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Riemann existence theorem and construction of real algebraic curves (*)

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Abstract. — We propose a method of construction of plane real algebraic curves given by $y^3+p(x)y+q(x)=0$ which has a prescribed arrangement on the affine plane. The construction is based on a consideration of the arrangement of $f^{-1}(\mathbf{RP}^1)$ on $\mathbf{CP}^1$ where $f:\mathbf{CP}^1\to\mathbf{CP}^1$ is the homogenized discriminant, i.e. the rational function defined by $f(x)=D/q^2$, $D=4p^3+27q^2$.

As examples of applications, we construct some M-curves of degrees 7 and 9 on $\mathbf{RP}^2$ whose realizability was unknown.

Résumé. — On propose une méthode de construction des courbes algébriques réelles planes données par $y^3+p(x)y+q(x)=0$, qui ont un arrangement prescrit sur le plan affine. La construction est basée sur la considération de l'arrangement de $f^{-1}(\mathbf{RP}^1)$ sur $\mathbf{CP}^1$, où $f:\mathbf{CP}^1\to\mathbf{CP}^1$ est le discriminant homogénéisé, i.e. la fonction rationnelle définie par $f(x)=D/q^2$, $D=4p^3+27q^2$.

Comme exemple d'applications, on construit certaines M-courbes de degrés 7 et 9 sur $\mathbf{RP}^2$ dont la réalisabilité n'était pas connue.

1. Introduction

In this paper we propose a method of construction of plane real algebraic curves given by $F(x,y)=0$, $\deg_y F=3$ (trigonal curves) which have a prescribed arrangement on the affine plane. This method allows one to obtain a complete classification of such curves (singular or not) up to fiberwise isotopies of the plane (we call an isotopy *fiberwise* if the image of any

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vertical line at any moment is a vertical line). In particular, this means that the same method may provide some restrictions for trigonal curves.

This result will be published in the next paper. Here we just illustrate the method of construction by a realization of a complex M-scheme[(2)] of degree 7 and some real M-schemes of degree 9 on $\mathbf{RP}^2$ whose realizability was previously unknown.

The proposed method was inspired by a construction of extremal polynomials for the Davenport's bound $\deg(p^3 - q^2) \geqslant 1 + (\deg p)/2$ in terms of so-called "dessins d'enfant" (see Sect. 3).

PROPOSITION 1.1. — *There exists an M-curve of degree 9 on $\mathbf{RP}^2$ whose real scheme is $\langle J \sqcup 1\langle 8\rangle \sqcup 1\langle 14\rangle \sqcup 4\rangle$.*

Following [3], we say that a curve of degree 7 on $\mathbf{RP}^2$ has a *jump* if it contains 5 ovals arranged with respect to some line as it is shown in Figure 1.

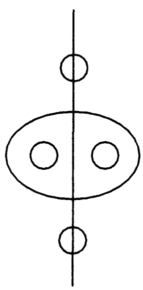

Figure 1

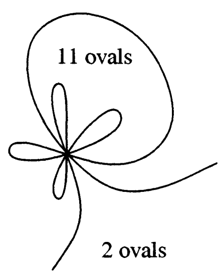

Figure 2

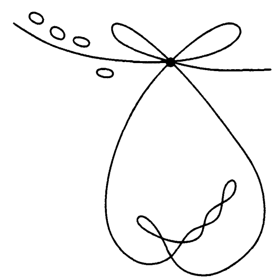

Figure 3

PROPOSITION 1.2. — *There exists an M-curve of degree 7 on $\mathbf{RP}^2$ without a jump whose complex scheme is $\langle J \sqcup 5_+ \sqcup 4_- \sqcup 1_+\langle 2_+ \sqcup 3_-\rangle\rangle$.*

Combined with the results of [13, 3, 9, 10, 4], Proposition 1.2 provides the classification of complex schemes of M-curves of degree 7 without a jump (for curves with jump, this classification is already completed in the papers cited above).

In Sect. 5 (Proposition 5.1), we prove the realizability of some other M-schemes of degree 9. All the curves in Proposition 5.1 are constructed by glueing affine sextics into a 6-fold singular point of a curve of degree 9.

(2) See [12] for the definition and notation of real and complex schemes.

We choose this example because the same glueing was used in [6] but our method allows us to obtain more curves of degree 9.

2. Preparation

To construct the curves from Propositions 1.1 and 1.2, we first construct singular curves depicted in Figures 2 and 3 and then perturb the singularities glueing (see [13]) the affine sextic depicted in Figure 4 (resp. quartic in Figure 5) into the 6-fold (resp. quadruple) point.

