqslant d(x) - C_1$, this gives

$$\rho(x) \geqslant \frac{1}{2}d(x) \geqslant C_1.$$

Hence,

$$\forall x \in G, \quad \rho(x) \geqslant \min\left\{c, \frac{1}{4}, \frac{C_1}{2}\right\}(1 + d(x)).$$

This proves the claim.

We now want to show that there exists a constant C such that

$$\forall x \in G, \quad \Gamma(\rho, \rho) \leqslant C, \quad |L^l \rho| \leqslant C. \tag{4.3}$$

Indeed, for any left-invariant vector field X, we have

$$X\rho(x) = \int d(y)[X\phi](yx)d\nu_r(y).$$

As

$$\int [X\phi](yx)d\nu_r(y) = X\int \phi(yx)d\nu_r(y) = 0$$

we get

$$X\rho(x) = \int (d(y) - d(x))[X\phi](yx)d\nu_r(y).$$

By the triangle inequality,

$$|d(y) - d(x)| \leqslant d(yx).$$

Thus

$$|X\rho(x)| \leqslant \int |d(y) - d(x)||X\phi|(yx)d\nu_r(y) \leqslant \int d(z)|X\phi(z)|d\nu_r(z) \leqslant C_1.$$

The desired inequality then follows from $\Gamma(\rho, \rho) = \sum_i |Y_i\rho|^2$ and the fact that ρ is a cylindric function hence $|Y_i\rho| \neq 0$ for finitely many i only. Similarly,

$$\begin{aligned}
|L^l\rho(x)| &= \left|\int d(y)[L^l\phi](yx)d\nu_r(y)\right| \\
&= \left|\int (d(y) - d(x))[L^l\phi](yx)d\nu_r(y)\right| \\
&\leqslant \int d(yx)|[L^l\phi](yx)|d\nu_r(y) = \int d(z)|L^l\phi(z)|d\nu_r(z) \leqslant C_2.
\end{aligned}$$

This proves the claim.

Now, consider a smooth function $f : [0, \infty) \to [0, 1]$ such that $f(x) = 1$ on $[0, 1]$, $f(x) = 0$ for $x \geqslant 2$, and set $f_R(x) = Rf(\rho(x)/R)$, $R \geqslant 1$. Then,

$$Y_i f_R = f'(\rho/R)Y_i\rho$$

and

$$Y_i^2 f_R = R^{-1} f''(\rho/R)|Y_i \rho|^2 + f'(\rho/R) Y_i^2 \rho.$$

It follows that

$$\Gamma(f_R, f_R) \leqslant C_3^2, \quad |L^l f_R| \leqslant C_3$$

for some constant C_3. This shows that we can use f_R/C_3 to obtain a lower bound on δ. Indeed, fix $x \in G$ with $\rho(x) \geqslant 2\rho(e)$. Set $R = \rho(x)/2$. Then

$$\delta(x) \geqslant C_3^{-1}(f_R(e) - f_R(x)) = \frac{\rho(x)}{2C_3}[f(2\rho(e)/\rho(x)) - f(2)] = \frac{\rho(x)}{2C_3}.$$

Hence there exist two constants $C_4, c_4 > 0$ such that

$$\delta(x) \geqslant c_4 \rho(x)$$

for all x such that $\rho(x) \geqslant C_4$. Finally, this shows that

$$d(x) \leqslant C(1 + \delta(x))$$

as desired.

5. Gaussian semigroups having prescribed behaviors

5.1. An existence statement

On a connected Lie group, consider a Gaussian semigroup $(\mu_t)_{t>0}$ which is either symmetric (hence gives rise to a self-adjoint semigroup on $L^2(G, \nu_r)$) or not symmetric but induces a self-adjoint semigroup on $L^2(G, d\nu_l)$ (as in Section 3.3). Then either $(\mu_t)_{t>0}$ is degenerate, that is, is singular with respect to Haar measure on G or $(\mu_t)_{t>0}$ is non-degenerate and it actually has a smooth positive density $x \mapsto \mu_t(x)$ (w.r.t. ν_l) which satisfies

$$c \leqslant t^{\kappa/2} \mu_t(e) \leqslant C$$

for all $t \in (0, 1)$ and some constants $0 < c < C < +\infty$. Here κ is an integer and one can show that κ belongs to the interval $[n, n(n+1)/2]$ where n is the topological dimension of the Lie group G. See [28]. When G is not a Lie group, but a locally compact connected locally connected metrizable group, it is not completely clear what behaviors one should expect from $\mu_t(e)$, assuming that the density $\mu_t(\cdot)$ does exist for all $t > 0$. This section is devoted to the proof of the following theorem which asserts that, on any such group G, there is a host of different behaviors that do appear.

THEOREM 5.1.— *Let G be a locally compact connected locally connected metrizable group. Assume that G is not a Lie group.*

1) *For any regularly varying function* ψ *of index* λ, $\lambda \in (0,\infty)$, *there exists a symmetric Gaussian convolution semigroup* $(\mu_t)_{t>0}$ *on* G *which has a continuous density* $\mu_t(\cdot)$ *w.r.t.* ν_l, *satisfies* (CKψ) *and such that*

$$\limsup_{t\to 0} \frac{\log \mu_t(e)}{\psi(1/t)} > 0.$$

2) *For any positive increasing function* ψ *such that* $\lim_\infty \psi = \infty$, *there exists a symmetric Gaussian convolution semigroup* $(\mu_t)_{t>0}$ *on* G *which has a continuous density* $\mu_t(\cdot)$ *w.r.t.* ν_l *and satisfies*

$$\forall t \in (0,1), \quad \log \mu_t(e) \leqslant C_\psi \log(1+1/t)\psi(1/t).$$

When G is abelian, these results are due to the first author [2]. When G is compact, see [7, Theorem 1.1]. Of course, we can see the above Theorem as stating the existence of left-invariant Dirichlet spaces on $L^2(G, d\nu_l)$ such that the associated Gaussian semigroup of measures $(\rho_t)_{t>0}$ admits a density $\rho_t(\cdot)$ satisfying the same conditions as $\mu_t(\cdot)$ above. See Lemma 3.9.

Statement 2) is optimal. More precisely, we have the following lemma.

