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Extension of Bochner-Lichnérowicz formula on spheres [(*)]

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Abstract. — Given a second order differential operator L, we define the vector space of "intrinsic bilinear operators" associated with it. They are constructed only from the operator L itself and the algebra structure given by the product of functions. When the operator is symmetric with respect to some positive measure, every positive quadratic form in this space provides information on the spectrum of the operator. The positiveness of a form relies only on the local structure of the operator.

The purpose of this paper to construct a sequence (R_k) of positive intrinsic quadratic forms on spheres (in this case, L is the Laplace-Beltrami operator) which carry all the information on the spectrum. More precisely, if f is an eigenvector of the Laplace-Beltrami operator associated to the eigenvalue λ and g is any smooth function, then, for the Riemann measure μ,

$$\int R_k(f,g)d\mu = \lambda(\lambda-\lambda_1)\cdots(\lambda-\lambda_{k-1})\int fgd\mu,$$

where $0,\lambda_1,\ldots,\lambda_{k-1}$ are the k first eigenvalues of the Laplace-Beltrami operator. This extends to the full spectrum the Bochner-Lichnérowicz formula which gives on the sphere a sharp lower bound on the first non-zero eigenvalue. An extension of this property is given for a family of operators which extends the ultraspherical operator on the real line.

Résumé. — On définit l'espace des formes bilinéaires intrinsèques associées à un opérateur L différentiel du second ordre. Ces formes sont définies uniquement à partir de l'opérateur L lui-même et de la structure d'algèbre sur les fonctions donnée par la multiplication. La positivité d'une de ces formes ne dépend que de la structure locale de l'opérateur. Pour des

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opérateurs symétriques par rapport à une mesure positive, chaque forme bilinéaire intrinsèque positive donne des informations sur le spectre de L.

Dans cet article, nous construisons une famille (R_k) de formes bilinéaires intrinsèques sur les sphères (dans ce cas, l'opérateur L est l'opérateur de Laplace-Beltrami) qui contient toute l'information sur le spectre. Plus précisément, si f est un vecteur propre de l'opérateur de Laplace-Beltrami de valeur propre λ, et si g est une fonction lisse quelconque, alors pour la mesure Riemannienne μ, on a

$$\int R_k(f,g)d\mu = \lambda(\lambda-\lambda_1)\cdots(\lambda-\lambda_{k-1})\int fgd\mu,$$

où $0, \lambda_1, \ldots, \lambda_{k-1}$ sont les k premières valeurs propres de l'opérateur de Laplace-Beltrami. C'est une extension à tout le spectre de la formule de Bochner-Lichnérowicz, qui donne une minoration précise de la première valeur propre non nulle. On étend ensuite cette propriété à tout une famille d'opérateurs du second ordre qui est une extension multidimensionnelle des opérateurs ultrasphériques en dimension 1.

1. Introduction

The famous Bochner-Lichnérowicz formula in Riemannian geometry asserts that on a Riemannian manifold of dimension n with Ricci curvature bounded below by a constant $\rho > 0$, the first non zero eigenvalue of the Laplace operator is bounded below by $\rho n/(n-1)$, and this inequality is optimal on spheres [3, 4, 8, 9]. This inequality is very simple to obtain: let Δ be the Laplace-Beltrami operator on the smooth manifold M (which we assume to be compact for simplicity), then, for any smooth function f on M, if we compute

$$\Gamma(f,f) = (1/2)\Delta(f^2) - f\Delta f,$$

we get

$$\Gamma(f,f) = |\nabla f|^2,$$

where $|\nabla f|$ is the lenght of the gradient of the function f computed in the Riemannian metric. Then, we may define the so-called iterated squared gradient by

$$\Gamma_2(f,f) = (1/2)\Delta(|\nabla f|^2) - \nabla f.\nabla\Delta f.$$

It turns out (and this is the Bochner-Lichnérowicz formula) that this may be computed from the Ricci tensor and is equal to

$$|\nabla\nabla f|^2 + \mathrm{Ric}(\nabla f, \nabla f),$$

where Ric denotes the Ricci tensor, $\nabla\nabla f$ denotes the (symmetric) tensor wich is the second covariant derivative of f, and $|M|^2$ is the square of the

Hilbert-Schmidt norm of the symmetric tensor M. Therefore, it is equivalent to say that the Ricci tensor is bounded below by ρ or to say that, for any smooth function f, $\Gamma_2(f,f) \geqslant \rho\Gamma(f,f)$.

This inequality is not by itself sufficient to produce the Bochner-Lichnerowicz lower bound, but if we recall that Δf is the trace of $\nabla\nabla f$, and that for any n-dimensional symmetric matrix, we have $|M|^2 \geqslant (1/n)(\operatorname{trace} M)^2$, then we get that a lower bound ρ on the Ricci tensor is equivalent to the fact that

$$\Gamma_2(f,f) \geqslant \rho\Gamma(f,f) + (1/n)(\Delta f)^2.$$

In fact, to get from this the lower bound on the non zero eigenvalues of $-\Delta$, let us introduce the Riemann measure μ and denote by $\langle f \rangle$ the integration of a function f with respect to it. We know that $-\Delta$ is symmetric with respect to μ, which means that

$$\langle g\Delta f\rangle = \langle f\Delta g\rangle,$$

for any pair of smooth functions, from which we deduce that

$$\langle \Delta f\rangle = 0$$

for any smooth function f. Now, if we call R_2 the positive quadratic form

$$R_2(f,f) = \frac{n}{n-1}[\Gamma_2(f,f) - \rho\Gamma(f,f) - (1/n)(\Delta f)^2],$$

and $R_2(f,g)$ the associated bilinear form. From what preceedes, we get, if $\Delta f = -\lambda f$, and for any smooth functions g,

$$\langle \Gamma(f,g)\rangle = \lambda\langle fg\rangle$$

and

$$\langle R_2(f,g)\rangle = \lambda(\lambda - \rho n/(n-1))\langle fg\rangle.$$

From the first one we get that the eigenvalues of $-\Delta$ are positive, and from the second one the fact that no eigenvalue lies between 0 and $\rho n/(n-1)$.

If we replace Δ by a general second order differential operator L, with no zero order term, we may follow the same construction, and define the operators Γ and Γ_2, which are built only from L and the product of two functions (we shall say later that those quadratic forms are intrinsic).