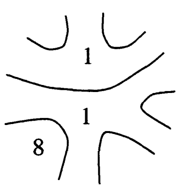

Figure 4 (see [6])

Figure 5

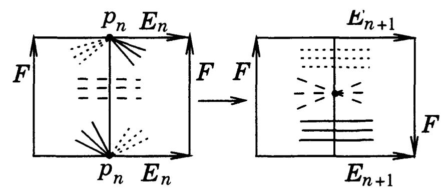

Figure 6

Denote by F_n the Hirzebruch surface of degree n and let E_n be the exceptional section (whose self-intersection is $-n$). In particular, $F_0 = \mathbf{P}^1{\times}\mathbf{P}^1$ and F_1 is the blown-up $\mathbf{P}^2$. Let $\pi_n : F_n \to \mathbf{P}^1$ be the fibration with fiber $\mathbf{P}^1$. The surfaces $F_1, F_2, \dots$ can be obtained from F_0 by successive birational transformations $F_0 \xrightarrow{\beta_0(p_0)} F_1 \xrightarrow{\beta_1(p_1)} F_2 \xrightarrow{\beta_2(p_2)} \dots$ where β_n is the blowup of a point $p_n \in E_n$ followed by the blowdown of the strict transform of the fiber $\pi_n^{-1}(\pi_n(p_n))$. If the points $p_0, p_1, \dots$ are real then all F_n are also real. Let us denote the set of real points of F_n by $\mathbf{R}F_n$.

We present $\mathbf{R}F_n$ in pictures as a rectangle obtained by cutting $\mathbf{R}F_n$ along E_n (horizontal edges) and a fiber (vertical edges). The interior of such a rectangle corresponds to an affine coordinate system on F_n where a generic smooth curve C is defined by a polynomial whose Newton polygon is $(0,0)$-$(l+kn,0)$-(l,k)-$(0,k)$ where k (resp. l) is the intersection of C with a fiber (resp. with E_n). We call (k,l) the *bidegree* of C. The action of β_n (for an even n) on $\mathbf{R}F_n$ is shown in Figure 6. We see in this picture that $\mathbf{R}F_n$ is a torus for an even n and a Klein bottle for an odd n.

Since $\mathbf{R}F_1$ is the blown-up $\mathbf{RP}^2$, the existence of the curve in Figure 2 (resp. Figure 3) follows from the existence of a curve of bidegree $(3,6)$ (resp. $(3,4)$) arranged on $\mathbf{R}F_1$ as it is shown in Figure 7 (resp. Figure 8).

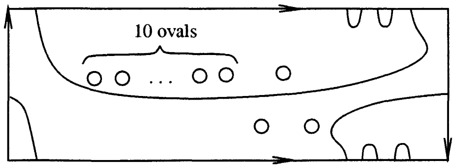

Figure 7

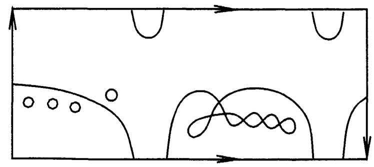

Figure 8

If a curve C of bidegree (k, l) on F_n has multiplicity m at the point p_n then the strict transform of C on F_{n+1} under $\beta_n(p_n)$ has bidegree $(k, l-m)$. Hence, applying $\beta_1(p_1) \circ \ldots \circ \beta_h(p_h)$ to the curve C in Figure 7 for $h = 6$ and $\{p_1 \ldots p_h\} = C \cap E_1$, we obtain the curve of bidegree $(3, 0)$ on F_7 whose real part is depicted in the upper part of Figure 9. Analogously, for $h = 4$, we obtain Figure 10 from Figure 8. The isolated points in Figure 9 and Figure 10 are simple double points with imaginary tangents (like $x^2 + y^2 = 0$). The curves in Figure 9 and Figure 10 will be constructed in Sect. 4.

