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Hypercontractivity for perturbed diffusion semigroups

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ABSTRACT. — μ being a nonnegative measure satisfying some Log-Sobolev inequality, we give conditions on F for the Boltzmann measure $\nu = e^{-2F}\mu$ to also satisfy some Log-Sobolev inequality. This paper improves and completes the final section in [6]. A general sufficient condition and a general necessary condition are given and examples are explicitly studied.

RÉSUMÉ. — μ étant une mesure positive satisfaisant une inégalité de Sobolev logarithmique, nous donnons des conditions sur F pour que la mesure de Boltzmann $\nu = e^{-2F}\mu$ satisfasse également une telle inégalité (améliorant et complétant ainsi la dernière partie de [6]. Les conditions obtenues sont illustrés par des exemples.

1. Introduction and framework

In [6] we have introduced a pathwise point of view in the study of classical inequalities. The last two sections of this paper were devoted to the transmission of Log-Sobolev and Spectral Gap inequalities to perturbed measures, without any explicit example. In the present paper we shall improve the results of section 8 in [6] and study explicit examples. Except for one point, the present paper is nevertheless self-contained. In order to describe the contents of the paper we have first to describe the framework.

Framework. — For a nonnegative measure μ on some measurable space E, let us first consider a μ symmetric diffusion process $(\mathbb{P}_x)_{x \in E}$ and its associated semi-group $(P_t)_{t \geqslant 0}$ with generator A. Here by a diffusion process

we mean a strong Markov family of probability measures $(\mathbb{P}_x)_{x \in E}$ defined on the space of continuous paths $\mathcal{C}^0(\mathbb{R}^+, E)$ for some, say Polish, state space E, such that there exists some algebra $\mathbb{D}$ of uniformly continuous and bounded functions (containing constant functions) which is a core for the extended domain $D_e(A)$ of the generator (see [7]).

One can then show that there exists a countable orthogonal family (C^n) of local martingales and a countable family (∇^n) of operators s.t. for all $f \in D_e(A)$

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_s) \, ds = \sum_n \int_0^t \nabla^n f(X_s) \, dC_s^n, \quad (1.1)$$

in $\mathbb{M}^2_{loc}(\mathbb{P}_\eta)$ (local martingales) for all probability measures η on E.

One can thus define the "carré du champ" Γ by $\Gamma(f, g) = \sum_n \nabla^n f \, \nabla^n g \overset{\text{def}}{=} (\nabla f)^2$, so that the martingale bracket is given by $< M^f >_t = \int_0^t \Gamma(f, f)(X_s) \, ds$. In terms of Dirichlet forms, all this, in the symmetric case, is roughly equivalent to the fact that the local pre-Dirichlet form $\mathcal{E}(f, g) = \int \Gamma(f, g) \, d\mu \quad f, g \in \mathbb{D}$ is closable, and has a regular (or quasi-regular) closure $(\mathcal{E}, D(\mathcal{E}))$, to which the semigroup P_t is associated. Notice that with our definitions, for $f \in \mathbb{D}$

$$\mathcal{E}(f, f) = \int \Gamma(f, f) \, d\mu = -2 \int f \, Af \, d\mu = -\frac{d}{dt} \| P_t f \|^2_{\mathbb{L}^2(\mu)} \big|_{t=0}. \quad (1.2)$$

It is then easy to check that $\Gamma(f, g) = A(fg) - f \, Ag - g \, Af$, that $\mathbb{D}$ is stable for the composition with compactly supported smooth functions and satisfies the usual chain rule.

Content. — The aim of this paper is to give conditions on F for the perturbed measure $\nu_F = e^{-2F}\mu$ to satisfy some Logarithmic Sobolev inequality, assuming that μ does. As in the final section of [6] these conditions are first described in terms of some martingale properties in the spirit of the work by Kavian, Kerkyacharian and Roynette (see [13]) (see section 2 Theorem 2.5).

