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Galois representations [(*)]

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ABSTRACT. — In the first part of this paper we try to explain to a general mathematical audience some of the remarkable web of conjectures linking representations of Galois groups with algebraic geometry, complex analysis and discrete subgroups of Lie groups. In the second part we briefly review some limited recent progress on these conjectures.

RÉSUMÉ. — Dans la première partie nous essayons d'expliquer à un public mathématique général le remarquable faisceau de conjectures reliant les représentations Galoisiennes avec la géométrie algébrique, l'analyse complexe et les sous-groupes discrets des groupes de Lie. Dans la deuxième partie nous mentionnons des progrès récents mais limités sur ces conjectures.

0. Introduction

This is a longer version of my talk at the Beijing ICM. The version to be published in the proceedings of the ICM was edited in an attempt to make it meet restrictions on length suggested by the publishers. In this version those cuts have been restored and I have added technical justifications for a couple of results stated in the published version in a form slightly different from that which can be found in the literature.

The first four sections of this paper contain a simple presentation of a web of deep conjectures connecting Galois representations to algebraic geometry, complex analysis and discrete subgroups of Lie groups. This will be of no interest to the specialist. My hope is that the result is not too banal

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and that it will give the non-specialist some idea of what motivates work in this area. I should stress that nothing I write here is original. In the final section I briefly review some of what is known about these conjectures and *very briefly* mention some of the available techniques.

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1. Galois representations

We will let $\mathbb{Q}$ denote the field of rational numbers and $\overline{\mathbb{Q}}$ denote the field of algebraic numbers, the algebraic closure of $\mathbb{Q}$. We will also let $G_{\mathbb{Q}}$ denote the group of automorphisms of $\overline{\mathbb{Q}}$, that is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the absolute Galois group of $\mathbb{Q}$. Although it is not the simplest it is arguably the most natural Galois group to study. An important technical point is that $G_{\mathbb{Q}}$ is naturally a topological group, a basis of open neighbourhoods of the identity being given by the subgroups $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ as K runs over subextensions of $\overline{\mathbb{Q}}/\mathbb{Q}$ which are finite over $\mathbb{Q}$. In fact $G_{\mathbb{Q}}$ is a profinite group, being identified with the inverse limit of discrete groups $\lim_{\leftarrow} \mathrm{Gal}(K/\mathbb{Q})$, where K runs over finite normal subextensions of $\overline{\mathbb{Q}}/\mathbb{Q}$.

The Galois theory of $\mathbb{Q}$ is most interesting when one looks not only at $G_{\mathbb{Q}}$ as an abstract (topological) group, but as a group with certain additional structures associated to the prime numbers. I will now briefly describe these structures.

For each prime number p we may define an absolute value $|\ |_p$ on $\mathbb{Q}$ by setting

$$|\alpha|_p = p^{-r}$$

if $\alpha = p^r a/b$ with a and b integers coprime to p. If we complete $\mathbb{Q}$ with respect to this absolute value we obtain the field $\mathbb{Q}_p$ of p-adic numbers, a totally disconnected, locally compact topological field. We will write $G_{\mathbb{Q}_p}$ for its absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. The absolute value $|\ |_p$ has a unique extension to an absolute value on $\overline{\mathbb{Q}}_p$ and $G_{\mathbb{Q}_p}$ is identified with the group of automorphisms of $\overline{\mathbb{Q}}_p$ which preserve $|\ |_p$, or equivalently the group of continuous automorphisms of $\overline{\mathbb{Q}}_p$. For each embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ we obtain a closed embedding $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$ and as the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ varies we obtain a conjugacy class of closed embeddings $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$. Slightly abusively we shall consider $G_{\mathbb{Q}_p}$ a closed subgroup of $G_{\mathbb{Q}}$, suppressing the fact that the embedding is only determined up to conjugacy.

This can be compared with the situation 'at infinity'. Let $|\ |_\infty$ denote the usual Archimedean absolute value on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\ |_\infty$ is the field of real numbers $\mathbb{R}$ and its algebraic closure is $\mathbb{C}$ the field of complex numbers. Each embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ gives rise to a closed embedding

$$\{1, c\} = G_{\mathbb{R}} = \mathrm{Gal}\,(\mathbb{C}/\mathbb{R}) \hookrightarrow G_{\mathbb{Q}}.$$

As the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ varies one obtains a conjugacy class of elements $c \in G_{\mathbb{Q}}$ of order 2, which we refer to as complex conjugations.

There are however many important differences between the case of finite places (i.e. primes) and the infinite place $|\ |_\infty$. For instance $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ is an infinite extension and $\overline{\mathbb{Q}}_p$ is not complete. We will denote its completion by $\mathbb{C}_p$. The Galois group $G_{\mathbb{Q}_p}$ acts on $\mathbb{C}_p$ and is in fact the group of continuous automorphisms of $\mathbb{C}_p$.

The elements of $\mathbb{Q}_p$ (resp. $\overline{\mathbb{Q}}_p$, resp. $\mathbb{C}_p$) with absolute value less than or equal to 1 form a closed subring $\mathbb{Z}_p$ (resp. $\mathcal{O}_{\overline{\mathbb{Q}}_p}$, resp. $\mathcal{O}_{\mathbb{C}_p}$). These rings are local with maximal ideals $p\mathbb{Z}_p$ (resp. $\mathfrak{m}_{\overline{\mathbb{Q}}_p}$, resp. $\mathfrak{m}_{\mathbb{C}_p}$) consisting of the elements with absolute value strictly less than 1. The field $\mathcal{O}_{\overline{\mathbb{Q}}_p}/\mathfrak{m}_{\overline{\mathbb{Q}}_p} = \mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p}$ is an algebraic closure of the finite field with p elements $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$, and we will denote it by $\overline{\mathbb{F}}_p$. Thus we obtain a continuous map

$$G_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{F}_p}$$

which is surjective. Its kernel is called the inertia subgroup of $G_{\mathbb{Q}_p}$ and is denoted $I_{\mathbb{Q}_p}$. The group $G_{\mathbb{F}_p}$ is procyclic and has a canonical generator called the (geometric) Frobenius element and defined by

$$\mathrm{Frob}_p^{-1}(x) = x^p.$$

In many circumstances it is technically convenient to replace $G_{\mathbb{Q}_p}$ by a dense subgroup $W_{\mathbb{Q}_p}$, which is referred to as the Weil group of $\mathbb{Q}_p$ and which is defined as the subgroup of $\sigma \in G_{\mathbb{Q}_p}$ such that σ maps to

$$\mathrm{Frob}_p^{\mathbb{Z}} \subset G_{\mathbb{F}_p}.$$

We endow $W_{\mathbb{Q}_p}$ with a topology by decreeing that $I_{\mathbb{Q}_p}$ with its usual topology should be an open subgroup of $W_{\mathbb{Q}_p}$.

We will take a moment to describe some of the finer structure of $I_{\mathbb{Q}_p}$ which we will need for technical purposes later. First of all there is a (not quite canonical) continuous surjection

$$t : I_{\mathbb{Q}_p} \twoheadrightarrow \prod_{l \neq p} \mathbb{Z}_l$$

such that

$$t(\mathrm{Frob}_p \sigma \mathrm{Frob}_p^{-1}) = p^{-1} t(\sigma)$$

for all $\sigma \in I_{\mathbb{Q}_p}$. The kernel of t is a pro-p-group called the wild inertia group. The fixed field $\overline{\mathbb{Q}}_p^{\ker t}$ is obtained by adjoining $\sqrt[n]{p}$ to $\overline{\mathbb{Q}}_p^{I_{\mathbb{Q}_p}}$ for all n coprime to p. Moreover

$$\sigma \sqrt[n]{p} = \zeta_n^{t(\sigma)} \sqrt[n]{p},$$

for some primitive n^{th}-root of unity ζ_n (independent of σ, but dependent on t). Also there is a natural decreasing filtration $I_{\mathbb{Q}_p}^u$ of $I_{\mathbb{Q}_p}$ indexed by $u \in [0, \infty)$ and satisfying

- $I_{\mathbb{Q}_p}^0 = I_{\mathbb{Q}_p}$,
- $\bigcup_{u>0} I_{\mathbb{Q}_p}^u$ is the wild inertia group,
- $\bigcap_{u<v} I_{\mathbb{Q}_p}^u = I_{\mathbb{Q}_p}^v$.
- $\bigcap_u I_{\mathbb{Q}_p}^u = \{1\}$,

This is called the upper numbering filtration. We refer the reader to [Se1] for the precise definition.

In my opinion the most interesting question about $G_{\mathbb{Q}}$ is to describe it together with the distinguished subgroups $G_{\mathbb{R}}$, $G_{\mathbb{Q}_p}$, $I_{\mathbb{Q}_p}$ and the distinguished elements $\mathrm{Frob}_p \in G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$.

I want to focus here on attempts to describe $G_{\mathbb{Q}}$ via its representations. Perhaps the most obvious representations to consider are those representations

$$G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$$

with open kernel, and these so called Artin representations are already very interesting. However one obtains a richer theory if one considers representations

$$G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

which are continuous with respect to the l-adic topology on $GL_n(\overline{\mathbb{Q}}_l)$. We refer to these as l-adic representations.

One justification for considering l-adic representations is that they arise naturally from geometry. Here are some examples of l-adic representations.

1. A choice of embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$ establishes a bijection between isomorphism classes of Artin representations and isomorphism classes of l-adic representations with open kernel. Thus Artin representations are a special case of l-adic representations.

2. There is a a unique character
$$\chi_l : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_l^\times \subset \overline{\mathbb{Q}}_l^\times$$
such that
$$\sigma\zeta = \zeta^{\chi_l(\sigma)}$$
for all l-power roots of unity ζ. This is called the l-adic *cyclotomic character.*

3. If $X/\mathbb{Q}$ is a smooth projective variety (and we choose an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$) then the natural action of $G_{\mathbb{Q}}$ on the cohomology
$$H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l) \cong H^i_{\text{et}}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_l)$$
is an l-adic representation. For instance if $E/\mathbb{Q}$ is an elliptic curve then we have the concrete description
$$H^1_{\text{et}}(E \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_l) \cong \mathrm{Hom}_{\mathbb{Z}_l}(\lim_{\leftarrow r} E[l^r](\overline{\mathbb{Q}}), \overline{\mathbb{Q}}_l) \cong \overline{\mathbb{Q}}_l^2,$$
where $E[l^r]$ denotes the l^r-torsion points on E. We will write $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l(j))$ for the twist
$$H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l) \otimes \chi_l^j.$$

Before discussing l-adic representations of $G_{\mathbb{Q}}$ further, let us take a moment to look at l-adic representations of $G_{\mathbb{Q}_p}$. The cases $l \neq p$ and $l = p$ are very different. Consider first the much easier case $l \neq p$. Here l-adic representations of $G_{\mathbb{Q}_p}$ are not much different from representations of $W_{\mathbb{Q}_p}$ with open kernel. More precisely define a Weil-Deligne (or simply, WD-) representation of $W_{\mathbb{Q}_p}$ over a field E of characteristic zero to be a pair
$$r : W_{\mathbb{Q}_p} \longrightarrow GL(V)$$
and
$$N \in \mathrm{End}\,(V),$$
where V is a finite dimensional E-vector space, r is a representation with open kernel and N is a nilpotent endomorphism which satisfies
$$r(\phi)Nr(\phi^{-1}) = p^{-1}N$$

for every lift $\phi \in W_{\mathbb{Q}_p}$ of Frob_p. The key point here is that there is no reference to a topology on E, indeed no assumption that E is a topological field. Given r there are up to isomorphism only finitely many choices for the pair (r, N) and these can be explicitly listed without difficulty. A WD-representation (r, N) is called *unramified* if $N = 0$ and $r(I_{\mathbb{Q}_p}) = \{1\}$. It is called *Frobenius semi-simple* if r is semi-simple. Any WD-representation (r, N) has a canonical *Frobenius semi-simplification* $(r, N)^{\mathrm{ss}}$, which may be defined as follows. Pick a lift ϕ of Frob_p to $W_{\mathbb{Q}_p}$ and decompose $r(\phi) = \Phi_s\Phi_u = \Phi_u\Phi_s$ where Φ_s is semi-simple and Φ_u is unipotent. The semi-simplification $(r, N)^{\mathrm{ss}}$ is obtained by keeping N and $r|_{I_p}$ unchanged and replacing $r(\phi)$ by Φ_s. In the case that $E = \overline{\mathbb{Q}}_l$ we call (r, N) *l-integral* if all the eigenvalues of $r(\phi)$ have absolute value 1. This is independent of the choice of Frobenius lift ϕ.

If $l \neq p$, then there is an equivalence of categories between l-integral WD-representations of $W_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}_l$ and l-adic representations of $G_{\mathbb{Q}_p}$. To describe it choose a Frobenius lift $\phi \in W_{\mathbb{Q}_p}$ and a surjection $t_l : I_{\mathbb{Q}_p} \twoheadrightarrow \mathbb{Z}_l$. Up to natural isomorphism the equivalence does not depend on these choices. We associate to an l-integral WD-representation (r, N) the unique l-adic representation sending

$$\phi^n\sigma \longmapsto r(\phi^n\sigma)\exp(t_l(\sigma)N)$$

for all $n \in \mathbb{Z}$ and $\sigma \in I_{\mathbb{Q}_p}$. The key point is Grothendieck's observation that for $l \neq p$ any l-adic representation of $G_{\mathbb{Q}_p}$ must be trivial on some open subgroup of the wild inertia group. We will write $\mathrm{WD}_p(R)$ for the WD-representation associated to an l-adic representation R. Note that $\mathrm{WD}_p(R)$ is unramified if and only if $R(I_{\mathbb{Q}_p}) = \{1\}$. In this case we call R *unramified.*

The case $l = p$ is much more complicated because there are many more p-adic representations of $G_{\mathbb{Q}_p}$. These have been extensively studied by Fontaine and his coworkers. They single out certain p-adic representations which they call de Rham representations. I will not recall the somewhat involved definition here (see however [Fo2] and [Fo3]), but note that 'most' p-adic representations of $G_{\mathbb{Q}_p}$ are not de Rham. To any de Rham representation R of $G_{\mathbb{Q}_p}$ on a $\overline{\mathbb{Q}}_p$-vector space V they associate the following.