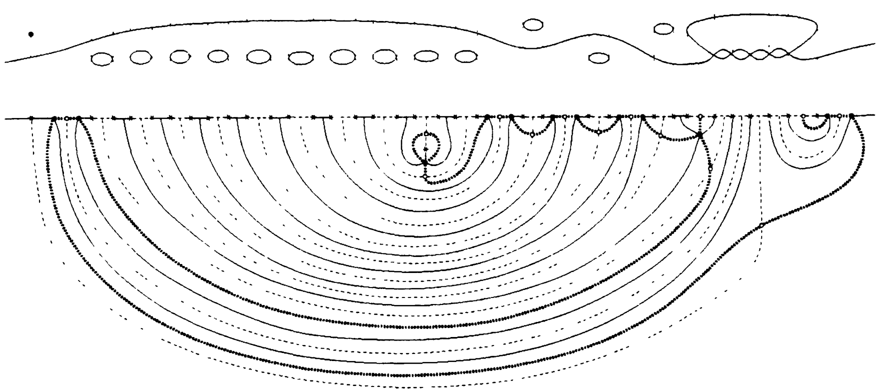

Figure 9

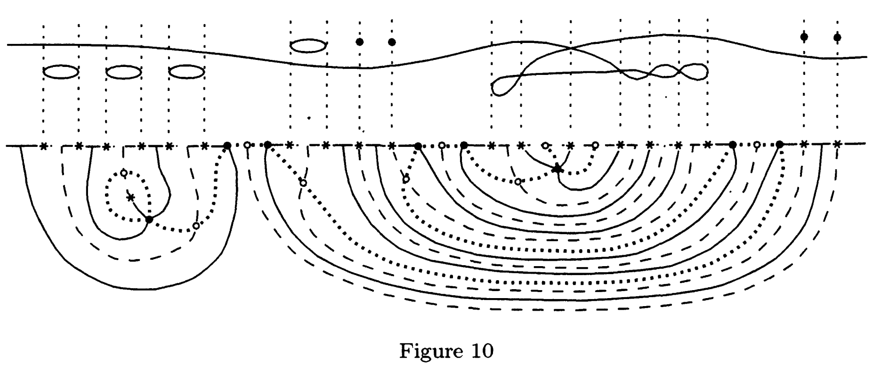

Figure 10

3. Degression

Let $p(t), q(t) \in \mathbf{C}[t]$, $\deg p = 2k$, $\deg q = 3k$ and set

$$r(t) = p^3 - q^2. \tag{3.1}$$

Suppose r is not identically zero. How small can be $\deg r$? This question was posed in [1] in 1965. The same year Davenport [2] had shown that $\deg r \geqslant k+1$ but it was unknown if the estimate is sharp. This estimate is very natural. Indeed, if we write $p = t^{2k} + a_1 t^{2k-2} + \ldots + a_{2k-1}$, $q = t^{3k} + b_1 t^{3k-2} + \ldots + b_{3k-1}$ with indeterminate coefficients then the condition $\deg r \leqslant k+1$ imposes $5k-2$ equations that is equal exactly to the number of the unknowns. However, it is very hard to show algebraically that this system has solutions other than $p = s^2$, $q = s^3$, $s = t^k + c_1 t^{k-2} + \ldots + c_{k-1}$.

Stothers [11] proved the sharpness of Davenport's estimate for any k. His result was rediscovered by Zannier [14]. A. Zvonkin gave another (?) elegant proof but he did not publish it because he claims that his proof coincides with the Zannier's one. However, he kindly permitted us to publish his proof and we do it in this section.

The main idea is to divide the both sides of (3.1) by q^2. Denote the obtained rational function by f. Then $f(t) = r/q^2 = p^3/q^2 + 1$. This means that

(*i*) f has $3k$ poles of multiplicity 2 at the roots of q,

(*ii*) the equation $f = 1$ has $2k$ triple roots at the roots of p,

and if $\deg r = k+1$ then

(*iii*) f has a zero of multiplicity $5k-1$ at $t = \infty$.

Conversely, any rational function f of degree $6k$ satisfying (*i*)–(*iii*), defines the required p and q.

From the topological point of view f is a branched covering $\mathbf{CP}^1 \to \mathbf{CP}^1$. Denote the preimage of the real segment $[1, +\infty]$ by Γ. If (*i*)–(*iii*) hold then Γ is a graph on $\mathbf{CP}^1$ whose vertices $f^{-1}(\infty)$ have valence 2 (white vertices) and the vertices $f^{-1}(1)$ have valence 3 (black vertices). The graph Γ cuts $\mathbf{CP}^1$ into polygons homeomorphic to a disc, one of which should have $5k-1$ white vertices and $5k-1$ black ones.

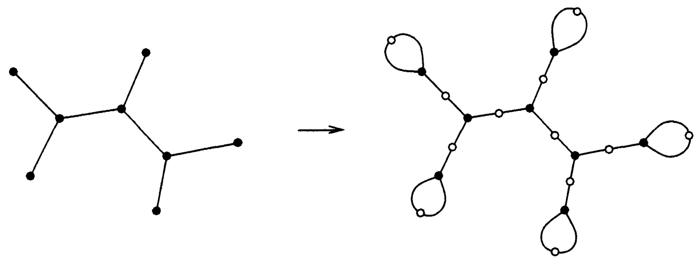

Figure 11

Now we are ready to construct f. Let us start with any binary tree in S^2 with $k-1$ triple vertices and $k+1$ ends and transform it to the graph Γ as it is shown in Figure 11. Define the mapping $\Gamma \to [1, \infty]$ which takes the white vertices to ∞ and the black vertices to 1, and extend it continuously to a mapping $f : S^2 \to \mathbf{CP}^1$ which maps each bigon homeomorphically onto $\mathbf{CP}^1 \setminus [1, \infty]$ and whose restriction onto the polygonal component of $S^2 \setminus \Gamma$ is a branched cyclic covering ramified in a single point t_0 such that $f(t_0) = 0$. Pull back the complex structure from $\mathbf{CP}^1$ to S^2. By Riemann's theorem, the obtained surface is isomorphic to $\mathbf{CP}^1$. Choose the isomorphism so that it takes t_0 into ∞. Than f becomes a rational function satisfying (i)–(iii).

