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Removability of singularities of harmonic maps into pseudo-Riemannian manifolds [(*)]

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ABSTRACT. — We consider harmonic maps into pseudo-Riemannian manifolds. We show the removability of isolated singularities for continuous maps, i.e. that any continuous map from an open subset of $\mathbb{R}^m$ into a pseudo-Riemannian manifold which is two times continuously differentiable and harmonic everywhere outside an isolated point is actually smooth harmonic everywhere.

RÉSUMÉ. — Nous considérons des applications harmoniques à valeurs dans une variété pseudo-riemannienne. Nous démontrons l'absence de singularités isolées pour les applications harmoniques continues, à savoir plus précisément que toute application harmonique entre un ouvert de $\mathbb{R}^m$ et une variété pseudo-riemannienne continuement différentiable deux fois et harmonique en dehors d'un point isolé est en fait régulière partout.

1. Introduction

Given $n \in \mathbb{N}^*$ and two nonnegative integers p and q such that $p+q=n$, a pseudo-Riemannian manifold $(\mathcal{N}, h)$ of dimension n and of signature (p,q) is a smooth n-dimensional manifold $\mathcal{N}$ equipped with a pseudo-Riemannian metric h, i.e. a section of $T^*\mathcal{N} \odot T^*\mathcal{N}$ (where $\odot$ is the symmetrised tensor product), such that $\forall M \in \mathcal{N}$, h_M is a non degenerate bilinear form of signature (p,q). Any local chart $\phi : \mathcal{N} \supset U \longrightarrow V \subset \mathbb{R}^n$ allows us to use local coordinates $(y^1, \cdots, y^n) \in V$: we then denote by $h_{ij}(y) := h_{\phi^{-1}(y)} \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right)$.

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We say that $(\mathcal{N}, h)$ is of class $\mathcal{C}^k$ if and only if h_{ij} is $\mathcal{C}^k$. We define the Christoffel symbol by

$$\Gamma^i_{jk}(y) := \frac{1}{2}h^{il}(y)\left(\frac{\partial h_{lk}}{\partial y^j}(y) + \frac{\partial h_{jl}}{\partial y^k}(y) - \frac{\partial h_{jk}}{\partial y^l}(y)\right),$$

where, as a matrix, (h^{ij}) is the inverse of (h_{ij}). Then for any open subset Ω of $\mathbb{R}^m$ and for any $\mathcal{C}^2$ map u from Ω to $\mathcal{N}$, if we note

$$u \simeq \phi \circ u = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} \quad \text{and} \quad \Gamma_{jk} := \begin{pmatrix} \Gamma^1_{jk} \\ \vdots \\ \Gamma^n_{jk} \end{pmatrix}$$

and if we set $\Delta u := \sum_{\alpha=1}^m \frac{\partial^2 u}{(\partial x^\alpha)^2}$ and $\Gamma(u)(\nabla u \otimes \nabla u) := \sum_{\alpha=1}^m \Gamma_{jk}(u)\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\alpha}$, we say that u is *harmonic into* $(\mathcal{N}, h)$ if and only if

$$\Delta u + \Gamma(u)(\nabla u \otimes \nabla u) = 0. \tag{1.1}$$

Equivalently we may say that u is a critical point of

$$\mathcal{A}[u] := \int_\Omega h_{ij}(u(x)) \sum_{\alpha=1}^m \frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\alpha} dx^1 \cdots dx^m$$

among $\mathcal{C}^2$ maps from Ω to $\mathcal{N}$. The purpose of this paper is to prove the following result.

THEOREM 1.1. — *Let $(\mathcal{N}, h)$ be a pseudo-Riemannian manifold of class $\mathcal{C}^2$, Ω an open subset of $\mathbb{R}^m$, where $m \geqslant 2$, $a \in \Omega$ and u a map from Ω to $\mathcal{N}$ such that*

- *u is continuous*
- *u is $\mathcal{C}^2$ and harmonic on $\Omega \setminus \{a\}$*

Then u is $\mathcal{C}^2$ and harmonic on Ω.

Such a result would be a consequence of standard results if the map u had a finite energy and if $(\mathcal{N}, h)$ was Riemannian: indeed one could prove then that u is weakly harmonic (because the capacity of a point vanishes) and obtain the same conclusion by using the continuity of u, thanks to results in [8] and [7] (with present form due to S. Hildebrandt). In dimension 2 the same *finite energy* and *Riemannian target* hypotheses lead to the same conclusion but without using the fact that u is continuous as proved in [9]. However the difficulty here comes from the fact that the target manifold is pseudo-Riemannian. In particular even if we would assume that the map u had a finite energy, it would not help much.

This result answers a question posed by F. Pedit. It is related to the construction of spectral curves associated to any torus in the sphere S^4, a work in progress by F. Burstall, D. Ferus, K. Leschke, F. Pedit and U. Pinkall (see [2] for an exposition of these ideas). Using Theorem 1.1 these authors are able to prove various results about Willmore surfaces (recall that the right notion of Gauss maps for Willmore surfaces is the *conformal Gauss map* which takes values into a pseudo-Riemannian homogeneous manifold, see e.g. [1], [5] or [2]).

Note that in the hypotheses of the Theorem it is necessary to assume that u is continuous. An instance is the map $x \longmapsto (\cos\log|x|, \sin\log|x|, 0)$ from the unit disk in $\mathbb{R}$ to the unit sphere S^2 in $\mathbb{R}^3$: it is smooth harmonic outside 0, but is discontinuous at the origin. Also an interesting question would be to know whether it is possible to replace the singular point by a codimension 2 submanifold: our proof does not generalize obviously to such a case, since in our method we need to enclose the singular set by arbitrary small hypersurfaces on which the map is smooth.

Comments on the proof. — Our proof is based on applications of the maximum principle. The strategy consists roughly of the following: on the one hand we construct a smooth harmonic map which agrees with the initial one on the boundary of a small ball centered at the singularity, on the other hand we prove a uniqueness theorem for harmonic maps which takes values in a neighbourhood of a point. The uniqueness result follows from the maximum principle Theorem 5.1 which is inspired by [6] (see also [4]). This reduces the uniqueness problem to an estimate on solutions of elliptic linear PDE's on a punctured domain, given in Lemma 6.1, the result where we exploit the fact that the capacity of a point vanishes. The existence result is obtained through a fixed point argument in Hölder spaces in Theorem 4.2. However in this result we need a uniform estimate in the Hölder topology of the inital harmonic map. This is the subject of our key result, Theorem 3.1, where the uniform Hölder continuity is established by using the maximum principle. In the course of this paper we give the proof of more or less standard results for the convenience of the reader: Lemma 2.1 is classical in Riemannian geometry, Lemmas 4.1 and 6.1 are certainly well-known to specialists but I did not find proofs of them in the litterature.

2. Preliminaries

2.1. An adaptated coordinate system

LEMMA 2.1. — *Let $(\mathcal{N}, h)$ be a pseudo-Riemannian manifold of class $\mathcal{C}^2$. Let M_0 be a point in $\mathcal{N}$, U be an open subset of $\mathcal{N}$ which contains M_0 and $\psi : U \longrightarrow W \subset \mathbb{R}^n$ a local $\mathcal{C}^2$ chart. Then there exists an open neighbourhood $U_0 \subset U$ of M_0 with the following property (denoting by $W_0 := \psi(U_0)$): $\forall M \in U_0$ there exists a smooth diffeomorphism $\Phi_M : W_0 \longrightarrow V_M \subset \mathbb{R}^n$ such that in the local coordinates $y \simeq \phi := \Phi_M \circ \psi \in V_M$,*

$$\phi(M) = 0, \tag{2.1}$$

$$h_{ij}(0) = \eta_{ij} := diag(1, \cdots, 1, -1, \cdots, -1), \tag{2.2}$$

$$\Gamma^i_{jk}(0) = 0, \tag{2.3}$$

and there exists constants $C_W, C_\Gamma > 0$ independent of $M \in U_0$ such that

$$\forall M \in U_0, \quad ||d\Phi_M||_{L^\infty(W_0)} + ||d\Phi_M^{-1}||_{L^\infty(V_M)} \leqslant C_W \tag{2.4}$$

and

$$||d\Gamma||_{L^\infty(V_M)} \leqslant C_\Gamma. \tag{2.5}$$

Proof. — We denote by $z = (z^1, \cdots, z^n) \in W$ the coordinates in the local chart $\psi : U \longrightarrow W$ and by $\widetilde{h}_{ij}(z) := h_{\psi^{-1}(z)}\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right)$ the expression of the metric. We look at a neighbourhood W_0 of $\psi(M_0)$ such that for all $M \in \psi^{-1}(W_0)$ there exists a map $\Phi_M : W_0 \longrightarrow \mathbb{R}^n$ such that, denoting by $z_M := \psi(M)$,

$$\forall y \in \mathbb{R}^n, \forall z \in W_0, \; y = \Phi_M(z) \quad \Longleftrightarrow \quad z^i - z^i_M = A^i_j y^j + \frac{1}{2} B^i_{jk} y^j y^k,$$

for some invertible matrix $A = (A^i_j) \in GL(n, \mathbb{R})$ and real coefficients B^i_{jk} satisfying $B^i_{jk} = B^i_{kj}$. This map is well-defined if we choose W_0 to be a sufficiently small neighbourhood of z_M. Let us compute the expression $h_{ij}(y)$ of the metric in the coordinates $y^i \simeq T^i_M \circ \psi$ in terms of $\widetilde{h}_{ij}(z)$:

$$h_{kl}(y) = \widetilde{h}_{ij}(z(y)) \frac{\partial z^i}{\partial y^k} \frac{\partial z^j}{\partial y^l} = \widetilde{h}_{ij}(z(y)) \left(A^i_k + B^i_{kp} y^p\right) \left(A^j_l + B^j_{lq} y^q\right).$$

In order to achieve (2.2) it suffices to choose A^j_j such that $\widetilde{h}_{ij}(z_M) A^i_k A^j_l = \eta_{kl}$ (requiring also that A is symmetric and positive definite ensures uniqueness). Next we compute that:

$$\frac{\partial h_{kl}}{\partial y^p}(0) = \frac{\partial \widetilde{h}_{ij}}{\partial z^r}(z_M) A^r_p A^i_k A^j_l + \widetilde{h}_{ij}(z_M) \left(B^i_{kp} A^j_l + A^i_k B^j_{lp}\right), \quad \forall k, l, p.$$

And we deduce Γ^i_{jk} in function of $\widetilde{\Gamma}^i_{jk} := \frac{1}{2}\widetilde{h}^{il}\left(\frac{\partial \widetilde{h}_{lk}}{\partial z^j} + \frac{\partial \widetilde{h}_{jl}}{\partial z^k} - \frac{\partial \widetilde{h}_{jk}}{\partial z^l}\right)$ at 0:

$$\Gamma^i_{jk}(0) = \eta^{il}\left(\widetilde{h}_{pq}(z_M)\widetilde{\Gamma}^p_{rs}(z_M)A^q_l A^r_j A^s_k + \widetilde{h}_{pq}(z_M)A^p_l B^q_{jk}\right).$$

Since A^p_l and $\widetilde{h}_{pq}(z_M)$ are of rank m and thanks to the relation $\widetilde{\Gamma}^p_{rs} = \widetilde{\Gamma}^p_{sr}$ we deduce that there exist unique coefficients B^i_{jk} satisfying $B^i_{jk} = B^i_{kj}$ such that (2.3) holds. Since Φ_M depends analytically on $\widetilde{h}_{ij}(z_M)$ and $\widetilde{\Gamma}^i_{jk}(z_M)$, Condition (2.4) is obtained by choosing W_0 sufficiently small. Then (2.5) is a consequence of (2.4). □

2.2. Notations

In the next two sections we will use spaces of Hölder continuous functions and of functions with higher derivatives which are Hölder continuous. We first recall some notations and results from [3]. For any point $x \in \mathbb{R}^m$ or $y \in \mathbb{R}^n$ we note $|x| := \sqrt{\sum_{\mu=1}^m (x^\mu)^2}$ and $|y| := \sqrt{\sum_{i=1}^n (y^i)^2}$.