1. A WD-representation $\mathrm{WD}_p(R)$ of $W_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}_p$ (see [Berg] and [Fo4]). (We recall some of the definition of $\mathrm{WD}_p(R)$. By the main result of [Berg] one can find a finite Galois extension $L/\mathbb{Q}_p$ such that, in the notation of [Fo3], $\underline{D}_{st,L}(R)$ is a free $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} L_0$-module of rank $\dim_{\mathbb{Q}_p} R$, where $L_0/\mathbb{Q}_p$ is the maximal unramified subextension of $L/\mathbb{Q}_p$. Then $\underline{D}_{st,L}(R)$ comes equipped with a semilinear action of $\mathrm{Gal}\,(L/\mathbb{Q}_p)$ (σ

acts $1 \otimes \sigma$-linearly), a $1 \otimes \mathrm{Frob}_p^{-1}$-linear automorphism ϕ and a nilpotent linear endomorphism N. The $\mathrm{Gal}\,(L/\mathbb{Q}_p)$-action commutes with ϕ and N and $\phi N \phi^{-1} = pN$. Define a linear action r of $W_{\mathbb{Q}_p}$ on $\underline{D}_{st,L}(R)$ with open kernel by setting $r(\sigma) = \phi^a \sigma$ if σ maps to Frob_p^a in $G_{\mathbb{F}_p}$. If $\tau : L_0 \hookrightarrow \overline{\mathbb{Q}}_p$ set $\mathrm{WD}_p(R)_\tau = (r, N) \otimes_{\overline{\mathbb{Q}}_p \otimes L_0, 1 \otimes \tau} \overline{\mathbb{Q}}_p$. The map ϕ provides an isomorphism from $\mathrm{WD}_p(R)_\tau$ to $\mathrm{WD}_p(R)_{\tau \mathrm{Frob}_p}$, and so up to equivalence $\mathrm{WD}_p(R)_\tau$ is independent of τ. Finally set $\mathrm{WD}_p(R) = \mathrm{WD}_p(R)_\tau$ for any τ.)

2. A multiset $\mathrm{HT}(R)$ of $\dim V$ integers, called the Hodge-Tate numbers of R. The multiplicity of i in $\mathrm{HT}(R)$ is

$$\dim_{\overline{\mathbb{Q}}_p} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(i))^{G_{\mathbb{Q}_p}},$$

where $G_{\mathbb{Q}_p}$ acts on $\mathbb{C}_p(i)$ via $\chi_p(\sigma)^i$ times its usual (Galois) action on $\mathbb{C}_p$.

A famous theorem of Cebotarev asserts that if $K/\mathbb{Q}$ is a Galois extension (possibly infinite) unramified outside a finite set of primes S (i.e. if $p \notin S$ the $I_{\mathbb{Q}_p}$ has trivial image in $\mathrm{Gal}\,(K/\mathbb{Q})$) then

$$\bigcup_{p \notin S} [\mathrm{Frob}_p]$$

is dense in $\mathrm{Gal}\,(K/\mathbb{Q})$. (Here $[\mathrm{Frob}_p]$ denotes the conjugacy class of Frob_p in $\mathrm{Gal}\,(K/\mathbb{Q})$.) It follows that a semi-simple l-adic representation R which is unramified outside a finite set S of primes is determined by $\{\mathrm{WD}_p(R)^{\mathrm{ss}}\}_{p \notin S}$.

We now return to the global situation (i.e. to the study of $G_{\mathbb{Q}}$). The l-adic representations of $G_{\mathbb{Q}}$ that arise from geometry, have a number of very special properties which I will now list. Let $R : G_{\mathbb{Q}} \longrightarrow GL(V)$ be a subquotient of $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l(j))$ for some smooth projective variety $X/\mathbb{Q}$ and some integers $i \geqslant 0$ and j.

1. (Grothendieck, [SGA4], [SGA5]) The representation R is unramified outside a finite set of primes.

2. (Fontaine, Messing, Faltings, Kato, Tsuji, de Jong, see e.g. [Il], [Bert]) The representation R is de Rham in the sense that its restriction to $G_{\mathbb{Q}_l}$ is de Rham.

3. (Deligne, [De3]) The representation R is pure of weight $w = i - 2j$ in the following sense. There is a finite set of primes S, such that

for $p \notin S$, the representation R is unramified at p and for every eigenvalue α of $R(\mathrm{Frob}_p)$ and every embedding $\iota : \overline{\mathbb{Q}}_l \hookrightarrow \mathbb{C}$

$$|\iota\alpha|_\infty^2 = p^w.$$

In particular α is algebraic (i.e. $\alpha \in \overline{\mathbb{Q}}$).

A striking conjecture of Fontaine and Mazur (see [Fo1] and [FM]) asserts that any irreducible l-adic representation of $G_\mathbb{Q}$ satisfying the first two of these properties arises from geometry in the above sense and so in particular also satisfies the third property.

CONJECTURE 1.1 (FONTAINE-MAZUR). — *Suppose that*

$$R : G_\mathbb{Q} \longrightarrow GL(V)$$

is an irreducible l-adic representation which is unramified at all but finitely many primes and with $R|_{G_{\mathbb{Q}_l}}$ de Rham. Then there is a smooth projective variety $X/\mathbb{Q}$ and integers $i \geqslant 0$ and j such that V is a subquotient of $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l(j))$. In particular R is pure of some weight $w \in \mathbb{Z}$.

We will discuss the evidence for this conjecture later. We will call an l-adic representation satisfying the conclusion of this conjecture *geometric.*

Algebraic geometers have formulated some very precise conjectures about the action of $G_\mathbb{Q}$ on the cohomology of varieties. We don't have the space here to discuss these in general, but we will formulate some of them as algebraically as possible.

CONJECTURE 1.2 (TATE). — *Suppose that $X/\mathbb{Q}$ is a smooth projective variety. Then there is a decomposition*

$$H^i(X(\mathbb{C}), \overline{\mathbb{Q}}) = \bigoplus_j M_j$$

with the following properties.

1. *For each prime l and for each embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$, $M_j \otimes_{\overline{\mathbb{Q}},\iota} \overline{\mathbb{Q}}_l$ is an irreducible subrepresentation of $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l)$.*

2. *For all indices j and for all primes p there is a WD-representation $\mathrm{WD}_p(M_j)$ of $W_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}$ such that*

$$\mathrm{WD}_p(M_j) \otimes_{\overline{\mathbb{Q}},\iota} \overline{\mathbb{Q}}_l \cong \mathrm{WD}_p(M_j \otimes_{\overline{\mathbb{Q}},\iota} \overline{\mathbb{Q}}_l)$$

for all primes l and all embeddings $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$.

3. *There is a multiset of integers* $\mathrm{HT}(M_j)$ *such that*
   (a) *for all primes l and all embeddings* $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$
   $$\mathrm{HT}(M_j \otimes_{\overline{\mathbb{Q}},\iota} \overline{\mathbb{Q}}_l) = HT(M_j)$$
   (b) *and for all* $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$
   $$\dim_{\mathbb{C}}((M_j \otimes_{\overline{\mathbb{Q}},\iota} \mathbb{C}) \cap H^{a,i-a}(X(\mathbb{C}),\mathbb{C}))$$
   is the multiplicity of a in $HT(M_j)$.

If one considers the whole of $H^i(X(\mathbb{C}),\overline{\mathbb{Q}})$ rather than its pieces M_j, then part 2. is known to hold up to Frobenius semisimplification for all but finitely many p and part 3. is known to hold (see [Il]). The whole conjecture is known to be true for $i = 0$ (easy) and $i = 1$ (where it follows from a theorem of Faltings [Fa] and the theory of the Albanese variety). The putative constituents M_j are one incarnation of what people call 'pure' motives.

If one believes conjectures 1.1 and 1.2 then 'geometric' l-adic representations should come in compatible families as l varies. There are many ways to make precise the notion of such a compatible family. Here is one.

By a *weakly compatible system of l-adic representations* $\mathcal{R} = \{R_{l,\iota}\}$ we shall mean a collection of semi-simple l-adic representations

$$R_{l,\iota} : G_{\mathbb{Q}} \longrightarrow GL(V \otimes_{\overline{\mathbb{Q}},\iota} \overline{\mathbb{Q}}_l),$$

one for each pair (l,ι) where l is a prime and $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$, which satisfy the following conditions.

- There is a multiset of integers $\mathrm{HT}(\mathcal{R})$ such that for each prime l and each embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$ the restriction $R_{l,\iota}|_{G_{\mathbb{Q}_l}}$ is de Rham and $\mathrm{HT}(R_{l,\iota}|_{G_{\mathbb{Q}_l}}) = \mathrm{HT}(\mathcal{R})$.
- There is a finite set of primes S such that if $p \notin S$ then $\mathrm{WD}_p(R_{l,\iota})$ is unramified for all l and ι.
- For all but finitely many primes p there is a Frobenius semi-simple WD-representation $\mathrm{WD}_p(\mathcal{R})$ over $\overline{\mathbb{Q}}$ such that for all primes $l \neq p$ and for all ι we have
  $$\mathrm{WD}_p(R_{l,\iota})^{\mathrm{ss}} \sim \mathrm{WD}_p(\mathcal{R}).$$

We make the following subsidiary definitions.

- We call $\mathcal{R}$ *strongly compatible* if the last condition (the existence of $\mathrm{WD}_p(\mathcal{R})$) holds for all primes p.
- We call $\mathcal{R}$ *irreducible* if each $R_{l,\iota}$ is irreducible.
- We call $\mathcal{R}$ *pure* of weight $w \in \mathbb{Z}$, if for all but finitely many p and for all eigenvalues α of $r_p(\mathrm{Frob}_p)$, where $\mathrm{WD}_p(\mathcal{R}) = (r_p, N_p)$, we have $\alpha \in \overline{\mathbb{Q}}$ and
$$|\iota\alpha|_\infty^2 = p^w$$
for all embeddings $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.
- We call $\mathcal{R}$ *geometric* if there is a smooth projective variety $X/\mathbb{Q}$ and integers $i \geqslant 0$ and j and a subspace
$$W \subset H^i(X(\mathbb{C}), \overline{\mathbb{Q}}(j))$$
such that for all l and ι, $W \otimes_{\overline{\mathbb{Q}},\iota} \overline{\mathbb{Q}}_l$ is $G_{\mathbb{Q}}$ invariant and realises $R_{l,\iota}$.

Conjectures 1.1 and 1.2 lead one to make the following conjecture.

CONJECTURE 1.3. —

1. *If $R : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{Q}}_l)$ is a continuous semi-simple de Rham representation unramified at all but finitely many primes then R is part of a weakly compatible system.*
2. *Any weakly compatible system is strongly compatible.*
3. *Any irreducible weakly compatible system $\mathcal{R}$ is geometric and pure of weight $(2/\dim \mathcal{R}) \sum_{h \in \mathrm{HT}(\mathcal{R})} h$.*

Conjectures 1.1 and 1.3 are known for one dimensional representations, in which case they have purely algebraic proofs based on class field theory (see [Se2]). Otherwise only fragmentary cases have been proved, where amazingly the arguments are extremely indirect involving sophisticated analysis and geometry. We will come back to this later.

2. L-functions

L-functions are certain Dirichlet series

$$\sum_{n=1}^{\infty} a_n/n^s$$

which play an important role in number theory. A full discussion of the role of L-functions in number theory is beyond the scope of this talk. However let us start with two examples in the hope of conveying some of their importance.

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$$

is the most celebrated example of a Dirichlet series. It converges to a non-zero holomorphic function in the half plane $\mathrm{Re}\, s > 1$. In its region of convergence it can also be expressed as a convergent infinite product over the prime numbers

$$\zeta(s) = \prod_p (1 - 1/p^s)^{-1}.$$

This is called an *Euler product* and the individual factors are called Euler factors. (This product expansion may easily be verified by the reader, the key point being the unique factorisation of integers as products of primes.) Lying deeper is the fact that $\zeta(s)$ has meromorphic continuation to the whole complex plane, with only one pole: a simple pole at $s = 1$. Moreover if we set

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

then Z satisfies the functional equation

$$Z(1-s) = Z(s).$$

Encoded in the Riemann zeta function is lots of deep arithmetic information. For instance the location of the zeros of $\zeta(s)$ is intimately connected with the distribution of prime numbers. Let me give another more algebraic example.

A big topic in algebraic number theory has been the study of factorisation into irreducibles in rings of integers in number fields, and to what extent it is unique. Particular attention has been paid to rings of cyclotomic integers $\mathbb{Z}[e^{2\pi i/p}]$ for p a prime, not least because of a relationship to Fermat's last theorem. In such a number ring there is a finite abelian group, the class group $\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])$, which 'measures' the failure of unique factorisation. It can be defined as the multiplicative semi-group of non-zero ideals in $\mathbb{Z}[e^{2\pi i/p}]$ modulo an equivalence relation which considers two ideals I and J equivalent if $I = \alpha J$ for some $\alpha \in \mathbb{Q}(e^{2\pi i/p})^\times$. The class group $\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])$ is trivial if and only if every ideal of $\mathbb{Z}[e^{2\pi i/p}]$ is principal, which in turn is true if and only if the ring $\mathbb{Z}[e^{2\pi i/p}]$ has unique factorisation. Kummer showed (by factorising $x^p + y^p$ over $\mathbb{Z}[e^{2\pi i/p}]$) that if $p \nmid \#\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])$ then Fermat's last theorem is true for exponent p.

But what handle does one have on the mysterious numbers $\#\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])$? The Galois group $\mathrm{Gal}\,(\mathbb{Q}(e^{2\pi i/p})/\mathbb{Q})$ acts on $\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])$ and on its Sylow p-subgroup $\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])_p$ and so we can form a decomposition

$$\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])_p = \bigoplus_{i=1}^{p-1} \mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])_p^{\chi_p^i}$$

into $\mathrm{Gal}\,(\mathbb{Q}(e^{2\pi i/p})/\mathbb{Q})$-eigenspaces. It turns out that if $\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])_p^{\chi_p^i} = (0)$ for all even i then $\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])_p = (0)$. Herbrand [Her] and Ribet [R1] proved a striking theorem to the effect that for any even positive integer n the special value $\zeta(1-n)$ is a rational number and that p divides the numerator of $\zeta(1-n)$ if and only if $\mathrm{Cl}\,(\mathbb{Z}[e^{2\pi i/p}])_p^{\chi_p^i} \neq (0)$. Note that $\zeta(s)$ is only defined at non-positive integers by analytic continuation.

Another celebrated example is the L-function of an elliptic curve E:

$$y^2 = x^3 + ax + b$$

(where $a, b \in \mathbb{Q}$ are constants with $4a^3 + 27b^2 \neq 0$). In this case the L-function is defined as an Euler product (converging in $\mathrm{Re}\, s > 3/2$)

$$L(E,s) = \prod_p L_p(E, p^{-s}),$$

where $L_p(E,X)$ is a rational function, and for all but finitely many p

$$L_p(E,X) = (1 - a_p(E)X + pX^2)^{-1},$$

with $p - a_p(E)$ being the number of solutions to the congruence

$$y^2 \equiv x^3 + ax + b \bmod p$$

in $\mathbb{F}_p^2$. It has recently been proved [BCDT] (see also section 5.4 below) that $L(E,s)$ can be continued to an entire function, which satisfies a functional equation

$$(2\pi)^{-s}\Gamma(s)L(E,s) = \pm N(E)^{1-s}(2\pi)^{s-2}\Gamma(2-s)L(E,2-s),$$

for some explicit positive integer $N(E)$. A remarkable conjecture of Birch and Swinnerton-Dyer [BSD] predicts that $y^2 = x^3 + ax + b$ has infinitely many rational solutions if and only if $L(E,1) = 0$. Again we point out that it is the behaviour of the L-function at a point where it is only defined by analytic continuation, which is governing the arithmetic of E. This conjecture has been proved when $L(E,s)$ has at most a simple zero at $s = 1$. (This combines work of Gross and Zagier [GZ] and of Kolyvagin [Kol1] with [BFH], [MM] and [BCDT]. See [Kol2] for a survey.)