Remark 3.1. — Given any k and coprime a, b, similar arguments allow to construct polynomials p, q, and $r = p^b - q^a$ such that $\deg p = ak$, $\deg q = bk$, $\deg r = (ab - a - b)k + 1$. Like is the case $a = 3$, $b = 2$, this is the minimal possible value for $\deg r$.

4. Construction

Let us construct a curve C of bidegree $(3, 0)$ on F_n ($n = 7$ or 5) whose real part is depicted in Figure 9 or Figure 10 respectively. It is defined by a polynomial whose Newton polygon is the triangle $(0,0)$-$(3n,0)$-$(0,3)$. Killing the coefficient of y^2, we rewrite C in the form

$$y^3 + p(x)y + q(x) = 0, \qquad \deg p = 2n, \quad \deg q = 3n. \tag{4.1}$$

The discriminant of (4.1) with respect to y is

$$D(x) = 4p^3 + 27q^2. \tag{4.2}$$

Let x_0 be a root of D ("$*$" in Figures 9 and 10). This means that either x_0 is the x-coordinate of a double point of C (then x_0 is a double root of D) or the vertical line $x = x_0$ is tangent to C (a simple root of D). Let

$F(y) = y^3 + p(x_0)y + q(x_0) = (y - y_1)(y - y_2)^2$. Since the coefficient of y^2 vanishes, y_1 and y_2 have opposite signs. Hence, $q(x_0) = F(0) > 0$ when $y_1 < y_2$, and $q(x_0) = F(0) < 0$ when $y_2 < y_1$. This means that the real roots of q ("$\circ$" in Figures 9 and 10) must separate the root of D where $y_1 < y_2$ from those where $y_2 < y_1$. Thus, to construct C, we need to find polynomials $p(x)$, $q(x)$, and $D(x)$, satisfying (4.1), (4.2) such that the real roots of D and q are arranged as in Figures 9 and 10.

Figure 12

Now we apply the main idea of Sect. 3: let us divide (4.2) by q^2. The result is a rational function $f(x) = D/q^2 = 4p^3/q^2 + 27$ whose poles are the roots of q taken with multiplicity 2, zeros are the roots of D, and the solutions of $f = 27$ are the roots of p taken with multiplicity 3. To construct f, consider the graph $\Gamma \subset S^2$ depicted in the lower parts of Figures 9 and 10 (since Γ is symmetric, we show only half of it). Let us map Γ onto $\mathbf{RP}^1$ according to the coloring in Figure 12 and then continue this mapping up to a branched covering $f : S^2 \to \mathbf{CP}^1$ sending homeomorphically each component of $S^2 \setminus \Gamma$ onto one of the half-planes of $\mathbf{CP}^1 \setminus \mathbf{RP}^1$ in an alternating order. The additional vertices (which are not mapped onto 0, 27, or ∞) are mapped to arbitrarily chosen points on the corresponding segments of $\mathbf{RP}^2$. Then the pull-back of the complex structure makes f to be a rational function which has the needed properties. Due to the symmetry principle, f becomes real in suitable coordinates.

In conclusion of this section, let us give a list of conditions on a colored embedded graph $\Gamma \subset S^2$ which are sufficient to construct a curve of bidegree $(3, 0)$ on F_n (these conditions are satisfied in the constructions of Section 5).

(1) The graph Γ is symmetric with respect to an equator of S^3 (which is included to Γ) and the coloring of the equator is imposed by the desired arrangement of the real algebraic curve as it is explained above;

(2) The valence of each "$\bullet$" is divisible by 6, and the incident edges are colored alternatively by the colors of the segments $[0, 27]$ and $[27, \infty]$;

(3) The valence of each "$\circ$" is divisible by 4, and the incident edges are colored alternatively by the colors of the segments $[27, \infty]$ and $[\infty, 0]$;

(4) The valence of each "$*$" is even, and the incident edges are colored alternatively by the colors of the segments $[\infty, 0]$ and $[0, 27]$;

(5) The valence of each non-colored vertex is even, and the incident edges are of the same color;

(6) The sum of the valences of all "$*$"-vertices is equal to $12n$ (together with the conditions (2)–(5), this implies that the sums of the valences of all "$\circ$"- and "$\bullet$"-vertices are also equal to $12n$);

(7) Each connected component of $S^3 \setminus \Gamma$ is homeomorphic to an open disk whose boundary (considered as the set of Carathéodory boundary elements) is colored as a covering of $\mathbf{RP}^1$. Moreover, the orientations of neighbouring disks induced by the coverings of their boundaries are opposite.