LEMMA 5.2.— *Let* G *be as in Theorem* 5.1. *Assume that* $(\mu_t)_{t>0}$ *is a symmetric Gaussian semigroup having a continuous density* $\mu_t(\cdot), t > 0$ *w.r.t* ν_l. *Then*

$$\lim_{t\to 0} \frac{\log \mu_t(e)}{\log(1+1/t)} = +\infty.$$

Proof.— As G is locally connected and G is not a Lie group, the sequence (G_α) such that $G = \lim\text{ proj } G_\alpha$ must have the property that the topological dimension n_α of G_α tends to infinity. Moreover, we have

$$\mu_t(e) \geqslant \mu_{\alpha,t}(e)$$

and for $t \in (0,1)$ (see [28])

$$\mu_{\alpha,t}(e) \geqslant c_\alpha t^{-n_\alpha/2}.$$

Thus

$$\log \mu_t(e) \geqslant \log c_\alpha + \frac{n_\alpha}{2} \log(1/t).$$

The lemma follows.

5.2. Proof of Theorem 5.1

As Theorem 5.1 is proved in [7, Theorem 1.1] for compact groups, it suffices to show that the general case follows from the compact case.

Let G be a locally compact connected locally connected metrizable group. By a result of Berestowskii and Plaut [9], there exist a simply connected connected Lie group L, a compact group K and a discrete subgroup H of $L \times K$ so that

$$G \cong (L \times K)/H.$$

In other words, G is covered by the direct product $L \times K$.

In what follows the neutral elements of different groups are all denoted by e. Define

$$\begin{aligned} H_L &= \{\ell \in L : (\ell, k) \in H \text{ for some } k \in K\} \\ H_K &= \{k \in K : (e, k) \in H\}. \end{aligned}$$

Note that H_K is a discrete normal subgroup of K. Hence H_K is finite. Let κ be its order. Note also that H_L is a discrete normal subgroup of L. Moreover, for any $\ell \in H_L$, and k, k' such that $(\ell, k), (\ell, k') \in H$, we have $k^{-1}k' \in H_K$. Thus, setting

$$\forall\, \ell \in H_L, \quad K_\ell = \{k : (\ell, k) \in H\},$$

we have

$$\forall\, \ell \in H_L, \quad \# K_\ell \leqslant \kappa.$$

Let $(\mu_t^L)_{t>0}$ be a symmetric Gaussian semigroup on L with infinitesimal generator $\sum_I Y_i^2$ where (Y_i) is a basis of the Lie algebra of L. By [28], this semigroup has a density satisfying

$$\forall\, t \in (0,1), \quad \log \mu_t^L(e) \approx \log(1 + 1/t)$$

where $\forall\, t \in (0,1)$, $f(t) \approx g(t)$ means there exist $0 < c < C < +\infty$ such that, for all $t \in (0,1)$, $c \leqslant f(t)/g(t) \leqslant C$.

Let $(\underline{\mu}_t^L)_{t>0}$ be the projection of $(\mu_t^L)_{t>0}$ onto L/H_L. As H_L is discrete, L and L/H_L have the same Lie algebra and the infinitesimal generator of $(\underline{\mu}_t^L)_{t>0}$ is again $\sum_I Y_i^2$. It follows that $(\underline{\mu}_t)_{t>0}^L$ has a smooth positive density given by

$$\underline{\mu}_t^L(x) = \sum_{\ell \in H_L} \mu_t^L(x\ell)$$

and satisfying (see, e.g., [28])

$$\forall\, t \in (0,1), \quad \log \underline{\mu}_t^L(e) \approx \log(1 + 1/t).$$

Let $(\mu_t^K)_{t>0}$ be a symmetric Gaussian semigroup on K having a continuous density $\mu_t^K(\cdot)$ for all $t>0$ (see [7]). Consider the Gaussian product-semigroup $(\overline{\mu}_t)_{t>0}$ on $L\times K$ defined by

$$\overline{\mu}_t = \mu_t^L \otimes \mu_t^K.$$

By projection, we obtain a Gaussian symmetric semigroup $(\mu_t)_{t>0}$ on $(L\times K)/H$ with (excessive) density $x\mapsto \mu_t(x)$ satisfying

$$\begin{aligned}
\mu_t(e) &= \sum_{h\in H}\overline{\mu}_t(h) = \sum_{\ell\in H_L}\mu_t^L(\ell)\sum_{k\in K_\ell}\mu_t^K(k)\\
&\leqslant \kappa\mu_t^K(e)\sum_{\ell\in H_L}\mu_t^L(\ell)\\
&= \kappa\mu_t^K(e)\underline{\mu}_t^L(e).
\end{aligned}$$

We also have

$$\begin{aligned}
\mu_t(e) &= \sum_{h\in H}\overline{\mu}_t(h) \geqslant \overline{\mu}_t(e)\\
&= \mu_t^K(e)\mu_t^L(e).
\end{aligned}$$

It follows that

$$\log\mu_t(e) \approx \log\mu_t^K(e) + \log(1+1/t). \tag{5.4}$$

From (5.4) it easily follows that Theorem 5.1 reduces to the compact case treated in [7].

6. Gaussian upper bounds

6.1. Gaussian upper bounds involving the intrinsic distance

For the purpose of this section, let us fix a group G as in Section 2 and a projective basis (X_i) of its Lie algebra. Fix also a non-negative symmetric matrix A and consider the Dirichlet space $(\mathcal{E}^l,\mathcal{F}^l)$ in $L^2(G,d\nu_l)$. The associated semigroup $(Q_t^A)_{t>0}$ is given by

$$Q_t^A f(x) = \int f(xy)\,d\rho_t^A(y).$$

As A will be fixed throughout the section, we will drop all references to it in our notation. We let d and δ be the corresponding intrinsic and relaxed distances

Assume that $(\rho_t)_{t>0}$ has a density $x\to\rho_t(x), t>0, x\in G$ w.r.t. $d\nu_l$. In the present setting, it is well known that one can choose the density

$x \mapsto \rho_t(x)$ as a lower semi-continuous function on G. This has the advantage of uniquely defining ρ_t point-wise. Introduce also the transition density $q(t,x,y)$ of the operator Q_t on $L^2(G,\nu_l)$, we have

$$q(t,x,y) = \rho_t(x^{-1}y).$$

Moreover, for all $t,s>0$ and all $x \in G$,

$$\rho_{t+s}(x) = \int_G \rho_t(xy)\rho_s(y^{-1})d\nu_l(y).$$

By construction, Q_t is self-adjoint on $L^2(G,d\nu_l)$. Hence $q(t,x,y) = q(t,y,x)$ and this implies that

$$\rho_t(x) = \rho_t(x^{-1}).$$

From this, we deduce that the density ρ_t is bounded if and only if

$$\rho_t(e) = \int_G |\rho_{t/2}(y)|^2 d\nu_l(y) < \infty.$$

Furthermore, the density ρ_t is bounded for all $t>0$ if and only if the Gaussian semigroup $(\rho_t)_{t>0}$ satisfies (CK).