There are now some very general conjectures along these lines about the special values of L-functions (see [BK]), but we do not have the space to discuss them here. We hope these two special cases give the reader an impression of what can be expected. We would like however to discuss the definition of L-functions in greater generality.

One general setting in which one can define L-functions is l-adic representations. Let us look first at the local setting. If (r,N) is a WD-representation of $W_{\mathbb{Q}_p}$ on an E-vector space V, where E is an algebraically closed field of characteristic zero, we define a local L-factor

$$L((r,N),X) = \det(1 - X\mathrm{Frob}_p)|^{-1}_{V^{I_{\mathbb{Q}_p},N=0}} \in E(X).$$

($V^{I_{\mathbb{Q}_p},N=0}$ is the subspace of V where $I_{\mathbb{Q}_p}$ acts trivially and $N = 0$.) One can also associate to (r,N) a conductor

$$f(r,N) = \operatorname{codim} V^{I_p,N=0} + \int_0^\infty \operatorname{codim} V^{I^u_{\mathbb{Q}_p}}\,du$$

which measures how deeply into $I_{\mathbb{Q}_p}$ the WD-representation (r,N) is non-trivial. It is known that $f(r,N) \in \mathbb{Z}_{\geqslant 0}$ (see [Se1]). Finally one has a local epsilon factor $\epsilon((r,N),\Psi_p) \in E$, which also depends on the choice of a non-trivial character $\Psi_p : \mathbb{Q}_p \to E^\times$ with open kernel (see [Tat]).

If $R : G_{\mathbb{Q}} \to GL(V)$ is an l-adic representation of $G_{\mathbb{Q}}$ which is de Rham at l and pure of some weight $w \in \mathbb{Z}$, and if $\iota : \overline{\mathbb{Q}}_l \hookrightarrow \mathbb{C}$ we will define an L-function

$$L(\iota R,s) = \prod_p L(\iota \mathrm{WD}_p(R), p^{-s}),$$

which will converge to a holomorphic function in $\operatorname{Re} s > 1 + w/2$. For example

$$L(1,s) = \zeta(s)$$

(where 1 denotes the trivial representation), and if $E/\mathbb{Q}$ is an elliptic curve then

$$L(\iota H^1(E(\mathbb{C}), \overline{\mathbb{Q}}_l), s) = L(E, s)$$

(for any ι). Note the useful formulae

$$L(\iota(R_1 \oplus R_2), s) = L(\iota R_1, s) L(\iota R_2, s) \quad \text{and} \quad L(\iota(R \otimes \chi_l^r), s) = L(\iota R, s+r).$$

Also note that $L(\iota R, s)$ determines $L(\mathrm{WD}_p(R), X)$ for all p and hence $\mathrm{WD}_p(R)$ for all but finitely many p. Hence by the Cebotarev density theorem $L(\iota R, s)$ determines R (up to semisimplification).

Write m_i^R for the multiplicity of an integer i in $\mathrm{HT}(R)$ and, if $w/2 \in \mathbb{Z}$, define $m_{w/2,\pm}^R \in (1/2)\mathbb{Z}$ by:

$$\begin{aligned} m_{w/2,+}^R + m_{w/2,-}^R &= m_{w/2}^R \\ m_{w/2,+}^R - m_{w/2,-}^R &= (-1)^{w/2}(\dim V^{c=1} - \dim V^{c=-1}). \end{aligned}$$

Also assume that $m_{w/2,+}^R, m_{w/2,-}^R \in \mathbb{Z}$, i.e. that $m_{w/2}^R \equiv \dim V \bmod 2$. Then we can define a Γ-factor

$$\begin{aligned} \Gamma(R, s) = {} & \Gamma_{\mathbb{R}}(s - w/2)^{m_{w/2,+}^R} \Gamma_{\mathbb{R}}(s - (w/2 - 1))^{m_{w/2,-}^R} \\ & \textstyle\prod_{i<w/2} \Gamma_{\mathbb{R}}(s - i)^{m_i^R} \prod_{i>w/2} \Gamma_{\mathbb{R}}(s - (w - 1 - i))^{m_i^R} \end{aligned}$$

and an ϵ-factor

$$\epsilon_\infty(R, e^{2\pi\sqrt{-1}x}) = \sqrt{-1}^{m_{w/2,-}^R} \prod_{i<w/2} \sqrt{-1}^{i m_i^R} \prod_{i>w/2} \sqrt{-1}^{(1+w+i) m_i^R},$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and where in each case we drop the factors involving $m_{w/2,\pm}^R$ if $w/2 \notin \mathbb{Z}$. Set

$$\Lambda(\iota R, s) = \Gamma(R, s) L(\iota R, s)$$

and

$$N(R) = \prod_p p^{f(\mathrm{WD}_p(R))}$$

(which makes sense as $f(\mathrm{WD}_p(R)) = 0$ whenever $\mathrm{WD}_p(R)$ is unramified) and

$$\epsilon(\iota R) = \epsilon_\infty(R, e^{2\pi\sqrt{-1}x}) \prod_p \iota\epsilon(\mathrm{WD}_p(R), \Psi_p),$$

where $\iota\Psi_p(x) = e^{-2\pi\sqrt{-1}x}$.

It is again worth noting that

$$\begin{array}{ll} \Lambda(\iota(R_1 \oplus R_2), s) = \Lambda(\iota R_1, s)\Lambda(\iota R_2, s) & \text{and } \Lambda(\iota(R \otimes \chi_l^r), s) = \Lambda(\iota R, s + r) \\ N(R_1 \oplus R_2) = N(R_1)N(R_2) & \text{and } N(R \otimes \chi_l^r) = N(R) \\ \epsilon(\iota(R_1 \oplus R_2)) = \epsilon(\iota R_1)\epsilon(\iota R_2) & \text{and } \epsilon(\iota(R \otimes \chi_l^r)) = \epsilon(\iota R)N(R)^{-r}. \end{array}$$

The following conjecture is a combination of conjecture 1.1 and conjectures which have become standard.

CONJECTURE 2.1. — *Suppose that R is an irreducible l-adic representation of $G_{\mathbb{Q}}$ which is de Rham and pure of weight $w \in \mathbb{Z}$. Then $m_p^R = m_{w-p}^R$ for all p, so that $m_{w/2} \equiv \dim V \bmod 2$. Moreover the following should hold.*

1. *$L(\iota R, s)$ extends to an entire function, except for a single simple pole if $R = \chi_l^{-w/2}$.*
2. *$\Lambda(\iota R, s)$ is bounded in vertical strips $\sigma_0 \leqslant \operatorname{Re} s \leqslant \sigma_1$.*
3. *$\Lambda(\iota R, s) = \epsilon(\iota R)N(R)^{-s}\Lambda(\iota R^*, 1 - s)$.*

It is tempting to believe that something like properties 1., 2. and 3. should characterise those Euler products which arise from l-adic representations. We will discuss a more precise conjecture along these lines in the next section. Why Galois representations should be **the** source of Euler products with good functional equations is a complete mystery.

Finally in this section let us discuss another Dirichlet series which predated and in some sense motivated L-functions for l-adic representations. Suppose that $X/\mathbb{Q}$ is a smooth projective variety. For some sufficiently large integer N we can choose a smooth projective model $\mathfrak{X}/\mathbb{Z}[1/N]$ for X and hence one can discuss the reduction $\mathfrak{X} \times \mathbb{F}_p$ for any prime $p \nmid N$ and its algebraic points $\mathfrak{X}(\overline{\mathbb{F}}_p)$. We will call two points in $\mathfrak{X}(\overline{\mathbb{F}}_p)$ equivalent if they are $G_{\mathbb{F}_p}$-conjugate. By the degree $\deg x$ of a point $x \in \mathfrak{X}(\overline{\mathbb{F}}_p)$, we shall mean the degree of the smallest extension of $\mathbb{F}_p$ over which x is defined. Then one defines the (partial) zeta function of X to be

$$\zeta_N(X, s) = \prod_{p \nmid N} \left(\prod_{x \in \mathfrak{X}(\overline{\mathbb{F}}_p)/\sim} (1 - p^{-s \deg x})^{-1} \right).$$

This will converge in some right half complex plane.

$\zeta_N(X, s)$ is clearly missing a finite number of Euler factors - those at the primes dividing N. There is no known geometric description of these missing Euler factors. However Grothendieck [G] showed that, for any ι,

$$\zeta_N(X,s) = \prod_{i=0}^{2\dim X} L_N(\iota H^i(X(\mathbb{C}),\overline{\mathbb{Q}}_l),s)^{(-1)^i}$$

where L_N indicates that the Euler factors at primes $p|N$ have been dropped. Thus it is reasonable to define

$$\zeta(X,s) = \prod_{i=0}^{2\dim X} L(\iota H^i(X(\mathbb{C}),\overline{\mathbb{Q}}_l),s)^{(-1)^i}$$

and

$$Z(X,s) = \prod_{i=0}^{2\dim X} \Lambda(\iota H^i(X(\mathbb{C}),\overline{\mathbb{Q}}_l),s)^{(-1)^i}.$$

For example the zeta function of a point is

$$\zeta(\text{ point}, s) = \zeta(s)$$

and the zeta function of an elliptic curve $E/\mathbb{Q}$ is

$$\zeta(E,s) = \zeta(s)\zeta(s-1)/L(E,s).$$

Conjecture 2.1 and Poincaré duality (and the expected semisimplicity of the action of Galois on $H^i(X(\mathbb{C}),\overline{\mathbb{Q}}_l)$, see conjecture 1.2) give rise to the following conjecture.

CONJECTURE 2.2. — *Suppose that $X/\mathbb{Q}$ is a smooth projective variety. Then $\zeta(X,s)$ has meromorphic continuation to the whole complex plane and satisfies a functional equation of the form*

$$Z(X,s) = \epsilon N^{-s} Z(X, 1+\dim X - s)$$

for some $N \in \mathbb{Q}_{>0}$ and $\epsilon \in \mathbb{R}$.

3. Automorphic forms

Automorphic forms may be thought of as certain smooth functions on the quotient $GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})$. We need several preliminaries before we can make a precise definition.

Let $\widehat{\mathbb{Z}}$ denote the profinite completion of $\mathbb{Z}$, i.e.

$$\widehat{\mathbb{Z}} = \lim_{\leftarrow N} \mathbb{Z}/N\mathbb{Z} = \prod_p \mathbb{Z}_p,$$

a topological ring. Also let $\mathbb{A}^\infty$ denote the topological ring of finite adeles

$$\mathbb{A}^\infty = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\widehat{\mathbb{Z}}$ is an open subring with its usual topology. As an abstract ring $\mathbb{A}^\infty$ is the subring of $\prod_p \mathbb{Q}_p$ consisting of elements (x_p) with $x_p \in \mathbb{Z}_p$ for all but finitely many p, however the topology is not the subspace topology. We define the topological ring of adeles to be the product

$$\mathbb{A} = \mathbb{A}^\infty \times \mathbb{R}.$$

Note that $\mathbb{Q}$ embeds diagonally as a discrete subring of $\mathbb{A}$ with compact quotient

$$\mathbb{Q}\backslash\mathbb{A} = \widehat{\mathbb{Z}} \times \mathbb{Z}\backslash\mathbb{R}.$$

We will be interested in $GL_n(\mathbb{A})$, the locally compact topological group of $n \times n$ invertible matrices with coefficients in $\mathbb{A}$. We remark that the topology on $GL_n(\mathbb{A})$ is the subspace topology resulting from the closed embedding

$$\begin{array}{rcl} GL_n(\mathbb{A}) & \hookrightarrow & M_n(\mathbb{A}) \times M_n(\mathbb{A}) \\ g & \mapsto & (g, g^{-1}). \end{array}$$

This is different from the topology induced from the inclusion $GL_n(\mathbb{A}) \hookrightarrow M_n(\mathbb{A})$. (For instance $GL_n(\widehat{\mathbb{Z}}) \times GL_n(\mathbb{R})$ is open in $GL_n(\mathbb{A})$ but not in $M_n(\mathbb{A})$.) The group $GL_n(\mathbb{Q})$ is a discrete subgroup of $GL_n(\mathbb{A})$ and the quotient $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})$ has finite volume (for the quotient of a (two sided) Haar measure on $GL_n(\mathbb{A})$ by the discrete measure on $GL_n(\mathbb{Q})$). If $U \subset GL_n(\widehat{\mathbb{Z}})$ is an open subgroup with $\det U = \widehat{\mathbb{Z}}^\times$, then the strong approximation theorem for SL_n tells us that

$$GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})/U = (GL_n(\mathbb{Q}) \cap U)\backslash GL_n(\mathbb{R}).$$

Note that $GL_n(\mathbb{Q}) \cap U$ is a subgroup of $GL_n(\mathbb{Z})$ of finite index. (For any open compact subgroup $U \subset GL_n(\mathbb{A}^\infty)$ we have

$$GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})/U = \coprod_{i=1}^{r} (GL_n(\mathbb{Q}) \cap g_i U g_i^{-1})\backslash GL_n(\mathbb{R})$$

for some integer $r \geqslant 1$ and some elements $g_i \in GL_n(\mathbb{A}^\infty)$.) Most of the statements we make concerning $GL_n(\mathbb{A})$ can be rephrased to involve only $GL_n(\mathbb{R})$, but at the expense of making them much more cumbersome. To achieve brevity (and because it seems more natural) we have opted to use the language of adeles. We hope this extra abstraction will not be too confusing for the novice.

Before continuing our introduction of automorphic forms let us digress to mention class field theory, which provides a concrete example of the presentational advantages of the adelic language. It also implies essentially all the conjectures we are considering in the case of one dimensional Galois representations. Indeed this article is about the search for a non-abelian analogue of class field theory. Class field theory gives a concrete description of the abelianisation (maximal continuous abelian quotient) $G_{\mathbb{Q}}^{\mathrm{ab}}$ of $G_{\mathbb{Q}}$ and $W_{\mathbb{Q}_p}^{\mathrm{ab}}$ of $W_{\mathbb{Q}_p}$ for all p. Firstly the local theory asserts that there is an isomorphism

$$\mathrm{Art}_p : \mathbb{Q}_p^\times \xrightarrow{\sim} W_{\mathbb{Q}_p}^{\mathrm{ab}}$$

with various natural properties, including the following.

- The image of the inertia group $I_{\mathbb{Q}_p}$ in $W_{\mathbb{Q}_p}^{\mathrm{ab}}$ is $\mathrm{Art}(\mathbb{Z}_p^\times)$.
- The induced map

  $$\mathbb{Q}_p^\times/\mathbb{Z}_p^\times \longrightarrow W_{\mathbb{Q}_p}^{\mathrm{ab}}/I_{\mathbb{Q}_p} \subset G_{\mathbb{F}_p}$$

  takes p to the geometric Frobenius element Frob_p.
- For $u > 0$, the image of the higher inertia group $I_{\mathbb{Q}_p}^u$ in $W_{\mathbb{Q}_p}^{\mathrm{ab}}$ is $\mathrm{Art}(1 + p^v\mathbb{Z}_p)$, where v is the least integer greater than or equal to u.