We shall then study in section 3 how this general criterion can be checked in the same general situation . Here again we are inspired by [13] (Well Method). It turns out that the Well Method can be generalized to other F-Sobolev inequalities (see [5]).

Since sections 2 and 3 are concerned with the hyperbounded point of view, and following the suggestion of an anonymous referee, we study in

section 4 the log-Sobolev point of view (i.e. the perturbation point of view is analyzed on log-Sobolev inequalities). We show that both point of view yield (almost) the same results.

In the final section we study some examples, namely Boltzmann measures on $\mathbb{R}^N$. Explicit examples and counter examples are given, and some comparison with existing results is done.

Some notation and general results

The material below can be found in many very good textbooks or courses see e.g. [3], [4], [9], [11], [12], [16].

We shall say that μ satisfies a Log-Sobolev inequality LSI if for some universal constants a and b and all $f \in \mathbb{D} \cap L^1(\mu)$,

$$ \int f^2 \log \left(\frac{f^2}{\| f \|^2_{\mathbb{L}^2(\mu)}} \right) d\mu \leqslant a \int \Gamma(f,f) \, d\mu + b \, \| f \|^2_{\mathbb{L}^2(\mu)} \, . \qquad (1.3) $$

When $b = 0$ we will say that the inequality is tight (TLSI), when $b > 0$ we will say that the inequality is defective (DLSI). So we never will use (LSI) without specifying (TLSI) or (DLSI).

Note that when μ is bounded (1.3) easily extends to any $f \in D(\mathcal{E})$. It is not the case when μ is not bounded, in which case it only extends to $f \in D(\mathcal{E}) \cap \mathbb{L}^1(\mu)$ or to $f \in D(\mathcal{E})$ but replacing log by $\log^+$ in the left hand side of (1.3). An example of such phenomenon is $f = (1+|x|)^{-\frac{1}{2}} \log^\alpha(e+|x|)$ for $1 < 2\alpha < 2$, $E = \mathbb{R}$ and $d\mu = dx$.

These inequalities are known to be related to continuity or contractivity of the semigroup P_t. We shall say that the semigroup is hyperbounded (resp. hypercontractive) if for some $t > 0$ and $p > 2$, P_t maps continuously $\mathbb{L}^2(\mu)$ into $\mathbb{L}^p(\mu)$ (resp. is a contraction). In this case we shall denote the corresponding norm $\| P_t \|_{\mathbb{L}^2(\mu) \to \mathbb{L}^p(\mu)}$, or simply $\| P_t \|_{2,p}$ when no confusion is possible. It is well known that hyperboundedness (resp. hypercontractivity) is equivalent to (DLSI) (resp. (TLSI)) (see e.g. [4] Theorem 3.6 or [6] Corollary 2.8). Gross theorem tells next that boundedness or contraction hold

for all $p > 2$ for some large enough t. Replacing p by $+\infty$ in the definition we get the notion of ultracontractivity extensively studied in the book by E.B. Davies [8]. Links with Log-Sobolev inequalities are especially studied in chapter 2 of [8].

Finally recall that (TLSI) is equivalent to (DLSI) plus some spectral gap condition (as soon as we will use spectral gap properties we shall assume that μ is a probability measure). The usual spectral gap (or Poincaré) inequality will be denoted by (SGP). A weaker one introduced by Röckner and Wang (see [18]) called the weak spectral gap property (WSGP) is discussed in [1] and in section 5 of [6]. In particular (DLSI)+(WSGP) implies (TLSI) originally due to Mathieu ([17]) is shown in [6] Proposition 5.13.

2. Hypercontractivity for general Boltzmann measures

We introduce in this section a general perturbation theory. In the framework of section 1 let F be some real valued function defined on E.

DEFINITION 2.1. — *The Boltzmann measure associated with F is defined as $\nu_F = e^{-2F}\,\mu$.*

When no confusion is possible we may not write the subscript F and simply write ν.

