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Multizetas, perinomal numbers, arithmetical dimorphy, and ARI/GARI


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Multizetas, perinomal numbers, arithmetical dimorphy, and ARI/GARI(*)

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ABSTRACT. — In a sprawling field like multizeta arithmetic, connected with intricate new structures and teeming with ‘special objects’ (functions, moulds etc), there is room for expositions of all formats: short, medium-sized, huge. Here is a survey on the tiniest scale possible, based on a talk given at the 2002 Luminy conference on Resurgent Analysis.

RÉSUMÉ. — Le texte qui suit, aussi ramassé que possible, reprend un exposé fait à Luminy en novembre 2002. Il présente un panorama des récents progrès en arithmétique ‘dimorphique’ des multizêtas et esquisse les théories (ARI/GARI, objets périnomaux, moules spéciaux) qui ont permis ces progrès.

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0. Overview. Some notations

We begin (§1, §2) with a few reminders about *arithmetical dimorphy* and then focus on the prototypical instance of dimorphy: the Q-ring of *multizetas*. The next two sections (§3, §4) outline, for future use, two special theories — one called forth by the study of dimorphy, the other predating it. They are the theory of *perinomal objects*; and the *flexion structure* — mainly the Lie algebra ARI and its group GARI. Next, we try (§5) to order the field of multizeta arithmetic as a hierarchy of increasingly arduous tasks, with a red thread running through everything: the search for *canonical irreducibles*. We then (§7 through §11) develop the tools (special moulds etc) which make it possible not only to explicitly decompose all multizetas into irreducibles (§12), but also to express these irreducibles *directly* and in a way that truly reflects their neutral position, half-way between the two natural bases of multizetas (§13). But before getting started, we must get a few definitions (about moulds, mould operations, and mould symmetries) out of the way.

Moulds $A^\bullet = \{A^\omega\} = \{A^{\omega_1,\ldots,\omega_r}\}$ are simply functions of a *variable number of variables*. These variables are noted as upper indices, with bold face reserved for sequences, which often get subsumed as a simple dot \(\bullet\). Mould addition is trivially defined, but mould multiplication is non-commutative and involves the breaking-up of sequences:

$$\{C^\bullet = A^\bullet \times B^\bullet\} \iff \{C^\omega = \sum_{\omega = \omega_1,\omega_2} A^\omega_1 B^\omega_2\} \quad (0.1)$$

Depending on the context, many other secondary operations may come into play. Moreover, useful moulds tend to fall into one or the other of a few *symmetry types*, which are either preserved by the basic operations, or transformed in transparent manner. Only six symmetry types will be relevant here, to wit: *symmetral/alternal*, *symmetrel/alternel*, *symmetril/alternil*.
A mould $A^*$ is said to be symmetral (resp alternal) or again symmetrel (resp. alternel) if the following identities hold for all $\omega^1, \omega^2$:

\[ \sum_{\omega \in \text{sha}(\omega^1, \omega^2)} A^\omega = A^{\omega^1} A^{\omega^2} \quad (\text{symmetral/alternal}) \quad (0.2) \]

\[ \sum_{\omega \in \text{she}(\omega^1, \omega^2)} A^\omega = A^{\omega^1} A^{\omega^2} \quad (\text{symmetrel/alternel}) \quad (0.3) \]

with $\text{sha}(\omega^1, \omega^2)$ (resp. $\text{she}(\omega^1, \omega^2)$) denoting the set of all ordinary (resp. contracting) shufflings of $\omega^1, \omega^2$. The last pair symmetrel/alternel applies only to moulds with a double-storeyed indexation, i.e. with indices of the form $\omega_i = (\omega_{i1}, \omega_{i2})$. It resembles symmetrel/alternel, except that the straight-forward addition $(\omega_i, \omega_j) \mapsto \omega_i + \omega_j$ makes way for the subtler contractions:

\[ \left( A^{\omega_{i1}, \omega_{i2}}, A^{\omega_{j1}, \omega_{j2}} \right) \mapsto \frac{1}{v_i - v_j} \left( A^{\omega_{i1}, \omega_{i2} + \omega_{j1}} - A^{\omega_{i1} + \omega_{j1}, \omega_{i2}, \omega_{j2}} \right) \quad (0.4) \]

Throughout, we shall use the following abbreviations for sums/differences:

\[ u_{i,j} := u_i + u_j \quad , \quad u_{i,j,k} := u_i + u_j + u_k \quad \text{etc} \quad ; \quad v_{i,j} := v_i - v_j \quad (0.5) \]

1. Multizeta numbers and numerical dimorphy

Some extremely important $\mathbb{Q}$-rings of transcendental numbers are dimorphic, i.e. possess two natural $\mathbb{Q}$-bases $\{ \alpha_m \}, \{ \beta_n \}$ with a simple conversion rule and two independent multiplication tables, all of which involve only rational coefficients and finite sums:

\[ \alpha_m = \sum^* H^n_m \beta_n, \]

\[ \beta_n = \sum^* K^n_n \alpha_m \quad (H^n_m, K^n_n \in \mathbb{Q}) \]

\[ \alpha_{n_1} \alpha_{n_2} = \sum^* A^{n_3}_{n_1, n_2} \alpha_{n_3}, \]

\[ \beta_{n_1} \beta_{n_2} = \sum^* B^{n_3}_{n_1, n_2} \beta_{n_3} \quad (A^{n_3}_{n_1, n_2}, B^{n_3}_{n_1, n_2} \in \mathbb{Q}) \]

The simplest, most basic of all such rings is $\text{Zeta}$, which is not only multiplicatively generated but also linearly spanned by the so-called multizetas.

---

(2) under ordinary/contracting shufflings, adjacent indices $\omega_i, \omega_j$ stemming from different sequences are forbidden/allowed to merge into $\omega_{i,j} := \omega_i + \omega_j$. Thus, for a pair $\omega^1 = (\omega_1,\omega_2) = (\omega_2,\omega_3)$, we have $\text{sha}(\omega^1, \omega^2) = \{ (\omega_1, \omega_2, \omega_3), (\omega_2, \omega_1, \omega_3), (\omega_2, \omega_3, \omega_1) \}$ but $\text{she}(\omega^1, \omega^2) = \{ (\omega_1, \omega_2, \omega_3), (\omega_2, \omega_1, \omega_3), (\omega_2, \omega_3, \omega_1), (\omega_1 + \omega_2, \omega_3), (\omega_1, \omega_2 + \omega_3) \}$.

(3) with some natural countable indexation $\{ m \}, \{ n \}$, not necessarily on $\mathbb{N}$ or $\mathbb{Z}$.

(4) or $\text{MZV}$, short for multiple zeta values.
In the first basis, the multizetas are given by polylogarithmic integrals:

$$W_\alpha^{\alpha_1, \ldots, \alpha_l} := \left(-1\right)^{l_0} \int_0^1 \frac{dt_1}{(\alpha_l - t_1)} \cdots \int_0^{t_3} \frac{dt_2}{(\alpha_2 - t_2)} \int_0^{t_2} \frac{dt_1}{(\alpha_1 - t_1)}$$  \hspace{1cm} (1.1)

with indices $\alpha_j$ that are either 0 or unit roots.

In the second basis, multizetas are expressed as familiar-looking sums:

$$Z_\epsilon^{\epsilon_1, \ldots, \epsilon_r} := \sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \cdots n_r^{-s_r} e_1^{-n_1} \cdots e_r^{-n_r}$$  \hspace{1cm} (1.2)

with $s_j \in \mathbb{N}^*$ and unit roots $e_j := \exp(2\pi i \epsilon_j)$ with ‘logarithms’ $\epsilon_j \in \mathbb{Q}/\mathbb{Z}$.

The stars * means that the integrals or sums are provisionally assumed to be convergent or semi-convergent: for $W_\alpha^{\alpha_1, \ldots, \alpha_l}$ this means that $\alpha_1 \neq 0$ and $\alpha_l \neq 1$, and for $Z_\epsilon^{\epsilon_1, \ldots, \epsilon_r}$ this means that $(\epsilon_1) \neq (0)$ i.e. $(\epsilon_1) \neq (1)$.

The corresponding moulds $W_\alpha^{\ast}$ and $Z_\epsilon^{\ast}$ turn out to be respectively symmetrical and symmetrel:

$$W_\alpha^{\ast, \ast} = \sum_{\alpha \in \text{sha}(\alpha^1, \alpha^2)} W_\alpha^{\alpha} \hspace{1cm} \forall \alpha^1, \forall \alpha^2$$  \hspace{1cm} (1.3)

$$Z_\epsilon^{\ast, \ast} = \sum_{(\epsilon') \in \text{she}(r^1, r^2)} Z_\epsilon^{\epsilon'} \hspace{1cm} \forall (r^1), \forall (r^2)$$  \hspace{1cm} (1.4)

These are the so-called “quadratic relations”, which express dimorphy. As for the conversion rule, it reads:

$$W_\alpha^{e_1, 0^{[s_1 - 1]}, \ldots, e_r, 0^{[s_r - 1]}} := Z_\epsilon^{e_r, e_{r-1}, \ldots, e_1, 0^{[s_1 - 1]}} \hspace{1cm} (1.5)$$

$$Z_\epsilon^{s_1, s_2, \ldots, s_r} := W_\alpha^{e_1, 0^{[s_1 - 1]}, \ldots, e_r, 0^{[s_r - 1]}} \hspace{1cm} (1.6)$$

There happen to be unique extensions $W_\alpha^{\ast} \rightarrow W_\alpha^{\ast}$ and $Z_\epsilon^{\ast} \rightarrow Z_\epsilon^{\ast}$ to the divergent case that keep our moulds symmetrical/symmetrel while conforming to the ‘initial conditions’ $W_\alpha^0 = W_\alpha^1 = 0$ and $Z_\epsilon^{(1)} = 0$. The only price to pay is a slight modification of the conversion rule: see §2 infra.