We may say that L satisfies a curvature-dimension inequality $CD(\rho,n)$ if, for any smooth function f,

$$\Gamma_2(f,f) \geqslant \rho\Gamma(f,f) + (1/n)(Lf)^2. \tag{1.1}$$

(Here, n does not need to be an integer and is only assumed to be positive, and may be infinite). If L is symmetric with respect to some measure μ, then the $CD(\rho, n)$ inequality carries the same information as before about the first non zero eigenvalue, but it says a lot more. For example, the $CD(\rho, \infty)$ inequality implies the celebrated logarithmic-Sobolev inequality, the gaussian isoperimetry, the property of concentration of measures, while the $CD(\rho, n)$ inequality for finite n implies the Sobolev inequality, and therefore the compactness of the resolvant, upper bounds on the diameter, upper bounds on the heat kernel, etc. [2].

Therefore, as we saw before, for the Laplace-Beltrami operator of a Riemannian manifold, it is equivalent to say that the Ricci curvature is bounded below by some constant ρ or to say that the (intrinsic) bilinear form $\Gamma_2(f, f) - \frac{1}{n}(\Delta f)^2 - \rho\Gamma(f, f)$ is positive.

The purpose of this note is to show that, for any integer k, we may construct on the n-dimensional sphere an intrinsic quadratic form defined on smooth functions, say $R_k(f, f)$, which is positive ($\forall f$, $R_k(f, f) \geqslant 0$), and which is such that if $\Delta f = -\lambda f$, for any smooth function g, we have

$$\langle R_k(f, g)\rangle = \lambda(\lambda - \lambda_1) \cdots (\lambda - \lambda_{k-1})\langle f, g\rangle,$$

where $\lambda_k = k(k + n - 1)$ is the k^{th} eigenvalue of the operator $-\Delta$. Those inequalities are for each k a kind of $CD(\rho, n)$ inequality, but at an higher order.

Therefore, the positivity of this sequence of operators R_k encodes the full spectrum of the sphere. More precisely, the R_k are constructed on the sphere by a recurrence formula only by means of algebraic manipulations on the Laplace-Beltrami operator Δ of the sphere (see definition 3.1 below). Assume that some elliptic operator L is defined on a smooth compact manifold M, and that it is self-adjoint with respect to some measure finite measure . Let $R_k(L)$ be the family of bilinear operators constructed from L in the same way, replacing Δ by L. Then, if, for $i = 1, \ldots, n$, $R_i(L)(f, f) \geqslant 0$ for any smooth f on the manifold, the spectrum of $-L$ must lie in $\{0, \lambda_1, \ldots, \lambda_{n-1}\} \cup [\lambda_{n-1}, \infty[$, where the λ_i are the eigenvalues of the spherical Laplace-Beltrami operator. (See proposition 2.2 in the next section.)

We then extend this sequence of sharp inequalities to a family of operators which generalizes in dimension n the one dimensional operator which is associated to the ultraspherical polynomials in dimension one.

2. Intrinsic bilinear operators

In what follows, we shall adopt a naïve point of view which avoids all problems which may occur on a non compact Riemannian manifold when dealing with the spectrum of the Laplace-Beltrami operator.

Let M be a set and $\mathcal{A}$ be an algebra of real valued functions on M. To fix the ideas, in most of the cases, M is a smooth manifold, and $\mathcal{A}$ is the set of compactly supported smooth functions on M, or may be some other set of smooth functions with a growth condition at infinity in the non compact case.

We consider a linear operator L from $\mathcal{A}$ into $\mathcal{A}$. If Q is a symmetric bilinear application mapping $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, we may define three new symmetric bilinear applications, $LQ(f,g)$, $Q(f,Lg) + Q(g,Lf)$, $Q(Lf,Lg)$.

Among those constructions, we shall distinguish the operation $[L,Q]$ as follows

$$[L,Q](f,g) = (1/2)[L(Q(f,g)) - Q(f,Lg) - Q(Lf,g)].$$

The vector space of bilinear operations which closed under those three operations and which contains the bilinear form $Q_0(f,g) = fg$ shall be called the space of intrinsic bilinear operators: they are contructed only from the operator L itself and the algebra structure. We shall call this space $\mathcal{I}(L)$. Those bilinear applications are just the bilinear members of the Lie algebra associated with L introduced by Ledoux [7].

We shall say that a element Q of $\mathcal{I}(L)$ is positive whenever, for each f in $\mathcal{A}$, one has $Q(f,f) \geqslant 0$.

To understand the link between the positivity of such quadratic forms and the spectrum of L, we shall assume that we are given on the set M a measure space structure, that $\mathcal{A}$ is made of measurable functions, and that a positive measure μ is given such that each element of $\mathcal{A}$ is integrable with respect to μ. The basic assumption is that L is symmetric in $L^2(\mu)$, that is, for each pair (f,g) in $\mathcal{A}^2$, one has

$$\int fLg\, d\mu = \int gLf\, d\mu.$$

We shall moreover assume that $\int Lf\, d\mu = 0$, for any f in $\mathcal{A}$. It is always the case if $\mathcal{A}$ contains the constant functions and if $L(1) = 0$. We shall call this measure μ a reversible measure associated to L. It needs not to be unique, but this shall be the case under mild conditions when it exists.

Those conditions are always fulfilled when L is the Laplace-Beltrami operator of a Riemannian manifold if we choose for μ the Riemann measure, but there are many other examples (we shall see some of those examples later on).

Under those conditions, to each Q in $\mathcal{I}(L)$, one may associate a real polynomial P_Q which has the following property:

if $f \in \mathcal{A}$ satisfies $Lf = -\lambda f$, then, for any $g \in \mathcal{A}$, one has

$$\int Q(f,g)\, d\mu = P_Q(\lambda) \int fg\, d\mu.$$

To see that, it is clear that it is the case for $Q_0(f,g) = fg$, with $P_{Q_0}(\lambda) = 1$, and the polynomials associated to $LQ(f,g)$, $Q(Lf,g) + Q(Lg,f)$ and $Q(Lf,Lg)$ are repectively 0, $-2\lambda P_Q$, $\lambda^2 P_Q$.

In particular, $P_{[L,Q]}(\lambda) = \lambda P_Q(\lambda)$.

(Formally, we do not need the reversible measure μ to associate those polynomials to a Q in $\mathcal{I}(L)$: it would be enough to describe exactly how the quadratic form Q is constructed from the basic multiplication Q_0.)