For any $\alpha \in (0,1)$ and for all open subset $\Omega \subset \mathbb{R}^m$, we define $\mathcal{C}^{0,\alpha}(\Omega)$ to be the set of functions f on Ω which are α-Hölder continuous on Ω, i.e. such that

$$\text{for all compact } K \subset \Omega, \quad \sup_{x,y\in K, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

For $k \in \mathbb{N}$, we let $\mathcal{C}^{k,\alpha}(\Omega)$ to be the set of $\mathcal{C}^k$ functions f on Ω such that $D^k f \in \mathcal{C}^{0,\alpha}(\Omega)$. Here $\forall \mu = (\mu_1, \cdots, \mu_m) \in \mathbb{N}^m$ we write $|\mu| := \mu_1 + \cdots + \mu_m$ and

$$D^k f := \left(\frac{\partial^k f}{(\partial x^1)^{\mu_1}\cdots(\partial x^m)^{\mu_m}}\right)_{\mu_1,\cdots,\mu_m\in\mathbb{N}, |\mu|=k}.$$

We also will use the notation

$$|D^k f| := \sum_{\mu_1,\cdots,\mu_m\in\mathbb{N}, |\mu|=k} \left|\frac{\partial^k f}{(\partial x^1)^{\mu_1}\cdots(\partial x^m)^{\mu_m}}\right|.$$

For $x, y \in \Omega$, we denote by $d_x := \mathrm{dist}(x, \partial\Omega)$ and $d_{x,y} := \min(d_x, d_y)$ and if $\alpha \in (0,1)$, $\beta \in \mathbb{R}$, $k \in \mathbb{N}$, we set

$$\begin{aligned}
[u]^{(\beta)}_{k,0;\Omega} &:= [u]^{(\beta)}_{k;\Omega} := \sup_{x\in\Omega} d_x^{\beta+k}|D^k u(x)| \\
|u|^{(\beta)}_{k,0;\Omega} &:= |u|^{(\beta)}_{k;\Omega} := \sum_{j=1}^k [u]^{(\beta)}_{j;\Omega} \\
[u]^{(\beta)}_{k,\alpha;\Omega} &:= \sup_{x,y\in\Omega} d_{x,y}^{\beta+k+\alpha}\frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha} \\
|u|^{(\beta)}_{k,\alpha;\Omega} &:= |u|^{(\beta)}_{k;\Omega} + [u]^{(\beta)}_{k,\alpha;\Omega}.
\end{aligned}$$

We also define

$$\mathcal{C}^{(\beta)}_{k,\alpha;\Omega} := \{u \in \mathcal{C}^{k,\alpha}(\Omega)/|u|^{(\beta)}_{k,\alpha;\Omega} < \infty\}.$$

For all $a \in \mathbb{R}^m$, $r \in (0,\infty)$ we note:

$$B^m(a,r) := \{x \in \mathbb{R}^m/|x-a| < r\}.$$

We denote by ω_m the measure of the unit ball $B^m(0,1)$. We will use the following, which is a consequence of Lemma 6.20, Lemma 6.21 and Theorem 6.22 in [3]:

LEMMA 2.2. — *Let $\alpha, \beta \in (0,1)$ and Ω be a smooth $\mathcal{C}^2$ open domain of $\mathbb{R}^m$. Then there exists a constant $C > 0$ which depends only on n, Ω and α such that for all $f \in \mathcal{C}^{0,\alpha}(\Omega)$ such that $|f|^{(2-\beta)}_{0,\alpha;\Omega} < \infty$, there exists a unique map $u \in \mathcal{C}^{2,\alpha}(\Omega)$ which is solution of*

$$\left\{ \begin{array}{rcll} \Delta u & = & f & \text{on } \Omega \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

Moreover $\exists \delta > 0$ such that u satisfies the estimate

$$|u|^{(-\beta)}_{2,\alpha;\Omega} \leqslant \delta |f|^{(2-\beta)}_{0,\alpha;\Omega}.$$

We also recall that the interpolation result Lemma 6.35 in [3] implies the following: if $j,k \in \mathbb{N}$, $\alpha,\beta \in (0,1)$, $\gamma \in \mathbb{R}$ then

$$j+\alpha \leqslant k+\beta \quad \Longrightarrow \quad |u|^{(\gamma)}_{j,\alpha;\Omega} \leqslant \widetilde{C} |u|^{(\gamma)}_{k,\beta;\Omega}, \tag{2.6}$$

where $\widetilde{C}$ is a positive constant which depends only on α, β, j, k, n.

3. Hölder continuity of the map around a

In this section we prove that any map satisfying the hypotheses of Theorem 1.1 is uniformly Hölder continuous in a neighbourhood of a.

THEOREM 3.1. — *Let $(\mathcal{N}, h)$ be a pseudo-Riemannian manifold of class $\mathcal{C}^2$, Ω an open subset of $\mathbb{R}^m$, a a point in Ω and u a continuous map from Ω to $\mathcal{N}$ which is $\mathcal{C}^2$ and harmonic on $\Omega \setminus \{a\}$. Then there exists $\alpha \in (0,1)$ and an open ball $B^m(a,R_1) \subset \Omega$ such that u is in $\mathcal{C}^{0,\alpha}(B^m(a,R_1))$ and $|u|_{0,\alpha;B^m(0,R_1)}$ is bounded.*

Proof. — We apply Lemma 2.1 with $M_0 = u(a)$. Let U_0 be the open neighbourhood of $u(a)$ and $\psi : U_0 \longrightarrow W_0$ be the local chart as in Lemma 2.1. We can assume w.l.g. that Ω is the ball $B^m(a, 2R)$, where, since u is continuous, $R \in (0, \infty)$ can been chosen in such a way that $u(B^m(a, 2R)) \subset U_0$.

For any $x_0 \in B^m(a, 2R)$, let $\Phi_{u(x_0)} \circ \psi : U_0 \longrightarrow V_{u(x_0)}$ be the local chart centered at $u(x_0)$ given by Lemma 2.1. It follows from (2.3) and (2.5) that in the coordinate system given by $\Phi_{u(x_0)} \circ \psi$,

$$\forall y \in V_{u(x_0)}, \forall \xi \in \mathbb{R}^m, \quad \left|\Gamma_{jk}(y)\xi^j\xi^k\right| \leqslant C_\Gamma |y||\xi|^2, \tag{3.1}$$

where C_Γ is the same constant as in (2.5). We observe that, by replacing R by a smaller number if necessary and because of the continuity of u and of (2.4), we can suppose that

$$\forall x, x_0 \in B^m(a, 2R), \quad \left|\Phi_{u(x_0)} \circ \psi \circ u(x)\right| \leqslant \inf\left(\frac{1}{2C_\Gamma}, \frac{1}{4}\right). \tag{3.2}$$

We now fix some $x_0 \in B^m(a, R)$ and set

$$u \simeq \Phi_{u(x_0)} \circ \psi \circ u.$$

We also let, for $r \in (0, R]$,

$$||u||_{r,x_0} = ||u||_{L^\infty(B^m(x_0,r))} := \sup_{x \in B^m(x_0,r)} |u(x)|,$$

and

$$\lambda := 2C_\Gamma \sup_{x \in B^m(a,2R)} |u(x)|.$$

Note that, because of the inclusion $B^m(x_0, R) \subset B^m(a, 2R)$ and of (3.2),

$$2C_\Gamma ||u||_{R,x_0} \leqslant \lambda \leqslant 1. \tag{3.3}$$

For any $\nu \in S^{n-1} \subset \mathbb{R}^n$ we consider the following functions on $V_{u(x_0)} \subset \mathbb{R}^n$ (which contains $u(B^m(x_0, R))$):

$$f_+(y) = \langle \nu, y\rangle + \lambda\frac{|y|^2}{2} \quad \text{and} \quad f_-(y) = \langle \nu, y\rangle - \lambda\frac{|y|^2}{2}.$$

Using (1.1) and (3.1) we find that on $B^m(x_0, R) \setminus \{a\}$,

$$\begin{aligned} -\Delta\left(f_+(u)\right) &= -\langle \nu, \Delta u\rangle - \lambda\Delta\left(\tfrac{|u|^2}{2}\right) \\ &= \langle \nu, \Gamma(u)(\nabla u \otimes \nabla u)\rangle - \lambda|\nabla u|^2 + \lambda\langle u, \Gamma(u)(\nabla u \otimes \nabla u)\rangle \\ &\leqslant -\lambda|\nabla u|^2 + C_\Gamma\left(|\nabla u|^2 + \lambda|u|\,|\nabla u|^2\right)|u|. \end{aligned}$$

Since $||u||_{R,x_0} \leqslant \frac{1}{4}$ by (3.2) and because of (3.3) we have $\lambda|u(x)| \leqslant \frac{1}{4}$, $\forall x \in B^m(x_0,R)$. So

$$-\Delta\left(f_+(u)\right) \leqslant \left(\frac{5}{4}C_\Gamma||u||_{R,x_0} - \lambda\right)|\nabla u|^2 \leqslant 0, \quad \text{on } B^m(x_0,R)\setminus\{a\} \quad (3.4)$$

Similarly

$$-\Delta\left(f_-(u)\right) \geqslant 0, \quad \text{on } B^m(x_0,R)\setminus\{a\} \quad (3.5)$$

Now fix $r \in (0,R]$ such that $r \neq |x_0|$ and define D_ε as follows

- if $|x_0| < r$, for any $\varepsilon \in [0, r-|x_0|)$, $D_\varepsilon := B^m(a,\varepsilon) \subset B^m(x_0,r)$
- if $|x_0| > r$, we assume that $\varepsilon = 0$ and set $D_\varepsilon = D_0 := \emptyset$.