5. Other M-curves of degree 9

PROPOSITION 5.1. — *(a) There exist M-curves of degree 9 whose real schemes are*

2) $\langle J \sqcup \alpha \sqcup 1\langle\beta\rangle\rangle$, $\alpha = 27 - \beta$, $\beta = 17^*, 20, 21$

3) $\langle J \sqcup \alpha \sqcup 1\langle\beta\rangle \sqcup 1\langle\gamma\rangle\rangle$, $\alpha = 26 - \beta - \gamma$ *where*
$\beta = 1,\ \gamma = 20, 21$
$\beta = 2,\ \gamma = 11^*, 12^*, 13, 15, 16^*, 17, 19, 20$
$\beta = 3,\ \gamma = 14, 17$
$\beta = 4,\ \gamma = 9, 11, 13, 14, 17, 18$
$\beta = 5,\ \gamma = 12, 13^*, 14, 15, 16, 17$
$\beta = 6,\ \gamma = 11, 12, 14$
$\beta = 7,\ \gamma = 14, 15$
$\beta = 8,\ \gamma = 9, 11, 12, 13, 14$
$\beta = 9,\ \gamma = 9, 10, 11$

4) $\langle J \sqcup \alpha \sqcup 1\langle\beta\rangle \sqcup 1\langle\gamma\rangle \sqcup 1\langle\delta\rangle\rangle$, $\alpha = 25 - \beta - \gamma - \delta$ *where*
$(\beta,\gamma) = (1,1),\ \delta = 8, 12, 15, 16, 17, 22$
$(\beta,\gamma) = (1,3),\ \delta = 13, 14, 15, 16$
$(\beta,\gamma) = (1,5),\ \delta = 8, 12$
$(\beta,\gamma) = (1,6),\ \delta = 13$
$(\beta,\gamma) = (1,8),\ \delta = 9$
$(\beta,\gamma) = (1,9),\ \delta = 11, 12, 14$
$(\beta,\gamma) = (1,10),\ \delta = 13$
$(\beta,\gamma) = (2,3),\ \delta = 9$
$(\beta,\gamma) = (3,3),\ \delta = 8$
$(\beta,\gamma) = (3,5),\ \delta = 9$
$(\beta,\gamma) = (3,7),\ \delta = 8$
$(\beta,\gamma) = (4,5),\ \delta = 5$
$(\beta,\gamma) = (5,5),\ \delta = 8, 10$
$(\beta,\gamma) = (7,7),\ \delta = 10$

6) $\langle J \sqcup \alpha \sqcup 1\langle \beta \sqcup 1\langle \gamma \rangle\rangle\rangle$, $\alpha = 26 - \beta - \gamma$ *where*
$\gamma = 1$, $\beta = 1^*, 17, 22$
$\gamma = 3$, $\beta = 1^*, 4, 5^*, 8$

(b) There exist flexible[(3)] *M-curves of degree 9 whose real schemes are*

3) $\langle J \sqcup 3 \sqcup 1\langle 5\rangle \sqcup 1\langle 18\rangle\rangle$, *i.e.* $\beta = 5$, $\gamma = 18$

4) $\langle J \sqcup \alpha \sqcup 1\langle \beta\rangle \sqcup 1\langle \gamma\rangle \sqcup 1\langle \delta\rangle\rangle$, $\alpha = 25 - \beta - \gamma - \delta$ *where*
$(\beta, \gamma, \delta) = (1, 1, 19), (1, 7, 13), (5, 7, 9)$

6) $\langle J \sqcup \alpha \sqcup 1\langle \beta \sqcup 1\langle \gamma \rangle\rangle\rangle$, $\alpha = 26 - \beta - \gamma$ *where*
$\gamma = 5$, $\beta = 2, 3, 5, 8, 9$
$\gamma = 7$, $\beta = 1, 3, 4$

Remark 5.2. — The list of the real schemes in Proposition 5.1 is given is the same format as the list in [5; Theorem 6] (this explains, in particular, such a strange numbering of the series). We do not include here the M-schemes which are listed in [5] but we include those which are constructed in [6] (marked by the asterisk) and in Proposition 1.1.

Remark 5.3. — We found the following misprints in [5].

In [5; Theorem 6, Series 4], there should be "$(\beta, \gamma) = (5, 7)$, $\delta = 8, 10$" instead of "$(\beta, \gamma) = (5, 8)$, $\delta = 8, 10$".

In [5; Theorem 6, Series 5], there should be $(\alpha, \beta, \gamma, \delta) = (1, 1, 3, 15)$ instead of $(1, 1, 3, 13)$.

In [5; Theorem 7], "α, β, γ – even" in Series 4 means "each of α, β, γ is even" whereas "$\alpha, \beta, \gamma, \delta$ – even" in Series 6 means "one of α, β, γ is even".