The main interest of those construction is the following:

PROPOSITION 2.1. — *Assume that Q is a positive intrinsinc bilinear operator. Then, for any eigen value λ of L in $\mathcal{A}$, (i.e. there exists an non zero element in $\mathcal{A}$ such that $Lf = -\lambda f$), then $P_Q(\lambda) \geqslant 0$.*

Proof. — It is straight forward. We write

$$0 \leqslant \int Q(f,f)\, d\mu = P_Q(\lambda) \int f^2\, d\mu.$$

□

Therefore, it is interesting to look for positive bilinear forms associated to a given operator L. For example, we have

PROPOSITION 2.2. — *Assume that, for some sequence $0 < \lambda_1 < \lambda_2, \ldots < \lambda_{k-1}$, there exists a sequence of intrinsic bilinear operators $R_1, \ldots, R_k$ which are positive ($R_i(f,f) \geqslant 0, \forall f \in \mathcal{A}$, $\forall i = 1, \ldots, k$) and that $P_{R_i}(l) = \lambda(\lambda - \lambda_1) \ldots (\lambda - \lambda_{i-1})$. If for some $f \in \mathcal{A}$, $f \neq 0$, we have $Lf = -\lambda f$, then $\lambda \in \{0, \lambda_1, \ldots, \lambda_{k-1}\} \cup [\lambda_{k-1}, \infty[$.*

Proof. — The proof of this proposition is straightforward. From the previous proposition, we have $P_{R_i}(\lambda) \geqslant 0$, for any $i = 1, \ldots, k$, and the conclusion follows by an easy induction. □

In the next section, we shall show that it is precisely what happens for the spherical Laplacian.

In fact, the positivity of R_2 is exactly the $CD(n-1,n)$ inequality of the sphere (1.1). The positivity of R_2 has proven to carry a lot of important information on the operator L, far beyond spectral properties. But up to now, we do not see what kind of information on the operator L could be deduced from the positivity of R_3 (not to talk of the other ones) although we have the feeling that it should carry similar properties. (See [7] for example, where a similar analysis is carried in a case where no dimensional information is involved.)

3. The spherical case

In what follows, we consider an n dimensional sphere (that is a sphere of radius 1 in $\mathbb{R}^{n+1}$), and it's Laplace-Beltrami operator, which may be seen as the usual Euclidean Laplacian acting in $\mathbb{R}^{n+1}$ on functions which in a neighborood of the sphere are constant on the radius.

The eigenvalues of the Laplace-Beltrami operator Δ are $\lambda_k = -k(n+k-1)$. The eigenvectors are the restrictions to the sphere of homogeneous harmonic polynomials in $\mathbb{R}^{n+1}$. We refer to [10] for a complete analysis of the spectral properties of the sphere.

To define the sequence R_k of intrinsic bilinear operators acting on functions, we shall proceed in the following way:

DEFINITION 3.1. — $R_0(f,g) = fg$, $R_1(f,g) = [\Delta, R_0](f,g)$. *For* $k \geqslant 2$, *we shall proceed by induction and set*

$$\gamma_k R_{k+1}(f,g) = [\Delta, R_k](f,g) - \alpha_k R_k(f,g) - \beta_k R_{k-1}((\Delta+\lambda_{k-1}I)f, (\Delta+\lambda_{k-1}I)g),$$

where

$$\alpha_k = k(n+k-2); \lambda_k = k(n+k-1); \beta_k = \frac{k}{n+2k-2}; \gamma_k = \frac{n+k-2}{n+2k-2}.$$

(Recall that n *is the dimension of the sphere.)*

Then, the main result of this paper is the following:

THEOREM 3.2. — *Let* m *be the Riemann measure of the sphere and* R_k *be defined as in definition 3.1. Then,*

1. *For each pair of smooth function g, and each eigenvector f solution of $\Delta f = -\lambda f$, one has*

$$\int R_k(f,g)dm = P_{R_k}(\lambda)\int fgdm,$$

with $P_{R_k}(\lambda) = \lambda(\lambda-\lambda_1)\dots(\lambda-\lambda_{k-1})$.

2. *For each k, R_k is positive.*

Remark. — For the radial functions on the sphere (i.e. functions wich depends only on the distance to some given point), this result was already obtained in [1], where it may be seen as a result on ultraspherical operators on the unit interval on $\mathbb{R}$. In this case, through a change of variables, the operator may be seen as

$$L(f) = (1-x^2)f'' - nxf'$$

on the interval $[-1,1]$, and the operator $R_k(f,f) = a_k(1-x^2)^k(f^{(k)})^2$. The vector space generated by the k lowest eigenvalues of L is the space of polynomials of degree less than k (the eigenvectors are precisely the ultraspherical polynomials), and R_k is precisely 0 on the space generated by the first $k-1$ eigenvectors. A bit of algebraic computations in this case shows that this is the unique intrinsic bilinear operator satisfying this property, up to some constant, and which is in each argument a differential operator with degree less than k.

It is remarkable that the same construction, whith the same values of the coefficients, still produce positive bilinear maps on the spheres. Moreover, if we compare R_k to R_2, one has the feeling that the coefficients γ_k should appear as the ratio of the dimensions of some fiber bundles of symmetric tensors. We had been unable to give such an interpretation.

Proof. — (of Theorem 3.2). The first assertion is easy to prove by induction. If Q is an element of $\mathcal{I}(L)$ associated with the polynomial P_Q, then it is a direct consequence from the definitions that

$$P_{[\Delta,Q]}(\lambda) = \lambda P_Q(\lambda),$$

and that if $Q_1(f,f) = Q(\Delta f + \mu f, \Delta f + \mu f)$, then

$$P_{Q_1}(\lambda) = (\lambda-\mu)^2 P_Q.$$

Therefore, if P_k denotes the polynomial associated to R_k, we have

$$P_{k+1}(\lambda) = \frac{1}{\gamma_k}[\lambda P_k(\lambda) - \alpha_k P_k(\lambda) - \beta_k(\lambda-\lambda_{k-1})^2 P_{k-1}(\lambda).$$

Assume that, for $q \leqslant k$,

$$P_q(\lambda) = \prod_{i=0}^{q-1} (\lambda - \lambda_i),$$

then we get

$$P_{k+1}(\lambda) = \frac{1}{\gamma_k} P_k(\lambda)[\lambda - \alpha_k - \beta_k(\lambda - \lambda_{k-1})],$$

and the result is the consequence of the obvious identities

$$1 - \beta_k = \gamma_k \ ; \ \alpha_k - \beta_k \lambda_{k-1} = \gamma_k \lambda_k.$$

The formula is clearly true for R_0 and R_1, since $P_0 = 1$ and $P_1(\lambda) = \lambda$, and this proves the first part.