And we let u_+^ε and u_-^ε be the maps from $B^m(x_0,r)\setminus\overline{D_\varepsilon}$ to $V_{u(x_0)}$ which are the solutions of respectively

$$\left\{\begin{array}{ccll} u_+^\varepsilon & = & ||u||_{R,x_0} + \lambda\frac{||u||^2_{R,x_0}}{2} & \text{on } \partial D_\varepsilon \\ u_+^\varepsilon & = & f_+(u) & \text{on } \partial B^m(x_0,r) \\ -\Delta u_+^\varepsilon & = & 0 & \text{on } B^m(x_0,r)\setminus\overline{D_\varepsilon}, \end{array}\right.$$

$$\left\{\begin{array}{ccll} u_-^\varepsilon & = & -||u||_{R,x_0} - \lambda\frac{||u||^2_{R,x_0}}{2} & \text{on } \partial D_\varepsilon \\ u_-^\varepsilon & = & f_-(u) & \text{on } \partial B^m(x_0,r) \\ -\Delta u_-^\varepsilon & = & 0 & \text{on } B^m(x_0,r)\setminus\overline{D_\varepsilon}. \end{array}\right.$$

Since

$$\left\{\begin{array}{cccl} u_+^\varepsilon & \geqslant & f_+(u) & \text{on } \partial\left(B^m(x_0,r)\setminus\overline{D_\varepsilon}\right) \\ -\Delta u_+^\varepsilon = 0 & \geqslant & -\Delta f_+(u) & \text{on } B^m(x_0,r)\setminus\overline{D_\varepsilon}, \end{array}\right.$$

the maximum principle implies that

$$u_+^\varepsilon \geqslant f_+(u) \quad \text{on } B^m(x_0,r)\setminus\overline{D_\varepsilon}.$$

Now we fix an arbitrary compact $K \subset B^m(x_0,r)\setminus\{a\}$. Then for ε sufficiently small we have

$$u_+^\varepsilon \geqslant f_+(u) \quad \text{on } K.$$

We let ε goes to 0: since $\{a\}$ has a vanishing capacity, the restriction of u_+^ε to K converges in $L^1(K)$ to $u_+ := u_+^0$ (apply Lemma 6.1 to $\phi_\varepsilon := u_+^\varepsilon - u_+$). Hence

$$u_+ \geqslant f_+(u) \quad \text{on } K.$$

Since u and u_+ are continuous on $B^m(x_0,r)$ and since K is arbitrary we deduce that

$$f_+(u) \leqslant u_+ \quad \text{on } B^m(x_0,r). \quad (3.6)$$

Similarly we get

$$u_- \leqslant f_-(u) \quad \text{on } B^m(x_0, r). \tag{3.7}$$

We deduce from (3.6) and (3.7) that for any $\nu \in S^m$

$$\left\{ \begin{array}{rcl} u_- & \leqslant & \langle \nu, u\rangle - \lambda\frac{|u|^2}{2} \\ \langle \nu, u\rangle + \lambda\frac{|u|^2}{2} & \leqslant & u_+ \end{array} \right. \quad \text{on } B^m(x_0, r)$$

and thus

$$u_- \leqslant u_- + \lambda\frac{|u|^2}{2} \leqslant \langle \nu, u\rangle \leqslant u_+ - \lambda\frac{|u|^2}{2} \leqslant u_+, \quad \text{on } B^m(x_0, r). \tag{3.8}$$

Hence if we let $v : B^m(x_0, r) \longrightarrow V_{u(x_0)}$ be the solution of

$$\left\{ \begin{array}{rcll} v & = & u & \text{on } \partial B^m(x_0, r) \\ -\Delta v & = & 0 & \text{on } B^m(x_0, r), \end{array} \right.$$

and $Q : B^m(x_0, r) \longrightarrow \mathbb{R}$ be the solution of

$$\left\{ \begin{array}{rcll} Q & = & \frac{|u|^2}{2} & \text{on } \partial B^m(x_0, r) \\ -\Delta Q & = & 0 & \text{on } B^m(x_0, r), \end{array} \right.$$

Then, since actually $u_\pm = \langle \nu, v\rangle \pm \lambda Q$, (3.8) implies that

$$\forall \nu \in S^{n-1}, \quad |\langle \nu, u\rangle - \langle \nu, v\rangle| \leqslant \lambda Q, \qquad \text{on } B^m(x_0, r).$$

Hence since ν is arbitrary and using the maximum principle for Q we obtain

$$|u - v| \leqslant \lambda Q \leqslant \lambda\frac{||u||^2_{r,x_0}}{2}, \qquad \text{on } B^m(x_0, r). \tag{3.9}$$

This implies in particular that, since $u(x_0) = 0$,

$$|v(x_0)| \leqslant \lambda\frac{||u||^2_{r,x_0}}{2}. \tag{3.10}$$

Moreover since v is harmonic, for all $x \in B^m(x_0, \frac{r}{2})$ we have (observing that $B^m(x, \frac{r}{2}) \subset B^m(x_0, r)$)

$$\frac{\partial v}{\partial x^\mu}(x) = \frac{2^m}{\omega_m r^m}\int_{B^m(x,\frac{r}{2})}\frac{\partial v}{\partial x^\mu} = \frac{2^m}{\omega_m r^m}\int_{\partial B^m(x,\frac{r}{2})} v\nu_\mu ds,$$

where ν is the exterior normal vector to the boundary. Hence $\forall x \in B^m(x_0, \frac{r}{2})$,

$$\left|\frac{\partial v}{\partial x^\mu}(x)\right| \leqslant \frac{2}{r}\sup_{\partial B^m(x,\frac{r}{2})}|v| \leqslant \frac{2}{r}\sup_{B^m(x_0,r)}|v| \leqslant \frac{2}{r}||u||_{r,x_0},$$

where we used the maximum principle for v. Hence we deduce that

$$\forall x \in B^m(x_0, \frac{r}{2}), \quad |v(x) - v(x_0)| \leqslant 2||u||_{r,x_0} \frac{|x - x_0|}{r}. \tag{3.11}$$

And from (3.10) and (3.11) we get

$$\forall x \in B^m(x_0, \frac{r}{2}), \quad |v(x)| \leqslant 2||u||_{r,x_0} \frac{|x - x_0|}{r} + \lambda \frac{||u||^2_{r,x_0}}{2}. \tag{3.12}$$

Using this inequality together with (3.9) we obtain that

$$\forall x \in B^m(x_0, \frac{r}{2}), \quad |u(x)| \leqslant 2||u||_{r,x_0} \frac{|x - x_0|}{r} + \lambda ||u||^2_{r,x_0}. \tag{3.13}$$

We now choose any $\rho \in (0, \frac{r}{2}]$ and take the supremum of the left hand side of (3.13) over $B^m(x_0, \rho)$. It gives

$$\forall r \in (0, R], \forall \rho \in (0, \frac{r}{2}], \quad ||u||_{\rho,x_0} \leqslant 2||u||_{r,x_0} \frac{\rho}{r} + \lambda ||u||^2_{r,x_0}. \tag{3.14}$$

For $k \in \mathbb{N}$ let $r_k := R4^{-k}$ and apply (3.14) for $r = r_k$ and $\rho = r_{k+1}$:

$$||u||_{r_{k+1},x_0} \leqslant \frac{||u||_{r_k,x_0}}{2} + \lambda ||u||^2_{r_k,x_0}.$$

This implies, denoting by $a_k := \lambda ||u||_{r_k,x_0}$, that

$$a_{k+1} \leqslant \frac{a_k}{2} + a_k^2. \tag{3.15}$$

We observe that, because of its definition, a_k is a positive decreasing sequence and, as a consequence of (3.2) and (3.3),

$$a_k \leqslant \lambda ||u||_{R,x_0} \leqslant \frac{1}{4}. \tag{3.16}$$

We now let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(q) = q^2 - \frac{q}{2}$ and consider the smooth function $\phi : [0, \infty) \longrightarrow \mathbb{R}$ which is a solution of

$$\left\{ \begin{array}{rcl} \phi(0) & = & a_0 \\ \dfrac{d\phi}{dt} & = & f(\phi) = \phi^2 - \frac{\phi}{2} \quad \text{on } [0, \infty). \end{array} \right.$$

LEMMA 3.2. — *Let $(a_k)_{k \in \mathbb{N}}$ be a decreasing sequence in $[0, \frac{1}{4}]$ which satisfies (3.15) and ϕ be defined as above. Then $\forall k \in \mathbb{N}$,*

$$a_k \leqslant \phi(k). \tag{3.17}$$

Proof of the Lemma. — We show (3.17) by induction. This inequality is obviously true for $k = 0$. Let us assume that (3.17) is true for some value $k \in \mathbb{N}$. We first observe that, since $f(0) = 0$ and $f \leqslant 0$ on $[0, \frac{1}{2}]$, $0 \leqslant a_0 \leqslant \frac{1}{4}$ implies that

$$\forall t \in [0, \infty), \quad 0 \leqslant \phi(t) \leqslant \frac{1}{4}.$$

Hence

- the fact that $f < 0$ on $(0, \frac{1}{4}]$ implies that ϕ is decreasing on $[0, \infty)$
- the fact that f is decreasing on $[0, \frac{1}{4}]$ implies that $f \circ \phi$ is increasing on $[0, \infty)$.

Thus

$$\forall t \in [k, \infty), \quad f \circ \phi(t) \geqslant f \circ \phi(k).$$

And hence

$$\begin{aligned} \phi(k+1) - \phi(k) &= \int_k^{k+1} \dot{\phi}(t)\, dt \\ &= \int_k^{k+1} f \circ \phi(t)\, dt \\ &\geqslant \int_k^{k+1} f \circ \phi(k)\, dt = f \circ \phi(k). \end{aligned}$$

Thus

$$\phi(k+1) \geqslant \phi(k) + f(\phi(k)) = \frac{\phi(k)}{2} + \phi(k)^2.$$

Now since (3.17) is true for k, i.e. $a_k \leqslant \phi(k)$, we deduce that

$$a_{k+1} = \frac{a_k}{2} + a_k^2 \leqslant \frac{\phi(k)}{2} + \phi(k)^2 \leqslant \phi(k+1).$$

□

Back to the proof of Theorem 3.1. — An easy quadrature shows that $\phi(t) = \frac{a_0}{2a_0 + e^{t/2}(1-2a_0)}$. Hence Lemma 3.2 implies

$$a_k \leqslant \phi(k) \leqslant \frac{a_0}{1-2a_0} e^{-k/2}$$

and since $0 < a_0 \leqslant \frac{1}{4}$, we deduce that $a_k \leqslant \frac{e^{-k/2}}{2}$, i.e.

$$||u||_{R4^{-k}, x_0} \leqslant \frac{1}{2\lambda} e^{-k/2}, \quad \forall k \in \mathbb{N}. \tag{3.18}$$

We now choose any $r \in (0, R]$. Then $\exists! k \in \mathbb{N}$ such that $\frac{R}{4^{k+1}} < r \leqslant \frac{R}{4^k}$. Then on the one hand $r \leqslant \frac{R}{4^k}$ implies

$$||u||_{r,x_0} \leqslant ||u||_{R4^{-k},x_0} \leqslant \frac{1}{2\lambda} e^{-k/2}$$

by (3.18). On the other hand $\frac{R}{4^{k+1}} < r \Longleftrightarrow k > \frac{\log \frac{R}{r}}{\log 4} - 1$ implies

$$\frac{1}{2\lambda} e^{-k/2} < \frac{e^{1/2}}{2\lambda} \left(\frac{r}{R}\right)^{\frac{1}{2\log 4}}.$$

Hence we deduce that

$$\forall r \in (0, R], \quad ||u||_{r,x_0} \leqslant \frac{e^{1/2}}{2\lambda} \left(\frac{r}{R}\right)^{\frac{1}{2\log 4}}.$$

We conclude that, since λ is independent of x_0, u is uniformly Hölder continuous on $B^m(a, R)$. $\square$

4. Existence of a smooth solution around a

We start with the following (classical) preliminary.