This section describes what is known about Gaussian upper bounds, that is, bounds of the sort

$$\rho_t(x) \leqslant \exp\left(M(t) - \frac{\mathbf{dist}(x)^2}{Ct}\right)$$

where $\mathbf{dist}(x,y)$ is some invariant distance on G and $\mathbf{dist}(x)$ is the distance between x and the neutral element. Here, **dist** will be either the intrinsic distance d or the relaxed distance δ. In most statements available in the literature the function M is obtained in terms of the function

$$M_0(t) = \log\left(\rho_t(e)\right). \tag{6.5}$$

The first result is an application of Davies' Gaussian bounds. See [15], Chapter 3.

THEOREM 6.1.— *Fix $0 \leqslant \lambda < 1$ and let ψ be a regularly varying function of index λ. Assume that $(\rho_t)_{t>0}$ satisfies (CKψ) (resp. (CK$\psi*$)). Then for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that the continuous density $\rho_t(\cdot)$ of ρ_t satisfies*

$$\forall\ x \in G,\ t \in (0,1), \quad \rho_t(x) \leqslant \exp\left(C_\epsilon M(t) - \frac{d(x)^2}{4(1+\epsilon)t}\right)$$

where $M(t) = \psi(1/t)$ (resp. $\lim_{t\to 0} M(t)/\psi(1/t) = 0$).

Note that the case $\lambda = 0$ is not excluded in the statement above. However, if $\lambda = 0$, the hypothesis of the theorem can not be satisfied if the slowly varying function ψ grows too slowly at infinity. Indeed, on any Lie group, if $(\rho_t)_{t>0}$ has a continuous density then its behavior for small t is of the form $\rho_t(e) \approx t^{-n/2}$ for some integer n larger or equal to the dimension of G. See [28]. This shows that ψ must at least satisfy $\psi(u) \geqslant c\log(u), u \geqslant 1$, for some c.

Thanks to Lemma 3.9, Theorem 6.1 has the following corollary which is of interest in the case of non-unimodular groups.

THEOREM 6.2.— *Let $(\mu_t)_{t>0}$ be a symmetric Gaussian semigroup. Let d be the corresponding intrinsic distance. Assume that $(\mu_t)_{t>0}$ satisfies* (CKψ) *(resp.* (CK$\psi *$)*) for some regularly varying function ψ of index $0 \leqslant \lambda < 1$. Let $\mu_t(\cdot)$, be the density of μ_t w.r.t. $d\nu_l$. Then for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$\forall \ x \in G, \ t \in (0,1), \quad \mu_t(x) \leqslant m(x)^{-1/2} \exp\left(C_\epsilon M(t) - \frac{d(x)^2}{4(1+\epsilon)t}\right)$$

where $M(t) = \psi(1/t)$ (resp. $\lim_{t\to 0} M(t)/\psi(1/t) = 0$).

6.2. Gaussian upper bounds involving the relaxed distance

Let M_0 be as in (6.5). Theorem 6.1 gives Gaussian upper bounds on the density ρ_t when $M_0(t) \leqslant Ct^{-\lambda}$ for some $\lambda \in (0,1)$. These bounds involve the intrinsic distance d. As observed in [6], Gaussian bounds involving the intrinsic distance d are simply not true when the hypothesis on the behavior of $M_0(t)$ is relaxed to condition $(*)$, i.e., $tM_0(t) \to 0$ as t tends to 0. Instead, one can then obtain Gaussian upper bounds involving the relaxed distance δ.

In the present setting, the results of [6] yield the following important theorems and corollaries.

THEOREM 6.3.— *Assume that ρ_t has a continuous density $\rho_t(\cdot)$ for all* $t > 0$.

(1) If $(\rho_t)_{t>0}$ satisfies (CKWt^γ)*, for some $0 < \gamma < 1$, then*

$$\rho_t(x) \leqslant \exp\left[M(t) - \frac{\delta(x)^2}{12\,t} + \frac{\delta(x)}{6}\right] \tag{6.6}$$

where, for each $\gamma' > \gamma/(\gamma-1)$, we have

$$\sup_{t\in(0,1)} t^{\gamma'} \log M(t) < +\infty.$$

(2) *If* $(\rho_t)_{t>0}$ *satisfies* (CKψ) *(resp.* (CK$\psi*$) *) for some regularly varying function* ψ *of index* $0 \leqslant \lambda < +\infty$ *then* (6.6) *holds with*

$$\sup_{t\in(0,1)} \frac{M(t)}{\psi(1/t)} < +\infty \qquad \left(\text{resp. } \lim_{t\to 0} \frac{M(t)}{\psi(1/t)} = 0\right).$$

In particular, if $(\rho_t)_{t>0}$ *satisfies* (CK$*$) *then* (6.6) *holds with a function* M *such that* $\lim_{t\to 0} tM(t) = 0$.

THEOREM 6.4.— *Let* $(\mu_t)_{t>0}$ *be a symmetric Gaussian semigroup. Let* δ *be the associated relaxed distance. Assume that, for all* $t > 0$, μ_t *has a continuous density* $\mu_t(\cdot)$ *w.r.t.* $d\nu_l$.