The second assertion is more delicate and shall require some steps.

First, we shall compute an explicit formula for R_k, and to do so we shall use a specific system of coordinates on the sphere. Since everything is invariant under rotations, it is enough to prove the positivity of the R_k's on the upper half sphere.

Let $X \in S_n$ be on the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$, and let x be its orthogonal projection on $\mathbb{R}^n$ (removing the last coordinate of X to fix the ideas). Then, x belongs to the unit ball of $\mathbb{R}^n$ and this unit ball shall be our local system of coordinates for the upper half-sphere (as well as for the lower one, in fact). In this system of coordinates, the Laplace-Beltrami operator has a simple form

$$\Delta(f)(x) = \sum_{ij} (\delta^{ij} - x^i x^j) \frac{\partial^2 f}{\partial x^i \partial x^j} - n \sum_i x^i \frac{\partial f}{\partial x^i}.$$

In all what follows, we shall denote g^{ij} the associated cometric in those coordinates, that is $g^{ij} = \delta^{ij} - x^i x^j$.

For a multindex $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$, and a smooth function f defined on the unit ball, let $D_I(f)$ denotes the partial derivative along those coordinates

$$D_I^k(f) = \frac{\partial^k f}{\partial x^{i_1} \dots \partial x^{i_k}}.$$

Then, $D^k f$ shall denotes the symmetric k-tensor whose coordinates are $D_I^k(f)$ in this system of coordinates.

If M is a symmetric k-tensor (like $D^k f$), with $k \geqslant 2$, TM shall denote it's contraction along two of it's indices, with respect to the metric g^{ij}. This operator gives a symmetric $k-2$ tensor. In our system of coordinates, for a multi $(k-2)$-index $J = (i_1, \dots, i_{k-2})$, and if Jij denotes the k-index obtained by concatenation of J and (ij), this writes

$$TM_J = M_{ijJ} g^{ij}.$$

(We use here the Einstein convention about the summation over repeated indices).

Since M is symmetric, it is irrelevant to know on which indices we have made the contraction, and here we have chosen the first coordinates.

By convention, we shall write $TM = 0$ if $k = 1$ or $k = 0$.

Moreover, if M is any tensor of order k, we shall denote by $|M|^2$ it's norm in the metric g, i.e.

$$|M|^2 = g^{(I,J)} M_I M_J,$$

the sum running over all pairs of multindices $(I = \{i_1, \dots, i_k\}, J = \{j_1, \dots, j_k\})$ and for such a pair,

$$g^{(I,J)} = g^{i_1 j_1} g^{i_1 j_2} \dots g^{i_k j_k}.$$

Similarly, we shall denote $M \cdot N$ for the bilinear form associated to this quadratic norm.

PROPOSITION 3.3. — *For any smooth function in the unit ball of* $\mathbb{R}^n$,

$$R_k(f,f) = \sum_{q=0}^{[k/2]} a_q^k |T^q D^k f|^2,$$

where $[k/2]$ *denotes the integer part of* $k/2$, *and the* a_q^k *are defined by induction from* $a_0^1 = 1$ *and*

$$\left\{ \begin{array}{lcl} a_q^k & = & \dfrac{k(n+2k-2q-4)}{(k-2q)(n+k-3)} a_q^{k-1}; \\ a_q^k & = & -\dfrac{(k+2-2q)(k+1-2q)}{2q(n+2k-2q-2)} a_{q-1}^k. \end{array} \right. \tag{3.2}$$

(A simple verification shows that those two formulae are compatible and that the operation which raises the first and the second index do commute.)

Proof. — The proof of this fact is not entirely easy and requires some computations. We shall proceed by induction on k. The formula is clearly true for $k = 1$, since $R_1(f,f) = |Df|^2$.

Then, let $Q_{k,q}(f,f)$ be the bilinear map defined by

$$Q_{k,q}(f,f) = |T^q D^k f|^2.$$

Our main task shall be to compute

$$\hat{Q}_{k,q} = [\Delta, Q_{k,q}] - \alpha_k Q_{k,q} - \beta_k Q_{k-1,q}(\Delta + \lambda_{k-1} Id, \Delta + \lambda_{k-1} Id).$$

We begin by an elementary lemma

LEMMA 3.4. — *Let X be the operator $\sum_i x^i \frac{\partial}{\partial x^i}$, and I be a multiindex of lenght k. Then $XD_I^k = D_I^k(X - kId)$.*

In what follows, and to simplify the notations, $X^{(k)}$ shall denote the operator $X - kId$.

Proof. — First, we observe that, for any index i, $XD_i = D_i X - D_i$. The general case follows immediately by induction. □

First, we compute $[\Delta, Q_{k,q}]$.

First, for a k-symmetric tensor M, and for $q \leqslant [k/2]$, we shall rewrite $|T^q M|^2$ as

$$|T^q M|^2 = \sum_{I,J} g_q^{I,J} M_I M_J, \tag{3.3}$$

where the sum runs over all pairs of multindices $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$, and

$$g_q^{I,J} = \prod_{r=0}^{q-1} g^{i_{2r+1} i_{2r+2}} g^{j_{2r+1} j_{2r+2}} \prod_{s=2q+1}^{k} g^{i_s j_s}.$$

(Here, the contraction T^q acts on the first $2q$ indices.)

Then, we introduce the Riemannian connection ∇ on symmetric tensors. The inverse metric g_{ij} may be written in this coordinates as

$$g_{ij} = \delta_{ij} + \frac{x^i x^j}{1 - |x|^2},$$

where $|x|$ is the Euclidean norm of the vector x. We shall use the usual notation to raise and lower indices using the metrics g_{ij} and g^{ij}, and use again the Einstein convention on the sum over repeated indices. Then, in this coordinate system, and for a multindex $I = \{i_1, \dots, i_k\}$, the tensor ∇M (symmetric or not) may be written as

$$\nabla_i M_I = \frac{\partial M_I}{\partial x^i} - \sum_{r=1}^{k} \sum_j g_{ii_r} x^j M_{I \cup \{j\} \setminus i_r}.$$

(If the tensor M is not symmetric, we have to be a little careful in the previous notation, and $I \cup \{j\} \setminus i_r$ denotes the multindex where the index j has replaced the index i_r at place r.)