LEMMA 4.1. — *Let Ω be an open subset of $\mathbb{R}^m$ whose boundary is C^2. Let $\phi \in C^{0,\alpha}(\partial\Omega)$ and f be the solution of*

$$\begin{cases} -\Delta f &= 0, \quad \text{on } \Omega \\ f &= \phi, \quad \text{on } \partial\Omega. \end{cases}$$

Then f is $C^{2,\alpha}$ on Ω and

$$[f]^{(-\alpha)}_{0,\alpha;\Omega} + [f]^{(-\alpha)}_{1,\alpha;\Omega} + [f]^{(-\alpha)}_{2,\alpha;\Omega} \leqslant C_1 |\phi|_{0,\alpha;\partial\Omega}, \tag{4.1}$$

where C_1 is a positive constant which depends only on Ω.

Remark. — We do not have an estimate on $|f|^{(-\alpha)}_{2,\alpha;\Omega}$. Indeed this quantity is in general infinite because $|f|^{(-\alpha)}_{0,0;\Omega} = \sup_{x\in\Omega} d_x^{-\alpha} |f(x)|$ cannot be finite unless the trace of f on $\partial\Omega$ vanishes. However the maximum principle and (4.1) imply the following:

$$|f|^{(0)}_{0,\alpha;\Omega} \leqslant \left(1 + C_1 \left(\frac{\mathrm{diam}\Omega}{2}\right)^{\alpha}\right) |\phi|_{0,\alpha;\partial\Omega}. \tag{4.2}$$

Proof.— First step : Ω is a half space — We assume that $\Omega = \mathbb{R}^m_+ := \{x = (\vec{x}, t)/\vec{x} \in \mathbb{R}^{m-1}, t \in (0, \infty)\}$. We use Proposition 7 and Lemma 4 in Chapter V of [10]: there exists a constant $C_0' > 0$ such that

$$\sup_{\vec{x} \in \mathbb{R}^{m-1}} |Df(\vec{x}, t)| \leqslant C_0' t^{-1+\alpha} |\phi|_{0,\alpha:\mathbb{R}^{m-1}}, \quad \forall (\vec{x}, t) \in \mathbb{R}^m_+. \tag{4.3}$$

Moreover using the fact that Df is harmonic, $\forall x \in \mathbb{R}^m_+$ we have if $\rho := t/2$,

$$\frac{\partial Df}{\partial x^\mu}(x) = \frac{1}{\omega_m \rho^m} \int_{B^m(x,\rho)} \frac{\partial Df}{\partial x^\mu}(y) dy = \frac{1}{\omega_m \rho^m} \int_{\partial B^m(x,\rho)} Df(y) \nu_\mu ds(y),$$

which implies by (4.3)

$$\begin{aligned} \left| \frac{\partial Df}{\partial x^\mu}(x) \right| &\leqslant \frac{1}{\omega_m \rho^m} \int_{\partial B^m(x,\rho)} |Df(y)| ds(y) \\ &\leqslant \frac{m}{\rho} C_0' |\phi|_{0,\alpha;\mathbb{R}^{m-1}} \left(\frac{t}{2} \right)^{-1+\alpha} = 2^{2-\alpha} m C_0' |\phi|_{0,\alpha;\mathbb{R}^{m-1}} t^{-2+\alpha}. \end{aligned}$$

Hence we obtain that there exists a constant C_0'' such that

$$\sup_{\vec{x} \in \mathbb{R}^{m-1}} |D^2 f(\vec{x}, t)| \leqslant C_0'' |\phi|_{0,\alpha;\mathbb{R}^{m-1}} t^{-2+\alpha}, \quad \forall (\vec{x}, t) \in \mathbb{R}^m_+. \tag{4.4}$$

A similar reasonning starting from (4.4) leads to

$$\sup_{\vec{x} \in \mathbb{R}^{m-1}} |D^3 f(\vec{x}, t)| \leqslant C_0''' |\phi|_{0,\alpha;\mathbb{R}^{m-1}} t^{-3+\alpha}, \quad \forall (\vec{x}, t) \in \mathbb{R}^m_+, \tag{4.5}$$

for some constant $C_0''' > 0$.

Now using (4.3) and (4.4) we can estimate $[f]^{(-\alpha)}_{1,\alpha;\mathbb{R}^m_+}$ as follows: if $x = (\vec{x}, t)$ and $y = (\vec{y}, s)$ are in $\mathbb{R}^m_+$ let $d := \inf(t, s)$. Then if $|x - y| \leqslant 2d$ we have by (4.4)

$$\begin{aligned} \frac{|Df(x) - Df(y)|}{|x - y|^\alpha} &\leqslant \sup_{\tau > d} |D^2 f(\xi, \tau)| \, |x - y|^{1-\alpha} \\ &\leqslant C_0'' |\phi|_{0,\alpha;\mathbb{R}^{m-1}} d^{-2+\alpha} (2d)^{1-\alpha} = 2^{1-\alpha} C_0'' |\phi|_{0,\alpha;\mathbb{R}^{m-1}} d^{-1}. \end{aligned}$$

On the other hand if $|x - y| > 2d$ we have by (4.3)

$$\begin{aligned} \frac{|Df(x) - Df(y)|}{|x - y|^\alpha} &\leqslant \frac{|Df(x)| + |Df(y)|}{d^\alpha} 2^{-\alpha} \\ &\leqslant \frac{2C_0' d^{-1+\alpha} |\phi|_{0,\alpha;\mathbb{R}^{m-1}}}{d^\alpha} 2^{-\alpha} = 2^{1-\alpha} C_0' |\phi|_{0,\alpha;\mathbb{R}^{m-1}} d^{-1}. \end{aligned}$$

Thus taking into account both cases we find that

$$[f]^{(-\alpha)}_{1,\alpha;\mathbb{R}^m_+} \leqslant 2^{1-\alpha}\sup(C'_0, C''_0)|\phi|_{0,\alpha;\mathbb{R}^{m-1}}.$$

An analogous work with (4.4) and (4.5) instead of (4.3) and (4.4) leads to $[f]^{(-\alpha)}_{2,\alpha;\mathbb{R}^m_+} \leqslant 2^{1-\alpha}\sup(C''_0, C'''_0)|\phi|_{0,\alpha;\mathbb{R}^{m-1}}$.

The estimate for $[f]^{(-\alpha)}_{0,\alpha;\mathbb{R}^m_+}$ follows from a slightly different argument. Again let $x = (\vec{x}, t)$ and $y = (\vec{y}, s)$ be in $\mathbb{R}^m_+$ and let $d := \inf(t, s)$. If $|x - y| \leqslant 2d$ the same reasoning as above works using (4.3) and gives

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} \leqslant 2^{1-\alpha}C'_0|\phi|_{0,\alpha;\mathbb{R}^{m-1}}. \tag{4.6}$$

However if $|x-y| > 2d$, then we write $|f(\vec{x}, t) - f(\vec{y}, s)| \leqslant |f(\vec{x}, t) - f(\vec{x}, 0)| + |f(\vec{x}, 0) - f(\vec{y}, 0)| + |f(\vec{y}, 0) - f(\vec{y}, s)|$ and estimate separately each term. Using again (4.3):

$$\begin{aligned}|f(\vec{x}, t) - f(\vec{x}, 0)| &\leqslant \int_0^t |Df(\vec{x}, \tau)|d\tau \\ &\leqslant \int_0^t C'_0|\phi|_{0,\alpha:\mathbb{R}^{m-1}}\tau^{-1+\alpha}d\tau = \frac{C'_0}{\alpha}|\phi|_{0,\alpha;\mathbb{R}^{m-1}}t^\alpha.\end{aligned}$$

Similarly one gets

$$|f(\vec{y}, s) - f(\vec{y}, 0)| \leqslant \frac{C'_0}{\alpha}|\phi|_{0,\alpha;\mathbb{R}^{m-1}}s^\alpha.$$

Lastly using $|f(\vec{x}, 0) - f(\vec{y}, 0)| \leqslant |\phi|_{0,\alpha;\mathbb{R}^{m-1}}|\vec{x} - \vec{y}|^\alpha$, one concludes that

$$\begin{aligned}|f(\vec{x}, t) - f(\vec{y}, s)| &\leqslant |\phi|_{0,\alpha;\mathbb{R}^{m-1}}\left(\frac{C'_0}{\alpha}t^\alpha + \frac{C'_0}{\alpha}s^\alpha + |\vec{x} - \vec{y}|^\alpha\right) \\ &\leqslant |\phi|_{0,\alpha;\mathbb{R}^{m-1}}\sup\left(1, \frac{C'_0}{\alpha}\right)(t^\alpha + s^\alpha + |\vec{x} - \vec{y}|^\alpha).\end{aligned}$$

Assume for instance that $s < t$, so that $d = s$. Then by the Minkowski inequality $t^\alpha + s^\alpha = d^\alpha + ((t-s)+d)^\alpha \leqslant d^\alpha + (t-s)^\alpha + d^\alpha = 2d^\alpha + (t-s)^\alpha$. Hence

$$t^\alpha + s^\alpha + |\vec{x} - \vec{y}|^\alpha \leqslant 2d^\alpha + (t-s)^\alpha + |\vec{x} - \vec{y}|^\alpha \leqslant 2d^\alpha + 2|x-y|^\alpha < (2^{1-\alpha}+2)|x-y|^\alpha.$$

And we thus get

$$|f(\vec{x}, t) - f(\vec{y}, s)| \leqslant (2^{1-\alpha} + 2)|\phi|_{0,\alpha:\mathbb{R}^{m-1}}\sup\left(1, \frac{C'_0}{\alpha}\right)|x - y|^\alpha. \tag{4.7}$$

So (4.6) and (4.7) implies the result on $[f]^{(-\alpha)}_{0,\alpha;\mathbb{R}^m_+}$.