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(5) $l_0$ is the number of zeros in the sequence $\{\alpha_1, \ldots, \alpha_l\}$.

(6) with the usual shorthand for differences: $\epsilon_{i,j} := \epsilon_i - \epsilon_j$. 

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Basic gradations/filtrations: Four parameters dominate the discussion:
- the weight $s := \sum s_i$ (in the $\text{Ze}^*$-encoding) or $:= l$ (in the $\text{Wa}^*$-encoding)
- the length $r :=$ number of $\epsilon_i$'s or $s_i$'s or non-zero $\alpha_i$'s.
- the degree $d := s - r = $ number of zeros in the $\text{Wa}^*$-encoding.\(^{15}\)
- the root order $p :=$ smallest $p$ such that all $\epsilon_i$ are in $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$.

Only $s$ defines an (additive and multiplicative) gradation; the other parameters merely induce filtrations.

2. Generating series/functions

The natural encodings $\text{Wa}^*$ and $\text{Ze}^*$ being unwieldy and too heterogeneous in their indexations, we must replace them by suitable generating series, so chosen as to preserve the simplicity of the two quadratic relations and that of the conversion rule. This essentially imposes the following definitions:

$$\text{Zag}^{(\epsilon_1, \ldots, \epsilon_r)} := \sum_{1 \leq s_j} \text{Wa}^{(\epsilon_1, \ldots, \epsilon_r)} v_1^{s_1-1} v_2^{s_2-1} \cdots u_1^{s_1-1} \cdots u_r^{s_r-1}$$  \hspace{1em} (2.1)

$$\text{Zig}^{(\epsilon_1, \ldots, \epsilon_r)} := \sum_{1 \leq s_j} \text{Ze}^{(\epsilon_1, \ldots, \epsilon_r)} u_1^{s_1-1} \cdots v_r^{s_r-1}$$  \hspace{1em} (2.2)

These power series are actually convergent: they define generating functions\(^{7}\) that are meromorphic, with multiple poles at simple locations. These functions, in turn, verify simple difference equations, and admit an elementary mould factorisation:

$$\text{Zag}^\bullet := \lim_{k \to \infty} (\text{doZag}^\bullet_k \times \text{coZag}^\bullet_k)$$  \hspace{1em} (2.3)

$$\text{Zig}^\bullet := \lim_{k \to \infty} (\text{coZig}^\bullet_k \times \text{doZig}^\bullet_k)$$  \hspace{1em} (2.4)

with dominant parts $\text{doZag}^\bullet / \text{doZig}^\bullet$ that carry the $u/v$-dependence\(^{8}\):

$$\text{doZag}^\bullet_k^{(\epsilon_1, \ldots, \epsilon_r)} := \sum_{1 \leq m_i \leq k} e_1^{-m_1} \cdots e_r^{-m_r} P(m_1 - u_1)P(m_1,2 - u_1,2) \cdots P(m_1..r - u_1..r)$$  \hspace{1em} (2.5)

$$\text{doZig}^\bullet_k^{(\epsilon_1, \ldots, \epsilon_r)} := \sum_{k \geq n_1 > n_2 > \cdots > n_r \geq 1} e_1^{-n_1} \cdots e_r^{-n_r} P(n_1 - v_1)P(n_2 - v_2) \cdots P(n_r - v_r)$$  \hspace{1em} (2.6)

\(^{15}\) $d$ is called degree, because under the correspondence scalars $\rightarrow$ generating series, the multizetas become coefficients of monomials of total degree $d$. See (2.1),(2.2).

\(^{7}\) still denoted by the same symbols.

\(^{8}\) with the usual abbreviations $m_{i,j} := m_i + m_j$, $m_{i,j,k} := m_i + m_j + m_k$ etc.
and corrective parts coZag\(^\bullet\)/coZig\(^\bullet\) that reduce to constants:

\[
\text{coZag}_k^{(u_1, \ldots, u_r)} := (-1)^r \sum_{1 \leq m_i \leq k} P(m_1)P(m_{1,2}) \ldots P(m_{1 \ldots r}) \quad (2.7)
\]

\[
\text{coZig}_k^{(v_1, \ldots, v_r)} := (-1)^r \sum_{k \geq n_1 \geq n_2 \geq \ldots \geq n_r \geq 1} \mu^{n_1, \ldots, n_r} P(n_1)P(n_2) \ldots P(n_r) \quad (2.8)
\]

\[
\text{coZag}_k^{(\epsilon_1, \ldots, \epsilon_r)} := 0 \quad \text{if} \quad (\epsilon_1, \ldots, \epsilon_r) \neq (0, \ldots, 0) \quad (2.9)
\]

\[
\text{coZig}_k^{(\epsilon_1, \ldots, \epsilon_r)} := 0 \quad \text{if} \quad (\epsilon_1, \ldots, \epsilon_r) \neq (0, \ldots, 0) \quad (2.10)
\]

with \(P(t) := 1/t\) (here and throughout) and with \(\mu^{n_1, n_2, \ldots, n_r} := \frac{1}{r_1!r_2! \ldots r_l!}\)
if the non-increasing sequence \((n_1, \ldots, n_r)\) attains \(r_1\) times its highest value,
\(r_2\) times its second highest value, etc.

Setting \(\text{Mini}_k^\bullet := \text{Zig}_k^\bullet \|_{v=0}\) we find:

\[
\text{Mini}_k^{(0, \ldots, 0)} := \sum_{1 \leq r \leq r/2} (-1)^{r-l} \mu^{r_1, \ldots, r_l} \frac{(P(n_1))^{r_1}}{r_1} \ldots \frac{(P(n_l))^{r_l}}{r_l} \quad (2.11)
\]

\[
\text{Mini}_k^{(\epsilon_1, \ldots, \epsilon_r)} := 0 \quad \text{if} \quad (\epsilon_1, \ldots, \epsilon_r) \neq (0, \ldots, 0) \quad (2.12)
\]

We have an exact equivalence between old and new symmetries:

\[
\{\text{Wa}^\bullet \text{ symmetrical}\} \iff \{\text{Zag}^\bullet \text{ symmetrical}\} \quad (2.13)
\]

\[
\{\text{Ze}^\bullet \text{ symmetrical}\} \iff \{\text{Zig}^\bullet \text{ symmetrical}\} \quad (2.14)
\]

and the old conversion rule for scalar multizetas\(^{10}\) becomes:

\[
\text{Zig}^\bullet = \text{Mini}^\bullet \times \text{swap}(\text{Zag})^\bullet \quad (2.15)
\]

\[
\text{swap}(\text{Zig}^\bullet) = \text{Zag}^\bullet \times \text{Mana}^\bullet \quad (2.16)
\]

with the involution \text{swap} defined as in (4.1) \text{ infra} and with elementary moulds \(\text{Mana}^\bullet/\text{Mini}^\bullet := \lim_{k \to \infty} \text{Mana}_k^\bullet/\text{Mini}_k^\bullet\) whose only non-zero components:

\[
\text{Mana}^{(u_1, \ldots, u_r)} \equiv \text{Mini}^{(v_1, \ldots, v_r)} \equiv \text{Mono}_r \quad (2.17)
\]

\(^{9}\) if we had no factor \(\mu^{n_1, \ldots, n_r}\) in (2.8), we would have \(\text{Zig}_k^\bullet \|_{v=0} = 0\) and therefore no \(\text{Mini}_k^\bullet\) terms. But the mould \(\text{Zig}_k^\bullet\) would fail to be \text{symmetrical}, as required. Here lies the origin of the corrective terms in the conversion rule.

\(^{10}\) some modified form of (1.5),(1.6).
due to (2.11) are expressible in terms of monozetas:

\[ 1 + \sum_{r \geq 2} \text{Mono}_r t^r := \exp \left( \sum_{s \geq 2} (-1)^{s-1} \zeta(s) \frac{t^s}{s} \right) \]  

(2.18)

To these relations one must add the so-called self-consistency relations:

\[ \text{Zag} \left( \begin{array}{c} u_1 \\ \vdots \\ u_r \end{array} \right) = \sum_{q e^*_i = q e_i} \text{Zag} \left( \begin{array}{c} q u_1 \\ \vdots \\ q u_r \end{array} \right) \]  

(2.19)

\[ \forall q | p, \forall u_i \in \mathbb{C}, \forall e_i, e^*_i \in \frac{1}{p} \mathbb{Z}/\mathbb{Z} \]

which merely reflect trivial identities between unit roots of order \( p \).

3. Perinomal objects: equations, functions, numbers

Let \( \text{Sl}_r(\mathbb{Z}) \) denote the ‘special group’ (integer entries, unit determinant) with its natural action \( M : f \mapsto M f \) on functions of \( r \) variables:

\[ M f(\ldots, x_i, \ldots) := f(\ldots, \sum M_{i,j} x_j, \ldots) \quad \forall M \in \text{Sl}_r(\mathbb{Z}) \]  

(3.1)

A perinomal system is a system of equations of the form:

\[ \left\{ M f \equiv^i \varphi(x, f), \ldots, M^s f \equiv^s \varphi(x, f) \right\} \quad (f \text{ unknown}, \ M \in \text{Sl}_r(\mathbb{Z})) \]  

(3.2)

with data \( \varphi(x, f) \) usually linear or affine in \( f \) and ‘elementary’ in \( x \).

A perinomal function is a solution of such a system.

A function is said to have finite perinomal degrees \( d_{i,j} \) if \( f(x_1, \ldots, x_i + k x_j, \ldots, x_r) \) is polynomial in \( k \) of degree \( d_{i,j} \).