We shall call *central blocks* the rectangles depicted in the upper parts of Figures 13.1–13.5 or their images under the reflection with respect to a vertical or a horizontal line.

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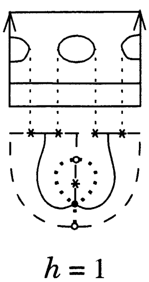
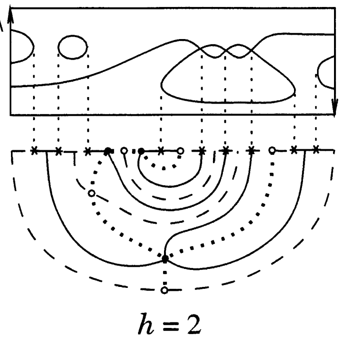
(3) See [12] for the definition of a flexible curve.





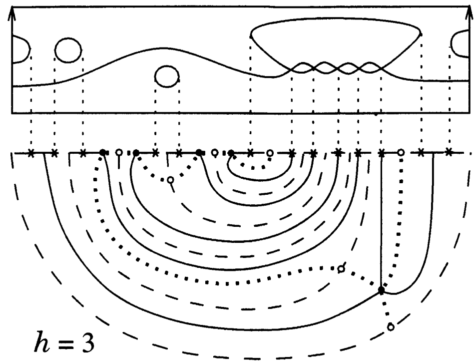
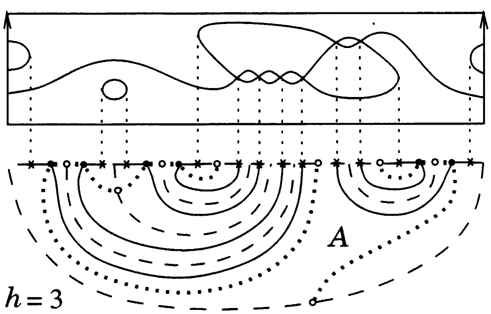


Figure 13.1

Figure 13.2

Figure 13.3



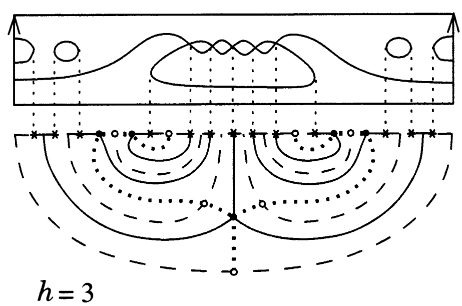
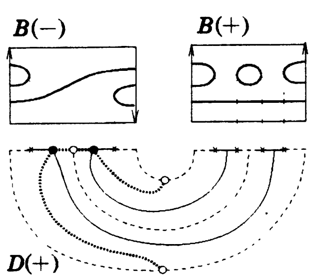


Figure 13.4

Figure 13.5



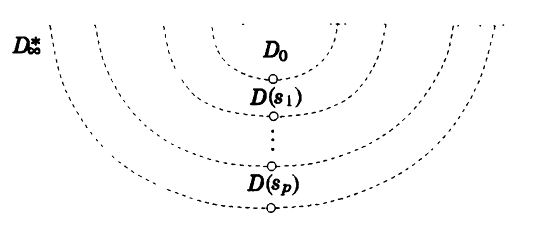


Figure 14

Figure 15

LEMMA 5.4. — *Let B_0, B_∞ be two central blocks and h_0, h_∞ the corresponding values of the parameter h (indicated in 13.1–13.5). Then for any sequence of signs $s_1, \dots, s_p$, $p \geqslant 0$, there exists a real algebraic curve C of bidegree $(3,0)$ on F_n, $n = p + h_0 + h_\infty$ such that the pair $(F_n \setminus E_n, C)$ (recall that $E_n \subset F_n$, $E_n^2 = -n$) is obtained (up to a fiberwise isotopy, see the Introduction) by the successive cyclic glueing (according to the arrows) of the blocks B_0, $B(s_1), \dots, B(s_p)$, B_∞, $B(-s_p), \dots, B(-s_1)$, where $B(+)$, $B(-)$ are shown in the upper part of Figure 14.*

Proof. — Let D_0 and D_∞ be the half-discs which are shown in Figures 13.1–13.5 under the blocks B_0 and B_∞, and let D_∞^* be the inversion image of D_∞. Let $D(+)$ be the half-annulus shown in the lower part of Figure 14 and $D(-)$ be its mirror image. Let us fill the lower half-plane by the domains $D_0, D(s_1), \dots, D(s_p), D_\infty^*$ according to Figure 15, and do symmetrically the upper half-plane. Let us deal with the obtained graph Γ in the same way as in Sect. 4 (in the case 13.4, to construct the mapping of the region A, one should take a double covering branched at a single point). □

Example. — We show in Figure 16 how the curve in Figure 9 can be obtained from the central blocks 13.1 and 13.3 by applying Lemma 5.4 followed by a contracting of an oval into an isolated double point (see Lemma 5.7 below).