(1) *If* $(\mu_t)_{t>0}$ *satisfies* (CKWt^γ), *for some* $0 < \gamma < 1$, *then*

$$\mu_t(x) \leqslant m(x)^{-1/2} \exp\left[M(t) - \frac{\delta^2(x)}{12\,t} + \frac{\delta(x)}{6}\right] \tag{6.7}$$

where, for each $\gamma' > \gamma/(\gamma - 1)$,

$$\sup_{t\in(0,1)} t^{\gamma'} \log M(t) < \infty.$$

(2) *If* $(\mu_t)_{t>0}$ *satisfies* (CKψ) *(resp.* (CK$\psi*$)*) for some regularly varying function* ψ *of index* $0 \leqslant \lambda < +\infty$ *then* (6.7) *holds with*

$$\sup_{t\in(0,1)} \frac{M(t)}{\psi(1/t)} < \infty \qquad \left(\text{resp. } \lim_{t\to 0} \frac{M(t)}{\psi(1/t)} = 0\right).$$

In particular, if $(\mu_t)_{t>0}$ *satisfies* (CK$*$) *then* (6.7) *holds with a function* M *such that* $\lim_{t\to 0} tM(t) = 0$.

Remark.— Concerning the second statement of each of the two last theorems, observe that Theorems 6.1 6.2 give much better results when $0 \leqslant \lambda < 1$.

We now state two important corollaries of the Gaussian estimates (6.6), (6.7). See [6, Corollary 3.9].

COROLLARY 6.5.— *Let* A *be a symmetric non-negative matrix. Let* $(\rho_t)_{t>0}$ *and* $(\mu)_{t>0}$ *be the two Gaussian convolution semigroups associated to* A *by Theorems 3.4, 3.8. Assume that* $(\rho_t)_{t>0}$ *(equivalently* $(\mu)_{t>0}$*) satisfies* (CK$*$). *Then they both satisfy* (CK$\#$). *In particular,* (CK$*$) *implies* (CK$\#$) *for all symmetric Gaussian semigroups.*

COROLLARY 6.6. — *Let A be a symmetric non-negative matrix. Let $(\rho_t)_{t>0}$ and $(\mu)_{t>0}$ be the two Gaussian convolution semigroups associated to A by Theorems 3.4, 3.8.*

- *Assume that $(\rho_t)_{t>0}$ (equivalently $(\mu)_{t>0}$) satisfies* (CKψ) *for some regularly varying function ψ of index $\lambda \in (0,1)$. Then the intrinsic distance d associated to A is continuous.*
- *Assume that $(\rho_t)_{t>0}$ (equivalently $(\mu)_{t>0}$) satisfies* (CKWt^γ) *for some $\gamma \in (0,1)$. Then the relaxed distance δ associated to A is continuous.*

Proof. — We only prove the first statement since the two proofs are similar (see also [6]). By Theorem 6.1, we have

$$d(x)^2 \leqslant C\left(t\psi(1/t) - t\log\rho_t(x)\right).$$

Thus

$$\limsup_{x\to e} d(x)^2 \leqslant C\left(t\psi(1/t) - t\log\rho_t(e)\right).$$

By hypothesis, $t\psi(1/t)$ and $t\log\rho_t(e)$ tend to zero as t tends to zero. Thus $\limsup_{x\to e} d(x)^2 = 0$. It follows that d is continuous.

6.3. Application to coverings

Consider a covering $\overline{G}$ of a group $G = \overline{G}/H$ where H is a discrete normal subgroup of $\overline{G}$. Let π be the canonical projection. Observe that the Lie algebras of $\overline{G}$ and G can be identified through the bijection $d\pi$. Let $(\overline{\mu}_t)_{t>0}$ and $(\mu_t)_{t>0}$ be two Gaussian semigroups on $\overline{G}$ and G respectively such that μ_t is the image of $\overline{\mu}_t$ under the canonical projection π. Our next theorem relates the respective behaviors of these two Gaussian semigroups under some additional hypotheses.

THEOREM 6.7. — *Let $\overline{G}$ be a connected locally connected locally compact metrizable group. Assume that H is a discrete normal subgroup of $\overline{G}$ and let $\pi : \overline{G} \to G = \overline{G}/H$ be the canonical projection. Identify the Lie algebras of $\overline{G}$ and G through the isomorphism $d\pi$ and fix a projective basis. Let A be a symmetric non-negative matrix. Let $(\overline{\rho}_t)_{t>0}$ and $(\overline{\mu_t})_{t>0}$ (resp. $(\rho_t)_{t>0}$) and $(\mu_t)_{t>0}$) be the two Gaussian convolution semigroups on $\overline{G}$ (resp. on G) associated to A by Theorems 3.4, 3.8. Then $(\rho_t)_{t>0}$ (resp. $(\mu_t)_{t>0}$) is the projection of $\overline{\rho}_t)_{t>0}$ (resp. $(\overline{\mu}_t)_{t>0}$) and:*

(1) For any regularly varying function ψ of index λ, $0 \leqslant \lambda < +\infty$, the Gaussian semigroup $(\overline{\rho}_t)_{t>0}$ (equivalently $(\overline{\mu}_t)_{t>0}$) has property

(CKψ) *(resp* (CK$\psi*$)*) if and only if the same is true for* $(\rho_t)_{t>0}$) *(equivalently* $(\mu_t)_{t>0}$*).*

(2) If ψ *is a regularly varying function of index* $0 \leqslant \lambda < \infty$ *and if* $(\overline{\mu}_t)_{t>0}$ *has a continuous density satisfying*

$$\log \overline{\rho}_t(\overline{e}) \approx \psi(1/t) \quad (\textit{equivalently}, \ \ \log \overline{\mu}_t(\overline{e}) \approx \psi(1/t))$$

then

$$\log \rho_t(e) \approx \psi(1/t) \quad (\textit{equivalently}, \ \ \log \mu_t(e) \approx \psi(1/t)).$$

(3) If the Gaussian semigroup $(\overline{\rho}_t)_{t>0}$) *(equivalently* $(\overline{\mu}_t)_{t>0}$*) satisfies* (CKWt^γ) *for some* $\gamma \in (0,1)$*, then* $(\rho_t)_{t>0}$ *(equivalently* $(\mu_t)_{t>0}$*) satisfies* (CKW$t^{\gamma'}$) *for all* $\gamma' > \gamma(\gamma-1)$.