The transmission of Log-Sobolev or Spectral Gap inequalities to Boltzmann measures has been extensively studied in various contexts. The first classical result goes back to Holley and Stroock.

PROPOSITION 2.2. — *Assume that μ is a probability measure and F is bounded. Then if μ satisfies (DLSI) with constants (a, b), ν_F satisfies (DLSI) with constants $\left(a\,e^{Osc(F)},\, b\,e^{Osc(F)}\right)$ where $Osc(F) = \sup(F) - \inf(F)$.*

This result is often stated with $2\,Osc(F)$ i.e. with an useless factor 2 (see [20] Proposition 3.1.18).

When F is no more bounded, general (though too restrictive) results have been shown by Aida and Shigekawa [2] (also see [6] section 7). Other results can be obtained through the celebrated Bakry-Emery criterion. As

in section 8 of [6] we shall follow a beautiful idea of Kavian, Kerkyacharian and Roynette (see [13]) in order to get better results (with a little bit more regularity). The main idea in [13] is that ultracontractivity for a Boltzmann measure built on $\mathbb{R}^N$ with μ the Lebesgue measure and F regular enough, reduces to check the boundedness of <u>one</u> and <u>only one</u> function.

The aim of this section is to improve these results. First let us state the hypotheses we need for F.

ASSUMPTIONS 2.3 H(F). —

(1) ν_F is a probability measure, $F \in D(\mathcal{E})$,

(2)   for all $f \in \mathbb{D}$, $\mathcal{E}_F(f, f) = \int \Gamma(f, f)\, d\nu_F < +\infty$,

(3)   for all $f \in \mathbb{D}$, $Af \in \mathbb{L}^1(\nu_F)$,

(4) $\int \Gamma(F, F)\, d\nu_F < +\infty$.

The Girsanov martingale Z_t^F is then defined as

$$Z_t^F \;=\; \exp\{-\int_0^t \nabla F(X_s).dC_s - \frac{1}{2}\int_0^t \Gamma(F, F)(X_s)\, ds\}. \quad (2.1)$$

When H(F) holds, we know that $Z_\cdot^F$ is a $\mathbb{P}_x$ martingale for ν_F, hence μ almost all x. Furthermore ν_F is then a symmetric measure for the perturbed process $\{Z_\cdot^F \, \mathbb{P}_x\}_{x \in E}$, which is associated with $\mathcal{E}_F$ (see (??.2)). For all this see [6] (especially Lemma 7.1 and section 2).

If in addition $F \in D(A)$, it is enough to apply Ito's formula in order to get another expression for Z_t^F, namely

$$Z_t^F \;=\; \exp\{F(X_0) - F(X_t) + \int_0^t \big(AF(X_s) - \frac{1}{2}\Gamma(F, F)(X_s)\big)ds\}. \quad (2.2)$$

If P_t^F denotes the associated (ν_F symmetric) semi-group, it holds ν_F a.s.

$$(P_t^F h)(x) = e^F(x)\, \mathbb{E}^{\mathbb{P}_x}\big[h(X_t)\, e^{-F(X_t)}\, M_t\big], \quad\quad (2.3)$$

with

$$M_t = \exp\Big(\int_0^t \big(AF(X_s) - \frac{1}{2}\Gamma(F, F)(X_s)\big)ds\Big).$$

When μ is a probability measure, $e^F \in \mathbb{L}^2(\nu_F)$, and a necessary condition for ν_F to satisfy (DLSI) is thus

$$P_t^F(e^F) = e^F \, \mathbb{E}^{\mathbb{P}_x}[M_t] \in \mathbb{L}^p(\nu_F) \qquad (2.4)$$

for all (some) $p > 2$ and t large enough. When μ is no more bounded one can formulate similar statements. For instance, if $e^F \in \mathbb{L}^r(\nu_F)$ for some $r > 1$, then (2.4) has to hold for some (all) $p > r$ and t large enough. One can also take $r = 1$ in some cases. Since the exact formulation depends on the situation we shall not discuss it here.