Secondly the global theory asserts that there is an isomorphism

$$\mathrm{Art} : \mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}_{>0}^\times \xrightarrow{\sim} G_{\mathbb{Q}}^{\mathrm{ab}}$$

such that the restriction of Art to $\mathbb{Q}_p^\times$ coincides with the composition of Art_p with the natural map $W_{\mathbb{Q}_p}^{\mathrm{ab}} \to G_{\mathbb{Q}}^{\mathrm{ab}}$. Thus Art is defined completely from a knowledge of the Art_p (and the fact that Art takes $-1 \in \mathbb{R}^\times$ to complex conjugation) and global class field theory can be thought of as a determination of the kernel of $\prod_p \mathrm{Art}_p$. (In the case of $\mathbb{Q}$ these assertions can be derived without difficulty from the Kronecker-Weber theorem that $G_{\mathbb{Q}_p}^{\mathrm{ab}} = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cycl}}/\mathbb{Q}_p)$ and $G_{\mathbb{Q}}^{\mathrm{ab}} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{cycl}}/\mathbb{Q})$, where K^{cycl} denotes the extension of K obtained by adjoining all roots of unity.) A similar direct description of the whole of $W_{\mathbb{Q}_p}$ or $G_{\mathbb{Q}}$ would be wonderful, but such a description seems to be too much to hope for.

We now return to our (extended) definition of automorphic forms. We will let $O(n) \subset GL_n(\mathbb{R})$ denote the orthogonal group consisting of matrices h for which ${}^t hh = I_n$. We will let $\mathfrak{gl}_n$ denote the complexified Lie algebra of $GL_n(\mathbb{R})$, i.e. $\mathfrak{gl}_n$ is $M_n(\mathbb{C})$ with Lie bracket $[X,Y] = XY - YX$. We will let $\mathfrak{z}_n$ denote the centre of the universal enveloping algebra of $\mathfrak{gl}_n$. (The universal enveloping algebra of $\mathfrak{gl}_n$ is an associative $\mathbb{C}$-algebra with a $\mathbb{C}$-linear map from $\mathfrak{gl}_n$ which takes the Lie bracket to commutators, and which is universal for such maps.) By an action of $\mathfrak{gl}_n$ on a complex vector space V we shall mean a $\mathbb{C}$-linear map $\mathfrak{gl}_n \to \mathrm{End}\,(V)$ which takes the Lie bracket to commutators. Thus a $\mathfrak{gl}_n$ action on V gives rise to a homomorphism $\mathfrak{z}_n \to \mathrm{End}\,(V)$, whose image commutes with the image of $\mathfrak{gl}_n$.

There is an isomorphism (the Harish-Chandra isomorphism, see for example [Dix])

$$\gamma_{\mathrm{HC}} : \mathfrak{z}_n \xrightarrow{\sim} \mathbb{C}[X_1, ..., X_n]^{S_n},$$

where S_n is the symmetric group on n-letters acting on $\mathbb{C}[X_1, ..., X_n]$ by permuting $X_1, ..., X_n$. Note that homomorphisms

$$\mathbb{C}[X_1, ..., X_n]^{S_n} \longrightarrow \mathbb{C}$$

are parametrised by multisets of cardinality n of complex numbers. Given such a multiset $H = \{x_1, ..., x_n\}$, we define

$$\begin{array}{rcl} \theta_H : \mathbb{C}[X_1, ..., X_n]^{S_n} & \longrightarrow & \mathbb{C} \\ f & \longmapsto & f(x_1, ..., x_n). \end{array}$$

The Harish-Chandra isomorphism γ_{HC} may be characterised as follows. Suppose that ρ is the irreducible (finite dimensional) representation of $\mathfrak{gl}_n$ with highest weight

$$\mathrm{diag}(t_1, ..., t_n) \longmapsto a_1 t_1 + ... + a_n t_n$$

where $a_1 \geqslant a_2 ... \geqslant a_n$ are integers. Let

$$H(\rho) = \{a_1 + (n-1)/2, a_2 + (n-3)/2, ..., a_n + (1-n)/2\}.$$

Then if $z \in \mathfrak{z}_n$ we have

$$\rho(z) = \theta_{H(\rho)}(\gamma_{\mathrm{HC}}(z)).$$

Automorphic forms will be certain smooth functions of $GL_n(\mathbb{A})$. (By smooth we mean locally constant as a function on $GL_n(\mathbb{A}^\infty)$ and smooth as a function on $GL_n(\mathbb{R})$.) If f is a smooth function on $GL_n(\mathbb{A})$, $g \in GL_n(\mathbb{A})$ and $X \in \mathfrak{gl}_n$ then we define

$$(gf)(h) = f(hg)$$

and

$$(Xf)(h) = (d/dt f(he^{tX}))|_{t=0}.$$

Note that $[X,Y]f = X(Yf) - Y(Xf)$.

We are now in a position to define cusp forms on $GL_n(\mathbb{A})$. For each partition $n = n_1 + n_2$ let N_{n_1,n_2} denote the subgroup of GL_n consisting of matrices of the form

$$\begin{pmatrix} I_{n_1} & * \\ 0 & I_{n_2} \end{pmatrix}.$$

If H is a multiset of complex numbers of cardinality n, then the space of cusp forms with infinitesimal character H, $\mathcal{A}_H^\circ(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ is the space of smooth functions

$$f : GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}) \longrightarrow \mathbb{C}$$

satisfying the following conditions.

1. (K-finiteness) The translates of f under $GL_n(\widehat{\mathbb{Z}}) \times O(n)$ (a choice of maximal compact subgroup of $GL_n(\mathbb{A})$) span a finite dimensional vector space;

2. (Infinitesimal character H) If $z \in \mathfrak{z}_n$ then $zf = \chi_H(\gamma_{\mathrm{HC}}(z))f$;

3. (Cuspidality) For each partition $n = n_1 + n_2$,
$$\int_{N_{n_1,n_2}(\mathbb{Q})\backslash N_{n_1,n_2}(\mathbb{A})} f(ug)du = 0;$$

4. (Growth condition) f is bounded on $GL_n(\mathbb{A})$.

One would like to study $\mathcal{A}_H^\circ(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ as a representation of $GL_n(\mathbb{A})$, unfortunately it is not preserved by the action of $GL_n(\mathbb{R})$ (because the K-finiteness condition depends on the choice of a maximal compact subgroup $O(n) \subset GL_n(\mathbb{R})$). It does however have an action of $GL_n(\mathbb{A}^\infty) \times O(n)$ and of $\mathfrak{gl}_n$, which is essentially as good. More precisely it is a $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-module in the sense that it is a complex vector space with both an action of $GL_n(\mathbb{A}^\infty) \times O(n)$ and $\mathfrak{gl}_n$ such that

1. the stabiliser in $GL_n(\mathbb{A}^\infty)$ of any $f \in \mathcal{A}_H^\circ(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ is open;

2. the actions of $GL_n(\mathbb{A}^\infty)$ and $\mathfrak{gl}_n$ commute;

3. $k(Xf) = (kXk^{-1})(kf)$ for all $k \in O(n)$ and all $X \in \mathfrak{gl}_n$;

4. the vector space spanned by the $O(n)$-translates of any f is finite dimensional;

5. if $X \in \mathrm{Lie}\, O(n) \subset \mathfrak{gl}_n$ then

$$(Xf) = d/dt(e^{tX} f)|_{t=0}.$$

Moreover $\mathcal{A}^\circ_H(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ is admissible as a $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-module, in the sense that for any irreducible (finite dimensional) smooth representation W of $GL_n(\widehat{\mathbb{Z}}) \times O(n)$ the space

$$\mathrm{Hom}_{GL_n(\widehat{\mathbb{Z}}) \times O(n)}(W, \mathcal{A}^\circ_H(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})))$$

is finite dimensional.

In fact the space $\mathcal{A}^\circ_H(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ is a direct sum of irreducible admissible $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-modules each occurring with multiplicity one. These irreducible constituents are referred to as *cuspidal automorphic representations* of $GL_n(\mathbb{A})$ with infinitesimal character H, although they are not strictly speaking representations of $GL_n(\mathbb{A})$ at all.

For example consider the (unusually simple) case $n = 1$. Define

$$\begin{array}{rcl} || \; || : \mathbb{Q}^\times\backslash\mathbb{A}^\times & \longrightarrow & \mathbb{R}^\times_{>0} \\ (h_v) & \longmapsto & |h_\infty|_\infty \prod_p |h_p|_p. \end{array}$$

Then

$$\mathcal{A}^\circ_{\{s\}}(\mathbb{Q}^\times\backslash\mathbb{A}^\times) = \mathcal{A}^\circ_{\{0\}}(\mathbb{Q}^\times\backslash\mathbb{A}^\times) \otimes || \; ||^s$$

and $\mathcal{A}^\circ_{\{0\}}(\mathbb{Q}^\times\backslash\mathbb{A}^\times)$ is just the space of locally constant functions on the compact space

$$\widehat{\mathbb{Z}}^\times \xrightarrow{\sim} \mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times_{>0}.$$

Thus

$$\mathcal{A}^\circ_{\{0\}}(\mathbb{Q}^\times\backslash\mathbb{A}^\times) = \bigoplus \psi$$

as ψ runs over all continuous characters

$$\psi : \mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times_{>0} \cong \widehat{\mathbb{Z}}^\times \longrightarrow \mathbb{C}^\times.$$

Any such character factors through $(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})^\times = (\mathbb{Z}/N\mathbb{Z})^\times$ for some integer N. Thus in some sense cuspidal automorphic representations are generalisations of Dirichlet characters. However this does not really convey the analytic flavour of more general cuspidal automorphic representations.

The case $n = 2$ is somewhat more representative. In this case we have $\mathcal{A}^\circ_{\{s,t\}}(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})) = (0)$ unless $s - t \in i\mathbb{R}$, $s - t \in \mathbb{Z}$ or $s - t \in (-1, 1)$. It is conjectured that the third possibility can not arise unless $s = t$. Let us consider the case $s - t \in \mathbb{Z}_{>0}$ a little further. If $s - t \in \mathbb{Z}_{>0}$ then it turns out that the irreducible constituents of $\mathcal{A}^{\circ}_{\{s,t\}}(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$ are in bijection with the weight $1 + s - t$ holomorphic cusp forms on the upper half plane which are normalised newforms (see for example [Mi]). To be more precise let $U_1(N) \subset GL_2(\widehat{\mathbb{Z}})$ denote the subgroup of elements with last row congruent to $(0, 1)$ modulo N. Also define $j : SO(2) \to \mathbb{C}^{\times}$ by

$$j : \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longmapsto a - b\sqrt{-1}.$$

Then it turns out that if π is an irreducible constituent of $\mathcal{A}^{\circ}_{\{s,t\}}(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$ with $s - t \in \mathbb{Z}_{>0}$ then there is a unique positive integer N such that the set of $\phi \in \pi^{U_1(N)}$ with

$$\phi(gk) = j(k)^{t-s-1}\phi(g)$$

for all $g \in GL_2(\mathbb{A})$ and $k \in SO(2)$, is one dimensional. If we choose a nonzero ϕ in this one dimensional space, then the function

$$f_{\phi}(x + y\sqrt{-1}) = y^{-s-1/2}\phi \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

is a holomorphic newform of weight $1 + s - t$ and level N. If we choose ϕ so that f_{ϕ} is normalised and if we denote this f_{ϕ} by f_{π}, then $\pi \mapsto f_{\pi}$ gives the desired bijection. Thus in some sense cuspidal automorphic representations are are also generalisations of classical holomorphic normalised newforms.

Note that if ψ is a character of $\mathbb{A}^{\times}/\mathbb{Q}^{\times}\mathbb{R}^{\times}_{>0}$ and if π is an irreducible constituent of $\mathcal{A}^{\circ}_{H}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ then $\pi \otimes (\psi \circ \det)$ is also an irreducible constituent of $\mathcal{A}^{\circ}_{H}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$. Concretely we may realise it as the space of functions $f(g)\psi(\det g)$ where $f \in \pi$. Also note that if π is an irreducible constituent of $\mathcal{A}^{\circ}_{H}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ then its contragredient π^* is an irreducible constituent of $\mathcal{A}^{\circ}_{-H}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$, where $-H$ is the multiset of $-s$ for $s \in H$. Concretely we may realise π^* as the set of $f({}^t g^{-1})$ for $f \in \pi$.

One of the main questions in the theory of automorphic forms is to describe the irreducible constituents of $\mathcal{A}^{\circ}_{H}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$. If we are to do this we first need some description of all irreducible admissible $GL_n(\mathbb{A}^{\infty}) \times (\mathfrak{gl}_n, O(n))$-modules, and then we can try to say which occur in $\mathcal{A}^{\circ}_{H}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$.

To describe this we must quickly recall the local situation. By a *smooth* representation of $GL_n(\mathbb{Q}_p)$ we mean a representation of $GL_n(\mathbb{Q}_p)$ on a complex vector space V such that the stabiliser of every vector in V is open in

$GL_n(\mathbb{Q}_p)$. We call V *admissible* if V^U is finite dimensional for every open subgroup $U \subset GL_n(\mathbb{Q}_p)$, or equivalently if for every irreducible (smooth) representation W of $GL_n(\mathbb{Z}_p)$

$$\dim \operatorname{Hom}_{GL_n(\mathbb{Z}_p)}(W, V) < \infty.$$

Every irreducible smooth representation of $GL_n(\mathbb{Q}_p)$ is admissible. We call an irreducible smooth representation V of $GL_n(\mathbb{Q}_p)$ *unramified* if $V^{GL_n(\mathbb{Z}_p)} \neq (0)$. In this case $\dim V^{GL_n(\mathbb{Z}_p)} = 1$. By a $(\mathfrak{gl}_n, O(n))$-module we mean a complex vector space V with an action of $\mathfrak{gl}_n$ and an action of $O(n)$ such that

1. $k(Xv) = (kXk^{-1})(kv)$ for all $k \in O(n)$ and all $X \in \mathfrak{gl}_n$;
2. the vector space spanned by the $O(n)$-translates of any v is finite dimensional;
3. if $X \in \operatorname{Lie} O(n) \subset \mathfrak{gl}_n$ then

$$(Xv) = d/dt(e^{tX}v)|_{t=0}.$$

We call V *admissible* if for each irreducible $O(n)$-module W we have

$$\dim \operatorname{Hom}_{O(n)}(W, V) < \infty.$$

If $n > 1$ then most irreducible smooth $GL_n(\mathbb{Q}_p)$-modules and most irreducible admissible $(\mathfrak{gl}_n, O(n))$-modules are infinite dimensional. In fact the only finite dimensional irreducible smooth $GL_n(\mathbb{Q}_p)$-modules are one dimensional and of the form $\psi \circ \det$ for a homomorphism $\psi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ with open kernel.