Proof. — By Lemma 3.9, it suffices to work with $(\overline{\rho}_t)_{t>0}$ and $(\rho_t)_{t>0}$. Let us start by noting that the measure $\overline{\rho}_t$ is absolutely continuous with respect to a Haar measure on $\overline{G}$ if and only if ρ_t is absolutely continuous with respect to a Haar measure on G. Moreover, their lower continuous densities w.r.t. the left Haar measures on $\overline{G}$ and G are related by

$$\rho_t(x) = \sum_{h \in H} \overline{\rho}_t(\overline{x}h) \tag{6.8}$$

where $\pi(\overline{x}) = x$. Thus

$$\rho_t(e) = \sup_G \rho_t(\cdot) \geqslant \sup_{\overline{G}} \overline{\rho}_t(\cdot) = \overline{\rho}_t(e).$$

Moreover we obtain the following lemma.

LEMMA 6.8. — *Referring to the setting of Theorem 6.7, if* $(\rho_t)_{t>0}$ *(resp.* $(\mu_t)_{t>0}$*) satisfies* (CK) *then so does* $(\overline{\rho}_t)_{t>0}$ *(resp.* $(\overline{\mu}_t)_{t>0}$*). Moreover if these Gaussian semigroups satisfy* (CK) *then*

$$\rho_t(e) \geqslant \overline{\rho}_t(\overline{e}) \quad (\textit{resp.}\, \mu_t(e) \geqslant \overline{\mu}_t(\overline{e})).$$

In particular, if $(\rho_t)_{t>0}$ *(resp.* $(\mu_t)_{t>0}$*) satisfies any one of the properties* (CKWψ), (CKψ), (CK$\psi*$) *where* ψ *is a fixed positive non-decreasing function, then the same property holds for* $(\overline{\rho}_t)_{t>0}$ *(resp.* $(\overline{\mu}_t)_{t>0}$*).*

The rest of the proofs of each the three assertions in Theorem 6.7 are similar and use Theorems 6.1 and 6.3. We omit the proof of the two first assertions and give a detailed proof of the third. Assume that $(\overline{\rho}_t)_{t>0}$ satisfies (CKWt^γ) for some $\gamma \in (0,1)$. Then, applying Theorem 6.3, we have

$$\overline{\rho}_t(x) \leqslant \exp\left(M(t) - \frac{\overline{\delta}(x)^2}{12t} + \frac{\overline{\delta}(x)}{6}\right)$$

with $\sup_{t\in(0,1)}\{t^{\gamma'}\log M(t)\} < \infty$ for all $\gamma' > \gamma/(\gamma-1)$. Thus, for $0<t<1$, the density ρ_t on G is bounded by

$$\rho_t(e) \leqslant C\exp(M(t))\sum_{h\in H}\exp\left(-c\overline{\delta}(h)^2\right) \tag{6.9}$$

for some constant $C, c > 0$. We are left with the task to show that

$$\sum_{h\in H}\exp\left(-c\overline{\delta}(h)^2\right) < +\infty. \tag{6.10}$$

Let U be a compact neighborhood of $\overline{e}$ to be chosen later with the property that (a) $U = U^{-1}$ and (b) $H\cap U^2 = \{\overline{e}\}$. Set $V = U^2$ and for any $\overline{g}\in\overline{G}$, set

$$|\overline{g}| = \inf\{n : \overline{g}\in V^n\}.$$

As any neighborhood of the identity in $\overline{G}$ generates $\overline{G}$, $|\overline{g}|$ is finite for all $g\in\overline{G}$. We need the following volume growth Lemma.

Lemma 6.9. — *There exists $A_1, A_2 > 1$ such that*

$$\nu_l(V^n) \leqslant A_1^n$$

and

$$\#\{h\in H : |h|\leqslant k\} \leqslant A_2^k.$$

Proof. — Although this is well known, we do not know a precise reference. We include a proof for completeness. Let $X = \{x_1,\dots,x_N\}$ be a maximal subset of V^{n-1} such that $x_iU\cap x_jU = \emptyset$ if $i\neq j$. Then, as $\bigcup_i x_iU\subset V^n$, we have

$$N \leqslant \frac{\nu_l(V^n)}{\nu_l(U)}.$$

We claim that $\bigcup_i x_iV^3$ covers V^{n+1}. Indeed, consider $z = v_1\dots v_{n+1}\in V^{n+1}$, $v_i\in V$, and set $y = v_1\dots v_{n-1}$. Suppose that $y\notin\bigcup_i x_iV$. Then $yU\cap x_iU = \emptyset$ for all i (this use the assumption that $U = U^{-1}$). As $y\in V^{n-1}$ this contradicts the maximality of the set $\{x_1,\dots,x_N\}$. Hence $y\in\bigcup_i x_iV$ and it follows that $z\in\bigcup_i x_iV^3$ as desired. Thus we have

$$\nu_l(V^{n+1}) \leqslant N\nu_l(V^3) \leqslant \frac{\nu_l(V^3)}{\nu_l(U)}\nu_l(V^n).$$

This clearly proves the first assertion of Lemma 6.9 with $A_1 = \nu_l(V^3)/\nu_l(U)$. To prove the second assertion observe that, by our assumption (b) concerning U, $hU \cap h'U = \emptyset$ for all distinct $h, h' \in H$. Thus $\#\{h \in H : |h| \leqslant k\} \leqslant \nu_l(V^{k+1})/\nu_l(U)$. This ends the proof of Lemma 6.9.

We now return to the proof of (6.10). Recall that $\overline{G}$ is the projective limit of a sequence of Lie groups $(\overline{G}_\alpha)$. Let $\overline{d}_\alpha$ be the distance on $\overline{G}_\alpha$ induced by $(\overline{\rho}_{\alpha,t})_{t>0}$. As $(\overline{\rho}_t)_{t>0}$ is non-degenerate, the distances $\overline{d}_\alpha$ are continuous and have compact closed balls in $\overline{G}_\alpha$. Thus we can choose α and $r > 0$ such that the set

$$V = \{\overline{g} \in \overline{G} : \overline{d}_\alpha \circ \overline{\pi}_\alpha(\overline{g}) \leqslant r\}$$

intersects H only at the neutral element and set

$$U = \{\overline{g} \in \overline{G} : \overline{d}_\alpha \circ \overline{\pi}_\alpha(\overline{g}) \leqslant r/2\}.$$

This U is compact and satisfies the conditions (a) and (b) considered above.