A perinomal number is a number attached to a perinomal function – by integration, summation, or taking its Taylor coefficients at the origin, etc.

Perinomal systems are a cross between difference and \( q \)-difference systems, but they also commend themselves to our attention for a number of more specific reasons:

1. Finiteness properties: Important spaces of perinomal functions admit a natural gradation by a global degree \( d \), with a finite basis for any given \( d \). The subject is of course closely tied up with the theory of finite linear representations of \( \text{Sl}_r(\mathbb{Z}) \).
2. **Closure properties**: Perinomal functions tend to be stable under partial differentiation, multiplication, various types of convolution, etc.

3. **Self-duality under Fourier/Borel/Laplace**: This tends to be the case whenever any of these transforms applies. For instance, we have a correspondence between **homogeneous linear perinomal systems** of the form:

\[
\{ e^{x \cdot \partial_x} f(x) = L_{i,j}(f), \quad \forall i, j \} \quad \xrightarrow{\mathcal{F}/\mathcal{B}/\mathcal{L}} \quad \{ e^{-x \cdot \partial_x} \xi \cdot \hat{f}(\xi) = L_{i,j}(\hat{f}), \quad \forall i, j \}
\]

which is reminiscent of the self-duality properties for **homogeneous linear differential equations** with polynomial coefficients:

\[
\left( \sum a_{m,n} x^m \partial_x^n \right) f(x) \quad \overset{\mathcal{B}/\mathcal{L}}{\leftrightarrow} \quad \left( \sum (-1)^n a_{m,n} \partial_x^n \xi^n \right) \hat{f}(\xi)
\]

4. **Duality between perinomal meromorphic functions and their residues**: This applies in particular to **eupolar** meromorphic functions \( f \) with multipoles of maximal order, located ‘at’ multi-integers \( n \) and carrying multisizes \( \rho \). Thus, for a rather trivial type of eupolarity:

\[
f(x) \overset{\text{ess}}{=} \sum_{n_i \in \mathbb{Z}} \frac{\rho(n_1, \ldots, n_r)}{(n_1 - x_1) \cdots (n_r - x_r)} : \{ f \text{ perinomal} \} \Leftrightarrow \{ \rho \text{ perinomal} \}
\]

5. **Link with multizeta arithmetic**: Multizeta arithmetic makes extensive use of **perinomal numbers** \( \rho^\# \) attached to discrete perinomal functions \( \rho \) via the series:

\[
\rho^\#(s_1, \ldots, s_r) \overset{\text{ess}}{=} \sum_{n_i \in \mathbb{N}^*} \rho(n_1, \ldots, n_r) n_1^{-s_1} \cdots n_r^{-s_r}
\]

or, equivalently, attached to meromorphic perinomal functions \( f \) under the taking of Taylor coefficients.

To put some flesh on these definitions, let us give two simple examples:

**Example 3.1.** — Fix \((k_1, \ldots, k_r)\) in \(\mathbb{N}^r\). The perinomal function \( f \):

\[
f(x) \overset{\text{ess}}{=} \sum_{n_i \in \mathbb{Z}^r} \frac{\rho(n_1, \ldots, n_r)}{(n_1 - x_1) \cdots (n_r - x_r)}
\]

with

\[
\rho(n_1, \ldots, n_r) \begin{cases} |n_1|^{k_1} \cdots |n_r|^{k_r} & \text{if } (n_1, \ldots, n_r) \text{ coprime} \\ 0 & \text{otherwise} \end{cases}
\]

\[(12)\] in (3.5) and throughout the sequel, the warning “ess(entially)” shall mean: up to the addition of simple (usually constant) corrective terms that ensure absolute convergence, or after suitable regroupings that ensure semi-convergence.
has Taylor coefficients \( \rho^# \) of the form\(^{(13)}\):

\[
\rho^#(s_1, \ldots, s_r) = 2^r \frac{\zeta(s_1 - k_1) \ldots \zeta(s_r - k_r)}{\zeta(\sum s_i - \sum k_i)} \quad \text{if all } s_i \text{ are even}
\]
\[
= 0 \quad \text{otherwise.}
\]

Despite being given as infinite sums, these \( \rho^# \) are clearly rational whenever all data \( k_i \) are even integers. This phenomenon shall be pivotal to the construction of the rational-coefficiented bimould \( loma^*/lomi^* \).

**Example 3.2.** — Consider now the less simplistic perinomal function:

\[
f(x) := \sum_{n_i \in \mathbb{Z}^*} \frac{\rho(n_1, n_2)}{(n_1 - x_1)(n_2 - x_2)}
\]

\[
\rho(n_1 + n_2, n_2) = \rho(n_1, n_2) + 1, \quad \rho(n_1, n_1 + n_2) = \rho(n_1, n_2) - 1
\]

\[
\rho(n_1, n_2) = \text{sign}(n_1)\text{sign}(n_2) (c_1 - c_2 + c_3 - c_4 + \ldots)
\]

\[
\text{if } \left| \frac{n_1}{n_2} \right| = [c_1, c_2, c_3, c_4, \ldots] = \text{continued fraction}
\]

with residues \( \rho \) defined by the perinomal system \( (3.9) \). What about the arithmetical nature of the corresponding perinomal numbers?

\[
\rho^#(s_1, s_2) \overset{\text{ess}}{=} \sum_{n_i \in \mathbb{N}^*} \rho(s_1, s_2) n_1^{-s_1} n_2^{-s_2}
\]

Whether we start from \( (3.9) \) or \( (3.10) \), that nature is far from clear. But the moment we form the functions \( f_a \) and \( f_i \):

\[
\begin{align*}
f_a(u_1, u_2) &= f(x_1, x_2) \\
+f(u_1, u_2) - f(u_2, u_1) &\quad \| \quad +f(v_1, v_2) - f(v_2, v_1) \\
+f(u_1 + u_2, u_1) - f(u_1, u_1 + u_2) &\quad \| \quad +f(v_1 - v_2, v_2) - f(v_2, v_1 - v_2) \\
-f(u_1 + u_2, u_2) + f(u_2, u_1 + u_2) &\quad \| \quad -f(v_1, v_2 - v_1) + f(v_2 - v_1, v_1)
\end{align*}
\]

we see that the multiresidues simplify dramatically, and that the \( \rho^#(s_1, s_2) \) are in fact simple rational combinations of multizetas. Furthermore, \( f_a \) links the \( \rho^#(s_1, s_2) \) to the \( Wa^* \)-basis, while \( f_i \) links them to the \( Ze^* \)-basis. This example is but the tip of a mighty iceberg — namely the direct-impartial expression of the multizeta irreducibles.

\footnote{\( \text{(13)} \) for homogeneity reasons, \( \rho^#(s_1, \ldots, s_r) \) always denotes the coefficient of \( x_1^{s_1-1} \ldots x_r^{s_r-1} \).}
4. The adequate structure: ARI/GARI and AXI/GAXI

The starting point is the algebra BIMU. Its elements are bimoulds, ie moulds $A^\bullet = \{A^{w_1,\ldots,w_r}\} = \{A^{(u_1,\ldots,u_r)}\}$ with double-storeyed indices $w_i = (u_i, v_i)$. BIMU is endowed with the ordinary mould product $\times$, which is often noted $\mu$ to avoid confusion with a host of other operations on bimoulds. All these operations involve simultaneous additions of the $u_i$-variables and subtractions of the $v_i$-variable, which makes it expedient to systematically use the abbreviations $(0.5)$ for sums and differences.

There is on BIMU a basic involution, the *swap*, which exchanges both sets of variables:

$$B^\bullet = \text{swap}(A^\bullet) \iff B^{(u_1,\ldots,u_r)} = A^{(u_1, r, u_1, r-1, \ldots, u_2, 3, u_1, 2)}$$

and a basic shift operator\(^{(14)}\), the *push*, which acts as follows:

$$C^\bullet = \text{push}(A^\bullet) \iff C^{(u_1, \ldots, u_r)} = A^{(-u_1, r, u_1, v_1, \ldots, u_r, v_r)}$$

(4.1)

All further operations involve *sequence flexions* $a.b \rightarrow a|b$ or $a\parallel b$. Thus, relative to the factorisation $w = \ldots a.b... = (u_3, v_3, u_4, v_4)(u_5, v_5, u_6, v_6, u_7, v_7, u_8, v_8, u_9, v_9)\ldots$ the flexion markers $\parallel, |, \parallel$ should be interpreted as follows:

$$a := (u_3, u_4, u_5, 6, 7, 8, 9)\quad |b := (u_4, v_4, u_7, v_7, u_8, v_8)\quad (4.3)$$

$$a := (u_3, u_4, u_5)\quad |b := (u_3, u_4, u_5, 6, 7, 6, 8, 9)\quad (4.4)$$

The binary operation *ari* defined by the flexions\(^{(15)}\):

$$A^\bullet = \text{ari}(A^\bullet, B^\bullet) \iff C^w = \sum_{w=b,c} (A^b B^c - B^b A^c)$$

$$+ \sum_{w=b,c,d} (A^{c}B^{b}[d] - B^{c}[B^{d}]A^{b}) + \sum_{w=a,b,c} (A^{a}[cB^{b}] - B^{a}[cA^{b}])$$

(4.5)

turns BIMU\(^{(16)}\) into a Lie algebra known as ARI.

\(^{(14)}\) of order $r+1$ when restricted to components of length $r$.

\(^{(15)}\) with $b \neq \emptyset, c \neq \emptyset$ in all three sums; but $a$ and $d$ may be empty.