Just as a character $\psi : \mathbb{A}^\times \to \mathbb{C}^\times$ can be factored as

$$\psi = \psi_\infty \times \prod_p \psi_p$$

where $\psi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ (resp. $\psi_\infty : \mathbb{R}^\times \to \mathbb{C}^\times$) and $\psi_p(\mathbb{Z}_p^\times) = \{1\}$ for all but finitely many p, so irreducible admissible $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-modules can be factorised. More precisely suppose that π_∞ is an irreducible admissible $(\mathfrak{gl}_n, O(n))$-module and that for each prime p, π_p is an irreducible smooth representation of $GL_n(\mathbb{Q}_p)$ with π_p unramified for all but finitely many p. For all but finitely many p choose $0 \neq w_p \in \pi_p^{GL_n(\mathbb{Z}_p)}$ and define the restricted tensor product

$$\bigotimes_x{}' \pi_x$$

to be the subspace of $\bigotimes_x \pi_x$ spanned by vectors of the form $\otimes_x v_x$ with $v_p = w_p$ for all but finitely many p. Then $\bigotimes_x{}' \pi_x$ is an irreducible admissible $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-module, which up to isomorphism does not depend on the choice of vectors w_p. Moreover any irreducible admissible $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-module π arrises in this way for unique π_p and π_∞. Thus a description of all irreducible admissible $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-modules is a purely local question: describe all irreducible admissible $(\mathfrak{gl}_n, O(n))$-modules, describe all irreducible smooth $GL_n(\mathbb{Q}_p)$-modules and describe which have a $GL_n(\mathbb{Z}_p)$-fixed vector.

There is a rather explicit description of all irreducible admissible $(\mathfrak{gl}_n, O(n))$-modules which we will not describe in detail (see [Lan1]). Briefly the irreducible admissible $(\mathfrak{gl}_n, O(n))$-modules with infinitesimal character H are parametrised by partitions $H = \coprod H_j$ into sub-multisets of cardinality 1 or 2 such that if $H_j = \{a, b\}$ then $a - b \in \mathbb{Z}_{\neq 0}$, and by a choice of $\delta_j \in \{0, 1\}$ for each H_j of cardinality 1. If H is a multiset of n complex numbers, set $w(H) = 2/n \sum_{a \in H} \operatorname{Re} a$. It is known that if π is a cuspidal automorphic representation with infinitesimal character H and if π_∞ is parametrised by $H = \coprod H_j$ and $\{\delta_j\}$ then the following hold.

- If $H_j = \{a\}$ then $\operatorname{Re} a \in ((w(H) - 1)/2, (w(H) + 1)/2)$.
- The indices j for which $H_j = \{a\}$ with $\operatorname{Re} a \neq w(H)/2$ can be paired up so that for any pair (j, j') we have $\delta_j = \delta_{j'}$, $H_j = \{a\}$ and $H_{j'} = \{w(H) + a - 2\operatorname{Re} a\}$.
- If $H_j = \{a, b\}$ then $\operatorname{Re}(a + b) \in (w(H) - 1, w(H) + 1)$.
- The indices j for which $H_j = \{a, b\}$ with $\operatorname{Re}(a + b) \neq w(H)$ can be paired up so that for any pair (j, j') we have $H_j = \{a, b\}$ and $H_{j'} = \{w(H) + a - \operatorname{Re}(a + b), w(H) + b - \operatorname{Re}(a + b)\}$.

A celebrated conjecture of Selberg predicts that if $H_j = \{a\}$ is a singleton then $\operatorname{Re} a = w(H)/2$, while if $H_j = \{a, b\}$ is a pair then $\operatorname{Re}(a + b) = w(H)$. This is equivalent to the assertion that for all $a, b \in H$ we have $a - b \in \frac{1}{2}\mathbb{Z}$.

Note that an irreducible $(\mathfrak{gl}_n, O(n))$-module π has a central character $\psi_\pi : \mathbb{R}^\times \to \mathbb{C}^\times$ defined by $\psi_\pi(-1) = \pi(-I_n)$ (where $-I_n \in O(n)$) and $\psi_\pi(t) = e^{\pi((\log t) I_n)} = t^{nw(H)/2}$ for $t \in \mathbb{R}^\times_{>0}$ (where $(\log t) I_n \in \mathfrak{gl}_n$ and where H parametrises the infinitesimal character of π). To any irreducible irreducible admissible $(\mathfrak{gl}_n, O(n))$-module π corresponding to $H = \coprod H_j$ and $\{\delta_j\}$ one can attach an Γ-factor

$$\Gamma(\pi, s) = \prod_{H_j = \{a_j\}} \Gamma_{\mathbb{R}}(s + a_j + \delta_j) \prod_{H_j = \{a_j, b_j\}} \Gamma_{\mathbb{R}}(\max(a_j, b_j) + s) \Gamma_{\mathbb{R}}(\max(a_j, b_j) + 1 + s),$$

and an ϵ constant

$$\epsilon(\pi, e^{2\pi\sqrt{-1}x}) = \prod_{H_j=\{a_j\}} \sqrt{-1}^{\delta_j} \prod_{H_j=\{a_j,b_j\}} \sqrt{-1}^{1+|a_j-b_j|}.$$

(See [J]).

Any irreducible smooth representation π of $GL_n(\mathbb{Q}_p)$ has a central character $\psi_\pi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. If π is unramified then $\psi_\pi(\mathbb{Z}_p^\times) = \{1\}$. One may (see [J]) also associate to π an L-factor

$$L(\pi, X) \in \mathbb{C}(X),$$

a conductor $f(\pi) \in \mathbb{Z}$ and an ϵ-factor

$$\epsilon(\pi, \Psi_p) \in \mathbb{C}^\times$$

(where $\Psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ is a non-trivial character with open kernel). If π is unramified and $\ker \Psi_p = \mathbb{Z}_p$ then $f(\pi) = 0$ and $\epsilon(\pi, \Psi_p) = 1$. Let $U_1(p^m)$ denote the subgroup of matrices in $GL_n(\mathbb{Z}_p)$ with last row congruent to $(0, ..., 0, 1) \bmod p^m$. Then for instance, the conductor $f(\pi)$ is the minimal non-negative integer f such that $\pi^{U_1(p^f)} \neq (0)$.

Thus to an irreducible admissible $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-module $\pi = \bigotimes'_x \pi_x$ one may associate

- a central character $\psi_\pi = \prod_x \psi_{\pi_x} : \mathbb{A}^\times \to \mathbb{C}^\times$;
- an L-function $L(\pi, s) = \prod_p L(\pi_p, p^{-s})$, which may or may not converge;
- an extended L-function $\Lambda(\pi, s) = \Gamma(\pi_\infty, s)L(\pi, s)$;
- a conductor $N(\pi) = \prod_p p^{f(\pi_p)} \in \mathbb{Z}_{>0}$;
- and an epsilon constant $\epsilon(\pi) = \prod_x \epsilon(\pi_x, \Psi_x) \in \mathbb{C}^\times$, where $\Psi_\infty(t) = e^{2\pi\sqrt{-1}t}$ and $\Psi_p(t) = e^{-2\pi\sqrt{-1}(t \bmod \mathbb{Z}_p)}$.

The following theorem and conjecture describe the (expected) relationship between automorphic forms and L-functions with Euler product and functional equation. We suppose $n > 1$. A similar theorem to theorem 3.1 is true for $n = 1$, except that $L(\pi, s)$ may have one simple pole. In this case it was due to Dirichlet. Conjecture 3.2 becomes vacuous if $n = 1$.

THEOREM 3.1 (GODEMENT-JACQUET, [GoJ]). — *Suppose that* π *is an irreducible constituent of* $\mathcal{A}^{\circ}_{H}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ *with* $n > 1$. *Then* $L(\pi, s)$ *converges to a holomorphic function in some right half complex plane* $\operatorname{Re} s > \sigma$ *and can be continued to a holomorphic function on the whole complex plane so that* $\Lambda(\pi, s)$ *is bounded in all vertical strips* $\sigma_1 \geqslant \operatorname{Re} s \geqslant \sigma_2$. *Moreover* $L(\pi, s)$ *satisfies the functional equation*

$$\Lambda(\pi, s) = \epsilon(\pi) N(\pi)^{-s} \Lambda(\pi^*, 1 - s).$$

CONJECTURE 3.2 (COGDELL-PIATETSKI-SHAPIRO, [CPS1]). — *Suppose that* π *is an irreducible admissible* $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$*-module such that the central character of* π *is trivial on* $\mathbb{Q}^\times$ *and such that* $L(\pi, s)$ *converges in some half plane. Suppose also that for all characters* $\psi : \mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times_{>0} \to \mathbb{C}^\times$ *the L-function* $\Lambda(\pi \otimes (\psi \circ \det), s)$ *(which will then converge in some right half plane) can be continued to a holomorphic function on the entire complex plane, which is bounded in vertical strips and satisfies the functional equation*

$$\Lambda(\pi\otimes(\psi\circ\det), s) = \epsilon(\pi\otimes(\psi\circ\det))N(\pi\otimes(\psi\circ\det))^{-s}\Lambda(\pi^*\otimes(\psi^{-1}\circ\det), 1-s).$$

($\Lambda(\pi^*\otimes(\psi^{-1}\circ\det), s)$ *also automatically converges in some right half plane.) Then there is a partition* $n = n_1 + \ldots + n_r$ *and cuspidal automorphic representations* π_i *of* $GL_{n_i}(\mathbb{A})$ *such that*

$$\Lambda(\pi, s) = \prod_{i=1}^{r} \Lambda(\pi_i, s).$$

Theorem 3.1 for $n = 2$ was proved in many cases by Hecke [Hec] and in full generality by Jacquet and Langlands [JL]. Conjecture 3.2 is known to be true for $n = 2$ ([We], [JL]) and $n = 3$ ([JPSS1]). For $n > 3$ a weaker form of this conjecture involving twisting by higher dimensional automorphic representations is known to hold (see [CPS1], [CPS2]).

This is a good place to mention the following results ([JS]) which will be useful later.

THEOREM 3.3. —

1. *Suppose that* π *and* π' *are two cuspidal automorphic representations of* $GL_n(\mathbb{A})$ *with* $\pi_p \cong \pi'_p$ *for all but finitely many* p. *Then* $\pi = \pi'$.

2. *Suppose that* $\pi_1, \ldots, \pi_r$ *and* $\pi'_1, \ldots, \pi'_s$ *are cuspidal automorphic representations with* $|\psi_{\pi_i,\infty}| = |\psi_{\pi'_j,\infty}|$ *independent of* i *and* j. *Suppose* S

is a finite set of primes containing all those at which some π_i *or* π'_j *is ramified. Suppose that we can write*

$$\prod_{i=1}^{r} L_S(\pi_i, s) = \prod_{j=1}^{s} L_S(\pi'_j, s) \prod_{p \notin S} \prod_{k=1}^{n} (1 - \alpha_{p,k}/p^s)^{-1},$$

where $\prod_{p \notin S} \prod_{k=1}^{n} (1 - \alpha_{p,k}/p^s)^{-1}$ *converges in some right half plane. Then*

$$L_S(\mathrm{ad}\,, s) = \prod_{p \notin S} \prod_{i,j} (1 - \alpha_i \alpha_j^{-1} p^{-s})^{-1}$$

converges in some right half plane and has meromorphic continuation to the entire complex plane. Suppose further that $L_S(\mathrm{ad}\,, s)$ *has a simple pole at* $s = 1$. *Then there is an index* i *such that*

$$L(\pi_{i,p}, X) = \prod_{k=1}^{n} (1 - \alpha_{p,k} X)^{-1}$$

for all but finitely many p.

The reason for us introducing automorphic forms is because of a putative connection to Galois representations, which we will now discuss. But first let us briefly describe the local situation. It has recently been established ([HT], [Hen2]) that there is a natural bijection, rec_p, from irreducible smooth representations of $GL_n(\mathbb{Q}_p)$ to n-dimensional Frobenius semi-simple WD-representations of $W_{\mathbb{Q}_p}$ over $\mathbb{C}$. The key point here is that the bijection should be natural. We will not describe here exactly what this means, instead we refer the reader to the introduction of [HT]. It does satisfy the following.

- $\psi_\pi \circ \mathrm{Art}_p^{-1} = \det \mathrm{rec}_p(\pi)$,
- $L(\mathrm{rec}_p(\pi), X) = L(\pi, X)$,
- $f(\mathrm{rec}_p(\pi)) = f(\pi)$, and
- $\epsilon(\mathrm{rec}_p(\pi), \Psi_p) = \epsilon(\pi, \Psi_p)$.

Thus π is unramified if and only if $\mathrm{rec}_p(\pi)$ is unramified, and if $n = 1$ then $\mathrm{rec}_p \pi = \pi \circ \mathrm{Art}_p^{-1}$. Thus existence of rec_p can be seen as a non-abelian generalisation of local class field theory.

Now suppose that $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ and that R is a de Rham semi-simple l-adic representation of $G_{\mathbb{Q}}$ which is unramified at all but finitely many primes. Let

$w(R) = (2/\dim R)\sum_{a\in \mathrm{HT}(R)} a$ and suppose that $w(R) \in \mathbb{Z}$ and that $m_i^R = m_{w(R)-i}^R$ for all i. Let $\pi_\infty(R)$ be the irreducible, admissible $(\mathfrak{gl}_n, O(n))$-module with infinitesimal character $\mathrm{HT}(R)$ parametrised as follows. We decompose $\mathrm{HT}(R)$ into $\{i, w(R) - i\}$ with multiplicity m_i^R if $2i \neq w(R)$ and $\{w(R)/2\}$ with multiplicity $m_{w(R)/2}^R$ if $w(R)/2 \in \mathbb{Z}$. To $m_{w(R)/2,+}^R$ of the $\{w(R)/2\}$ we associate $\delta = 0$ and to $m_{w(R)/2,-}^R$ of them we associate $\delta = 1$. (Of course even without the assumptions that $w(R) \in \mathbb{Z}$ and that $m_i^R = m_{w(R)-i}^R$ for all i, one can fabricate some definition of $\pi_\infty(R)$, which equals this one whenever these assumptions are met. This however is rather pointless.) Then we can associate to R an irreducible, admissible $GL_n(\mathbb{A}^\infty)\times(\mathfrak{gl}_n, O(n))$-module

$$\pi(\iota R) = \pi_\infty(R) \otimes \prod_p \mathrm{rec}_p^{-1}(\iota \mathrm{WD}_p(R)).$$

By the Cebotarev density theorem R is completely determined by $\pi(\iota R)$.