Since ∇ is the Riemannian connection, $\nabla g = 0$ for the two tensors g_{ij} and g^{ij}, and we have

$$\Delta |T^q M|^2 = 2|\nabla T^q M|^2 + 2(T^q \nabla^i \nabla_i M) \cdot (T^q M).$$

We first compute the first term

LEMMA 3.5. —

$$\begin{aligned} |\nabla T^q D^k f|^2 = \; & |T^q D^{k+1} f|^2 \\ & -4q T^q D^{k+1} f \cdot T^{q-1} D^{k-1} X^{(k-1)} f \\ & -2(k-2q) T^{q+1} D^{k+1} f \cdot T^q D^{k-1} X^{(k-1)} f \\ & +4q^2 |T^{q-1} D^{k-1} X^{(k-1)} f|^2 \\ & +(k-2q)(k+2q+n-1) |T^q D^{k-1} X^{(k-1)} f|^2 \end{aligned}$$

Proof. — Let $I = \{i_1, \dots, i_k\}$ be a multiindex of lenght k and iI denotes the concatenation of i and I.

$$\begin{aligned} (\nabla D^k f)_{iI} &= D^{k+1}_{iI} f - \sum_{l=1}^{k} g_{ii_l} x^p D^k_{(I \setminus i_l)p} f \\ &= D_{iI} f - \sum_{l=1}^{k} g_{ii_l} X D^{k-1}_{I \setminus i_l} f \\ &= D_{iI} f - \sum_{l=1}^{k} g_{ii_l} D^{k-1}_{I \setminus i_l} X^{(k-1)} f. \end{aligned}$$

We have then

$$|\nabla T^q D^k f|^2 = g^{ij} g_q^{I,J} (\nabla D^k f)_{iI} \cdot (\nabla D^k f)_{jJ}, \tag{3.4}$$

the sum running over all pairs of multiindices (I, J).

Using the formula above for $\nabla D^k f$, this sum decomposes as $A-2B+D$, where

$$A = g^{ij} g_q^{IJ} D_{iI}^{k+1} f D_{jJ}^{k+1} f = |T^q D^{k+1} f|^2;$$

$$B = \sum_l g^{ij} g_q^{IJ} g_{ii_l} D_{I\backslash i_l}^{k-1} X^{(k-1)} f D_{jJ}^{k+1)} f;$$

$$C = \sum_{l,l'} g^{ij} g_q^{IJ} g_{ii_l} g_{jj_{l'}} D_{I\backslash i_l}^{k-1} X^{(k-1)} f D_{J\backslash j_{l'}}^{k-1} X^{(k-1)} f.$$

For simplicity, for a $(k-1)$-multiindex I let us write M_I instead of $D_I^{k-1} X^{(k-1)} f$, and let us denote by M the corresponding tensor. To compute the B term, we notice that

$$g^{ij} g_{ii_l} = \delta_{i_l}^j,$$

and therefore

$$B = \sum_l g_q^{IJ} M_{I\backslash i_l} D_{i_l J}^{k+1} f.$$

We have then to decompose this sum again according to the fact that $l \leqslant 2q$ or $l > 2q$. Each term with $l \leqslant 2q$ gives rise to $T^q D^{k+1} f \cdot T^{q-1} M$, and each term with $l > 2q$ gives rise to $T^{q+1} D^{k+1} f \cdot T^q M$.

The C term is a bit more complicated. First, we observe that

$$g^{ij} g_{ii_l} g_{jj_{l'}} = g_{i_l j_{l'}}.$$

Then

$$C = \sum_{l,l'} g_q^{IJ} g_{i_l j_{l'}} M_{I\backslash i_l} M_{J\backslash j_{l'}}.$$

Each term of the sum with l and l' less than or equal to $2q$ gives $|T^{q-1} D^{k-1} X^{(k-1)}|^2$.

Then, each term with $l \leqslant 2q$ and $l' > 2q$ or $l > 2q$ and $l' \leqslant 2q$ also gives $|T^{q-1} M|^2$.

The same is true for the terms with $l > 2q$, $l' > 2q$ with $l \neq l'$, but the terms with $l = l' > 2q$ give $n|T^{q-1} M|^2$, since then we have

$$\begin{aligned} g_q^{I,J} g_{i_l i_l} M_{I\backslash i_l} M_{J\backslash j_l} &= g_{q-1}^{I\backslash i_l, J\backslash j_l} g^{i_l j_l} g_{i_l j_l} M_{I\backslash i_l} M_{J\backslash j_l} \\ &= n g_{q-1}^{I\backslash i_l, J\backslash j_l} M_{I\backslash i_l} M_{J\backslash j_l}. \end{aligned}$$

We get the final result summing up all those quantities. $\square$

The next step is to compare $T^q \nabla^i \nabla_i D^k f$ to $T^q D^k \Delta f$. This is done in the following lemma.

LEMMA 3.6. —

$$\nabla_i \nabla^i D_I^k - D_I^k \Delta = \sum_{l \neq l'} g_{i_l i_{l'}} D^{k-2}_{I \setminus i_l \setminus i_{l'}} X^{(k-2)} X^{(k-1)} + k(n+k-2) D_I^k.$$

Proof. — To do this computation, we commute separately both terms.

First, from the definition of ∇_i, and for a multiindex I, we have thanks to lemma 3.4

$$\nabla_i D_I = D_{iI} - \sum_l g_{ii_l} D_{I \setminus i_l} X^{(k-1)}.$$

Then, we have

$$D_i g_{jl} = x^p (g_{ik} g_{pl} + g_{il} g_{pk}),$$

which may be seen directly or from the fact that $\nabla_i g_{jl} = 0$.