\(^{(16)}\) i.e. the set of all $A^\bullet$ with vanishing length-$0$ component ($A^0 = 0$).
Likewise, the binary operation \( gari \) defined by the flexions \(^{(17)}\):
\[
C^* = gari(A^*, B^*) \quad \iff \quad C^w = \sum_{w=n^1.b^1.c^1...a^s.b^s.c^s.n^s+1} A[\beta^1]...[\beta^s]B^a B^a_{s+1} B^c B^c_{s} \quad (4.6)
\]
turns \( BIMU^* \) into a Lie group \( GARI \), with \( ARI \) as its Lie algebra.

**Central bimoulds**, by definition, \( gari \)-commute with, and \( ari \)-annihilate, everyone else. They are of the form:
\[
C^{w_1,...,w_r} := c(r) \in \mathbb{C} \quad \text{if} \quad (v_1,...,v_r) = (0,...,0) \quad (\forall (u_1,...,u_r)) \quad (4.7)
\]
\[
:= 0 \quad \text{if} \quad (v_1,...,v_r) \neq (0,...,0) \quad (\forall (u_1,...,u_r)) \quad (4.8)
\]

The following are important subalgebras/subgroups of \( GARI/ARI \):

- \( ARI_{push} := \{ A^* ; A^* \text{ push-invariant} \} \)
- \( ARI_{al} := \{ A^* ; A^* \text{ alternating} \} \)
- \( ARI_{al/\text{al}} := \{ A^* ; A^* \text{ alternating} \} \quad \text{swap}(A^*) \text{ alternating}, IP \}
- \( ARI_{al/\text{il}} := \{ A^* ; A^* \text{ alternating} \} \quad \text{swap}(A^*) \text{ alternil}, IP \}
- \( GARI_{spush} := \{ A^* ; A^* \text{ spush-invariant} \} \)
- \( GARI_{as} := \{ A^* ; A^* \text{ symmetric} \} \)
- \( GARI_{as/\text{as}} := \{ A^* ; A^* \text{ symmetric} \} \quad \text{swap}(A^*) \text{ symmetric}, IP \}
- \( GARI_{as/\text{is}} := \{ A^* ; A^* \text{ symmetric} \} \quad \text{swap}(A^*) \text{ symmetril}, IP \}

**Initial parity clause (IP)**: in the above definitions, we demand that bimoulds in \( ARI_{al/\text{al}} \) or \( ARI_{al/\text{il}} \) (resp. \( GARI_{as/\text{as}} \) or \( GARI_{as/\text{is}} \)) should have as their length-1 component an **even** function of \( w_1 \), but we allow for the addition of (resp. multiplication by) a **central bimould** \( C^* \) before taking the **swap**.

We have the non-trivial inclusions and isomorphisms:

\[
\begin{align*}
ARI_{push} \supset ARI_{al/\text{al}} \quad \text{algebra isom.} \quad ARI_{al/\text{il}} \\
\downarrow \text{expari} \quad \downarrow \text{expari} \\
GARI_{spush} \supset GARI_{al/\text{as}} \quad \text{group isom.} \quad GARI_{as/\text{is}}
\end{align*}
\]

and a non-trivial action \( ari/gari \) of \( ARI/GARI \) on the \( mu \)-algebra \( BIMU \).

\(^{(17)}\) with \( s \geq 2 \) and all factor sequences \( b^i \neq 0 \) and \( c^i.a^{i+1} \neq 0 \). The factors \( c^i \) et \( a^{i+1} \) may turn empty but **separately** so and the extreme factors \( a^1,c^s,a^{s+1} \) may also turn empty, **separately or jointly**. As for \( B^* \), it denotes the inverse \( invmu(B^*) \) of \( B^* \) with respect to ordinary mould multiplication \( mu \) (same as \( \times \)).

\(^{(18)}\) i.e. the set of all \( A^* \) with unit length-0 component \( (A^0 = 1) \).
Though ARI/GARI traces its origins to singularity theory, its double series of variables $u_2$ and $v_i$ as well as its property of accommodating and reproducing double symmetries, makes it an ideal tool for investigating arithmetical dimorphy. ARI/GARI is actually part of a larger umbrella structure, AXI/GAXI, which regroups all flexion derivations/automorphisms\(^{(19)}\) of BIMU.

5. Multizeta arithmetic: the main steps

\textit{R1. Formalisation: from numbers to symbols.}

Formalising the scalar multizetas means replacing the familiar systems of numbers $Wa^*/Ze^*$ by symbols $wa^*/ze^*$ subject to the same quadratic relations, conversion rule, and self-consistency constraints. In terms of generating series, it means replacing $Zag^*/Zig^*$ by the most general pair $zag^*/zig^*$ of symmetral/symmetril bimoulds connected under the swap:\(^{(20)}\)

\[
\text{swap}(zig)^* = \text{gari}(zag^*, mana^*) \quad \text{(with mana^* central)} \quad (5.1)
\]

and subject to the old self-consistency constraints (2.19) but with components that are arbitrary power series instead of well-defined meromorphic functions.

\textit{R2. Free generation.}

It says that the $\mathbb{Q}$-rings of (scalar) formal multizetas are polynomial rings in countably many indeterminates – the so-called irreducibles.

\textit{R3. From the atomic to the subatomic level.}

It means re-interpreting the apparently unbreakable irreducibles as elements of a Lie algebra $\text{ARI}_{\text{reg,disc}}^{\text{reg,disc}}$\(^{(21)}\), which opens the way to a finer analysis.

\textit{R4. Free subgeneration.}

It says that the finer and truly ultimate building blocks\(^{(22)}\), or subgenerators, are either free (for small values of the root order $p$) or “very nearly free” (for larger $p$). The various dimensions (additive/multiplicative) that can be

\(^{(19)}\) i.e. of all those derivations or automorphisms of the $mu$-algebra BIMU that can be expressed in terms of the flexions (4.3),(4.4). Elements of AXI/GAXI are determined not by single bimoulds $A^*$, but by pairs $(A_L^*, A_R^*)$ consisting of a left and a right bimould.

\(^{(20)}\) due to $mana^*$ being a central bimould, we have in fact:

\[
gari(mana^*, zag^*) = gari(zag^*, mana^*) = mu(zag^*, mana^*) \quad (\text{but} \neq \mu(mana^*, zag^*))
\]

\(^{(21)}\) the upper index \textit{reg} means regular in $u$ at the origin, i.e. without Laurent terms etc; the upper index \textit{disc} for discrete means with $v$-variables in $\mathbb{Q}/\mathbb{Z}$; the underlining of both upper indices means subject to the self-consistency constraints (2.19).

\(^{(22)}\) in ARI.
attached to subspaces/subrings of multizetas and that depend on the various filtrations/gradations (by \( s, r, d, p \) etc) mostly follow from that.

**R5. Decomposition into irreducibles: constructive.**
It should be fully constructive, i.e. amenable to effective computation.

**R6. Decomposition into irreducibles: explicit.**
In a context such as this, if we are to maintain a meaningful distinction between *constructive* and *explicit*, the latter can mean only one thing, namely: given by direct formulas which, though inevitably complex, are nonetheless perspicuous enough\(^{23}\) and reasonably compact\(^{24}\). Above all, *explicit* means that we are not required to solve larger and larger linear systems as the natural filtration/gradation parameters \( (s, r, d, p \) etc) increase.

**R7. Decomposition into irreducibles: canonical.**
Though redolent of subjectivity, the notion of *canonicity* matters immensely. Here, we are fortunate in being able to construct, among all possible, more or less natural systems of irreducibles, one that is indisputably canonical.

**R8. Direct and ‘impartial’ expression of the irreducibles.**
It goes way beyond the mere reversing of the canonical-explicit decomposition of multizetas into irreducibles; rather, it asks for a direct and ‘impartial’ (i.e. ‘equidistant’ from the two competing bases \( wa^* \) and \( ze^* \)) expression of the irreducibles. This is where perinomal algebra comes in.

**R9. Materialisation: from symbols to numbers.**
That would mean: showing that the \( \mathbb{Q} \)-ring of ‘actual’ or ‘genuine’ multizetas is actually isomorphic to its formalisation. This is the one great challenge that still lies ahead. It apparently defies extant mathematical tools, but the advent of *direct*\(^{25}\) *numerical derivations*\(^{26}\) might change that.

### 6. The general scheme

To simplify, we set out the general procedure for *ordinary multizetas*\(^{27}\). The \( \mathbb{Q} \)-ring of multizetas splits into two/three factors rings:

\[
\text{Zeta} = \text{Zeta}_{I+II} \otimes \text{Zeta}_{III} \quad (6.1)
\]

\[
\text{Zeta} = \text{Zeta}_I \otimes \text{Zeta}_{II+III} \quad (6.2)
\]

\[
\text{Zeta} = \text{Zeta}_I \otimes \text{Zeta}_{II} \otimes \text{Zeta}_{III} \quad \text{with} \quad \text{Zeta}_I = \mathbb{Q}[\pi^2] \quad (6.3)
\]

---

\(^{(23)}\) to make the main features, symmetries etc of the objects at hand easily detectable.

\(^{(24)}\) both as mathematical text or as computation programmes.

\(^{(25)}\) all the emphasis here is on *direct*; other derivations are useless chaff.

\(^{(26)}\) that annihilate \( \mathbb{Q} \) but act non-trivially on \( \mathbb{Q} \)-rings of transcendental numbers

\(^{(27)}\) i.e. multizetas without unit roots.
The factor-ring $\mathcal{Zeta}_1$ is generated by $\pi^2$.
The factor-ring $\mathcal{Zeta}_{III}$ contains all irreducibles of even weight and ‘length’.
The factor-ring $\mathcal{Zeta}_{II}$ contains all irreducibles of odd weight and ‘length’.