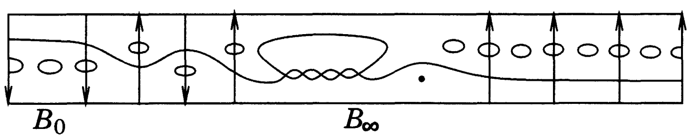


Figure 16

COROLLARY 5.5. — *Let $(s_1, \dots, s_p)$, $p \geqslant 1$, $s_i = \pm 1$, be an arbitrary sequence of signs such that $s_1 = 1$. Let $a_1, \dots, a_p$ be non-negative integers such that*

$$(1, s_1, \dots, s_p, 1, -s_p, \dots, -s_1) = (\underbrace{1, \dots, 1}_{a_1}, -1, \underbrace{1, \dots, 1}_{a_2}, -1, \dots, \underbrace{1, \dots, 1}_{a_p}, -1).$$

Then there exists a curve of bidegree $(3,0)$ on F_{p+2} arranges as in Figure 17 with $b_i = 2a_i + 1$, $i = 1, \dots, p$.

Proof. — Set $B_0 = B_\infty =$ [the block in Figure 13.1] in Lemma 5.4. □

COROLLARY 5.6. — *There exist curves of bidegree $(3,0)$ arranged on F_7 as in Figure 17 with $p = 5$ and $(b_1, \dots, b_5) = (15, 1, 1, 1, 1)$, $(11, 5, 1, 1, 1)$,* *(9, 5, 3, 1, 1), (9, 1, 7, 1, 1), (7, 5, 1, 1, 5), (7, 3, 5, 3, 1), (7, 1, 3, 1, 7), (5, 3, 5, 3, 3), or (5, 5, 5, 1, 3).* □

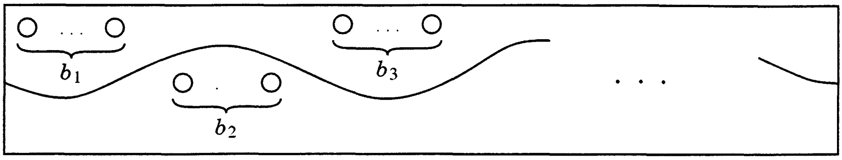


Figure 17

LEMMA 5.7. — *Let $A \subset F_{p+2}$ be a curve constructed in Lemma 5.4 (or in Corollaries 5.4 and 5.6). Then there exist a nodal curve $A' \subset F_{p+2}$ of the same bidegree obtained from A by applying of any number of transformations shown in Figure 18.*

Figure 18

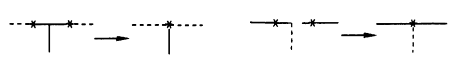


Figure 19

Proof. — Apply the transformations in Figure 19 to the graph Γ. □

Remark 5.8. — Corollary 5.5 can be easily proved by Viro's method using the subdivision of the triangle $(0,0)$-$(3p+6,0)$-$(0,3)$ into $n+2$ triangles $(3p,0)$-$(3p+3,0)$-$(0,3)$. However, it is not clear how to prove Lemma 5.7 in this way.

COROLLARY 5.9. — *There exist curves of degree 9 with a simple 6-fold singular point and 13 ovals distributed as in Figure 20 where a and b are the number of ovals in the corresponding regions, the exterior ovals are not shown, and $S_1 = \{(a,b) \mid a+b \leqslant 10$ and a, b are odd$\}$, $S_2 = S_1 \cup \{(1,11)\} \setminus \{(5,5)\}$.*

Proof. — Apply Lemma 5.7 and the transformation $\beta_1(p_1)\dots\beta_6(p_6)$ (see Sect. 2) to the curves from Lemma 5.4. The curves from Corollary 5.6 provide the upper two rows of Figure 20.1. In the other cases, one should choose the central blocks in Lemma 5.4 in the following way.

The curves in the lower row in Figure 20.1. The left: (13.1 and 13.3) or (13.1 and 13.5); the middle: (13.1 and 13.4); the right: (13.2 and 13.2).

The curves in Figure 20.2 and 20.3. The upper left curve in each of Figure 20.2 and 20.3: (13.1 and 13.4); the other curves: (13.1 and 13.2). □

Remark 5.10. — It is clear from the construction that any collection of the tangents at the singular point is realizable in the case of the left curve in the lower row of Figure 20.1 (it is marked by the asterisk). Unfortunately, in the other cases this is not so.