Step 2 — estimate on an arbitrary domain. — If Ω is a domain with a smooth $\mathcal{C}^2$ boundary, then using local chart and a partition of unity one can construct an extension $g \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ of $\phi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$ which satisfies

$$[g]^{(-\alpha)}_{0,\alpha;\Omega} + [g]^{(-\alpha)}_{1,\alpha;\Omega} + [g]^{(-\alpha)}_{2,\alpha;\Omega} \leqslant C_1' |\phi|_{0,\alpha;\partial\Omega}.$$

Then the harmonic extension of ϕ is $f = g + h$, where h is a function which vanishes on $\partial\Omega$ and which satisfies $-\Delta h = \Delta g$ on Ω. Because of the previous estimate on g, $[\Delta g]^{(2-\alpha)}_{0,\alpha;\Omega} \leqslant C_1'|\phi|_{0,\alpha;\partial\Omega}$. Now Lemma 2.2 implies that $|h|^{(-\alpha)}_{2,\alpha;\Omega} \leqslant \delta |\Delta g|^{(2-\beta)}_{0,\alpha;\Omega}$. Hence the estimate on f follows by summing the estimates on g and h. □

LEMMA 4.2. — *Let $(\mathcal{N}, h)$ be a pseudo-Riemannian manifold of class $\mathcal{C}^2$, Ω an open subset of $\mathbb{R}^m$, $a \in \Omega$ and u a continuous map from Ω to $\mathcal{N}$ which is $\mathcal{C}^2$ and harmonic on $\Omega \setminus \{a\}$. Then there exists $\alpha \in (0,1)$ and an open ball $B^m(a, R_2)$ such that $\forall r \in (0, R_2)$, there exists a map $\underline{u} \in \mathcal{C}^{0,\alpha}(\overline{B^m(a,r)}, \mathcal{N}) \cap \mathcal{C}^{2,\alpha}(B^m(a,r), \mathcal{N})$ which is a solution of*

$$\left\{ \begin{array}{rcll} \Delta \underline{u} + \Gamma(\underline{u})(\nabla \underline{u} \otimes \nabla \underline{u}) & = & 0 & \text{on } B^m(a,r) \\ \underline{u} & = & u & \text{on } \partial B^m(a,r). \end{array} \right. \tag{4.8}$$

I.e. $\underline{u}$ is a harmonic map with values in $\mathcal{N}$ which agrees with u on the boundary of $B^m(a,r)$.

Proof. — Again we start by applying Lemma 2.1 with $M_0 = u(a)$: it provides us with a local chart $\Phi_{u(a)} \circ \psi$ on $\mathcal{N}$ around $u(a)$. We denote by y^i, h_{ij} and Γ^i_{jk} respectively the coordinates, the metric and the Christoffel symbols in this chart. In the following we make the identification $u \simeq \Phi_{u(a)} \circ \psi \circ u$, so that we view u as a map from $B^m(a, \overline{R})$ to $\mathbb{R}^n$ such that $u(a) = 0$ and the majorations (2.3) and (2.5) hold.

For every $r \in (0, \overline{R}]$ we let $v : B^m(a,r) \longrightarrow \mathbb{R}^n$ be the harmonic extension of u inside $B^m(a,r)$, i.e.

$$\left\{ \begin{array}{rcll} \Delta v & = & 0 & \text{on } B^m(a,r) \\ v & = & u & \text{on } \partial B^m(a,r). \end{array} \right. \tag{4.9}$$

We first apply Theorem 3.1 which ensures us that the $\mathcal{C}^{0,\alpha}$ norm of u in a neighbourhood of u is bounded: $\exists \alpha \in (0,1)$, $\exists R_1 \in (0, \overline{R}]$ such that $|u|_{0,\alpha;B^m(a,R_1)}$ is finite. This allows us to use Lemma 4.1 in order to estimate v: we will use the notations

$$|v|^{(-\alpha)}_{k,\alpha;r} := |v|^{(-\alpha)}_{k,\alpha;B^m(a,r)}, \quad \forall k \in \mathbb{N}, \quad \text{and} \quad |[v]|_{\alpha;r} := |v|^{(0)}_{0,\alpha;r} + [v]^{(-\alpha)}_{1,\alpha;r} + [v]^{(-\alpha)}_{2,\alpha;r}$$

and then (4.1) and (4.2) imply

$$\forall r \in (0, R_1], \quad |[v]|_{\alpha;r} \leqslant C_2 |u|_{0,\alpha;R_1}, \tag{4.10}$$

where $C_2 = C_1 + 1 + C_1 \overline{R}^{\alpha}$. We will denote by

$$\Lambda := \sup_{r \in (0, R_1]} |[v]|_{\alpha;r}. \tag{4.11}$$

Our purpose is to construct the extension $\underline{u}$ satisfying (4.8). By writing

$$\underline{u} = v + w,$$

it clearly relies on finding a map $w \in \mathcal{C}^{2,\alpha}(B^m(a,r), \mathbb{R}^n)$ such that

$$\left\{ \begin{array}{rcll} -\Delta w & = & \Gamma(v+w)(\nabla(v+w) \otimes \nabla(v+w)) & \text{on } B^m(a,r) \\ w & = & 0 & \text{on } \partial B^m(a,r). \end{array} \right. \tag{4.12}$$

Let us denote by $\mathcal{C}^{(-\beta)}_{k,\alpha,r} := \mathcal{C}^{(-\beta)}_{k,\alpha;B^m(a,r)}$. We will construct w in $\mathcal{C}^{(-\alpha)}_{2,\alpha,r}$ by using a fixed point argument. We first observe that Lemma 2.2 can be rephrased (and specialized by choosing $\beta = \alpha$) by saying that there exists a continuous operator, denoted in the following by $(-\Delta)^{-1}$, from $\mathcal{C}^{(2-\alpha)}_{0,\alpha,r}$ to $\mathcal{C}^{(-\alpha)}_{2,\alpha,r}$ which to each $f \in \mathcal{C}^{(2-\alpha)}_{0,\alpha,r}$ associates the unique solution $\phi \in \mathcal{C}^{(-\alpha)}_{2,\alpha,r}$ of

$$\left\{ \begin{array}{rcll} -\Delta \phi & = & f & \text{on } B^m(a,r) \\ \phi & = & 0 & \text{on } \partial B^m(a,r). \end{array} \right.$$

We will denote by δ the norm of $(-\Delta)^{-1}$. Hence $w \in \mathcal{C}^{(-\alpha)}_{2,\alpha,r}$ is a solution of (4.12) if and only if

$$w = (-\Delta)^{-1} \left(\Gamma(v+w)(\nabla(v+w) \otimes \nabla(v+w)) \right).$$

Note that v does not belong to $\mathcal{C}^{(-\alpha)}_{2,\alpha,r}$ (because in particular the trace of v on $\partial B^m(a,r)$ does not vanish). However estimates (4.10) holds. This leads us to introduce the set

$$\mathcal{E}_{\alpha;r} := \{ f \in \mathcal{C}^{2,\alpha}(B^m(a,r), \mathbb{R}^n) / |[f]|_{\alpha;r} < \infty \} \supset \mathcal{C}^{(-\alpha)}_{2,\alpha,r},$$

where the inclusion here is a continuous embedding. We then have:

LEMMA 4.3. — *Let $r \in (0, R_1]$ and w_0, w_1, w_2 and w_3 be in $\mathcal{E}_{\alpha;r}$. Then $\Gamma(w_0)(\nabla w_2 \otimes \nabla w_3)$ and $\Gamma(w_1)(\nabla w_2 \otimes \nabla w_3) \in \mathcal{C}^{(2-\alpha)}_{0,\alpha;r}$ and*

$$\begin{aligned} &|(\Gamma(w_1) - \Gamma(w_0))(\nabla w_2 \otimes \nabla w_3)|^{(2-\alpha)}_{0,\alpha;r} \\ &\qquad \leqslant C_3 r^{\alpha} |[w_1 - w_0]|_{\alpha;r} |[w_2]|_{\alpha;r} |[w_3]|_{\alpha;r}, \end{aligned} \tag{4.13}$$

$$|\Gamma(w_1)(\nabla w_2 \otimes \nabla w_3)|^{(2-\alpha)}_{0,\alpha;r} \leqslant C_3 r^\alpha |[w_1]|_{\alpha;r} |[w_2]|_{\alpha;r} |[w_3]|_{\alpha;r}, \tag{4.14}$$

where C_3 is a positive constant.

Proof of Lemma 4.3. — It follows from the interpolation inequality (2.6) that $|Dw_a|^{(1-\alpha)}_{0,\alpha;r} \leqslant \widetilde{C}|[w_a]|_{\alpha;r}$, $\forall a = 0,1,2,3$. Moreover, for $r \in (0, R_1] \subset (0, \overline{R}]$, estimate (2.5) implies that

$$|(\Gamma(w_1) - \Gamma(w_0))|^{(0)}_{0,\alpha;r} \leqslant C_\Gamma |w_1 - w_0|^{(0)}_{0,\alpha;r} \leqslant \widetilde{C} C_\Gamma |[w_1 - w_0]|_{\alpha;r}.$$

Hence, using also the inequality $|f|^{(\beta+\gamma)}_{0,\alpha;\Omega} \leqslant |f|^{(\beta)}_{0,\alpha;\Omega} |g|^{(\gamma)}_{0,\alpha;\Omega}$, $\forall \beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma \geqslant 0$ (see 6.11 in [3]), we obtain that

$$\begin{aligned} &|(\Gamma(w_1) - \Gamma(w_0))\,(\nabla w_2 \otimes \nabla w_3)|^{(2-2\alpha)}_{0,\alpha;r} \\ &\quad\leqslant\ |\Gamma(w_1) - \Gamma(w_0)|^{(0)}_{0,\alpha;r} |Dw_2|^{(1-\alpha)}_{0,\alpha;r} |Dw_3|^{(1-\alpha)}_{0,\alpha;r} \\ &\quad\leqslant\ \widetilde{C}^3 C_\Gamma |[w_1 - w_0]|_{\alpha;r} |[w_2]|_{\alpha;r} |[w_3]|_{\alpha;r}. \end{aligned}$$

Thus (4.13) follows from the preceding inequality and from

$$|(\Gamma(w_1) - \Gamma(w_0))(\nabla w_2 \otimes \nabla w_3)|^{(2-\alpha)}_{0,\alpha;r} \leqslant r^\alpha |(\Gamma(w_1) - \Gamma(w_0))(\nabla w_2 \otimes \nabla w_3)|^{(2-2\alpha)}_{0,\alpha;r}.$$

And (4.14) is a straightforward consequence of (4.13) and of (2.3). □

Back to the proof of Theorem 4.2. — Lemma 4.3 allows us to define the operator

$$\begin{array}{rccc} T: & \mathcal{C}^{(-\alpha)}_{2,\alpha,r} & \longrightarrow & \mathcal{C}^{(-\alpha)}_{2,\alpha,r} \\ & w & \longmapsto & (-\Delta)^{-1}\left(\Gamma(v+w)(\nabla(v+w) \otimes \nabla(v+w))\right) \end{array}$$

and (4.14) implies

$$|T(w)|^{(-\alpha)}_{2,\alpha;r} \leqslant \delta C_3 r^\alpha \left(|[v]|_{\alpha;r} + |w|^{(-\alpha)}_{2,\alpha;r} \right)^3.$$

In particular, letting $\mathcal{B}_\Lambda := \{w \in \mathcal{C}^{(-\alpha)}_{2,\alpha,r} / |w|^{(-\alpha)}_{2,\alpha;r} \leqslant \Lambda\}$ (we recall that Λ was defined in (4.11)), we observe that for all $r \in (0, R_1] \cap (0, R'_1]$ where $R'_1 = (8\delta C_3 \Lambda^2)^{-1/\alpha}$, i.e. such that in particular $\delta C_3 r^\alpha (2\Lambda)^3 < \Lambda$,

$$\forall w \in \mathcal{B}_\Lambda, \quad |T(w)|^{(-\alpha)}_{2,\alpha;r} \leqslant \Lambda,$$

which means that T maps the closed ball $\mathcal{B}_\Lambda$ into itself.