With those two identities, we get

$$\begin{aligned} \nabla_j \nabla_i D_I &= D_{ijI} - \sum_l g_{ii_l} D_{I \setminus i_l} X^{(k-1)} \\ &\quad - g_{ij} D_I X^{(k)} - \sum_l g_{ii_l} D_{iI \setminus i_l} X^{(k)} \\ &\quad + \sum_{l \neq l'} g_{ii_l} g_{ji_{l'}} D_{I \setminus i_l \setminus i_{l'}} X^{(k-2)} X^{(k-1)}. \end{aligned}$$

From this, we may compute $\nabla_i \nabla^i D_I = g^{ij} \nabla_i \nabla_j D_I$, and we get

$$\begin{aligned} \nabla_i \nabla^i D_I &= g^{ij} D_{ijI} - D_I [(2k+n) X - k(n+2k-1) Id] \\ &\quad + \sum_{l \neq l'} g_{ii_l} g_{ji_{l'}} D_{I \setminus i_l \setminus i_{l'} X^{(k-2)} X^{(k-1)}}. \end{aligned}$$

In order to compute $D_I \Delta$, it is easier notice that

$$x^i x^j \frac{\partial^2}{\partial x^i \partial x^j} = X^2 - X,$$

which allows us to decompose Δ into

$$\Delta = \Delta_0 - X^2 - (n-1) X,$$

where Δ_0 is the usual (Euclidean) Laplacian, and gives

$$g^{ij} D_{ij} = \Delta_0 - X^2 + X.$$

Since Δ_0 commutes with D_I, one gets easily, using again lemma 3.4, we have

$$\begin{aligned} D_I\Delta &= (\Delta_0 - (X+k)^2)D_I - (n-1)D_IX \\ &= g^{ij}D_{ij}D_I - D_I[(2k+n)X - k(k+1)Id]. \end{aligned}$$

This gives the result. □

From this, writing $[\Delta, D^k]$ for $\nabla_i\nabla^i D^k - D^k\Delta$, we get the following

COROLLARY 3.7. —

$$\begin{aligned} T^q[\Delta, D^k]\cdot T^qD^k &= k(n+k-2)|T^qD^k|^2 \\ &\quad +2q(2k-2q+n-2)T^{q-1}D^{k-2}X^{(k-2)}X^{(k-1)}\cdot T^qD^k \\ &\quad +(k-2q)(k-1-2q)T^qD^{k-2}X^{(k-2)}X^{(k-1)}\cdot T^{q+1}D^k. \end{aligned}$$

Proof. — To see that, we use lemma 3.6, and we decompose as before according to the position of l and l' with respect to $2q$. If $l \leqslant 2q$, let $\hat{l}$ be the index coupled with l in the contraction T^q, that is $\hat{l} = l+1$ if l is odd and $\hat{l} = l-1$ if l is even.

Each of the terms with $l \leqslant 2q$, $l' \leqslant 2q$, $l' \neq \hat{l}$ gives rise to $T^{q-1}D^{k-2}X^{(k-2)}X^{(k-1)}\cdot T^qD^k$.

If $l' = \hat{l}$, this gives $nT^{q-1}D^{k-2}X^{(k-2)}X^{(k-1)}\cdot T^qD^k$.

Each of the terms with $l \leqslant 2q$, $l' > 2q$ or $l > 2q$, $l' \leqslant 2q$ also gives $T^{q-1}D^{k-2}X^{(k-2)}X^{(k-1)}\cdot T^qD^k$.

The terms with $l > 2q$ and $l' > 2q$ give $T^qD^{k-2}X^{(k-2)}X^{(k-1)}\cdot T^{q+1}D^k$.
□

It remains to compute the last term, which comes from $R_{k-1}(\Delta + \lambda_{k-1}Id, \Delta + \lambda_{k-1}Id)$. In the core of the proof of lemma 3.6, we have computed $D^k\Delta$.

A simple computation then gives

LEMMA 3.8. —

$$|T^qD^{k-1}(\Delta + \lambda_{k-1}Id)|^2 = |T^{q+1}D^{k+1} - (n+2k-2)T^qD^{k-1}X^{(k-1)}|^2.$$

To sum up those results, let a_q^k be a sequence of coefficients, with $a_q^k = 0$ if $q > 2k$ or $q < 0$, and let $R_k = \sum_q a_q^k Q_{k,q}$.

Then, let

$$\hat{R}_k(f,f) = [\Delta, R_k](f,f) - \alpha_k R_k(f,f) - \beta_k R_{k-1}((\Delta+\lambda_{k-1})f, (\Delta+\lambda_{k-1})f).$$

To simplify the notations, we set

$$\hat{D}^k = D^{k-1}X^{(k-1)} \text{ and } \bar{D}^k = D^{k-2}X^{(k-2)}X^{(k-1)}.$$

We obtain

$$\begin{aligned}
\hat{R}_k \;&=\; \sum_q (a_q^k - \beta_k a_{q-1}^{k-1})|T^q D^{k+1} f|^2 \\
&\quad - \sum_q 2[2(q+1)a_{q+1}^k + (k-2q)a_q^k - 2k a_q^{k-1}] T^q D^{k+1} f \cdot T^{q-1}\hat{D}^k f \\
&\quad + [4(q+1)^2 a_{q+1}^k + (k-2q)(k+2q+n-1)a_q^k \\
&\quad - k(n+2k-2)a_q^{k-1}]|T^q \hat{D}^k f|^2 + [2q(2k-2q+n-2)a_q^k \\
&\quad + (k+2-2q)(k+1-2q)a_{q-1}^k] T^{q-1}\bar{D}^k f \cdot T^q D^k f.
\end{aligned}$$

Now, if the coefficients a_q^k satisfy the two recurrence formulae given in 3.2, then it is a simple computation to see that

$$\hat{R}_k(f,f) = \gamma_k \sum_q a_q^{k+1} |T^q D^{k+1} f|^2 = \gamma_k R_{k+1}(f,f).$$

This completes the proof of proposition 3.3. □

It remains to show that all the R_k are positive.

To see that, at some given point x, let $\mathcal{S}_k$ be the space of symmetric tensors of order k, on which we consider the Euclidean metric that we already considered, which comes from the Riemannian metric at point x, that is, if a symmetric tensor S has coordinates S_I, where I varies along all multiindices of lenght k,

$$|S|^2 = \sum_{I,J} g^{I,J} S_I S_J,$$

where $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$, and $g^{I,J} = \prod_{s=1}^k g^{i_s j_s}$. Similarly, we denote by $S \cdot S'$ the scalar product of two symmetric tensors in this space.

If S' is a symmetric $k-2$ tensor, we may consider the symmetric k-tensor $JS' = g_{ij} \odot S'$, where $\odot$ denotes the symmetric tensor product. More precisely, for any multiindex $I = \{i_1, \ldots, i_k\}$

$$(JS')_I = \frac{1}{k!} \sum_\sigma g_{i_{\sigma(1)} i_{\sigma(2)}} S'_{i_{\sigma(3)} \cdots i_{\sigma(k)}},$$

the sum running over all permutations σ of k elements.