CONJECTURE 3.4. — *Suppose that H is a multiset of n* **integers** *and that π is an irreducible constituent of $\mathcal{A}_H^\circ(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$. Identify $\overline{\mathbb{Q}} \subset \mathbb{C}$. Then each $\mathrm{rec}_p(\pi_p)$ can be defined over $\overline{\mathbb{Q}}$ and there is an irreducible geometric strongly compatible system of l-adic representations $\mathcal{R}$ such that $\mathrm{HT}(\mathcal{R}) = H$ and $\mathrm{WD}_p(\mathcal{R}) = \mathrm{rec}_p(\pi_p)$ for all primes p.*

CONJECTURE 3.5. — *Suppose that*

$$R : G_{\mathbb{Q}} \longrightarrow GL(V)$$

is an irreducible l-adic representation which is unramified at all but finitely many primes and for which $R|_{G_{\mathbb{Q}_l}}$ is de Rham. Let $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$. Then $w(R) \in \mathbb{Z}$ and for all i we have $m_i^R = m_{w(R)-i}^R$. Moreover $\pi(\iota R)$ is a cuspidal automorphic representation of $GL_n(\mathbb{A})$.

These conjectures are essentially due to Langlands [Lan2], except we have used a precise formulation which follows Clozel [Cl1] and we have incorporated conjecture 1.1 into conjecture 3.5.

Conjecture 3.5 is probably the more mysterious of the two, as only the case $n = 1$ and fragmentary cases where $n = 2$ are known. This will be discussed further in the next section. Note the similarity to the main theorem of global class field theory that $\prod_p \mathrm{Art}_p : \mathbb{A}^\times \to G_{\mathbb{Q}}^{\mathrm{ab}}$ has kernel $\mathbb{Q}^\times$. (Namely that $\pi(\iota R)$ occurs in a space of functions on $GL_n(\mathbb{A})$ which are left invariant by $GL_n(\mathbb{Q})$.)

The following theorem provides significant evidence for conjecture 3.4.

THEOREM 3.6 ([KOT], [CL2], [HT]). — *Suppose that H is multiset of* n distinct *integers and that π is an irreducible constituent of $\mathcal{A}_H^\circ(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$. Let $\iota : \overline{\mathbb{Q}}_l \hookrightarrow \mathbb{C}$. Suppose moreover that $\pi^* \cong \pi \otimes (\psi \circ \det)$ for some character $\psi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$, and that either $n \leqslant 2$ or for some prime p the representation π_p is square integrable (i.e. $\mathrm{rec}_p(\pi_p)$ is indecomposable). Then there is a continuous representation*

$$R_{l,\iota} : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

with the following properties.

1. *$R_{l,\iota}$ is geometric and pure of weight $2/n \sum_{h\in H} h$.*
2. *$R_{l,\iota}|_{G_{\mathbb{Q}_l}}$ is de Rham and $\mathrm{HT}(R_{l,\iota}|_{G_{\mathbb{Q}_l}}) = H$.*
3. *For any prime $p \neq l$ there is a representation $r_p : W_{\mathbb{Q}_p} \to GL_n(\overline{\mathbb{Q}}_l)$ such that $\mathrm{WD}_p(R_{l,\iota})^{\mathrm{ss}} = (r_p, N_p)$ and $\mathrm{rec}_p(\pi_p) = (\iota r_p, N_p')$.*

This was established by finding the desired l-adic representations in the cohomology of certain unitary group Shimura varieties. It seems not unreasonable to hope that similar techniques might allow one to improve many of the technical defects in the theorem. However Clozel has stressed that in the cases where H does not have distinct elements or where $\pi^* \ncong \pi \otimes (\psi \circ \det)$, there seems in general to be no prospect of finding the desired l-adic representations in the cohomology of Shimura varieties. It seems we need a new technique.

(As theorem 3.6 is not explicitly in the literature we indicate how it can be deduced from theorem VII.1.9 of [HT]. Note that for $x \in \mathbb{R}_{>0}^\times$ we have $\psi(x) = x^N$ for some $N \in \mathbb{Z}$. By a standard descent argument (see for example the proof of theorem VII.1.9 of [HT]) it suffices to construct $R_{l,\iota}|_{G_L}$ for all imaginary quadratic fields L in which p splits. For this we apply theorem VII.1.9 of [HT] to $\pi_L \otimes \phi$, where π_L denotes the base change of π to $GL_n(\mathbb{A}_L)$ and where $\phi : \mathbb{A}_L^\times/L^\times \to \mathbb{C}^\times$ is a continuous character which satisfies

- $\phi\phi^c = \psi \circ \mathbf{N}_{L/\mathbb{Q}}$, and
- $\phi|_{\mathbb{C}^\times}(z) = z^N$ if n is odd and $z^{N+1}/|z|$ if n is even.

To construct such a character ϕ, choose any character ϕ_0 satisfying the second condition and look for $\phi = \phi_0\phi_1$ where $\phi_1 : \mathbb{A}_L^\times/L^\times\mathbb{C}^\times \to \mathbb{C}^\times$ is a continuous character satisfying

$$\phi_1\phi_1^c = \psi_1 \circ \mathbf{N}_{L/\mathbb{Q}}$$

with $\psi_1 = \psi\phi_0|_{\mathbb{A}^\times}^{-1} : \mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}_{>0}^\times \to \mathbb{C}^\times$, i.e. ϕ_1 should satisfy

$$\phi_1|_{\mathbf{N}_{L/\mathbb{Q}}\mathbb{A}_L^\times} = \psi_1.$$

Choose an open compact subgroup $U \subset (\mathbb{A}_L^\infty)^\times$ satisfying

- ${}^cU = U$,
- $L^\times \hookrightarrow (\mathbb{A}_L^\infty)^\times/U$, and
- $\psi_1(U \cap (\mathbb{A}^\infty)^\times) = \{1\}$.

It follows from the first two conditions that $\mathbb{A}^\times \cap (UL^\times\mathbb{C}^\times) = (U \cap \widehat{\mathbb{Z}}^\times)\mathbb{Q}^\times\mathbb{R}^\times$, so that $(\mathbf{N}_{L/\mathbb{Q}}\mathbb{A}_L^\times) \cap (UL^\times\mathbb{C}^\times) \subset (U \cap \widehat{\mathbb{Z}}^\times)\mathbb{Q}^\times\mathbb{R}_{>0}^\times$. Thus $\psi_1|_{\mathbf{N}_{L/\mathbb{Q}}\mathbb{A}_L^\times}$ extends to a character of $(\mathbf{N}_{L/\mathbb{Q}}\mathbb{A}_L^\times)L^\times U\mathbb{C}^\times$ which is trivial on $L^\times U\mathbb{C}^\times$. As $(\mathbf{N}_{L/\mathbb{Q}}\mathbb{A}_L^\times)L^\times U\mathbb{C}^\times$ is open of finite index in $\mathbb{A}_L^\times$, this character in turn extends to a character of $\mathbb{A}_L^\times$ which is trivial on $L^\times U\mathbb{C}^\times$. This will suffice for ϕ_1.)

4. Summary

Let us first summarise the various conjectures we have made. This summary will be less precise than the conjectures stated in the previous sections, but should convey the main thrust of those conjectures. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let H be a multiset of integers of cardinality $n > 1$. Then the following sets should be in natural bijection. One way to make precise the meaning of natural in this context is that to each object M in any of the sets below we can associate local L-factors (rational functions of a variable X) $L_p(M, X)$ for all but finitely many primes p. In each case these factors completely determine M. Two objects should correspond if and only if for all but finitely many p they give rise to the same local L-factors.

(AF) Irreducible constituents π of $\mathcal{A}_H^\circ(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$. In this case $L_p(\pi, X) = L(\pi_p, X)$.

(LF) Near equivalence classes of irreducible admissible $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-modules π with the following properties. (We call two $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-modules, π and π' nearly equivalent if $\pi_p \cong \pi'_p$ for all but finitely many primes p.)

  (a) π_∞ has infinitesimal character H.

  (b) The central character ψ_π of π is trivial on $\mathbb{Q}^\times \subset \mathbb{A}^\times$.

(c) For all characters $\psi : \mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times_{>0}$ the L-function $\Lambda(\pi \otimes (\psi \circ \det), s)$ converges in some right half plane, has holomorphic continuation to the entire complex plane so that it is bounded in vertical strips and satisfies the functional equation

$$\Lambda(\pi \otimes (\psi \circ \det), s)$$

$$= \epsilon(\pi \otimes (\psi \circ \det))N(\pi \otimes (\psi \circ \det))^{-s}\Lambda((\pi \otimes (\psi \circ \det))^*, 1-s).$$

(d) There is a finite set of primes S containing all primes p for which $\mathrm{rec}(\pi_p)$ is ramified, such that writing

$$L(\pi_p, X) = \prod_{i=1}^{n}(1-\alpha_{p,i}X)^{-1}$$

for $p \notin S$,

$$\sum_{p\notin S,i,j}\sum_{m=1}^{\infty}\alpha_{p,i}^m\alpha_{p,j}^{-m}/mp^{ms} + \log(s-1)$$

is bounded as $s \to 1$ from the right.

In this case $L_p(\pi, X) = L(\pi_p, X)$.

(lR) (Fix $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$.) Irreducible l-adic representations

$$R : G_{\mathbb{Q}} \longrightarrow GL(V)$$

which are unramified at all but finitely many primes and for which $R|_{G_{\mathbb{Q}_l}}$ is de Rham with $\mathrm{HT}(R|_{G_{\mathbb{Q}_l}}) = H$. In this case $L_p(R, X) = \iota L(\mathrm{WD}_p(R), X)$.

(WCS) Irreducible weakly compatible systems of l-adic representations $\mathcal{R}$ for which $\mathrm{HT}(\mathcal{R}) = H$. In this case $L_p(\mathcal{R}, X) = L_p(\mathrm{WD}_p(\mathcal{R}), X)$.

(GCS) Irreducible geometric strongly compatible systems of l-adic representations $\mathcal{R}$ with $\mathrm{HT}(\mathcal{R}) = H$. In this case $L_p(\mathcal{R}, X) = L_p(\mathrm{WD}_p(\mathcal{R}), X)$.

For $n = 1$ we drop the item (LF), because it would need to be modified to allow $L(\pi\otimes(\psi\circ\det), s)$ to have a simple pole, while in any case condition (LF) (b) would make the inclusion $(LF) \subset (AF)$ trivial. This being said in the case $n = 1$ all the other four sets are known to be in natural bijection (see [Se2]). This basically follows because global class field theory provides an isomorphism

$$\mathrm{Art} : \mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times_{>0} \xrightarrow{\sim} G^{\mathrm{ab}}_{\mathbb{Q}}.$$

I would again like to stress how different are these various sorts of objects and how surprising it is to me that there is any relation between them. Items (AF) and (LF) both concern representations of adele groups, but arising in rather different settings: either from the theory of discrete subgroups of Lie groups or from the theory of L-functions with functional equation. Items (lR) and (WCS) arise from Galois theory and item (GCS) arises from geometry.

So what do we know about the various relationships for $n > 1$?

Not much. Trivially one has $(GCS) \subset (WCS) \subset (lR)$. The inclusion $(AF) \subset (LF)$ is OK by theorem 3.1. As discussed in section 3 we have significant partial results in the directions $(LF) \subset (AF)$ and $(AF) \subset (GCS)$, but both seem to need new ideas. (Though I should stress that I am not really competent to discuss converse theorems.)

One way to establish the equivalence of all five items would be to complete the passages $(LF) \subset (AF)$ and $(AF) \subset (GCS)$ and to establish the passage $(lR) \subset (AF)$. It is these inclusions which have received most study, though it should be pointed out that in the function field case the equivalence of the analogous objects was established by looking at the inclusions

$$(lR) \subset (LF) \subset (AF) \subset (GCS).$$

(The proof of the inclusion $(lR) \subset (LF)$ was proved by Grothendieck [G] and Laumon [Lau]. It is rooted in the study of l-adic cohomology, and it is this which is most special to the function field case. The inclusion $(LF) \subset (AF)$ uses a converse theorem due to Piatetski-Shapiro [PS], and the inclusion $(AF) \subset (GCS)$ is due to Drinfeld [Dr] and Lafforgue [Laf]. Please note that this thumb-nail sketch is not precise in a number of respects. For instance (LF) has to be modified to allow for twists by more automorphic forms and the definition of geometric in (GCS) needs modifying.) However, it is striking, that in the case of number fields, **all** known inclusions of items (lR), (WCS) or (GCS) in (LF) go via (AF).

For the rest of this article we will concentrate on what still seems to be the least understood problem: the passage from (lR) or (WCS) to (AF) or (LF). Although the results we have are rather limited one should not underestimate their power. Perhaps the most striking illustration of this is that the lifting theorems discussed in section 5.4 (combined with earlier work using base change and converse theorems) allowed Wiles [Wi] to finally prove Fermat's last theorem.

5. Automorphy of Galois representations

In this section we will discuss some results which shed some light on the passage from (lR), (WCS) or (GCS) to (AF) or (LF). The discussion will of necessity be somewhat more technical. In particular we will need to discuss automorphic forms, l-adic representations and so on over general number fields (i.e. finite extensions of $\mathbb{Q}$) other than $\mathbb{Q}$. We will leave it to the reader's imagination exactly how such a generalisation is made. In this connection we should remark that if L/K is a finite extension of number fields and if R is a semi-simple de Rham l-adic representation of G_L which is unramified at all but finitely many primes, then (see [A])

$$L(R, s) = L(\mathrm{Ind}\,_{G_L}^{G_K} R, s)$$

(formally if the L-functions don't converge). In fact this is true Euler factor by Euler factor and similar results hold for conductors and ϵ-factors (see [Tat]). This observation can be extremely useful.

5.1. Brauer's theorem

The result I want to discuss is a result of Brauer [Br] about finite groups.

THEOREM 5.1 (BRAUER). — *Suppose that r is a representation of a finite group G. Then there are nilpotent subgroups $H_i < G$, one dimensional representations ψ_i of H_i and integers n_i such that as virtual representations of G we have*

$$r = \sum_i n_i \mathrm{Ind}\,_{H_i}^{G} \psi_i.$$

As Artin [A] had realised this theorem has the following immediate consequence. (Indeed Brauer proved his theorem in response to Artin's work.)

COROLLARY 5.2. — *Let $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$. Suppose that*

$$R : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

is an l-adic representation with **finite image**. *Then the L-function $L(\iota R, s)$ has meromorphic continuation to the entire complex plane and satisfies the expected functional equation.*

Artin's argument runs as follows. Let G denote the image of R and write

$$R = \sum_i n_i \mathrm{Ind}\,_{H_i}^{G} \psi_i$$

as in Brauer's theorem. Let $L/\mathbb{Q}$ be the Galois extension with group G cut out by R and let $K_i = L^{H_i}$. Then one has almost formal equalities

$$\begin{aligned} L(\iota R, s) &= \prod_i L(\iota \mathrm{Ind}\,_{G_{K_i}}^{G_{\mathbb{Q}}} \psi_i, s)^{n_i} \\ &= \prod_i L(\iota\psi_i, s)^{n_i}. \end{aligned}$$

By class field theory for the fields K_i, the character ψ_i is automorphic on $GL_1(\mathbb{A}_{K_i})$ and so $L(\iota\psi_i, s)$ has holomorphic continuation to the entire complex plane (except possibly for one simple pole if $\psi_i = 1$) and satisfies a functional equation. It follows that $L(\iota R, s)$ has meromorphic continuation to the entire complex plane and satisfies a functional equation. The problem with this method as it stands, is that some of the integers n_i will usually be negative so that one can only conclude the meromorphy of $L(\iota R, s)$, not its holomorphy.