Let us now prove that, for r small enough, the restriction of T on $\mathcal{B}_\Lambda$ is also contracting: writing that, $\forall w, \widetilde{w} \in \mathcal{B}_\Lambda$,

$$\begin{aligned} T(w) - T(\widetilde{w}) \quad = \quad & (-\Delta)^{-1}\left[(\Gamma(v+w) - \Gamma(v+\widetilde{w}))(\nabla(v+w)\otimes\nabla(v+w)\right] \\ & + (-\Delta)^{-1}\left(\Gamma(v+\widetilde{w})(\nabla(v+w)\otimes\nabla(w-\widetilde{w})\right) \\ & + (-\Delta)^{-1}\left(\Gamma(v+\widetilde{w})(\nabla(w-\widetilde{w})\otimes\nabla(v+\widetilde{w})\right) \end{aligned}$$

and using (4.13) we obtain, assuming that $r \leqslant R_1$,

$$\begin{aligned} |T(w) - T(\widetilde{w})|^{(-\alpha)}_{2,\alpha;r} \quad \leqslant \quad & \delta C_3 r^\alpha |w-\widetilde{w}|^{(-\alpha)}_{2,\alpha;r}\left(|v+w|^{(-\alpha)}_{2,\alpha;r}\right)^2 \\ & + \delta C_3 r^\alpha |w-\widetilde{w}|^{(-\alpha)}_{2,\alpha;r}|v+w|^{(-\alpha)}_{2,\alpha;r}|v+\widetilde{w}|^{(-\alpha)}_{2,\alpha;r} \\ & + \delta C_3 r^\alpha |w-\widetilde{w}|^{(-\alpha)}_{2,\alpha;r}\left(|v+\widetilde{w}|^{(-\alpha)}_{2,\alpha;r}\right)^2 \\ \leqslant \quad & 3\delta C_3 r^\alpha (2\Lambda)^2 |w-\widetilde{w}|^{(-\alpha)}_{2,\alpha;r}. \end{aligned}$$

Hence T is contracting if we further assume that $r < R_1''$, where $R_1'' := (12\delta C_3\Lambda^2)^{-1/\alpha}$, because it implies that $3\delta C_3 r^\alpha (2\Lambda)^2 < 1$. In conclusion (observing that actually $R_1'' < R_1'$) if we let $R_2 := \inf(R_1, R_1'')$, then for all $r \in (0, R_2)$, T maps the closed ball $\mathcal{B}_\Lambda$ into itself and is contracting. Hence it admits a unique fixed point $w \in \mathcal{B}_\Lambda$ which is a solution of (4.12). □

5. A maximum principle

THEOREM 5.1. — *Let $(\mathcal{N}, h)$ be a pseudo-Riemannian manifold of class $\mathcal{C}^2$ and M_0 be a point in $\mathcal{N}$. There exists an open neighbourhood U_{M_0} of M_0, a local chart $\phi : U_{M_0} \longrightarrow \mathbb{R}^n$ and constant $\alpha > 0$ such that for any open subset Ω of $\mathbb{R}^m$ and for any pair of harmonic mappings $u, v : \Omega \longrightarrow (U_{M_0}, h_{ij})$ (i.e. which satisfy (1.1)), then the function $f : \Omega \longrightarrow \mathbb{R}$ defined (using $u \simeq \phi\circ u$ and $v \simeq \phi \circ v$) by*

$$f(x) := (\alpha^2 + |u(x)|^2)(\alpha^2 + |v(x)|^2)\frac{|u(x)-v(x)|^2}{2}, \quad \forall x \in \Omega, \qquad (5.1)$$

satisfies the inequality

$$-div(\rho\nabla f) \leqslant 0, \quad on\ \Omega, \qquad (5.2)$$

where

$$\rho(x) := \frac{1}{\alpha^2 + |u(x)|^2}\frac{1}{\alpha^2 + |v(x)|^2}, \quad \forall x \in \Omega.$$

Remark. — Note that here $|\cdot|$ is an Euclidean norm on U_{M_0} which has nothing to do with the metric h on $\mathcal{N}$. More precisely, assuming that

$\phi(M_0) = 0$, for any points $M, \widetilde{M} \in U_{M_0}$ we set $\langle M, \widetilde{M}\rangle := \langle \phi(M) - \phi(M_0), \phi(\widetilde{M}) - \phi(M_0)\rangle = \langle \phi(M), \phi(\widetilde{M})\rangle$, $|M|^2 := |\phi(M)|^2 = \langle \phi(M), \phi(M)\rangle$ and $|M - \widetilde{M}|^2 := |\phi(M) - \phi(\widetilde{M})|^2$.

Proof of Theorem 5.1. — Again we first apply Lemma 2.1 around M_0: it provides us with a local chart $\phi : U'_{M_0} \longrightarrow \mathbb{R}^n$ such that $\phi(M_0) = 0$ and estimates (2.3) and (2.5) on the Christoffel symbols Γ^i_{jk} hold. We fix some $\alpha \in (0, \infty)$ which is temporarily arbitrary and whose value will be chosen later. Then given a pair of harmonic maps $u, v : \Omega \longrightarrow (U'_{M_0}, h_{ij})$ we compute $\mathrm{div}(\rho\nabla f)$, where f is given by (5.1). We first find that

$$\rho\nabla f = \langle u - v, \nabla(u - v)\rangle + |u - v|^2 \left(\frac{\langle u, \nabla u\rangle}{\alpha^2 + |u|^2} + \frac{\langle v, \nabla v\rangle}{\alpha^2 + |v|^2}\right).$$

Hence (by using the notations $\langle\cdot,\cdot\rangle$ for the scalar product in $\mathbb{R}^n$ and $\cdot$ for the scalar product in $\mathbb{R}^m$)

$$\begin{aligned}
\mathrm{div}(\rho\nabla f) \;&=\; |\nabla(u - v)|^2 + \langle u - v, \Delta(u - v)\rangle \\
&\quad + 2\langle u - v, \nabla(u - v)\rangle \cdot \left(\frac{\langle u, \nabla u\rangle}{\alpha^2 + |u|^2} + \frac{\langle v, \nabla v\rangle}{\alpha^2 + |v|^2}\right) \\
&\quad + |u - v|^2 \left(\frac{|\nabla u|^2}{\alpha^2 + |u|^2} + \frac{|\nabla v|^2}{\alpha^2 + |v|^2} + \frac{\langle u, \Delta u\rangle}{\alpha^2 + |u|^2} + \frac{\langle v, \Delta v\rangle}{\alpha^2 + |v|^2}\right. \\
&\qquad \left. -2\frac{|\langle u, \nabla u\rangle|^2}{(\alpha^2 + |u|^2)^2} - 2\frac{|\langle v, \nabla v\rangle|^2}{(\alpha^2 + |v|^2)^2}\right) \\
&=\; G_1 + G_2 + B_1 + B_2 + B_3 + B_4,
\end{aligned}$$

where the "good" terms are

$$G_1 := |\nabla(u - v)|^2, \quad G_2 := |u - v|^2 \left(\frac{|\nabla u|^2}{\alpha^2 + |u|^2} + \frac{|\nabla v|^2}{\alpha^2 + |v|^2}\right),$$

and the "bad" terms are

$$\begin{aligned}
B_1 \;&:=\; \langle u - v, \Delta(u - v)\rangle \\
B_2 \;&:=\; 2\langle u - v, \nabla(u - v)\rangle \cdot \left(\frac{\langle u, \nabla u\rangle}{\alpha^2 + |u|^2} + \frac{\langle v, \nabla v\rangle}{\alpha^2 + |v|^2}\right) \\
B_3 \;&:=\; |u - v|^2 \left(\frac{\langle u, \Delta u\rangle}{\alpha^2 + |u|^2} + \frac{\langle v, \Delta v\rangle}{\alpha^2 + |v|^2}\right) \\
B_4 \;&:=\; -2|u - v|^2 \left(\frac{|\langle u, \nabla u\rangle|^2}{(\alpha^2 + |u|^2)^2} + \frac{|\langle v, \nabla v\rangle|^2}{(\alpha^2 + |v|^2)^2}\right).
\end{aligned}$$

We now need to estimate the bad terms in terms of the good ones. We let $R \in (0, \infty)$ such that $\phi(U'_{M_0}) \subset B^n(0, R)$. We shall assume in the following that

$$|u| \leqslant r \quad \text{and} \quad |v| \leqslant r \quad \text{for some } r \in (0, R), \tag{5.3}$$

where r has not yet been fixed. In the following we will first choose α as a function of C_Γ and R, and second we will choose r as a function of α, C_Γ and R.

Estimation of B_1

We have

$$\begin{aligned}
-\Delta(u-v) &= \Gamma(u)(\nabla u\otimes\nabla u)-\Gamma(v)(\nabla v\otimes\nabla v)\\
&= \Gamma(u)(\nabla u\otimes\nabla(u-v))+\Gamma(u)(\nabla(u-v)\otimes\nabla v)\\
&\quad+(\Gamma(u)-\Gamma(v))(\nabla v\otimes\nabla v).
\end{aligned}$$

And because of (2.3) and (2.5) which implies $|\Gamma(y)|\leqslant C_\Gamma|y|$ and $|\Gamma(y)-\Gamma(y')|\leqslant C_\Gamma|y-y'|$ on U'_{M_0}, we deduce that

$$\begin{aligned}
|\Delta(u-v)| &\leqslant C_\Gamma|u|\left(|\nabla u|\,|\nabla(u-v)|+|\nabla v|\,|\nabla(u-v)|\right)\\
&\quad+C_\Gamma|u-v|\,|\nabla v|^2.
\end{aligned}$$

Using (5.3) and a symmetrization in u and v one is led to

$$|\Delta(u-v)|\leqslant C_\Gamma R|\nabla(u-v)|\left(|\nabla u|+|\nabla v|\right)+\frac{C_\Gamma}{2}|u-v|\left(|\nabla u|^2+|\nabla v|^2\right).$$

Hence we deduce using Young's inequality that

$$\begin{aligned}
|B_1| &\leqslant C_\Gamma R|u-v|\,|\nabla(u-v)|\left(|\nabla u|+|\nabla v|\right)+\frac{C_\Gamma}{2}|u-v|^2\left(|\nabla u|^2+|\nabla v|^2\right)\\
&\leqslant \frac{|\nabla(u-v)|^2}{4}+2C_\Gamma^2R^2|u-v|^2\left(|\nabla u|^2+|\nabla v|^2\right)\\
&\quad+\frac{C_\Gamma}{2}|u-v|^2\left(|\nabla u|^2+|\nabla v|^2\right).
\end{aligned}$$