This splitting of the ring $\mathcal{Zeta}$ stems from a corresponding factorisation of the generating functions:

$$
\begin{align*}
\text{zag}^\bullet &= \text{gari}(\text{zag}^\bullet_{II}, \text{zag}^\bullet_{III}) \quad (\text{zag}^\bullet_{III} \in \text{GARI}_{as/ls}^{o,l}) \\
\text{zag}^\bullet &= \text{gari}(\text{zag}^\bullet_{II}, \text{zag}^\bullet_{II+III}) \quad (\text{zag}^\bullet_{II+III} \in \text{GARI}_{as/ls}^{o}) \\
\text{zag}^\bullet &= \text{gari}(\text{zag}^\bullet_{II}, \text{zag}^\bullet_{III}, \text{zag}^\bullet_{III}) \quad (\text{zag}^\bullet_{II} \in \text{GARI}_{as/ls}^{e,l})
\end{align*}
$$

The factors $\text{zag}^\bullet_{II}$ and $\text{zag}^\bullet_{III}$ carry only terms of even weight. As a consequence, their components of even/odd length are even/odd functions of $u$.

The factor $\text{zag}^\bullet_{III}$ carries only terms of odd weight. As a consequence, its components of even/odd length are odd/even functions of $u$.

The factor $\text{zag}^\bullet_{I}$ is symmetrical/ill$^{(28)}$ but doesn’t verify the initial parity condition $IP$ (see §4). Therefore, its gari-logarithm is not alternal/ill$^{(29)}$. The factors $\text{zag}^\bullet_{II}$ and $\text{zag}^\bullet_{III}$, on the other hand, do verify that condition and so belong to the proper symmetrical/ill group $\text{GARI}_{as/ls}$. Consequently, their gari-logarithms $\text{lozago}_{II}$ and $\text{lozago}_{III}$ belong to the proper alternal/ill algebra $\text{ARI}_{al/ill}$ and can be further analysed therein. Actually, it turns out that $\text{lozago}_{II}$ and $\text{lozago}_{III}$ can be uniquely generated by the $u$-homogeneous parts of a crucial bimould, $\text{loma}^\bullet \in \text{ARI}_{al/ill}$, with well-defined coefficients in front of the multibrackets. These coefficients are none other than the formal irreducibles.

Isolating the factor $\text{zag}^\bullet_{III}$ from the other two is quite easy. Indeed$^{(30)}$:

$$
\text{gari}(\text{zag}^\bullet_{III}, \text{zag}^\bullet_{III}) = \text{gari}(\text{nepar}(\text{invgari}(\text{zag}^\bullet_{III})), \text{zag}^\bullet_{III}) 
$$

But separating $\text{zag}^\bullet_{I}$ from $\text{zag}^\bullet_{II}$ is a far more arduous undertaking, especially if we insist, as we do, on getting a ‘canonical’ separation. This requires an elaborate construction, with the introduction of three special bimoulds, leading to a subfactorisation of $\text{zag}^\bullet_{I}$:

$$
\begin{align*}
\text{zag}^\bullet_{I} &= \text{gari}^\bullet(\text{tal}^\bullet, \text{invgari}(\text{pal}^\bullet), \text{expari}(\text{roma}^\bullet)) \\
&= \text{gari}^\bullet(\text{tal}^\bullet, \text{expari}(\text{viroma}^\bullet), \text{invgari}(\text{pal}^\bullet))
\end{align*}
$$

$^{(28)}$ i.e. symmetrical and with a symmetrical swappee.

$^{(29)}$ i.e. more exactly, it is alternal all right, but its swappee is not.

$^{(30)}$ invgari denotes the gari-inversion; and nepar multiplies each length-$r$ component by $(-1)^r$ while changing the signs of all $u_i$’s and $v_i$’s.
Multizetas, perinomal numbers, arithmetical dimorphy, and ARI/GARI

- with a ’eupolar’ factor \( \text{pal}^* \in \text{GARI}_{as/as} \) but \( \not\in \text{GARI}_{as/as} \) (see §7);
- with a ’eutrigonometric’ factor \( \text{tal}^* \in \text{GARI}_{as/as} \) but \( \not\in \text{GARI}_{as/as} \) (see §7);
- with a ’corrective factor’ \( \text{roma}^* \in \text{ARI}_{al/al} \) or its variant \( \text{viroma}^* \in \text{ARI}_{al/al} \).

In sum, everything begins with the construction of two special bimoulds\(^{31}\):
- \( \text{pal}^*/\text{pit}^* \) and \( \text{tal}^*/\text{til}^* \)
- both symmetral/symmetral (as bimoulds)
- both relatively elementary (as functions of \( u \))

but the really sensitive part consists in constructing and understanding two further, even more crucial, and highly non-elementary, bimoulds:
- \( \text{loma}^*/\text{lomi}^* \) and \( \text{roma}^*/\text{romi}^* \)
- both alternal/alternil (as bimoulds)
- both with rational coefficients\(^{32}\) (as formal series in \( u \))
- both strongly transcendental (as meromorphic functions of \( u \)) and actually of perinomal and eupolar type.

7. The bisymmetral bimoulds \( \text{pal}^*/\text{pit}^* \) and \( \text{tal}^*/\text{til}^* \)

The two semi-elementary factors in the decomposition of \( \text{Zag}^* \) are built from the following simple ingredients:

\[
\begin{align*}
P(t) & := \frac{1}{t} & (7.1) \\
Q(t) & := \frac{\pi}{\tan(\pi t)} & (7.2) \\
Qa^{v_0} = Qa^{(v_0)} & := \frac{1}{p} \sum_{1 \leq k \leq p} e^{2\pi ikv_0} Q(u_0 + \frac{k}{p}) \quad \text{if} \quad v_0 \in \frac{1}{p} \mathbb{Z}/\mathbb{Z} & (7.3) \\
Ea^{v_0} = Ea^{(v_0)} & := \pi \quad \text{if} \quad v_0 = 0 \mathbb{Q}/\mathbb{Z} \quad \text{(resp.} \quad 0 \quad \text{if} \quad v_0 \not= 0 \mathbb{Q}/\mathbb{Z}) & (7.4)
\end{align*}
\]

The simpler bimould, \( \text{pal}^* \), depends only on the \( u \)-variables. It is called eupolar because of the very specific form of its poles. Its length-\( r \) component is a homogeneous polynomial of total degree \( r \) in simple \( P \)-expressions:

\[
\text{pal}^{u_1,\ldots,u_r} \in \mathbb{Q}[P(u_1), P(u_2), \ldots, P(u_r), P(u_{1,2}), P(u_{1,2,3}), \ldots, P(u_{1,\ldots,r})]
\]

The second bimould, \( \text{tal}^* \), is called eutrigonometric. For vanishing \( v \)-variables, it closely resembles the eupolar bimould since its length-\( r \) component

\(^{31}\) as usual, the swappee of a bimould bears the same name, with \( i \) instead of \( a \).

\(^{32}\) in the case of \( \text{roma}^*/\text{romi}^* \), the coefficients become rational after an elementary rescaling \( \pi^2 \mapsto 1 \).
is again a homogeneous polynomial\(^{33}\) of degree \(r\) in \(Q\)-expressions and \(\pi\):
\[
tal^{w_1, \ldots, w_r} \in \mathbb{Q}[\pi, Q(u_1), Q(u_2), \ldots, Q(u_r), Q(u_{1,2}), Q(u_{1,2,3}), \ldots, Q(u_{1,\ldots,r})]
\]
For general \(v\)-variables, \(tal^{w_1, \ldots, w_r}\) is still a homogeneous polynomial of total
degree \(r\), but in the variables \(Eaw^i\) and \(Qaw^i\), with double indices \(w^i_1 = (u^i_1)\)
subject to \(\sum u^i_1 v^i_1 = \sum u_i v_i\) and of the form \((u^i_j)\) or \((u^i_j v^i_k)\) or \((u^i_1 \ldots v^i_r)\).

Both bimoulds verify the \textit{self-consistency constraints} and both are \textit{bissymmetrical}\(^{34}\) with, in the case of \(tal^*\), an elementary connection factor \(mana^*\)
much like the ‘global’ \(mana^*\) in (2.15) but carrying only even powers of \(\pi\):
\[
\text{swap}(pil^*) = pal^* = \text{symmetral} \quad (7.5)
\]
\[
\text{swap}(til^*) = gari(tal^*, mana^*) = gari(mana^*, tal^*) = \text{symmetral} \quad (7.6)
\]

These properties\(^{35}\) completely determine \(pal^*/pil^*\) and \(tal^*/til^*\). These
bimoulds admit fully explicit expressions and enjoy an incredible number of
properties. As far as multizeta algebra is concerned, they intervene at three
critical junctures:
- \(tal^*\) describes ‘almost all’ poles of \(Zag_I^*\). It is therefore the mainstay of the
  \textit{canonical-rational Drinfel’d associator}.
- \(pa^*\) provides us with a canonical-explicit isomorphism between the two
  ‘doubly symmetric’ or ‘dimorphic’ algebras \(ARI_{al/al}\) and \(ARI_{al/il}\).
- \(pa^*\) enables us to construct the so-called “singulators” and these in turn
  make it possible to remove all the unwanted singularities at \(u = 0\) which
  appear during the inductive construction of the moulds \(loma^*\) and \(roma^*\).