As H is a discrete subgroup, it is easy to check that $\inf_{H\setminus\{e\}} \overline{\delta}(h) > 0$ and, by the choice of α and r, $\inf_{H\setminus\{e\}} \overline{\delta}_\alpha \circ \overline{\pi}_\alpha(h) > r$. Thus Theorem 4.6 yields

$$\overline{\delta}_\alpha \circ \overline{\pi}_\alpha(h) \approx \overline{d}_\alpha \circ \overline{\pi}_\alpha(h).$$

For any $h \in H$, we also have

$$r|h| \leqslant r + \overline{d}_\alpha \circ \overline{\pi}_\alpha(h).$$

Indeed, set $t = \overline{d}_\alpha \circ \overline{\pi}_\alpha(h)$ and let $\gamma : [0,t] \mapsto \overline{G}_\alpha$ be a $\overline{d}_\alpha$ minimizing curve from e_α to $\overline{\pi}_\alpha(h)$, parametrized by arc-length. Set $\lfloor t/r \rfloor = k-1$ and $h_{\alpha,i} = \gamma(ir)$, $i = 0, 1, \ldots, k-1$, and $h_{\alpha,k} = \overline{\pi}_\alpha(h)$. Set $h_0 = \overline{e}$, $h_k = h$ and, for $i = 1, \ldots, k-1$, pick h_i to be any element in $\overline{\pi}_\alpha^{-1}(h_{\alpha,i})$. Then, by construction,

$$\overline{d}_\alpha \circ \overline{\pi}_\alpha(h_{i-1}^{-1} h_i) \leqslant r.$$

Hence, $|h| \leqslant k \leqslant 1 + t/r = 1 + \overline{d}_\alpha \circ \overline{\pi}_\alpha(h)/r$ as desired. We conclude that there exists a constant C such that

$$\forall\, h \in H, \quad r|h| \leqslant r + \overline{d}_\alpha \circ \overline{\pi}_\alpha(h) \leqslant C\overline{\delta}_\alpha \circ \overline{\pi}_\alpha(h) \leqslant C\overline{\delta}(h).$$

It follows from this and Lemma 6.9 that

$$\sum_{h\in H} \exp(-c\overline{\delta}(h)^2) \leqslant \sum_k A_2^{k+1} \exp(-\epsilon k^2) < \infty$$

with $\epsilon > 0$. This proves (6.10). Together, (6.9) and (6.10) show that $(\rho_t)_{t>0}$ has property (CKW$t^{\gamma'}$) for all $\gamma' > \gamma/(\gamma-1)$, proving the third assertion of Theorem 6.7.

Remark. — Theorem 6.7 is not entirely satisfactory. Indeed, it is natural to ask whether or not as long as $(\rho_t)_{t>0}$ satisfies (CK), we have

$$\forall t \in (0,1), \quad \rho_t(e) \approx \overline{\rho}_t(\overline{e})$$

without further hypothesis on $(\rho_t)_{t>0}$. In particular, a Fourier transform argument gives such a result in the abelian case, see [6, Proposition 6.1]. However, a general result seems out of reach if one uses Gaussian bound techniques to prove (6.10) as above.

7. Examples

7.1. The Iwasawa example G_ρ

Consider the infinite dimensional torus $\mathbb{T}^\infty = (\mathbb{R}/\mathbb{Z})^\infty$ and the topological space $G = \mathbb{R} \times \mathbb{R} \times \mathbb{T}^\infty$. Elements of this space are denoted by (x, y, θ) where x, y are reals and $\theta = (\theta_1, \theta_2, \ldots) \in \mathbb{T}^\infty$. Let us also denote by p the canonical projection from $\mathbb{R}$ to $\mathbb{T}$ and by p_∞ the projection from $\mathbb{R}^\infty$ to $\mathbb{T}^\infty$.

For $g = (x, y, \theta)$, $g' = (x', y', \theta') \in G$, define the product

$$gg' = (x + x', y + y', \theta + \theta' + p_\infty((xy' - x'y)\rho/2))$$

where $\rho = (\rho_1, \rho_2, \ldots)$ is a vector in $\mathbb{R}^\infty$. This product is compatible with the topology of G and turns G into a locally compact connected locally connected metric group. Note that the group $G = G_\rho$ actually depends on the choice of ρ. This example appears in Iwasawa's work [17] and in [9]. Let $\mathcal{G}$ denote the Lie algebra of G. As a vector space, $\mathcal{G}$ can be identified with $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^\infty$ where the first line corresponds to $\frac{\partial}{\partial x}$ at the identity, the second line to $\frac{\partial}{\partial y}$ and the i-th line in $\mathbb{R}^\infty$ corresponds to $\frac{\partial}{\partial \theta_i}$. Here, we are viewing $\mathcal{G}$ as the tangent space at the origin. Let us now describe the left-invariant vector fields on G corresponding to these vectors. With obvious notation, for $g = (x, y, \theta)$ we have

$$\begin{aligned} X(g) &= \frac{\partial}{\partial x} - \frac{y}{2} \sum_i \rho_i \frac{\partial}{\partial \theta_i} \\ Y(g) &= \frac{\partial}{\partial x} + \frac{x}{2} \sum_i \rho_i \frac{\partial}{\partial \theta_i} \\ \Theta_i(g) &= \frac{\partial}{\partial \theta_i}. \end{aligned}$$

In particular, this allows us to compute the Lie bracket which is 0 except for

$$[X, Y] = \sum_i \rho_i \frac{\partial}{\partial \theta_i}.$$

From this it follows that the topological commutator group $[G,G]$ is the closure in $\mathbb{T}^\infty$ of $\{g = p_\infty(t\rho) : t \in \mathbb{R}\}$. Iwasawa [17, pg. 550-551] takes the coordinates ρ_i of ρ to be linearly independent over the rationals. This implies that $[G,G] = \mathbb{T}^\infty$, a fact that lets Iwasawa conclude that, in this case, G cannot split globally as a direct product of a Lie group and a compact group. Another interesting choice is $\rho = (1, 1/q, 1/q^2, \ldots)$ where $q \geqslant 2$ is an integer. In this case, $[G,G]$ is a q-adic solenoid in $\mathbb{T}^\infty$ and, repeating Iwasawa's argument [17], one sees that G does not split.