We choose $\alpha\in(0,\infty)$ sufficiently small so that $\frac{1}{2\alpha^2}\geqslant 4\left(2C_\Gamma^2R^2+\frac{C_\Gamma}{2}\right)$ and we impose also that $r\leqslant\alpha$. Then by (5.3)

$$|u|,\,|v|\leqslant r\leqslant\alpha\quad\Longrightarrow\quad\frac{1}{\alpha^2+|u|^2},\frac{1}{\alpha^2+|v|^2}\geqslant\frac{1}{2\alpha^2}\geqslant 4\left(2C_\Gamma^2R^2+\frac{C_\Gamma}{2}\right)$$

and thus

$$|B_1|\leqslant\frac{G_1}{4}+\frac{|u-v|^2}{4}\left(\frac{|\nabla u|^2}{\alpha^2+|u|^2}+\frac{|\nabla v|^2}{\alpha^2+|v|^2}\right)=\frac{G_1}{4}+\frac{G_2}{4}.\qquad(5.4)$$

Estimation of B_2

Using again Young's inequality we obtain

$$\begin{aligned}
|B_2| &\leqslant 2|u-v|\,|\nabla(u-v)| \left(\frac{|u|\,|\nabla u|}{\alpha^2+|u|^2} + \frac{|v|\,|\nabla v|}{\alpha^2+|v|^2}\right) \\
&\leqslant \frac{|\nabla(u-v)|^2}{2} + 2|u-v|^2 \left(\frac{|u|\,|\nabla u|}{\alpha^2+|u|^2} + \frac{|v|\,|\nabla v|}{\alpha^2+|v|^2}\right)^2 \\
&\leqslant \frac{G_1}{2} + |u-v|^2 \left(\frac{4|u|^2|\nabla u|^2}{(\alpha^2+|u|^2)^2} + \frac{4|v|^2|\nabla v|^2}{(\alpha^2+|v|^2)^2}\right).
\end{aligned}$$

We further impose that $r \leqslant \frac{\alpha}{4}$. Then by (5.3)

$$\frac{4|u|^2}{\alpha^2+|u|^2}, \frac{4|v|^2}{\alpha^2+|v|^2} \leqslant \frac{4r^2}{\alpha^2} \leqslant \frac{1}{4} \tag{5.5}$$

and

$$|B_2| \leqslant \frac{G_1}{2} + |u-v|^2 \left(\frac{1}{4}\frac{|\nabla u|^2}{\alpha^2+|u|^2} + \frac{1}{4}\frac{|\nabla v|^2}{\alpha^2+|v|^2}\right) = \frac{G_1}{2} + \frac{G_2}{4}. \tag{5.6}$$

Estimation of B_4

We first write

$$|B_4| \leqslant 2|u-v|^2 \left(\frac{|u|^2}{\alpha^2+|u|^2}\frac{|\nabla u|^2}{\alpha^2+|u|^2} + \frac{|v|^2}{\alpha^2+|v|^2}\frac{|\nabla v|^2}{\alpha^2+|v|^2}\right),$$

and using the fact that $\frac{|u|^2}{\alpha^2+|u|^2}, \frac{|v|^2}{\alpha^2+|v|^2} \leqslant \frac{1}{16}$ because of (5.5), we deduce that

$$|B_4| \leqslant \frac{|u-v|^2}{8} \left(\frac{|\nabla u|^2}{\alpha^2+|u|^2} + \frac{|\nabla v|^2}{\alpha^2+|v|^2}\right) = \frac{G_2}{8}. \tag{5.7}$$

Estimation of B_3

We use that

$$|\Delta u| = |\Gamma(u)(\nabla u \otimes \nabla u)| \leqslant C_\Gamma |u|\,|\nabla u|^2$$

and thus $|\langle u, \Delta u\rangle| \leqslant |u|\,|\Delta u| \leqslant C_\Gamma |u|^2 |\nabla u|^2$ and similarly $|\langle v, \Delta v\rangle| \leqslant C_\Gamma |v|^2|\nabla v|^2$. Hence by (5.3)

$$\begin{aligned}
|B_3| &\leqslant C_\Gamma |u-v|^2 \left(|u|^2 \frac{|\nabla u|^2}{\alpha^2+|u|^2} + |v|^2 \frac{|\nabla v|^2}{\alpha^2+|v|^2}\right) \\
&\leqslant C_\Gamma r^2 |u-v|^2 \left(\frac{|\nabla u|^2}{\alpha^2+|u|^2} + \frac{|\nabla v|^2}{\alpha^2+|v|^2}\right).
\end{aligned}$$

We further require on r that $C_\Gamma r^2 \leqslant \frac{1}{4}$. Then

$$|B_3| \leqslant \frac{G_2}{4}. \tag{5.8}$$

Conclusion

By choosing

$$\alpha \leqslant \frac{1}{2\sqrt{2C_\Gamma^2 R^2 + C_\Gamma}}, \quad r \leqslant \inf\left(R, \frac{\alpha}{4}, \frac{1}{2\sqrt{C_\Gamma}}\right), \tag{5.9}$$

we obtain using (5.4), (5.6), (5.7) and (5.8) that

$$\begin{aligned} \operatorname{div}(\rho\nabla f) &= G_1 + G_2 + B_1 + B_2 + B_3 + B_4 \\ &\geqslant G_1 + G_2 - \left(\frac{G_1}{4} + \frac{G_2}{4}\right) - \left(\frac{G_1}{2} + \frac{G_2}{4}\right) - \frac{G_2}{4} - \frac{G_2}{8} \\ &= \frac{G_1}{4} + \frac{G_2}{8} \geqslant 0. \end{aligned}$$

Hence (5.2) follows by choosing $U_{M_0} := \{M \in U'_{M_0} / |\phi(M)| < r\}$ where r satisfies (5.9). $\square$

6. A result related to capacity

We prove here the following result.

LEMMA 6.1. — *Let Ω be an open subset of $\mathbb{R}^m$, for $m \geqslant 2$. Let $\rho \in \mathcal{C}^1(\Omega, \mathbb{R})$ be a function satisfying $0 < A \leqslant \rho \leqslant B < \infty$. Let $a \in \Omega$, $\varepsilon_0 > 0$ such that $\overline{B^m(a, \varepsilon_0)} \subset \Omega$ and, for all $\varepsilon \in (0, \varepsilon_0)$, $\Omega_\varepsilon := \Omega \setminus \overline{B^m(a, \varepsilon)}$.*

Let $(\phi_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ be a family of functions $\phi_\varepsilon \in \mathcal{C}^2(\Omega_\varepsilon) \cap \mathcal{C}^0(\overline{\Omega_\varepsilon})$ such that

$$\left\{ \begin{array}{rcll} \phi_\varepsilon &=& M & \textit{on } \partial B^m(a,\varepsilon) \\ \phi_\varepsilon &=& 0 & \textit{on } \partial\Omega \\ -\mathit{div}(\rho\nabla\phi_\varepsilon) &=& 0 & \textit{on } \Omega_\varepsilon, \end{array} \right. \tag{6.1}$$

where $M > 0$ is a constant independant of ε. Then for all compact $K \subset \Omega \setminus \{a\}$ and for $\varepsilon \in (0, \varepsilon_0)$ such that $K \subset \Omega_\varepsilon$, the restriction of ϕ_ε on K converges to 0 in $L^1(K)$ when ε tends to 0.

Proof. — A first step consists in proving that the energy of ϕ_ε,

$$E_\varepsilon := \mathcal{A}_\varepsilon[\phi_\varepsilon] := \int_{\Omega_\varepsilon} \rho |\nabla \phi_\varepsilon|^2 \, dx$$

converges to 0 when ε tends to 0. This is a very standard result which can be checked as follows: we know that ϕ_ε is energy minimizing and hence that $E_\varepsilon \leqslant \mathcal{A}_\varepsilon[f]$, for all $f \in \mathcal{C}^2(\Omega_\varepsilon) \cap \mathcal{C}^0(\overline{\Omega_\varepsilon})$ such that $f = M$ on $\partial B^m(a,\varepsilon)$ and $f = 0$ on $\partial\Omega$. One can choose for f $v_\varepsilon(x) := M\chi(x)G_\varepsilon(x-a)$, where $\chi \in \mathcal{C}^2(\Omega_\varepsilon)$ satisfies $0 \leqslant \chi \leqslant 1$, $\mathrm{supp}\chi \subset B^m(a,\varepsilon_0)$, $\chi = 1$ on $B^m(a,\varepsilon_0/2)$ and $|\nabla\chi| \leqslant 4/\varepsilon_0$ and where G_ε is the Green function on $\mathbb{R}^m$ ($G_\varepsilon(x) := \frac{\log|x|}{\log\varepsilon}$ if $m = 2$ and $G_\varepsilon(x) := \frac{\varepsilon^{m-2}}{|x|^{m-2}}$ if $m \geqslant 3$). Then a straightforward computation shows that $\lim_{\varepsilon\to 0} \mathcal{A}_\varepsilon[v_\varepsilon] = 0$. Hence

$$\lim_{\varepsilon\to 0} E_\varepsilon = 0. \tag{6.2}$$

Second we consider, for all $s \in [0,M]$, the level sets

$$\Omega_\varepsilon^s := \{x \in \Omega_\varepsilon / \phi_\varepsilon(x) > s\}.$$

Note that the maximum principle implies that ϕ_ε takes values in $[0,M]$. Sard's Theorem implies that the set $V_c := \{s \in [0,M] / \exists x \in \Omega_\varepsilon$ such that $\phi_\varepsilon(x) = s$ and $\nabla\phi_\varepsilon(x) = 0\}$ (critical values) is negligible (moreover it is also closed[(1)]). And $\forall s \in [0,M] \setminus V_c$, $\partial\Omega_\varepsilon^s = \{x \in \Omega_\varepsilon / \phi_\varepsilon(x) = s\}$ is a smooth submanifold. We let Γ^s be the exterior part of $\partial\Omega_\varepsilon^s$ so that we have the splitting

$$\partial\Omega_\varepsilon^s = \Gamma^s \cup \partial B^m(a,\varepsilon).$$

Using the equation (6.1) we observe that, $\forall s,s' \in [0,M] \setminus V_c$ such that $s < s'$,