8. From the atomic to subatomic level. Free generation
and subgeneration

\(ARI_{al/al}\) is easily shown to be closed under the \(ari\) bracket. Under the
adjoint action of \(pa^*\) in \(ARI\), this closure property carries over to \(ARI_{al/il}\):
\[
\text{adari}(pa^*) : ARI_{al/al} \xrightarrow{\text{isom.}} ARI_{al/il} \quad (8.1)
\]
Further, using the factorisation (or stability) property:
\[
GARI_{as/is} = GARI_{as/is} \cdot GARI_{as/is} \quad (8.2)
\]
\[
zag^* = gari(zag_0^*, \expari(A^*)) \quad \text{with} \quad A^* \in ARI_{ent/diag}^{al/il} \quad (8.3)
\]
\(^{33}\) but with only even powers of \(\pi\) in it.
\(^{34}\) however, \(pa^*, tal^* \notin GARI_{as/as}\) since \(pal^{w_1}\) and \(tal^{w_1}\) are odd functions of \(w_1\).
\(^{35}\) together with the initial conditions \(pal^{w_1} = -\frac{1}{2} Qa w_1\) and \(tal^{w_1} = -\frac{1}{2} Qa w_1\).
we get the general zig\(^\bullet\) verifying all the constraints of \(RI\) in §5 by postcomposing any particular solution\(^{36}\) zig\(\mathcal{Z}_0^\bullet\) by the \(ari\)-exponential of the generic element \(A^\bullet\) of \(ARI_{\text{ent}/\text{disc}}^*\). Expanding \(A^\bullet := \sum c_J A^\bullet_J\) along any rational linear basis \(\{A_J\}\) of \(ARI_{\text{ent}/\text{disc}}^*\), we get one degree of freedom per basis element. The corresponding scalar coefficients \(c_J\) are none other than the sought-after irreducibles. Thus, the \(\mathbb{Q}\)-ring \(\text{Zeta}^{\text{form}}\) is seen to be isomorphic to the polynomial ring \(\mathbb{Q}[\pi^2] \otimes \mathbb{Q}[c_{J_1}, c_{J_2}, \ldots]\): this is the free generation theorem.

Now, as scalars, the irreducibles \(a_J\) are ‘atoms’, i.e. incapable of further analysis. But they correspond one-to-one to dual objects \(A^\bullet_J\) which, being elements of the Lie algebra \(ARI_{\text{ent}/\text{disc}}^*\), may be broken down as Lie brackets of still ‘simpler’ and much less numerous ‘subatoms’ \(B^\bullet_J\). Moreover, for small values of the root-order \(p\), in particular for \(p = 1, 2, 3\), the subatoms in question freely generate \(ARI_{\text{ent}/\text{disc}}^*\) as an algebra. This is the so-called free subgeneration theorem. And even for general values of \(p\), the relations between the subatoms \(B^\bullet_J\) are few in number and easy to describe, so that we may speak of a nearly free subgeneration. In all instances, knowledge of the subatoms implies knowledge of all the relevant dimensions: additive, multiplicative, etc (see §5).

We already saw how the change scalars \(\rightarrow\) generating series/functions entails a far-going restoration of symmetry between the two fundamental encodings, and a welcome ‘compactification’ of the conversion rule and quadratic constraints. We now register another, more decisive gain: the possibility of moving from the atomic to the subatomic level, leading to a complete understanding of the irreducibles. As for the further shift, from generating series to generating functions, the specific reward it brings is the notion of perinomal function, with the attendant direct-impartial description of the irreducibles.

9. Constructing \(loma^\bullet/lomi^\bullet\) and \(roma^\bullet/romi^\bullet\): the easy steps

This leaves us with two main tasks:
- constructing in \(ARI_{\text{ent}/\text{disc}}^*\) a mould \(loma^\bullet := \sum loma^\bullet_s\), regular in \(u\) at the origin, and with weight-homogeneous summands\(^{37}\) \(loma^\bullet_s\) that shall

---

\(^{36}\) For example the ready-made \(\text{Zag}^\bullet\) constructed from the genuine multizetas, or the factor \(\text{zig}_I^\bullet\) in (6.5).

\(^{37}\) the summand \(loma_{\geq s}^\bullet\) of weight \(s\) has \(s\) non-vanishing components, and its component of length \(r\) is a polynomial \(loma_{\geq s}^{u_1, \ldots, u_r} \in \mathbb{Q}[u_1, \ldots, u_r]\) of total degree \(d := s - r\).
yield the afore-mentioned sub-atoms $B_J$ and support the description of all irreducibles but $\pi^2$.

- constructing in $\text{ARI}_{\text{sing/disc}}^{\text{al/il}}$ a mould $\text{roma}^*$, singular in $u$ at the origin, but with singularities that exactly compensate those of the factors $\text{pal}^*$ and $\text{tal}^*$ in (6.8),(6.9) so as to produce a regular factor $\text{zag}^2_I$, leading to an automatic separation of the ‘rogue’ irreducible $\pi^2$ from all others.

Both constructions rely on an induction on the component length $r^{38}$ but there is a sharp dichotomy between easy steps:

- for $\text{loma}^*$: going from length $r$ odd to length $r+1$ even
- for $\text{roma}^*$: going from length $r$ even to length $r+1$ odd

which are automatic under the ARI/GARI machinery, and tricky steps (the alternate ones!) which involve a complex compensation mechanism.

Here is a table setting forth the easy induction steps for $\text{loma}^*$:

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{loma}^*</td>
<td></td>
<td>_r$</td>
</tr>
<tr>
<td>$\downarrow \text{adari}(\text{pal}^*)^{-1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{viloma}^*</td>
<td></td>
<td>_r$</td>
</tr>
<tr>
<td>$\downarrow \text{trivial extension}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{viloma}^*</td>
<td></td>
<td>_{r+1}$</td>
</tr>
<tr>
<td>$\downarrow \text{adari}(\text{pal}^*)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{loma}^*</td>
<td></td>
<td>_{r+1}$</td>
</tr>
</tbody>
</table>

The trivial extension $\text{viloma}^*||_r \mapsto \text{viloma}^*||_{r+1}$ consists of course in setting the $r+1^{th}$ component of $\text{viloma}^*||_{r+1}$ equal to 0.

The same basic procedure holds for $\text{roma}^*$, but with even/odd exchanged and with the regular/singular dichotomy applying not to $\text{roma}^*$ itself, but to the product $\text{zag}^2_I$.

\(^{38}\) in the sequel, $M^*||_r$ denotes $M^*$ defined modulo all components of length $> r$. 
10. Singulators and the removal of singularities: the tricky steps

If we apply the same procedure for the ‘tricky steps’, the machinery will work all right and still produce an extension \( loma_{r + 1} \) with the proper symmetries, but with unwanted singularities at the origin. To removed these, some finely honed terms have to be added. The key notion here is that of singulator. These are operators \( \text{slank}_r \) and \( \text{slang}_r \) that turn elementary bimoulds (regular in \( u \) at 0, and with a single non-zero component, for \( r = 1 \)) into bimoulds with the proper symmetries (either \( \text{al/ai} \) or \( \text{al/il} \)) and the proper singularities (‘eupolar’) at the component of length \( r_0 \).

Here is how they are defined:

\[
\text{slank}_r : \quad \text{H}^* := \text{neginvar} \cdot \text{leng}_0 \cdot \text{adari(pal)}^{-1} \cdot \text{mu(mupaj, leng}_1 \cdot \text{H}^*, \text{paj}) \equiv \text{pushinvar}_r \cdot \text{leng}_r \cdot \text{mu(anti.pal, garit(pal))}.
\]

\[
\text{slang}_r : \quad \text{H}^* := \text{adari(pal)} \cdot \text{slank}_r \text{H}^* \quad (10.3)
\]

\[
\text{slank}_r \text{H}^* \in \text{ARI}_{\text{al/al}} \quad; \quad \text{slang}_r \text{H}^* \in \text{ARI}_{\text{al/il}} \quad (r_0 \in \mathbb{N}^*) \quad (10.4)
\]

and here is how they affect their argument \( \text{H}^* \):

\[
\begin{align*}
\text{H}^* & \in \text{ARI}_{\text{reg/disc}}^{	ext{al/al}} \quad \| \quad \text{sole non-zero component for } r = 1, \\
\text{slank}_r \text{H}^* & \in \text{ARI}_{\text{sing/disc}}^{	ext{al/al}} \quad \| \quad \text{sole non-zero component for } r = r_0, \\
\text{slang}_r \text{H}^* & \in \text{ARI}_{\text{sing/disc}}^{	ext{al/il}} \quad \| \quad \text{sole non-zero components for } r \geq r_0,
\end{align*}
\]

In the above relations \( \text{leng}_r, \text{neginvar}, \text{pushinvar} \) denote projectors on \( \text{BIMU} \):

- \( \text{leng}_r \) retains the component of length \( r \) and annihilates all others.
- \( \text{neginvar} \) turns any bimould into one that is an even function of \( w \).
- \( \text{pushinvar} \) turns any bimould into one that is push-invariant (see (4.2)).

The elementary symmetrical bimoulds \( \text{paj}^*, \text{mupaj}^* \) are mutually inverse under the ordinary mould product \( \text{mu} \) and depend only on the \( u \)-variables:

\[
\begin{align*}
\text{paj}^{u_1, \ldots, u_r} &= P(u_1)P(u_{1,2}) \ldots P(u_{1,\ldots,r}) \quad (10.5) \\
\text{mupaj}^{u_1, \ldots, u_r} &= (-1)^rP(u_{1,\ldots,r})P(u_{2,\ldots,r}) \ldots P(u_r) \quad (10.6)
\end{align*}
\]
The operator \( \text{pari} \) multiplies components of length \( r \) by \((-1)^r\) and anti reverses the order in index sequences. Lastly, \( \text{garit} \) denotes the natural action of GARI on BIMU.\(^{39}\)

11. **Explicit formulae for \( \text{loma}^* / \text{lomi}^* \)**

Starting from the two elementary bimoulds \( H^*_s \) and \( K^*_n \) with length-1 components of the form \( H^*_s := u_{i_1}^{s_1-1} \) and \( K^*_n := P(n-u_1) \) and applying \( \text{slang}_r \), we get two series of bimoulds:

\[
H^*_r := \text{slang}_r H^*_s \in \text{ARI}^{\text{sing/disc}}_{al/iz} \quad \text{and} \quad K^*_r := \text{slang}_r K^*_n \in \text{ARI}^{\text{sing/disc}}_{al/iz}
\]

that make it possible to compensate the unwanted singularities produced at each trickly step of the induction. This leads to two parallel expansions.