In [9], Berestowskii and Plaut use Iwasawa example to illustrate their result concerning the global splitting of a cover of a general locally compact connected locally connected group. Indeed, set $\overline{G} = L \times K$ where $L = \mathbb{H}$ is the Heisenberg group, that is $\mathbb{R}^3$ equipped with the product

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z' + (xy' - yx')/2)$$

and $K = \mathbb{T}^\infty$. Consider the map $\pi : \overline{G} \to G$ given by

$$\begin{aligned}\pi(x,y,z,\theta) = \; & (x, y, (p(z\rho_1), p(z\rho_2) \\ & +\theta_1, p(z\rho_3) + \theta_2, \ldots, p(z\rho_i) + \theta_{i-1}, \ldots)).\end{aligned} \tag{7.1}$$

Assuming that $\rho_1 \neq 0$, one can check (see [9]) that π is a surjective group homorphism. The kernel H of this projection is

$$H = \{(0, 0, k/\rho_1, -(p(k\rho_2/\rho_1), p(k\rho_3/\rho_1), \ldots) : k \in \mathbb{Z}\} \tag{7.2}$$

which is clearly a discrete subgroup of $\overline{G}$, that is $(\overline{G}, \pi)$ is a covering of G.

We now introduce more precise notation concerning the Lie algebras $\mathcal{G}$ and $\overline{\mathcal{G}}$ of G and $\overline{G}$, respectively. We first look at these two Lie algebras independently and then write down the identification provided by the map $d\pi$, assuming that $\rho_1 \neq 0$. As explained above, $\mathcal{G} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^\infty$. Denote by $(E_i)_1^\infty$ the canonical basis of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^\infty$ and, using a common abuse of notation, let also E_i denote the left-invariant vector field on G corresponding to $E_i \in \mathcal{G}$. Then we have

$$E_1 = X, \;\; E_2 = Y, \;\; E_3 = \Theta_1, \;\; E_4 = \Theta_2, \;\; \cdots$$

and

$$Z = [X,Y] = \sum_1^\infty \rho_i E_{i+2}.$$

For $\overline{\mathcal{G}}$, we have $\overline{\mathcal{G}} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^\infty$ (as vector spaces) with basis $(F_1, F_2, F_3, F_4, F_5, \ldots)$ where (F_1, F_2, F_3) is the natural basis of the Heisenberg Lie algebra. In particular $[F_1, F_2] = F_3$. The covering map π induces an isomorphism $d\pi$ between the Lie algebras $\overline{\mathcal{G}}$ and $\mathcal{G}$. To write down this isomorphism concretely, we need to observe that $(E_1, E_2, Z, E_4, E_5, \ldots)$ is another basis of $\mathcal{G}$. In this basis, the isomorphism $d\pi$ is simply given by

$$d\pi(F_1) = E_1,\ d\pi(F_2) = E_2,\ d\pi(F_3) = Z,\ d\pi(F_4) = E_4, \ldots.$$

We will use these facts and notation below.

7.2. Gaussian semigroups on $G = G_\rho$

Let $(\mu_t)_{t>0}$ be a symmetric Gaussian semigroup on G. By Theorem 2.4, there exists a unique symmetric Gaussian semigroup $(\overline{\mu}_t)_{t>0}$ on $\overline{G}$ such that

$$\mu_t(V) = \overline{\mu}_t(\pi^{-1}(V)),$$

for all $t > 0$ and all measurable sets V. As $\overline{G} = L \times K = \mathbb{H} \times \mathbb{T}^\infty$, we can project $(\overline{\mu}_t)_{t>0}$ to obtain a symmetric Gaussian semigroup on $K = \mathbb{T}^\infty$ which we denote by $(\overline{\mu}_t^K)_{t>0}$. This allows us to state the following conjecture.

Conjecture. — The short-time properties of $(\mu_t)_{t>0}$, $(\overline{\mu}_t)_{t>0}$, $(\overline{\mu}_t^K)_{t>0}$ are the same. More precisely, any of these symmetric Gaussian semigroups satisfy property (CK) if and only if the two others do and we have

$$\forall\, t \in (0,1), \quad \log \mu_t(e) \approx \log \overline{\mu}_t(e) \approx \log \overline{\mu}_t^K(e).$$

Note that the effect of taking the log in the last sequence of comparisons is to make the contribution of the Lie group factor irrelevant. We would like to stress that, at this stage, we are very far to have a proof of this conjecture. In [6, Cor. 6.3], the authors use a simple Fourier transform argument to treat a similar problem in the purely abelian case where $\overline{G} = \mathbb{R}^n \times \mathbb{T}^\infty$, $G = \mathbb{T}^n \times \mathbb{T}^\infty$. Even in this simpler situation, only the comparison $\log \mu_t(e) \approx \log \overline{\mu}_t(e)$ is known to hold whereas the part $\log \overline{\mu}_t(e) \approx \log \overline{\mu}_t^K(e)$ is an open problem.

In connection with this conjecture, it is interesting to describe more concretely the lifting from $(\mu_t)_{t>0}$ to $(\overline{\mu}_t)_{t>0}$ when $(\mu_t)_{t>0}$ is associated with the symmetric non-negative matrix $A = (a_{i,j})_{i,j}$ in the basis $E = (E_i)$ of $\mathcal{G}$. This means that the infinitesimal generator of $(\mu_t)_{t>0}$ is

$$-L = \sum a_{i,j} E_i E_j.$$

To lift this operator to $\overline{G}$, we need to change basis and use the basis

$$\widetilde{E} = (\widetilde{E}_i) = (E_1, E_2, Z, E_4, E_5, \ldots).$$

Let $\widetilde{A}=(\widetilde{a}_{i,j})$ be the matrix such that

$$-L=\sum \widetilde{a}_{i,j}\widetilde{E}_i\widetilde{E}_j.$$

Then, setting $\rho_{-1}=\rho_0=0$, we have

$$\widetilde{a}_{i,j}=\begin{cases} a_{i,j}-a_{3,j}\frac{\rho_{j-2}}{\rho_1}-a_{i,3}\frac{\rho_{i-2}}{\rho_1}+a_{3,3}\frac{\rho_{i-2}\rho_{j-2}}{\rho_1^2} & \text{if } i\neq 3,\ j\neq 3\\ a_{3,j}\frac{1}{\rho_1}-a_{3,3}\frac{\rho_{j-2}}{\rho_1} & \text{if } i=3,\ j\neq 3\\ a_{i,3}\frac{1}{\rho_1}-a_{3,3}\frac{\rho_{i-2}}{\rho_1} & \text{if } i\neq 3,\ j=3\\ a_{3,3}\frac{1}{\rho_1^2} & \text{if } i=j=3.\end{cases} \tag{7.3}$$