A remarkable fact is that the (almost always) necessary condition (2.4) is also a sufficient one. The next two theorems explain why. Though the proof of the first one is partly contained in [6] (Proposition 8.8) we shall give here the full proof for completeness.

THEOREM 2.4. — *Assume that P_t is ultracontractive with $\| P_t \|_{p,\infty} = K(t,p)$ for all $p \geqslant 1$. Assume that $H(F)$ is in force, $F \in D(A)$ and M_t is bounded by some constant $C(t)$. Then a sufficient condition for ν_F to satisfy (DLSI) is that*

$$P_t^F(e^F) \; = \; e^F \, \mathbb{E}^{\mathbb{P}_x}[M_t] \in \mathbb{L}^q(\nu_F)$$

for some $t > 0$ and some $q > 2$.

Proof. — Pick some $f \in \mathbb{D}$. Since $|f| e^{-F} \in \mathbb{L}^2(\mu)$ and using the Markov property, for $t > 0$, $q > 2$ it holds

$$\int (P_{t+s}^F(|f|))^q \, d\nu_F = \int e^{qF} \left(\mathbb{E}^{\mathbb{P}_x}[M_t \, \mathbb{E}^{\mathbb{P}_{X_t}}[M_s \left(e^{-F} \, |f| \right)(X_s')]] \right)^q d\nu_F \, ,$$

$$\leqslant \int e^{qF} \, (C(s))^q \left(E^{\mathbb{P}_x}[M_t \left(P_s(|f| \, e^{-F}) \right)(X_t)] \right)^q d\nu_F$$

$$\leqslant (C(s))^q \, (\| P_s \|_{2,\infty})^q \, \| f \|_{\mathbb{L}^2(\nu_F)}^q \int \left(e^F \, \mathbb{E}^{\mathbb{P}_x}[M_t] \right)^q d\nu_F \, .$$

Hence

$$\| P_{t+s}^F \|_{2,q} \leqslant C(s) K(s,2) \, \| e^F \, \mathbb{E}^{\mathbb{P}_x}[M_t] \|_{\mathbb{L}^q(\nu_F)} \, , \qquad (2.5)$$

and we are done. $\qquad \square$

Recall that if in addition either μ is a probability measure, or $e^F \in \mathbb{L}^p(\nu_F)$ for some $p > 1$, condition in the Theorem is also necessary.

When P_t is only hyperbounded, the previous arguments are no more available and one has to work harder to get the following analogue of Theorem 2.4

THEOREM 2.5. — *Assume that P_t is hyperbounded. Assume that H(F) is in force, $F \in D(A)$ and that M_t is bounded by some constant $C(t)$. Assume in addition that $e^F \in \mathbb{L}^r(\nu_F)$ for some $r > 1$ (we may choose $r = 2$ when μ is a Probability measure).*

Then a necessary and sufficient condition for ν_F to satisfy (DLSI) is that
$$P_t^F(e^F) = e^F \, \mathbb{E}^{\mathbb{P}_x}[M_t] \in \mathbb{L}^p(\nu_F)$$
for some $p > 2$ and some $t > 0$ large enough.

Proof. — The proof is based on the following elementary consequence of Girsanov theory and the variational characterization of relative entropy (see [6] section 2) : if $\int f^2 \, d\nu_F = 1$ and f is nonnegative, then

$$\int \Big(\sum_j \log h_j \Big) f^2 d\nu_F \leqslant \frac{t}{2} \mathcal{E}_F(f, f) + \log \int f^2 \, h_1 \, P_t^F(h_2) \, d\nu_F . \qquad (2.6)$$