5.2. Base change

Suppose that L/K is a finite extension and that R is an irreducible de Rham l-adic representation of G_K ramified at only finitely many primes. Then $R|_{G_L}$ is a semi-simple de Rham l-adic representation of G_L ramified at only finitely many primes. Suppose moreover that L/K is Galois and cyclic and that σ is a generator of Gal (L/K). Then an irreducible de Rham l-adic representation r of G_L which is ramified at only finitely many primes arises in this way if and only if $r \cong {}^{\sigma}r$.

If one believes conjectures 3.4 and 3.5, one might expect that if L/K is an extension of number fields and if π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_K)$ then there are cuspidal automorphic representations Π_i of $GL_{n_i}(\mathbb{A}_L)$ such that for all places v of L one has

$$\oplus_i \mathrm{rec}_v(\Pi_{i,v}) = \mathrm{rec}_{v|_K}(\pi_{v|_K})|_{W_{L_v}}.$$

Moreover if L/K is Galois and cyclic with Gal (L/K) generated by σ and if Π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ with $\Pi = \Pi \circ \sigma$ then one might expect that there is a cuspidal automorphic representation π of $GL_n(\mathbb{A}_K)$ such that for all places v of L we have

$$\mathrm{rec}_v(\Pi_v) = \mathrm{rec}_{v|_K}(\pi_{v|_K})|_{W_{L_v}}.$$

If $n = 1$ then this is true. For the first assertion one can take $\Pi = \pi \circ \mathbf{N}_{L/K}$. The second assertion follows from class field theory, the key point being that $\mathbf{N}_{L/K} L^\times = K^\times \cap \mathbf{N}_{L/K}\mathbb{A}_L^\times$. For $n > 1$ the second part is known and the first part is known if L/K is Galois and soluble. The argument (due to Langlands [Lan3] if $n = 2$ and to Arthur and Clozel [AC] if $n > 2$) is much less direct. They only need treat the case that L/K is cyclic and here they use the trace formula. It seems to be essential for the method that there is a simple characterisation of the image of the base change map.

One draw back of the second part of this result is that (even in the case $n = 1$), given Π there is no complete recipe for π: at the primes v of K which are inert in L, we don't know which extension of $\mathrm{rec}_v(\Pi_v)$ to take. This can be surprisingly serious. If however we know how to associate irreducible l-adic representations $R(\pi)$ to π and $R(\Pi)$ to Π and if R is any l-adic representation of G_K with $R|_{G_L} \sim R(\Pi)$, then $R \sim R(\Pi \otimes (\psi \circ \det))$ for some character ψ of $\mathbb{A}_K^\times / K^\times \mathbf{N}_{L/K} \mathbb{A}_L^\times$.

5.3. Converse theorems

Converse theorems are theorems along the lines of conjecture 3.2, which tell one that L-functions with good arithmetic properties come from automorphic forms.

As Cogdell and Piatetski-Shapiro point out, conjecture 3.2 has very important consequences for Galois representations, some of which we will now discuss. We stress that in the examples below we are **assuming** conjecture 3.2. In a very few cases the known cases of this conjecture give unconditional results which we will mention at the end.

5.3.1. Automorphic induction

Suppose that L/K is an extension of number fields and that R is an irreducible, de Rham l-adic representation of G_L ramified at only finitely many primes, then $\mathrm{Ind}_{G_L}^{G_K} R$ is a semi-simple, de Rham l-adic representation of G_K ramified at only finitely many primes. Thus if one believes conjectures 3.4 and 3.5 one might expect that if Π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ then there is a partition $n[L : K] = n_1 + ... + n_r$ and cuspidal automorphic representations π_i of $GL_{n_i}(\mathbb{A}_K)$ with

$$L(\Pi, s) = \prod_{i=1}^{r} L(\pi_i, s).$$

In many cases this would follow from conjecture 3.2. For simplicity we will just consider the case $K = \mathbb{Q}$. One can form an irreducible $GL_n(\mathbb{A}^\infty) \times (\mathfrak{gl}_n, O(n))$-module $\mathrm{Ind}_L^{\mathbb{Q}}\Pi$ such that for every finite order character ψ of $\mathbb{A}^\times / \mathbb{Q}^\times \mathbb{R}_{>0}^\times$ we have $L((\mathrm{Ind}_L^{\mathbb{Q}}\Pi) \otimes \psi, s) = L(\Pi \otimes (\psi \circ \mathbf{N}_{L/\mathbb{Q}}), s)$, with similar formulae for ϵ constants, conductors etc. (This is a purely local question and one can for instance just make use of the local reciprocity maps rec_p - see section 3.) If $n > 1$ or if $n = 1$ and Π does not factor through $\mathbf{N}_{L/\mathbb{Q}}$, one then simply applies conjecture 3.2 to $\mathrm{Ind}_L^{\mathbb{Q}}\Pi$ and deduces the existence of $\pi_1, ..., \pi_r$. Conjecture 3.2 would also allow one to treat some other cases when $n = 1$ and Π does factor through $\mathbf{N}_{L/\mathbb{Q}}$, for instance if the normal closure of $L/\mathbb{Q}$ is soluble or perfect. In this case we may assume that Π is the trivial representation and hence may apply the Artin conjecture (see below) to $\mathrm{Ind}_{G_L}^{G_\mathbb{Q}} 1$.

The existence of $\pi_1, ..., \pi_r$ is known if $n = 1$ and $[L : K] \leqslant 3$ by the converse theorems of [JL] and [JPSS1]. It follows from the theory of Arthur-Clozel [AC] discussed below (see section 5.2) if L/K is Galois and soluble. This was extended by Harris [Har] to some cases where L/K is only assumed to have soluble normal closure. Harris' result is however restricted to cases in which one can attach l-adic representations to all the automorphic representations occurring in his argument.

5.3.2. Artin's conjecture

The "strong form" of this conjecture asserts that if K is a number field, if $R : G_K \to GL_n(\overline{\mathbb{Q}}_l)$ is an irreducible l-adic representation with **finite image** and if $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ then there is a cuspidal automorphic representation π of $GL_n(\mathbb{A}_K)$ with $L(\pi, s) = L(\iota R, s)$. In particular it implies that $L(\iota R, s)$ is entire, except possibly for one simple pole if $n = 1$. Many cases (including those where the image of R is either perfect or soluble) of this conjecture would follow from conjecture 3.2. More precisely suppose one can write

$$R = \sum_i n_i \mathrm{Ind}_{G_{L_i}}^{G_K} \chi_i$$

where $n_i \in \mathbb{Z}$ and where χ_i is a one dimensional representation of G_{L_i} which does not extend to G_K. The cases of automorphic induction implied by conjecture 3.2 would show that there are integers m_i and cuspidal automorphic representations π_i of $GL_{r_i}(\mathbb{A}_K)$ with

$$L(\iota R, s) = \prod_i L(\pi_i, s)^{m_i}.$$

As R is irreducible, $R \otimes R^*$ contains the trivial representation exactly once. By Brauer's theorem 5.1, we can write

$$R \otimes R^* = \sum_i a_i \mathrm{Ind}_{G_{K_i}}^{G_K} \psi_i$$

where $a_i \in \mathbb{Z}$ and ψ_i is a one dimensional representation of G_{K_i}. The multiplicity of the trivial representation in $R \otimes R^*$ is just the sum of the a_i for which $\psi_i = 1$. Thus

$$\begin{aligned} \mathrm{ord}_{s=1}L(\iota R \otimes R^*, s) &= \sum_i a_i \mathrm{ord}_{s=1}L(\iota\psi_i, s) \\ &= -\sum_{\psi_i=1} a_i \\ &= -1, \end{aligned}$$

and it follows from theorem 3.3 that there is a cuspidal automorphic representation π of $GL_n(\mathbb{A}_K)$ with $L(\pi_v, s) = L(\iota \mathrm{WD}_v(R), s)$ for all but finitely many places v of K. Because both $L(\pi, s)$ and $L(R, s)$ satisfy functional equations of the same form one may deduce that $L(\pi, s) = L(R, s)$ (see for instance corollary 4.5 of [Hen1]).

One is left with the following (rather artificial) question in finite group theory, to which I do not know the answer. Suppose that R is an irreducible representation of a finite group G with $\dim R > 1$. When can one find subgroups $H_i < G$, integers n_i, one dimensional representations χ_i of H_i which do not extend to G such that

$$R = \sum_i n_i \mathrm{Ind}\,_{H_i}^G \chi_i?$$

The answer is 'always' if G is perfect.
(In fact for any finite group G, if $R \neq 1$ then one can find an expression

$$R = \sum_i n_i \mathrm{Ind}\,_{H_i}^G \chi_i$$

in which $\chi_i \neq 1$ for all i. Write

$$R = \sum_i n_i \mathrm{Ind}\,_{H_i}^G \psi_i$$

as in Brauer's theorem with each H_i nilpotent. As R does not contain the trivial representation,

$$\sum_{\psi_i=1} n_i \mathrm{Ind}\,_{H_i}^G \mathrm{Ind}\,_{\{1\}}^{H_i} 1 = (\sum_{\psi_i=1} n_i) \mathrm{Ind}\,_{\{1\}}^G 1 = 0.$$

Thus we can replace each $\psi_i = 1$ by $1 - \mathrm{Ind}\,_{\{1\}}^{H_i} 1$, which is minus a sum of non-trivial irreducible representations of H_i. As H_i is nilpotent, each of these is in turn induced from a non-trivial character of a subgroup of H_i. Substituting this into our expression for R, our claim follows. This result seems to be due to van der Waall [vW].)
The answer is also 'always' if G is soluble.
(One can argue by induction on $\#G$. Let r be an irreducible representation of a soluble group G with $\dim r > 1$. If r is induced from a proper subgroup we are done by the inductive hypothesis. In particular we may suppose that G is not nilpotent and or even the semidirect product of an abelian group by a nilpotent group. Moreover we may suppose that the restriction of r to any normal subgroup is isotypical. Suppose that G has a non-trivial normal subgroup N such that G/N is not nilpotent. By Brauer's theorem we may write the trivial representation of G/N as

$$\sum_i n_i \mathrm{Ind}\,_{H_i/N}^{G/N} \chi_i$$

where $n_i \in \mathbb{Z}$, H_i/N is a proper subgroup of G/N and where χ_i is a character of H_i/N. Thus

$$r = \sum_i n_i \mathrm{Ind}\,_{H_i}^{G}(r|_{H_i}\chi_i),$$

and by the inductive hypothesis we are done as long as for all i the representation $r|_{H_i}\chi_i$ does not contain a character of G. If it did then there would be a character ψ of G such that $(r \otimes \psi)|_N$ is trivial and again we are done by applying the inductive hypothesis to G/N. Thus we may suppose that every proper quotient of G is nilpotent. As we are supposing that G is not nilpotent, it follows that G has a unique minimal normal subgroup M and that G/M is nilpotent. As G is soluble, M must be an elementary abelian p-group for some prime p. Then G has a unique Sylow-p-subgroup, which we will denote by S. Let H denote a Sylow-p-complement in G, so that G is the semi-direct product of S by H and H is nilpotent. By the minimality of M, M must be an irreducible G/M-module. In particular S acts trivially on M, i.e. M is contained in the centre of S, and M is an irreducible H-module. If $h \in H$ we see that there is $m_h \in \mathrm{Hom}\,(S, M)$ such that

$$hsh^{-1} = sm_h(s)$$

for all $s \in S$. If we let H act on $\mathrm{Hom}\,(S, M)$ via $h(\phi)(s) = h\phi(s)h^{-1}$ then we see that $h \mapsto m_h$ gives a 1-cocycle on H valued in $\mathrm{Hom}\,(S, M)$. As $H^1(H, \mathrm{Hom}\,(S, M)) = (0)$ we see that there is an element $\phi \in \mathrm{Hom}\,(S, M)$ such that

$$hsh^{-1}s^{-1} = h\phi(s)h^{-1}\phi(s)^{-1}$$

for all $h \in H$ and $s \in S$. Thus $\ker \phi$ is a normal in S and centralises H. If M were a trivial H-module then we would have $G = S \times H$ and G would be nilpotent. Thus we may assume that $M \cap \ker \phi = \{1\}$ so that $S = M \times \ker \phi$. Thus G is the semidirect product of M by the nilpotent group $H \times \ker \phi$, and we are done.)

Without assuming conjecture 3.2 only a few cases of Artin's conjecture are known. For instance combining the base change results discussed section 5.2 with results deriving from the converse theorem for GL_3 (see [JPSS1], [GeJ], [JPSS2]) Langlands [Lan3] and Tunnell [Tu] deduced the strong Artin conjecture for two dimensional representations of G_K with soluble image.

5.3.3. Galois descent

Let $\iota : \overline{\mathbb{Q}}_l \hookrightarrow \mathbb{C}$. Let $K/\mathbb{Q}$ be a finite, totally real Galois extension. Suppose that Π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_K)$ such that for each place $v|\infty$ the infinitesimal character of Π_v is parametrised by a multiset of n distinct integers and such that for some finite place w of K the representation Π_w is square integrable. Suppose also that

$$R : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

is an l-adic representation such that $R \sim R^* \otimes \psi$ for some character ψ of $G_{\mathbb{Q}}$, and such that $R|_{G_K}$ is irreducible. Suppose finally that $R|_{G_K}$ and Π are associated, in the sense that for all but finitely places v of K we have

$$\iota L(\mathrm{WD}_v(R|_{G_K}), X) = L(\Pi_v, X).$$

Then it would follow from conjecture 3.2 that there is a regular algebraic cuspidal automorphic representation π of $GL_n(\mathbb{A})$ associated to R in the same sense. Roughly speaking this tells us that to check the automorphy of an l-adic representation of $G_{\mathbb{Q}}$ it would suffice to do so after a finite, totally real Galois base change. (If $n = 2$ one can drop the assumption that Π is square integrable at some finite place. We remind the reader that an irreducible representation π_v of $GL_n(K_v)$ is called square integrable if for for all $x \in \pi_v$ and all $f \in \pi_v^*$

$$\int_{GL_n(K_v)/K_v^\times} |f(gx)|^2 |\psi_{\pi_v}(\det g)|^{-2/n} dg$$

converges. It turns out that π_v is square integrable if and only if $\mathrm{rec}_v(\pi_v)$ is indecomposable.)