Now, the Gaussian semigroup $(\overline{\mu}_t)_{t>0}$ is exactly the Gaussian semigroup associated to the matrix $\widetilde{A}$ in the basis $F=(F_i)$ of $\overline{\mathcal{G}}$, i.e., the Gaussian semigroup with generator

$$\sum \widetilde{a}_{i,j}F_iF_j.$$

Indeed, for any smooth cylindric function f on G, we have

$$\begin{aligned}\left(\sum \widetilde{a}_{i,j}F_iF_j\right) f\circ\pi &= \left(\sum \widetilde{a}_{i,j}d\pi(F_i)d\pi(F_j)f\right)\circ\pi\\ &= \left(\sum \widetilde{a}_{i,j}\widetilde{E}_i\widetilde{E}_jf\right)\circ\pi=\left(\sum a_{i,j}E_iE_jf\right)\circ\pi.\end{aligned}$$

Our next result shows that the conjecture formulated above holds true at least for a subclass of symmetric Gaussian semigroups, namely, those which lift to a product Gaussian semigroup on $\overline{G}=L\times K=\mathbb{H}\times\mathbb{T}^\infty$.

THEOREM 7.1. — *Referring to the notation introduced above, assume that the symmetric Gaussian semigroup $(\overline{\mu}_t)_{t>0}$ has the form*

$$\overline{\mu}_t=\overline{\mu}_t^L\otimes\overline{\mu}_t^K$$

where $(\overline{\mu}_t^L)_{t>0}$ is a symmetric Gaussian semigroup on $L=\mathbb{H}$ and $(\overline{\mu}_t^K)_{t>0}$ is a symmetric Gaussian semigroup on $K=\mathbb{T}^\infty$. Then $(\mu_t)_{t>0}$ satisfies (CK) *if and only if $(\overline{\mu}_t)_{t>0}$ does and*

$$\forall t\in(0,1),\quad \log\mu_t(e)\approx\log\overline{\mu}_t(e)\approx\log\overline{\mu}_t^K(e).$$

Proof. — Recall the well known general fact (see, e.g., [8, Lem. 3.3]) that $(\mu_t)_{t>0}$ is absolutely continuous with respect to Haar measure if and only if $(\overline{\mu}_t)_{t>0}$ is. Hence, things boil down to the comparisons between $\mu_t(e)$, $\overline{\mu}_t(e)$ and $\overline{\mu}_t^K(e)$. In particular, in all cases of interest for this theorem, $(\overline{\mu}_t^L)_{t>0}$ has a smooth positive density bounded above by

$$\overline{\mu}_t^L((x,y,z))\leqslant C\overline{\mu}_t^L(e)\exp(-c(|x|^2+|y|^2+|z|))$$

for all $t \in (0,1)$ and all $(x,y,z) \in L = \mathbb{H}$, for some constants $C, c > 0$. See, e.g., [28] (the absence of square in $|z|$ in this formula is not a typo).

As noticed earlier, we always have

$$\mu_t(e) \geqslant \overline{\mu}_t(e) = \overline{\mu}_t^L(e)\overline{\mu}_t^K(e).$$

and

$$\mu_t(e) = \sum_{h \in H} \overline{\mu}_t(h)$$

where H is the kernel of π given at (7.2). Hence, for $t \in (0,1)$,

$$\begin{aligned} \mu_t(e) &= \sum_{k \in \mathbb{Z}} \overline{\mu}_t^L((0,0,k/\rho_1))\overline{\mu}_t^K(-p_\infty((k\rho_2/\rho_1, k\rho_3/\rho_1, \ldots))) \\ &\leqslant C\overline{\mu}_t^L(e)\overline{\mu}_t^K(e) \sum_{k \in \mathbb{Z}} \exp\left(-c\frac{|k|}{\rho_1}\right) \\ &\leqslant C_1\overline{\mu}_t^L(e)\overline{\mu}_t^K(e). \end{aligned}$$

Thus, we have

$$\mu_t(e) \approx \overline{\mu}_t(e) = \overline{\mu}_t^L(e)\overline{\mu}_t^K(e).$$

Given that the behavior of $\overline{\mu}_t^L(e)$ is under control, this suffices to prove Theorem 7.1.

Next we apply this result to give very explicit examples of behaviors of symmetric Gaussian semigroups on $G = G_\rho$. The proof is a direct application of (7.3), Theorem 7.1 and the results of [1], [2].

THEOREM 7.2.— *Fix a sequence $\mathbf{a} = (a_i)_1^\infty$ of non-negative numbers. Assume that $a_3 = 0$ but $a_i > 0$ for $i \neq 3$. Set $A = (a_{i,j})$ with $a_{i,i} = a_i$ and $a_{i,j} = 0$ if $i \neq j$. Set also*

$$N(s) = \#\{i : a_i < s\}.$$

Let $(\mu_t)_{t>0}$ be the symmetric Gaussian semigroup on G whose infinitesimal generator is $-L = \sum_i a_i E_i^2$. Then we have:

(i) The following properties are equivalent

1. *The Gaussian semigroup $(\mu_t)_{t>0}$ is absolutely continuous with respect to Haar measure.*
2. *The Gaussian semigroup $(\mu_t)_{t>0}$ satisfies* (CK).
3. $\lim_{s\to\infty} s^{-1}\log N(s) = 0.$

(ii) The Gaussian semigroup $(\mu_t)_{t>0}$ *satisfies* (CK$*$) *if and only if*

$$\lim_{s\to\infty} s^{-1}N(s) = 0.$$

(iii) If $N \approx \psi$ *at infinity and* ψ *is a regularly varying function of index* $\lambda > 0$ *then*

$$\forall t \in (0,1), \quad \log \mu_t(e) \approx \psi(1/t).$$

(iv) If $N \approx \psi$ *at infinity and* ψ *is slowly varying then*

$$\forall t \in (0,1), \quad \log \mu_t(e) \approx \psi^{\#}(1/t) = \int_{1/2}^{1/t} \psi(s)\frac{ds}{s}.$$

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