Then, we may consider the subspace $J\mathcal{S}_{k-2}$ of $\mathcal{S}_k$, the image of $\mathcal{S}_{k-2}$ under J. For any tensor $S \in \mathcal{S}_k$, let $\pi(S)$ be it's orthogonal projection over $J\mathcal{S}_{k-2}$. We have

PROPOSITION 3.9. — *For any smooth function f and at every point x, in our system of coordinates*

$$R_k(f,f) = a_0^k |D^k f - \pi D^k f|^2.$$

Since by our recurrence formulae 3.2 it is clear that a_0^k is positive, the proof of the theorem shall follow immediately from this proposition.

Proof. — To understand this identity, we shall compute $\pi(S)$ for any symmetric tensor S, and show that it is a linear combination of the $J^q T^q S$, where q ranges from 1 to $[k/2]$.

To see that, let us first recall the operator T, which maps $\mathcal{S}_k$ into $\mathcal{S}_{k-2}$:

$$(TS)_I = g^{ij} S_{ijI}.$$

We begin by a lemma.

LEMMA 3.10. —

1. *If $S \in \mathcal{S}_k$ and $S' \in \mathcal{S}_{k-2}$, we have*

$$S \cdot JS' = TS \cdot S'.$$

2. *On $\mathcal{S}_k$, one has*

$$TJ = \frac{2(n+2k)}{(k+1)(k+2)} Id + \frac{k(k-1)}{(k+1)(k+2)} JT. \tag{3.5}$$

Proof. — The first assertion comes directly from the definitions.

For the second one, we have, for any symmetric tensor S of order k and any mulitindex I

$$(TJS)_I = \frac{1}{(k+2)!} \sum_{\sigma \in \pm_{k+2}} g^{i_1 i_2} g_{i_{\sigma(1)} i_{\sigma(2)}} S_{i_{\sigma(3)} \cdots i_{\sigma(k+2)}}.$$

We have to decompose this sum in different terms.

1. If $\{\sigma(1), \sigma(2)\} = \{1,2\}$, then we obtain nS_I. There are $2k!$ such terms.

2. If $\sigma(1) \in \{1,2\}$ and $\sigma(2) \notin \{1,2\}$, or if $\sigma(1) \notin \{1,2\}$ and if $\sigma(2) \in \{1,2\}$, we get S_I. There are $4kk!$ such terms.

3. After symmetrization, all the other terms give $(JTS)_I$. There are $k(k-1)k!$ such terms.

$\square$

From this we get immediatly by induction:

COROLLARY 3.11. — *On $\mathcal{S}_k$, and for $q = 1, \ldots, [k/2]$,*

$$T^q J = A_q^k T^{q-1} + B_q^k J T^q,$$

with

$$A_q^k = \frac{2q(n+2k+2-2q)}{(k+1)(k+2)} \; ; \; B_q^k = \frac{(k+1-2q)(k+2-2q)}{(k+1)(k+2)}.$$

We shall set in the following $A_q^k = B_q^k = 0$ for $q > [k/2]$. From the corollary, we have

COROLLARY 3.12. — *If $S \in \mathcal{S}_k$,*

$$\pi(S) = \sum_{q=1}^{[k/2]} \alpha_q^k J^q T^q S,$$

where

$$\alpha_1^k = \frac{1}{A_1^{k-2}} \; ; \; \alpha_q^k = -\alpha_{q-1}^k \frac{B_{q-1}^{k-2}}{A_q^{k-2}} \text{ for } q \geqslant 2.$$

Proof. — We check that, for any tensor $S' \in \mathcal{S}_{k-2}$, one has

$$\sum_q \alpha_q^k J^q T^q S \cdot JS' = S \cdot JS'.$$

But we have

$$\begin{aligned} J^q T^q S. \cdot JS' &= T^q S \cdot T^q JS' \\ &= T^q S \cdot [A_q^{k-2} T^{q-1} S' + B_q^{k-2} J T^q S'] \\ &= A_q^{k-2} T^q S \cdot T^{q-1} S' + B_q^{k-2} T^{q+1} S \cdot T^q S'. \qquad \square \end{aligned}$$

The previous corrollary gives immediately that

$$|S-\pi(S)|^2 = |S|^2 - \sum_{q=1}^{[k/2]} \alpha_q^k |T^q S|^2. \tag{3.6}$$

□

Now, if we set $\alpha_0^k = 1$, we observe that the coefficients α_q^k follow the same recurrence relation than the coefficients a_q^k (second line of the formula 3.2), including for $q = 1$, and this completes the proof of proposition 3.9. □

4. An Extension

In this section, we shall produce another example of an operator L acting on the algebra of smooth functions on the unit ball for which there exists a sequence of positive bilinear applications R_k whose associated polynomials are $\lambda(\lambda-\lambda_1)\ldots(\lambda-\lambda_{k-1})$, where the λ_i's are the eigenvalues of $-L$. This operator plays in dimension n the same rôle that the ultraspherical generator in dimension 1.

Let B be the unit ball in $\mathbb{R}^n$, and let q be a parameter larger that n. We define L_q as

$$L_q(f)(x) = (\delta^{ij} - x^i x^j)\frac{\partial^2 f}{\partial x^i \partial x^j}(x) - q x^i \frac{\partial f}{\partial x^i}(x).$$

This operator is reversible with respect to the measure defined on the unit ball by

$$d\mu_q(x) = c_{n,q}(1-|x|^2)^{(q-n-1)/2} dx_1 \ldots dx_n,$$

which is finite as soon as $q > n-1$.

The ball is a manifold with boundary, which is at a finite distance if we equip it with the metric inherited from the spherical metric. Therefore, if we have to consider L as a self adjoint operator, we shall impose Neumann boundary conditions. (In fact, a more precise analysis shows that this operator is essentially self adjoint as soon as $q > n+1$ but this is irrelevant for our purpose.)

In what follows, we shall restrict our attention to the case where $q \geqslant n$, although it would be interesting to find similar results for $n-1 < q < n$.

If we want to describe the eigenvectors of this operator, it is simpler to notice that the operator L_q maps polynomials into polynomials, and moreover polynomials of degree less than k into polynomials of degree less than k. Therefore, we may choose the space of polynomials to be the algebra $\mathcal{A}$.

PROPOSITION 4.1. — *The eigenvalues of L_q are exactly $\lambda_k^q = -k(q+k-1)$. The eigenspace associated to λ_q^k is the space $\mathcal{H}_k$ of polynomials of degree less than k which are orthogonal in $L^2(\mu_q)$ to polynomials of degree less than $k-1$.*

Proof. — To compute the eigenvalues of L_q, we first observe that, if I is a multiindex of lenght k, then

$$D_I L_q = (L_{q+2k} - \lambda_k^q Id) D_I.$$

This comes immediately by induction from the case $k = 1$.