**First expansion of \( \text{loma}^* \) and each \( \text{loma}_s^* \): as power series in \( u \).**

\[
\text{act}(\text{loma}_s^*) := +\text{act}(H^*_i) \quad (r = 3) \\
\| + \sum_{\begin{array}{c} s_1 + s_2 = s \\ 1 + 2 = 3 \end{array}} \beta^{[s_1, s_2]} \text{act}(H^*_{i_1}) \text{act}(H^*_{i_2}) \\
\| \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quadratic residues \quad rho^* that are discrete perinomial functions of the integers n_i.

(39) which of course is distinct from the adjoint action of GARI in ARI.
The first expansion is not unique, but becomes so if we want it to coincide with the second one. It then involves well-defined coefficients $\beta^\bullet$ which are, unexpectedly but crucially, rational numbers. Each one of them is expressible as a ratio of finitely many hyper-bernoullian numbers. In fact, for $r = 3$, they are quotients $\beta' / \beta''$ of just three ordinary Bernoulli numbers.\(^{(40)}\)

As moulds, both $\rho^\bullet$ and $\beta^\bullet$ are alternal.

We have similar expansions for the bimould $\text{roma}^\bullet$.

12. Explicit-canonical decomposition of multizetas into irreducibles

Let $\tau^s$ be the projector on $\text{ARI}^{\text{reg/disc}}_{\text{al/i}}$ which, when applied to a bimould $M^\bullet$, retains only the part of weight $s$. For a component of length $r$, this means retaining only the part of degree $d = s - r$ in the $u$-variables.

$$\tau^s M^{(u_1, \ldots, u_r)} := M^{(u_1, \ldots, u_r)}|_{u\text{-part of degree } s - r} \quad (12.1)$$

For the generating series of the multizetas, whether ‘genuine’\(^{(41)}\) or formal\(^{(42)}\), this leads to unique decompositions, with a $\text{loma}^\bullet$-part that carries only rational coefficients, and coefficients $\text{Irr}^\bullet / \text{irr}^\bullet$ that absorb all the transcendence:

$$\text{act}(Zag^\bullet_{\text{II}}) = 1 + \sum \text{Irr}_{\text{II}}^{s_1, \ldots, s_r} \text{act}(\tau^{s_1}\text{loma}^\bullet) \ldots \text{act}(\tau^{s_r}\text{loma}^\bullet) \quad (12.2)$$

$$\text{act}(Zag^\bullet_{\text{III}}) = 1 + \sum \text{Irr}_{\text{III}}^{s_1, \ldots, s_r} \text{act}(\tau^{s_1}\text{loma}^\bullet) \ldots \text{act}(\tau^{s_r}\text{loma}^\bullet) \quad (12.3)$$

$$\text{act}(zag^\bullet_{\text{II}}) = 1 + \sum \text{irr}_{\text{II}}^{s_1, \ldots, s_r} \text{act}(\tau^{s_1}\text{loma}^\bullet) \ldots \text{act}(\tau^{s_r}\text{loma}^\bullet) \quad (12.4)$$

$$\text{act}(zag^\bullet_{\text{III}}) = 1 + \sum \text{irr}_{\text{III}}^{s_1, \ldots, s_r} \text{act}(\tau^{s_1}\text{loma}^\bullet) \ldots \text{act}(\tau^{s_r}\text{loma}^\bullet) \quad (12.5)$$

In all four sums, the indices $s_i$ are odd integers $\geq 3$ and “act” denotes any transitive action of ARI/GARI in BIMU – it doesn’t matter which. The $\text{gari}$-factorisations between symmetrical bimoulds:

$$Zag^\bullet_{\text{II,III}} := \text{gari}^\bullet(Zag^\bullet_{\text{II}}, Zag^\bullet_{\text{III}}) \quad (12.6)$$

$$zag^\bullet_{\text{II,III}} := \text{gari}^\bullet(zag^\bullet_{\text{II}}, zag^\bullet_{\text{III}}) \quad (12.7)$$

\(^{(40)}\) the step $3 \mapsto 4$ requiring no corrections, these harmless quotients $\beta' / \beta''$ already yield the explicit-canonical decomposition of all multizetas of length $r \leq 4$ and of any weight $s$, up to infinity!

\(^{(41)}\) with upper-case initials.

\(^{(42)}\) with lower-case initials.
induce corresponding \(\mu\)-factorisations for the symmetral, scalar moulds:

\[
\text{Irr}_{\text{II+III}}^\bullet := \text{Irr}_\text{II}^\bullet \times \text{Irr}_\text{III}^\bullet \quad (12.8)
\]

\[
\text{irr}_{\text{II+III}}^\bullet := \text{irr}_\text{II}^\bullet \times \text{irr}_\text{III}^\bullet \quad (12.9)
\]

Moreover, due to the \emph{parity} which governs everything here, the only non-zero components in the mould logarithms \(\logmu(\text{Irr}_\text{II}^\bullet)\) and \(\logmu(\text{irr}_\text{II}^\bullet)\) (resp. \(\logmu(\text{Irr}_\text{III}^\bullet)\) and \(\logmu(\text{irr}_\text{III}^\bullet)\)) are those of even (resp. odd) length \(r\). As an easy consequence, the mould \(\text{irr}_{\text{II+III}}^\bullet\), or \(\text{irr}^\bullet\) for short, actually determines its two factors \(\text{irr}_\text{II}^\bullet\) and \(\text{irr}_\text{III}^\bullet\). Summing up, we may say:

\[\text{Together with the symbol } \text{irr}_\text{II}^\bullet \sim \pi^2, \text{ the symmetral mould } \text{irr}_{\text{II+III}}^\bullet \text{ constitutes a system, both complete and free }^{44}, \text{ of canonical irreducibles for the (ordinary or \textquote{rootless}) formal multizetas. More precisely, any such multizeta may be uniquely linearised as a sum:}\]

\[
\sum_{r \geq 0} \sum_{s_0 \text{ even } \geq 0} \gamma_{s_0; s_1, \ldots, s_r} \pi^{s_0} \text{irr}_{s_1, \ldots, s_r}^\bullet \quad (\gamma^\bullet \in \mathbb{Q}). \quad (12.11)
\]

Analogous results hold for the \emph{rooted} multizetas.

\[\text{13. \textquote{Impartial} expression of the irreducibles as perinominal numbers}\]

Plugging (6.8),(12.4),(12.5) into the factorisation (6.6) and then picking the Taylor coefficients of either \(\text{zag}^\bullet\) or \(\text{zig}^\bullet\), we get the formal multizetas, in both encodings \(\text{wa}^\bullet\) and \(\text{ze}^\bullet\), automatically\(^{45}\), uniquely, and explicitly expanded as finite sums of irreducibles \(\text{irr}^\bullet\). The process of course may be reversed, yielding expressions of \(\text{irr}^\bullet\) in terms of either \(\text{wa}^\bullet\) and \(\text{ze}^\bullet\), but

\[\text{(43) i.e. subject to no other constraints than symmetrality.}\]

\[\text{(44) that is, free up to the symmetrality constraints. These constraints could easily be removed by working with the alternal mould } \logmu(\text{irr}^\bullet) \text{ and picking some Lyndon basis in the corresponding Lie algebra, but that would entail a slight loss of canonicity. The truth of the matter is that the irreducibles spontaneously present themselves in the shape of a symmetral mould – and there is no going against that. The whole point of the reduction, of course, lies in the change from a mould } \text{ze}^\bullet \text{ with a double symmetry and indices } s_i \text{ running through } \{1, 2, 3, 4, \ldots\}, \text{ to a mould } \text{irr}^\bullet \text{ with a single symmetry and indices } s_i \text{ running through } \{3, 5, 7, 9, \ldots\}.\]

\[\text{(45) via the } \text{ARI/GARI machinery}\]

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these reverse formulae are trebly defective: they are not particularly explicit; they are decidedly non-unique; and they are ‘partial’, in the sense of leaning towards one or the other of the two natural encodings. To remove these blemishes, we require expansions which, like (12.2)-(12.3), express $\text{Zag}_{II}, \text{Zag}_{III}$ in terms of $\text{loma}^*$, but treating these bimoulds as perinomial-meromorphic functions and no longer as power series. So, instead of breaking up $\text{loma}^*$ under the projectors $\tau^s$, we now apply the following dilation automorphisms \[46\)
\[\delta^n M^{(u_1,\ldots,u_r)} := n^{-r} M^{(u_1/n,\ldots,u_r/n)} \tag{13.1}\]
The new expansions read (with all $n_i$ running through $\mathbb{N}^*$):
\[
\text{act}(\text{Zag}_{II}^*) \overset{\text{ess}}{=} 1 + \sum \text{Urr}_{II} n_1^{s_1} \cdots n_r^{s_r} \text{act}(\delta^n \text{loma}^*) \ldots \text{act}(\delta^n \text{loma}^*) \tag{13.2}
\]
\[
\text{act}(\text{Zag}_{III}^*) \overset{\text{ess}}{=} 1 + \sum \text{Urr}_{III} n_1^{s_1} \cdots n_r^{s_r} \text{act}(\delta^n \text{loma}^*) \ldots \text{act}(\delta^n \text{loma}^*) \tag{13.3}
\]
with symmetral moulds $\text{Urr}_{II}^*, \text{Urr}_{III}^*$ that are rational-valued perinomial functions of the integers $n_i$.