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon^s \setminus \Omega_\varepsilon^{s'}} -\mathrm{div}\,(\rho\nabla\phi_\varepsilon)\,dx \\ &= \int_{\Gamma^s} -\rho\langle\nabla\phi_\varepsilon,\nu\rangle d\mathcal{H}^{m-1} + \int_{\Gamma^{s'}} \rho\langle\nabla\phi_\varepsilon,\nu\rangle d\mathcal{H}^{m-1} \\ &= \int_{\Gamma^s} \rho|\nabla\phi_\varepsilon| d\mathcal{H}^{m-1} - \int_{\Gamma^{s'}} \rho|\nabla\phi_\varepsilon| d\mathcal{H}^{m-1}, \end{aligned}$$

where $d\mathcal{H}^{m-1}$ is the $(m-1)$-dimensional Hausdorff measure. Here we have used in the last line the fact that, on Γ^s and $\Gamma^{s'}$, $\nabla\phi_\varepsilon$ is parallel (and of opposite orientation) to the normal vector ν. This implies that the function

$$[0,M] \setminus V_c \ni s \longmapsto \int_{\Gamma^s} \rho|\nabla\phi_\varepsilon| d\mathcal{H}^{m-1} \quad \text{is constant.} \tag{6.3}$$

We now use the coarea formula to obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} \rho|\nabla\phi_\varepsilon|^2 dx &= \int_0^M ds \int_{\Gamma^s} \rho|\nabla\phi_\varepsilon|^2 \frac{d\mathcal{H}^{m-1}}{|\nabla\phi_\varepsilon|} \\ &= \int_0^M ds \int_{\Gamma^s} \rho|\nabla\phi_\varepsilon| d\mathcal{H}^{m-1}. \end{aligned}$$

(1) and $V_c \subset (0,M)$ because of the Hopf maximum principle

Thus we deduce that, using (6.3),

$$\forall s \in [0,M] \setminus V_c, \quad \int_{\Gamma^s} \rho|\phi_\varepsilon| d\mathcal{H}^{m-1} = \frac{1}{M}\int_{\Omega_\varepsilon} \rho|\nabla\phi_\varepsilon|^2 dx = \frac{E_\varepsilon}{M}. \tag{6.4}$$

We now let $F_\varepsilon : [0,M] \longrightarrow [0,\infty)$ be the function defined by

$$F_\varepsilon(s) := |\Omega_\varepsilon^s \cup \overline{B^m(a,\varepsilon)}|, \quad \text{the Lebesgue measure of } \Omega_\varepsilon^s \cup \overline{B^m(a,\varepsilon)}.$$

Obviously F_ε is a decreasing function and so F'_ε is a nonpositive measure. We can decompose this measure as $F'_\varepsilon = (F'_\varepsilon)_a + (F'_\varepsilon)_s$, where $(F'_\varepsilon)_a$ is the absolutely continuous part of F'_ε and $(F'_\varepsilon)_s$ is the singular part of F'_ε. Moreover F_ε is differentiable on $[0,M] \setminus V_c$ with

$$\forall s \in [0,M] \setminus V_c, \quad F'_\varepsilon(s) = -\int_{\Gamma^s} \frac{d\mathcal{H}^{m-1}}{|\nabla\phi_\varepsilon|}$$

and $\operatorname{supp}(F'_\varepsilon)_s \subset V_c$. We deduce from this identity and from (6.4), by using the Cauchy–Schwarz inequality, that $\forall s \in [0,M] \setminus V_c$,

$$\begin{aligned}
|\Gamma^s| &= \int_{\Gamma^s} d\mathcal{H}^{m-1} = \int_{\Gamma^s} \sqrt{|\nabla\phi_\varepsilon|}\frac{d\mathcal{H}^{m-1}}{\sqrt{|\nabla\phi_\varepsilon|}} \\
&\leqslant \sqrt{\int_{\Gamma^s} |\nabla\phi_\varepsilon| d\mathcal{H}^{m-1}}\sqrt{\int_{\Gamma^s} \frac{d\mathcal{H}^{m-1}}{|\nabla\phi_\varepsilon|}} \\
&\leqslant \sqrt{\frac{1}{A}\int_{\Gamma^s} \rho|\nabla\phi_\varepsilon| d\mathcal{H}^{m-1}}\sqrt{-F'_\varepsilon(s)} \\
&\leqslant \sqrt{\frac{E_\varepsilon}{AM}}\sqrt{-F'_\varepsilon(s)}.
\end{aligned}$$

Hence

$$\forall s \in [0,M] \setminus V_c, \quad |\Gamma^s|^2 \leqslant -\frac{E_\varepsilon}{AM}F'_\varepsilon(s). \tag{6.5}$$

We observe that this inequality extends on the whole interval $[0,M]$ in the sense of measure: V_c is Lebesgue negligeable and if s is a singular point of F'_ε, then the above inequality holds since the left hand side is a function. We next exploit (6.5) together with the isoperimetric inequality for the subset $\Omega_\varepsilon^s \cup \overline{B^m(a,\varepsilon)} \subset \mathbb{R}^m$ and its boundary Γ^s:

$$m^{m-1}\omega_m F_\varepsilon(s)^{m-1} = m^{m-1}\omega_m |\Omega_\varepsilon^s \cup \overline{B^m(a,\varepsilon)}|^{m-1} \leqslant |\Gamma^s|^m. \tag{6.6}$$

Then (6.5) and (6.6) imply

$$F'_\varepsilon + k_\varepsilon F_\varepsilon^{2(m-1)/m} \leqslant 0, \quad \text{with } k_\varepsilon := \frac{AM\left(m^{m-1}\omega_m\right)^{2/m}}{E_\varepsilon}, \tag{6.7}$$

in the sense of measure on $[0,M]$.

The case $m = 2$

Equation (6.7) then implies

$$\forall s \in [0, M], \quad F_\varepsilon(s) \leqslant F_\varepsilon(0)e^{-k_\varepsilon s} = |\Omega| e^{-k_\varepsilon s}.$$

Hence for any compact $K \subset \Omega \setminus \{a\}$ and for ε small enough, by using the coarea formula, we have

$$||\phi_\varepsilon||_{L^1(K)} = \int_0^M |K \cap \Omega_\varepsilon^s| ds \leqslant \int_0^M F_\varepsilon(s) ds \leqslant \frac{|\Omega|}{k_\varepsilon}\left(1 - e^{-k_\varepsilon M}\right).$$

which implies that $||\phi_\varepsilon||_{L^1(K)}$ tends to 0 when $\varepsilon \longrightarrow 0$ because k_ε tends to ∞, because of (6.2). Hence the Lemma is proved in this case.

The case $m \geqslant 2$

Let us denote by $\beta := 2^{\frac{m-1}{m}} - 1 \in (0, 1)$. We deduce analogously to the preceding case that

$$\forall s \in [0, M], \quad F_\varepsilon(s) \leqslant \frac{|\Omega|}{(1 + \beta|\Omega|^\beta k_\varepsilon s)^{1/\beta}}$$

and thus

$$||\phi_\varepsilon||_{L^1(K)} \leqslant \frac{|\Omega|^{1-\beta}}{(1-\beta)k_\varepsilon}\left(1 - \frac{1}{(1 + \beta|\Omega|^\beta k_\varepsilon M)^{1/\beta - 1}}\right),$$

which leads to the same conclusion. $\square$

7. The proof of the main theorem

We conclude this paper by proving Theorem 1.1. Let u be a continuous map from Ω to $\mathcal{N}$ and assume that u is $\mathcal{C}^2$ and harmonic with values in $(\mathcal{N}, h_{ij})$ on $\Omega \setminus \{a\}$. Using Theorem 5.1 with $M_0 = u(a)$ we deduce that there exists a neighbourhood $U_{u(a)}$ of $u(a)$ in $\mathcal{N}$ such (5.2) holds for any pair of harmonic maps into $(U_{\phi(a)}, h_{ij})$. We hence can restrict u to a ball $B^m(a, R)$, where R is chosen so that $u(B^m(a, R)) \subset U_{u(a)}$. Then we use the existence result 3: we deduce that there exists some $R_2 \in (0, R)$ such that, for any $r \in (0, R_2)$ there exists a map $\underline{u} \in \mathcal{C}^{2,\alpha}(B^m(a, r)) \cap \mathcal{C}^{0,\alpha}(\overline{B^m(a, r)})$ which is harmonic into $(U_{u(a)}, h_{ij})$ and which coincides with u on $\partial B^m(a, r)$. Then we choose some $r \in (0, R_2)$ and we identify $u \simeq \phi \circ u$ and $\underline{u} \simeq \phi \circ \underline{u}$

as in Theorem 5.1. Note that it is clear that there exists some $A \in (0, \infty)$ such that $|u|, |\underline{u}| \leqslant A$ on $B^m(a,r)$. Now let

$$f(x) := (\alpha^2 + |u(x)|^2)(\alpha^2 + |\underline{u}(x)|^2)\frac{|u(x) - \underline{u}(x)|^2}{2}, \quad \forall x \in B^m(a,r).$$

where α has been chosen as in Theorem 5.1. For any $\varepsilon > 0$ such that $\overline{B^m(a,\varepsilon)} \subset B^m(a,r)$, we consider the map $\phi_\varepsilon \in \mathcal{C}^2(B^m(a,r) \setminus \overline{B^m(a,\varepsilon)}) \cap \mathcal{C}^0(\overline{B^m(a,r)} \setminus B^m(a,\varepsilon))$ which is the solution to

$$\left\{ \begin{array}{rcll} \phi_\varepsilon & = & M & \text{on } \partial B^m(a,\varepsilon) \\ \phi_\varepsilon & = & 0 & \text{on } \partial B^m(a,r) \\ -\mathrm{div}\,(\rho\nabla\phi_\varepsilon) & = & 0 & \text{on } B^m(a,r) \setminus \overline{B^m(a,\varepsilon)}, \end{array} \right.$$

where

$$\rho(x) := \frac{1}{\alpha^2 + |u(x)|^2}\frac{1}{\alpha^2 + |\underline{u}(x)|^2}, \quad \forall x \in B^m(a,r)$$

and

$$M := 2A^2(\alpha^2 + A^2)(\alpha^2 + A^2).$$

Clearly we have $f \leqslant \phi_\varepsilon$ on $\partial(B^m(a,r) \setminus \overline{B^m(a,\varepsilon)})$ and Theorem 5.1 implies that $-\mathrm{div}\,(\rho\nabla f) \leqslant 0 = -\mathrm{div}\,(\rho\nabla\phi_\varepsilon)$ on $B^m(a,r) \setminus \overline{B^m(a,\varepsilon)}$. Hence the maximum principle implies that $f \leqslant \phi_\varepsilon$ on $B^m(a,r) \setminus \overline{B^m(a,\varepsilon)}$. Now if we fix a compact subset $K \subset B^m(a,r) \setminus \{a\}$ and suppose that ε is sufficiently small so that $K \subset B^m(a,r) \setminus \overline{B^m(a,\varepsilon)}$, the inequality $f \leqslant \phi_\varepsilon$ on K implies

$$||f||_{L^1(K)} \leqslant ||\phi_\varepsilon||_{L^1(K)}.$$

Letting ε tend to 0 and using Lemma 6.1 we deduce that $||f||_{L^1(K)} = 0$. Since K is arbitrary and f is continuous on $B^m(a,r)$, we conclude $f = 0$ on $B^m(a,r)$. Hence u coincides with $\underline{u}$ on $B^m(a,r)$. Thus u is $\mathcal{C}^{2,\alpha}$ on $B^m(a,r)$. □

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