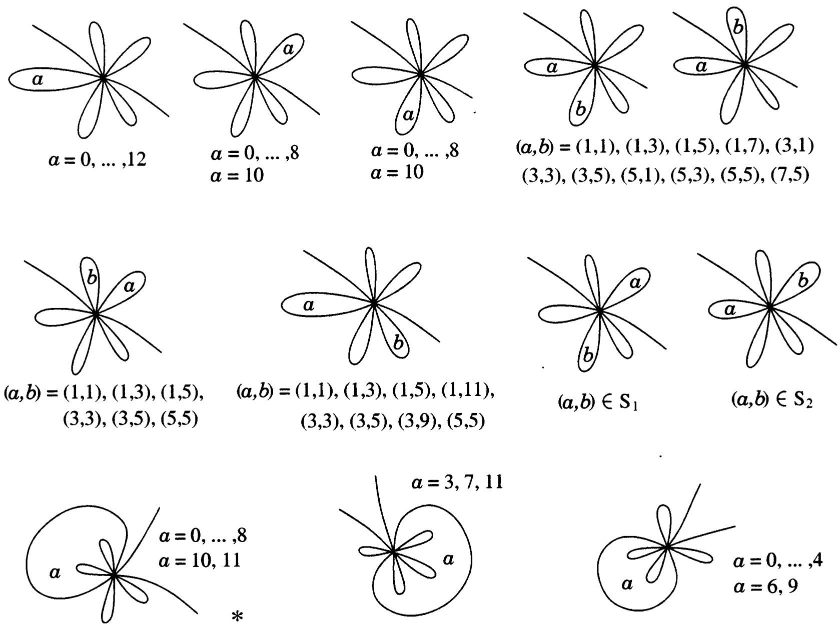


Figure 20.1

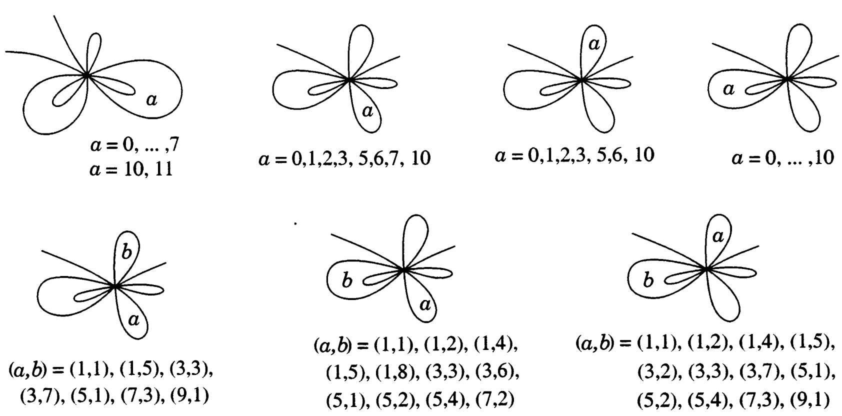


Figure 20.2

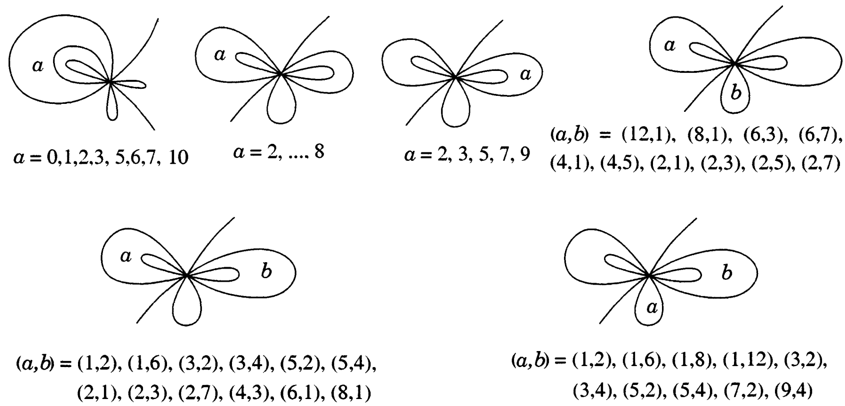


Figure 20.3

Remark 5.11. — Using the braid-theoretical methods (the Garside normal form of the braid from [7]), one can prove that Figure 20.1–20.3 contain all the isotopy types of curves of degree 9 which have a simple six-fold point and whose perturbation can provide an M-curve of degree 9.

Proof of Proposition 5.1. — (a). Maximal dissipation (see [13]) of a simple 6-fold singular point are described in [6]. Two more dissipations $B_2(1,8,1)$ and $A_3(0,5,5)$ are constructed in [8]. Applying all the dissipations of the series A (resp. B or C) to all the curves in Figure 20.1 (resp. Figure 20.2 or Figure 20.3) one obtains all the required algebraic curves.

(b). Flexible dissipations of the types $A_4(1,4,5)$, $B_2(1,4,5)$, and $C_2(1,3,6)$ are constructed in [7; Sect. 7.2]. Applying them to the singular point of the curves in Figure 20.1–Figure 20.3, one obtains all the required flexible curves.

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