A first fallout (from inverting (??)) is yet another expansion for $\text{loma}^*$:
\[
\text{act}(\text{loma}^*) \overset{\text{ess}}{=} \sum \text{Orr}_{III} n_1^{s_1} \cdots n_r^{s_r} \text{act}(\delta^n \text{Zag}_{III}^* - 1^*) \ldots \text{act}(\delta^n \text{Zag}_{III}^* - 1^*) \tag{13.4}
\]
which is often referred to as “wasteful-useful” 47. But the main consequence is a direct-impartial expression for the irreducibles. Indeed, if for any index $X$ of the form $II$, $III$ or $II+III$ we set:
\[
\text{Yrr}_{X}^{s_1,\ldots,s_r} \overset{\text{ess}}{=} \sum_{n_i \geq 1} \text{Urr}_{X} n_1^{s_1} \cdots n_r^{s_r} \tag{13.5}
\]
we get three parallel identities between symmetral moulds:
\[
\text{Zag}_{II+III}^* = \text{gari}^*(\text{Zag}_{II}^*, \text{Zag}_{III}^*) \tag{13.6}
\]
\[
\text{Urr}_{II+III}^* := \text{Urr}_{II}^* \times \text{Urr}_{III}^* \tag{13.7}
\]
\[
\text{Yrr}_{II+III}^* := \text{Yrr}_{II}^* \times \text{Yrr}_{III}^* \tag{13.8}
\]
and we find that the moulds $\text{Irr}_{X}^*$ and $\text{Yrr}_{X}^*$ actually coincide, though they vastly differ as to the way they are defined.

\[46\] they are automorphisms of $\text{ARI}_{al/al}$ and, thanks to the factor $n^{-r}$, of $\text{ARI}_{al/II}$ as well.

\[47\] “wasteful”, because it derives an object with sparse poles and rational Taylor coefficients from one with “dense” poles and transcendental Taylor coefficients; “useful”, because it automatically transports important properties (like invariance under the digonal involution: see [E2], Appendix) from upper-case $\text{Zag}^*$ and $\text{Zag}_{III}^*$ over to $\text{loma}^*$ and then, via (6.8),(12.4),(12.5), back to lower-case $\text{zag}^*$, thus proving that these properties (digonal invariance etc) are algebraically implied by the quadratic relations.
Remark 13.1. — There is also a notion of formal perinomal numbers, parallel to that of formal multizetas (but relative to constraints altogether different from the quadratic relations) and the above relations translate into a direct-impartial expression of the irreducibles $\text{irr}_X$ attached to the formal multizetas.

The whole theory also carries over to the case of rooted multizetas, but with the new phenomenon of retroaction. Results are particularly simple for the root orders $p = 2$ and $p = 3$. If anything, the case of Eulerian multizetas ($p = 2$) is even simpler than that of ordinary multizetas ($p = 1$). Perinomal functions still rule the roost. They fall into six main classes:

<table>
<thead>
<tr>
<th>Root Order</th>
<th>Bimoulds in $A_{\text{reg}}^{\text{disc}}(\text{al/it})$</th>
<th>Eupolar Functions (meromorphic-perinomal)</th>
<th>Residues $\rho$ (discrete-perinomal)</th>
<th>Taylor Coefficients $\rho^*$ (perinomal numbers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 1$</td>
<td>$\text{loma}^*$</td>
<td>“dense” multipoles</td>
<td>well-defined perinomal degrees $d_{i,j}$</td>
<td>rational and “Bernoullian”</td>
</tr>
<tr>
<td>$p = 2$</td>
<td>$\text{loma}^*$</td>
<td>“sparse” multipoles</td>
<td>well-defined perinomal degrees $d_{i,j}$</td>
<td>rational and “Bernoullian”</td>
</tr>
<tr>
<td>$p \geq 1$</td>
<td>$\log_\text{ari}(\mathbb{Z}^*)$</td>
<td>“dense” multipoles</td>
<td>well-defined perinomial degrees $d_{i,j}$</td>
<td>multizetaic and “impartial”</td>
</tr>
</tbody>
</table>

14. Conclusion. Looking back/ahead/sideways

Looking back: Despite interesting work by M. Petitot, J. van der Hoeven, Minh, etc, and some vigorous numerical exploration by Broadhurst, Borwein, Kreimer etc, plus some inspired conjectures based on these numerical data, the main problems in multizeta arithmetic were still open by the end of the 90s. The intervention in the last months of 2000 of the (pre-existing) ARI/GARI apparatus unfroze the situation. It yielded at once the free generation theorem and the basic dimorphic special bimoulds, leading to the canonical-explicit decomposition into irreducibles. Later still, in 2002, the recourse to perinomal objects opened the way for the direct-impartial expansion of irreducibles.

Looking ahead: Here are four possible avenues for further research:
- going through the list of known multizetaic constraints and showing that each of them is algebraically derivable from the quadratic relations. This

---

(48) which means that, for a fixed weight $s$, some of the constraints binding the multizetas of a given length $r_0 < s$ do not result from the double symmetries written down for $r = 1, \ldots, r_0$: the full sequence $r = 1, \ldots, s$ must be taken into account.

(49) we say that a meromorphic-eupolar functions on $\mathbb{C}^r$ with multipoles “at” the multiintegers $\mathbb{Z}^r$ has “dense” (resp. “sparse”) multipoles if the latters’ number inside a ball of radius $l$ grows like $O(l^r)$ (resp. $o(l^r)$).
has already been done in special instances. The trouble with that process is of course its open-endedness...

- attempting to establish once and for all the rigorous arithmetical isomorphism of formal and genuine multizetas. The most promising approach here is that of direct numerical derivations patterned on the alien derivations of analysis, which have largely systematised the proving of transcendence results for resurgent functions.

- pursuing the investigation of dimorphy beyond multizetas, in the next main dimorphic Q-rings \( \subset \mathbb{C} \) that have already been identified, beginning of course with suitable formalisations of these rings.

- exploring the trigebra \( \mathcal{N}_a \) of natural analysable germs\(^{50}\) and its numerical accompaniment, the ring \( \mathcal{N}_a \) of natural numbers, as the (probably) broadest setting for the understanding of numerical dimorphy.

**Looking sideways:** strangely, multizeta theory is rife with mistakes and misconceptions, some of which persist long after exposure. Here is a sample:

- the conversion rule (2.15)-(2.17): though long known and immediate to derive (see §2), it sometimes receives needlessly convoluted proofs.

- the meromorphic continuation of \( \text{Ze}^* \) in the \( s \)-variables: this almost self-evident fact (two lines of proof!) has been questioned, even denied, and then given clumsy, roundabout proofs\(^{51}\). Nor is there any awareness of the existence of another \( s \)-continuable bimould \( \text{Ze}^* \) but linked to the \( \text{Wa}^* \) encoding.\(^{52}\) The only difficult, and still open, question in this context pertains to the irreducibles \( \text{Irr}^* \): these are easily definable as holomorphic functions for large positive \( s \)-variables, but do they admit a meromorphic continuation on the negative side?

- status of the self-consistency constraints (2.19): it has been variously stated or suggested that they don’t exhaust the additional constraints for rooted multizetas\(^{53}\). In fact the exact opposite is the case: they are always sufficient and for small values of the root order \( p \) they are even redundant.

- status of ARI/GARI: it has been likened to various constructs, in particular the Ihara algebra, which is downright absurd, if only because:

(i) the Ihara algebra lacks the dual set of variables \( u/v \) which is indispensable for a ‘symmetrical’ treatment of dimorphy\(^{54}\)

(ii) it cannot accommodate the singular functions\(^{55}\) which fit effortlessly into the ARI/GARI framework and on which everything revolves

\(^{(50)}\) with its two sets of exotic derivations: alien and foreign.

\(^{(51)}\) which fall well short of a full description of the multipoles and their residues.

\(^{(52)}\) it is closely connected with the Taylor coefficients of the gari-inverse of \( \text{Zag}^* \).

\(^{(53)}\) i.e. the constraints that must be added to the quadratic relations and the conversion rule to get an (empirically) complete system of constraints.

\(^{(54)}\) as a consequence, even the bracket of ARI and that of the Ihara algebra are irredeemably non-isomorphic.

\(^{(55)}\) i.e. those bimoulds of ARI/GARI, which, as functions of \( u \) or \( v \), have poles at the origin, or away from it.
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(iii) it has no place for any of the sixty-odd special moulds\textsuperscript{56} which are essential to the construction and description of irreducibles.

- **status of numerical dimorphy:** when not ignored purely and simply, this central fact about transcendental numbers is often discussed within the quite uncongenial framework of "period theory" which, due to its partiality for integrals over series, distorts at the outset all the symmetries that underpin dimorphy.

Bibliography


[E3], [E4] shall be available as Orsay preprints in February 2005, and [E5], [E6] shall appear later on in the course of that same year. All four texts [E3,4,5,6] are extremely lengthy, but abridged versions shall be submitted to ordinary mathematical journals. Electronic files of the unabridged versions shall be put on the WEB, along with an assortment of Tables and Maple programmes.

\textsuperscript{56} the infatuation of algebraists with the Ihara apparatus must be the reason why, after more than a decade of fudging with multizetas, such conspicuous objects as pal\textsuperscript{2022} /pzl* and tale / tile had still escaped discovery, although these moulds were standing right there, tall and erect, and almost impossible to miss, at the very entrance of multizeta territory, as its guardians!