Since L_q is symmetric with respect to the measure μ_q, it maps the space H_k into itself. Let λ be an eigenvalue of the restriction of L_q to H_k, and P an eigenvector. There exists a multindex I of lenght k such that $D_I P$ is a constant c different from 0. Then, we write

$$\lambda c = D_I L_q P = L_{q+2k} D_I P - \lambda_k^q D_I P = -\lambda_q^k c.$$

□

For this operator, we may also find a sequence R_k^q of intrinsic positive quadratic maps associated to the polynomials $\lambda(\lambda - \lambda_1^q)\ldots(\lambda - \lambda_{k-1}^q)$. In fact, we shall recopy exactly theorem 3.2, and just change everywhere n to q.

PROPOSITION 4.2. — *Let $R_0^q(f,f) = f^2$, $R_1^q = [L_q, R_0]$, and, for any $k \geqslant 2$, let*

$$\gamma_k^q R_{k+1}^q(f,g) = [\Delta, R_k^q](f,g) - \alpha_k^q R_k^q(f,g) - \beta_k^q R_{k-1}^q((\Delta + \lambda_{k-1}^q I)f, (\Delta + \lambda_{k-1}^q I)g),$$

where

$$\alpha_k^q = k(q+k-2); \lambda_k^q = k(q+k-1); \beta_k^q = \frac{k}{q+2k-2}; \gamma_k^q = \frac{q+k-2}{q+2k-2}.$$

Then,

1. $P_{R_k^q}(\lambda) = \lambda(\lambda - \lambda_1^q)\cdots(\lambda - \lambda_{k-1}^q)$.
2. *For each k, R_k^q is positive.*

Proof. — The proof of the first assertion in theorem 3.2 was a purely algebraic computation and nothing is changed if we replace n by q everywhere.

For the second assertion, let us first begin by the case where q is an integer larger than n. Then, let f be a function on the unit ball of $\mathbb{R}^q$, which depends only on the first n coordinates. If we compute it's spherical Laplacian, in our system of coordinates, we get exactly $L_q f$. Therefore, L_q is the spherical Laplacian of dimension q acting on the functions depending only on the first n coordinates. We may then apply our main theorem 3.2 without any further computation.

To ge beyond the case where q is an integer, we first prove the extension of proposition 3.3:

PROPOSITION 4.3. — *For any smooth function in the unit ball of* $\mathbb{R}^n$,

$$R_k^q(f,f) = \sum_{p=0}^{[k/2]} a_p^{k,q} |T^p D^k f|^2,$$

where $[k/2]$ *denotes the integer part of* $k/2$, *and the* $a_p^{k,q}$ *are defined by induction from* $a_0^1 = 1$ *and*

$$\left\{\begin{array}{rcl} a_p^{k,q} & = & \dfrac{k(q+2k-2p-4)}{(k-2p)(q+k-3)} a_p^{k-1,q}; \\ a_p^{k,q} & = & -\dfrac{(k+2-2p)(k+1-2p)}{2p(q+2k-2p-2)} a_{p-1}^{k,q}. \end{array}\right. \tag{4.7}$$

Proof. — We first observe that, from the definition of R_k^q, it is clear that R_k^q is a rational expression with respect to q. It is also the case for the formula proposed for R_k^q. From what we have just seen, those two formulae coincide when q is an integer larger than n, since then we just apply the proposition 3.3 to functions on the unit ball of R^q depending on the first n coordinates.

Therefore, those expression coincide for all q. □

It remains to prove that the R_k^q are positive. For this, we may not just extend the expression of R_k as an orthogonal projection identity, since this has no meaning for a non integer q.

This shall be done in the next lemma

LEMMA 4.4. — *Let* E *be an Euclidean space of dimension* n *and define the norm* $|T^p M|^2$ *as in formula 3.3 for symmetric* k*-tensors* M *over* E.

Then, for $p \leqslant [k/2]$, define the sequence $a_p^{k,q}$ by $a_0^{k,q} = 1$ and

$$a_p^{k,q} = -\frac{(k+2-2p)(k+1-2p)}{2p(q+2k-2p-2)} a_{p-1}^{k,q}$$

for $p \geqslant 1$. Then, for any $q \geqslant n$, and any symmetric k-tensor M,

$$\sum_{q=0}^{[k/2]} a_p^{k,q} |T^p M|^2 \geqslant 0.$$

Proof. — Up to a change of coordinates, we may as well suppose that $g^{ij} = \delta^{ij}$. Then, we imbed E into $\hat{E} = E \times \mathbb{R}$, on which we put the standard Euclidean metric $\hat{g}^{ij} = \delta^{ij}$. The tensor M shall be extended to a symmetric k-tensor $\hat{M}$ which is such that $\hat{M}_I = M_I$ if all the components of the multiindex I are less than or equal to n, and 0 otherwise.

We chose $\hat{g}_{ij}$ to be δ_{ij} if $i, j \leqslant n$, and $\hat{g}_{n+1,n+1} = q - n$, all other coefficient being 0.

For the new metric $\hat{g}$, and for a symmetric k tensor M on E, we have $T^q \hat{M} = \hat{T^q M}$.

Then, for a symmetric k-tensor M on E, we set $\hat{J}M = \hat{g}_{ij} \odot \hat{M}$. Let $\hat{\pi}M$ be the orthogonal projection of $\hat{M}$ over the subspace of the tensors of the form $\hat{J}M$, where M ranges over all symmetric $k-2$ tensors over E.

$$\sum_{q=0}^{[k/2]} a_p^{k,q} |T^p M|^2 = |\hat{M} - \hat{\pi} M|^2.$$

In fact, if we look at the proof of formula 3.9, which gave the case $q = n$, we see that the only place where the parameter n appears is formula 3.5. There, n comes from $g_{ij} g^{ij} = n$. Here, we have replaced g_{ij} by a matrix which is not the inverse of $\hat{g}^{ij}$, but which satisfy $\hat{g}_{ij} \hat{g}^{ij} = q$.

From this, following the same argument, it is easy to see that we have, for symmetric k-tensors M,

$$T \hat{J} M = \frac{2(q+2k)}{(k+1)(k+2)} M + \frac{k(k-1)}{(k+1)(k+2)} \hat{J} T,$$

and this formula leads to the explicit computation of $\pi(M)$, and hence of $|M - \pi(M)|^2$. □

This completes the proof of the positivity of the bilinear maps R_k^q. □

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