We will sketch the argument. We may suppose that $n > 1$. One can first use the Langlands-Arthur-Clozel theory (see section 5.2) to check that if L is any subfield of K with $\mathrm{Gal}\,(K/L)$-soluble then there is a cuspidal automorphic representation Π_L of $GL_n(\mathbb{A}_L)$ associated to $R|_{G_L}$ (see section 5.2). By Brauer's theorem we can find subfields $L_i \subset K$ with $\mathrm{Gal}\,(K/L_i)$-soluble, characters ψ_i of $\mathrm{Gal}\,(K/L_i)$ and integers m_i such that the trivial representation of $\mathrm{Gal}\,(K/\mathbb{Q})$ equals

$$\sum_i m_i \mathrm{Ind}\,_{\mathrm{Gal}\,(K/L_i)}^{\mathrm{Gal}\,(K/\mathbb{Q})} \psi_i.$$

Moreover for each pair of indices i, j we can find intermediate fields L_{ijk} between L_j and K and characters ψ_{ijk} of $G_{L_{ijk}}$ such that

$$(\mathrm{Ind}\,_{G_{L_j}}^{G_{\mathbb{Q}}} \psi_j)|_{G_{L_i}} \cong \bigoplus_k \mathrm{Ind}\,_{G_{L_{ijk}}}^{G_{L_i}} \psi_{ijk}.$$

Thus

$$R = \sum_i m_i \mathrm{Ind}\,_{G_{L_i}}^{G_{\mathbb{Q}}} (R|_{G_{L_i}} \otimes \psi_i)$$

and

$$R^* \otimes R = \sum_{ijk} m_i m_j \mathrm{Ind}\,_{G_{L_{ijk}}}^{G_{\mathbb{Q}}} ((R|_{G_{L_{ijk}}} \otimes \psi_i|_{G_{L_{ijk}}})^* \otimes (R|_{G_{L_{ijk}}} \otimes \psi_{ijk})).$$

In particular, if $t_{ijk} = 1$ or 0 depending whether

$$R|_{G_{L_{ijk}}} \otimes \psi_i|_{G_{L_{ijk}}} \cong R|_{G_{L_{ijk}}} \otimes \psi_{ijk}$$

or not, then

$$1 = \sum_{ijk} m_i m_j t_{ijk}.$$

As
$$R = \sum_i m_i \mathrm{Ind}_{G_{L_i}}^{G_\mathbb{Q}} (R|_{G_{L_i}} \otimes \psi_i),$$
one can find cuspidal automorphic representations $\pi_1, ..., \pi_r$ and $\pi'_1, ..., \pi'_t$ such that
$$\prod_{i=1}^{r} L(\pi_i, s) = L(\iota R, s) \prod_{i=1}^{t} L(\pi'_i, s).$$
By theorem 3.3 it suffices to show that $L(R^* \otimes R, s)$ has a simple pole at $s = 1$. But, because $(R|_{G_{L_{ijk}}} \otimes \psi_i|_{G_{L_{ijk}}})$ and $(R|_{G_{L_{ijk}}} \otimes \psi_{ijk})$ are irreducible and are associated to cuspidal automorphic representations of $GL_n(\mathbb{A}_{L_{ijk}})$, we have
$$\begin{array}{rl} & \mathrm{ord}_{s=1} L(\iota R \otimes R^*, s) \\ = & -\sum_{ijk} m_i m_j \mathrm{ord}_{s=1} L((R|_{G_{L_{ijk}}} \otimes \psi_i|_{G_{L_{ijk}}})^* \otimes (R|_{G_{L_{ijk}}} \otimes \psi_{ijk}), s) \\ = & -\sum_{ijk} m_i m_j t_{ijk} \\ = & -1. \end{array}$$

5.4. Lifting theorems

To describe this sort of theorem we first remark that if $R : G_\mathbb{Q} \to GL_n(\overline{\mathbb{Q}}_l)$ is continuous then after conjugating R by some element of $GL_n(\overline{\mathbb{Q}}_l)$ we may assume that the image of R is contained in $GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ and so reducing we obtain a continuous representation
$$\overline{R} : G_\mathbb{Q} \longrightarrow GL_n(\overline{\mathbb{F}}_l).$$

The lifting theorems I have in mind are results of the general form if R and R' are l-adic representations of $G_\mathbb{Q}$ with R' automorphic and if $\overline{R} = \overline{R}'$ then R is also automorphic. Very roughly speaking the technique (pioneered by Wiles [Wi] and completed by the author and Wiles [TW]) is to show that $R \bmod l^r$ arises from automorphic forms for all r by induction on r. As $\ker(GL_n(\mathbb{Z}/l^r\mathbb{Z}) \twoheadrightarrow GL_n(\mathbb{Z}/l^{r-1}\mathbb{Z}))$ is an abelian group one is again led to questions of class field theory and Galois cohomology.

I should stress that such theorems are presently available only in very limited situations. I do not have the space to describe the exact limitations, but the sort of restrictions that are common are as follows.

1. If $R : G_\mathbb{Q} \to GL(V)$ then there should be a character $\mu : G_\mathbb{Q} \to GL_n(\overline{\mathbb{Q}}_l)$ and a non-degenerate bilinear form $(\ ,\ )$ on V such that

- $(R(\sigma)v_1, R(\sigma)v_2) = \mu(\sigma)(v_1, v_2)$ and
- $(v_2, v_1) = \mu(c)(v_1, v_2)$.

(This seems to be essential for the method of [TW]. It combines an "essentially self-dual" hypothesis and an "oddness" hypothesis.)

2. R should be de Rham with distinct Hodge-Tate numbers. (This again seems essential to the method of [TW], but see [BT].)

3. Either R and R' should be ordinary (i.e. their restrictions to $G_{\mathbb{Q}_l}$ should be contained in a Borel subgroup); or R and R' should be crytsalline (not just de Rham) at l with the same Hodge-Tate numbers and l should be large compared with the differences of elements of $\mathrm{HT}(R)$. (The problems here are connected with the need for an integral Fontaine theory, but they are not simply technical problems. There are some complicated results pushing back this restriction in isolated cases, see [CDT], [BCDT], [Sa], but so far our understanding is very limited. The results of [BCDT] did suffice to show that every rational elliptic curve is modular.)

4. The image of $\overline{R}$ should not be too small (e.g. should be irreducible when restricted to $\mathbb{Q}(e^{2\pi i/l})$), though in the case $n = 2$ there is beautiful work of Skinner and Wiles ([SW1] and [SW3]) dispensing with this criterion, which this author has unfortunately not fully understood.

In addition, all the published work is for the case $n = 2$. However there is ongoing work of a number of people attempting to dispense with this assumption. Using a very important insight of Diamond [Dia], the author, together with L.Clozel and M.Harris, has generalised to all n the so called minimal case (originally treated in [TW]) where R is no more ramified than $\overline{R}$. One would hope to be able to deduce the non-minimal case from this, as Wiles did in [Wi] for $n = 2$. In this regard one should note the work of Skinner and Wiles [SW2] and the work of Mann [Ma]. However there seems to be one missing ingredient, the analogue of the ubiquitous Ihara lemma, see lemma 3.2 of [Ih] (and also theorem 4.1 of [R2]). As this seems to be an important question, but one which lies in the theory of discrete subgroups of Lie groups, let us take the trouble to formulate it, in the hope that an expert may be able to prove it. It should be remarked that there are a number of possible formulations, which are not completely equivalent and any of which would seem to suffice. We choose to present one which has the virtue of being relatively simple to state.

CONJECTURE 5.3. — *Suppose that $G/\mathbb{Q}$ is a unitary group which becomes an inner form of GL_n over an imaginary quadratic field E. Suppose that $G(\mathbb{R})$ is compact. Let l be a prime which one may assume is large compared to n. Let p_1 and p_2 be distinct primes different from l with $G(\mathbb{Q}_{p_1}) \cong GL_n(\mathbb{Q}_{p_1})$ and $G(\mathbb{Q}_{p_2}) \cong GL_n(\mathbb{Q}_{p_2})$. Let U be an open compact subgroup of $G(\mathbb{A}^{p_1,p_2})$ and consider the representation of $GL_n(\mathbb{Q}_{p_1}) \times GL_n(\mathbb{Q}_{p_2})$ on the space $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/U, \overline{\mathbb{F}}_l)$ of locally constant $\overline{\mathbb{F}}_l$-valued functions on*

$$G(\mathbb{Q})\backslash G(\mathbb{A})/U = (G(\mathbb{Q}) \cap U)\backslash(GL_n(\mathbb{Q}_{p_1}) \times GL_n(\mathbb{Q}_{p_2}).$$

(Note that $G(\mathbb{Q}) \cap U$ is a discrete cocompact subgroup of $GL_n(\mathbb{Q}_{p_1}) \times GL_n(\mathbb{Q}_{p_2})$.) Suppose that $\pi_1 \otimes \pi_2$ is an irreducible sub-representation *of $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/U, \overline{\mathbb{F}}_l)$ with π_1 generic. Then π_2 is also generic.*

The most serious problem with applying such lifting theorems to prove an l-adic representation R is automorphic is the need to find some way to show that $\overline{R}$ is automorphic. The main success of lifting theorems to date, has been to show that if E is an elliptic curve over the rationals then $H^1(E(\mathbb{C}), \overline{\mathbb{Q}}_l)$ is automorphic, so that E is a factor of the Jacobian of a modular curve and the L-function $L(E,s)$ is an entire function satisfying the expected functional equation ([Wi], [TW],[BCDT]). This was possible because $GL_2(\mathbb{Z}_3)$ happens to be a pro-soluble group and there is a homomorphism $GL_2(\mathbb{F}_3) \longrightarrow GL_2(\mathbb{Z}_3)$ splitting the reduction map. The Artin representation

$$G_{\mathbb{Q}} \longrightarrow GL(H^1(E(\mathbb{C}), \mathbb{F}_3)) \longrightarrow GL_2(\mathbb{Z}_3)$$

is automorphic by the Langlands-Tunnell theorem alluded to in section 5.2.

5.5. Other techniques?

I would like to discuss one other technique which has been some help if $n = 2$ and may be helpful more generally. We will restrict our attention here to the case $n = 2$ and $\det R(c) = -1$. We have said that the principal problem with lifting theorems for proving an l-adic representation $R : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_l)$ is automorphic is that one one needs to know that $\overline{R}$ is automorphic. This seems to be a very hard problem. Nonetheless one can often show that $\overline{R}$ becomes automorphic over some Galois totally real field $K/\mathbb{Q}$. (Because K is totally real, if $\overline{R}(G_{\mathbb{Q}}) \supset SL_2(\mathbb{F}_l)$ and $l > 3$ then $\overline{R}(G_K) \supset SL_2(\mathbb{F}_l)$. So this 'potential automorphy' is far from vacuous). The way one does this is to look for an abelian variety A/K with multiplication by a number field F with $[F : \mathbb{Q}] = \dim A$, and such that $\overline{R}$ is realised on $H^1(A(\mathbb{C}), \mathbb{F}_l)[\lambda]$ for some prime $\lambda|l$, while for some prime $\lambda'|l' \neq l$ the image of G_K on

$H^1(A(\mathbb{C}),\mathbb{F}_{l'})[\lambda']$ is soluble. One then argues that $H^1(A(\mathbb{C}),\mathbb{F}_{l'})[\lambda']$ is automorphic, hence by a lifting theorem $H^1(A(\mathbb{C}),\mathbb{Q}_{l'})\otimes_{F_{l'}} F_{\lambda'}$ is automorphic, so that (tautologically) $H^1(A(\mathbb{C}),\mathbb{F}_l)[\lambda]$ is also automorphic, and hence, by another lifting theorem, $R|_{G_K}$ is automorphic. One needs K to be totally real, as over general number fields there seems to be no hope of proving lifting theorems, or even of attaching l-adic representations to automorphic forms. In practice, because of various limitations in the lifting theorems one uses, one needs to impose some conditions on the behaviour of a few primes, like l, in K and some other conditions on A. The problem of finding a suitable A over a totally real field K, comes down to finding a K-point on a twisted Hilbert modular variety. This is possible because we are free to choose K, the only restriction being that K is totally real and certain small primes (almost) split completely in K. To do this, one has the following relatively easy result.

PROPOSITION 5.4 ([MB],[P]). — *Suppose that $X/\mathbb{Q}$ is a smooth geometrically irreducible variety. Let S be a finite set of places of $\mathbb{Q}$ and suppose that X has a point over the completion of $\mathbb{Q}$ at each place in S. Let $\mathbb{Q}_S$ be the maximal extension of $\mathbb{Q}$ in which all places in S split completely (e.g. $\mathbb{Q}_{\{\infty\}}$ is the maximal totally real field). Then X has a $\mathbb{Q}_S$-point.*

In this regard it would have extremely important consequences if the following question had an affirmative answer. I do not know if it is reasonable to expect one.

QUESTION 5.5. — *Suppose that $X/\mathbb{Q}$ is a smooth geometrically irreducible variety. Let S be a finite set of places of $\mathbb{Q}$ and suppose that X has a point over the completion of $\mathbb{Q}$ at each place in S. Let $\mathbb{Q}_S^{\mathrm{sol}}$ be the maximal soluble extension of $\mathbb{Q}$ in which all places in S split completely. Does X necessarily have a $\mathbb{Q}_S^{\mathrm{sol}}$-point?*

Because of limitations in the lifting theorems available we can not at present successfully employ this strategy to all odd two dimensional l-adic representations. However we can apply it to all but finitely many elements in any compatible family. Thus for instance one can prove the following result.

THEOREM 5.6 ([TAY]). — *Suppose that $\mathcal{R}$ is an irreducible weakly compatible system of two dimensional l-adic representations with $\mathrm{HT}(\mathcal{R}) = \{n_1, n_2\}$ where $n_1 \neq n_2$. Suppose also that $\det R_{l,\iota}(c) = -1$ for one (and hence for all) pairs (l,ι). Then there is a Galois totally real field $K/\mathbb{Q}$ and a cuspidal automorphic representation π of $GL_2(\mathbb{A}_K)$ such that*

- *for all $v|\infty$, π_v has infinitesimal character H, and*

- *for all (l, ι) and for all finite places $v \not| l$ of K we have*

$$\mathrm{rec}(\pi_v) = \mathrm{WD}_v(R_{l,\iota}|_{G_K})^{\mathrm{ss}}.$$

In particular $\mathcal{R}$ is pure of weight $(n_1 + n_2)/2$. If $|n_1 - n_2| > 1$ then for each l and ι the l-adic representation $R_{l,\iota}$ is geometric. This conclusion also holds if $|n_1 - n_2| = 1$ but for distinct primes $l' \neq p$ and for an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$ the WD-representation $\mathrm{WD}_p(R_{l,\iota})$ has a nontrivial N.

Applying Brauer's theorem as in example 5.3.3 of section 5.3 we obtain the following corollary.

COROLLARY 5.7 ([TAY]). — *Keep the assumptions of the theorem. Then $\mathcal{R}$ is strongly compatible and*

$$L(\iota\mathcal{R}, s) = \prod_i L(\pi_i, s)^{n_i}$$

where $n_i \in \mathbb{Z}$ and where π_i is a cuspidal automorphic representation of $GL_2(\mathbb{A}_{K_i})$ for some totally real field K_i. The L-function $L(\iota\mathcal{R}, s)$ has meromorphic continuation to the entire complex plane and satisfies the expected functional equation.

We remark that conjecture 3.2 would imply that the compatible systems considered in theorem 5.6 are automorphic over $\mathbb{Q}$ (see example 5.3.3 of section 5.3).

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