Choose $j = 1, 2$, $h_1 = f^{\alpha-1}$ and $h_2 = f^\beta$. (2.6) becomes

$$\frac{(\alpha + \beta - 1)}{2} \int f^2 \log(f^2) \, d\nu_F \leqslant \frac{t}{2} \mathcal{E}_F(f, f) + \log \int f^{1+\alpha} \, P_t^F(f^\beta) \, d\nu_F . \qquad (2.7)$$

Let (q, s) a pair of conjugate real numbers. Then

$$P_t^F(f^\beta) \leqslant \big(P_t^F(f^{q\beta} \, e^{-\frac{q}{s} F}) \big)^{\frac{1}{q}} \big(P_t^F(e^F) \big)^{\frac{1}{s}} ,$$

and accordingly

$$\int f^{1+\alpha} P_t^F(f^\beta) \, d\nu_F \leqslant \int f^{1+\alpha} \big(P_t^F(f^{q\beta} \, e^{-\frac{q}{s} F}) \big)^{\frac{1}{q}} \big(P_t^F(e^F) \big)^{\frac{1}{s}} \, d\nu_F \qquad (2.8)$$

$$\leqslant \Big(\int f^{1+\alpha} e^{-q\delta F} \, P_t^F(f^{q\beta} \, e^{-\frac{q}{s} F}) \, d\nu_F \Big)^{\frac{1}{q}} \Big(\int f^{1+\alpha} \, e^{s\delta F} \, P_t^F(e^F) \, d\nu_F \Big)^{\frac{1}{s}}$$

$$\leqslant \Big(\int e^{-\frac{2q\delta}{1-\alpha} F} \big(P_t^F(f^{q\beta} e^{-\frac{q}{s} F}) \big)^{\frac{2}{1-\alpha}} \, d\nu_F \Big)^{\frac{1-\alpha}{2q}} \Big(\int e^{\frac{2s\delta}{1-\alpha} F} \big(P_t^F(e^F) \big)^{\frac{2}{1-\alpha}} \, d\nu_F \Big)^{\frac{1-\alpha}{2s}}$$

where we have used Hölder's inequality successively with $f^{1+\alpha} \, d\nu_F$ and $d\nu_F$, and we also used $\int f^2 \, d\nu_F = 1$ to get the last expression. We have of course

to choose $\alpha < 1$. We shall also choose $\beta = 1$. The first factor in the latter expression can be rewritten

$$\int e^{-\frac{2q\delta}{1-\alpha} F} \left(P_t^F (f^q e^{-\frac{q}{s} F}) \right)^{\frac{2}{1-\alpha}} d\nu_F$$

$$= \int e^{\theta F} \left(\mathbb{E}^{\mathbb{P}_x} (f^q(X_t) e^{-(1+\frac{q}{s}) F(X_t)} M_t) \right)^{\frac{2}{1-\alpha}} d\mu ,$$

with

$$\theta = -\frac{2q\delta}{1-\alpha} + \frac{2}{1-\alpha} - 2 .$$

Hence if we choose $\alpha = q\delta < 1$, $\theta = 0$. Furthermore $q = 1 + \frac{q}{s}$ and $f^q e^{-qF} \in \mathbb{L}^{\frac{2}{q}}(\mu)$ with norm 1, provided $q < 2$. Using our hypotheses we thus obtain

$$\int e^{-\frac{2q\delta}{1-\alpha} F} \left(P_t^F (f^q e^{-\frac{q}{s} F}) \right)^{\frac{2}{1-\alpha}} d\nu_F \leqslant \left(C(t) \parallel P_t \parallel_{\frac{2}{q}, \frac{2}{1-\alpha}} \right)^{\frac{2}{1-\alpha}} . \qquad (2.9)$$

For the second factor we choose

$$\frac{2s\delta}{1-\alpha} < r ,$$

and since $\alpha = q\delta$, this choice imposes

$$\delta < \frac{r}{2s+rq} \qquad \text{hence} \qquad \alpha < \frac{rq}{2s+rq} .$$

Note that the condition $\alpha < 1$ is then automatically satisfied. Applying Hölder again we get

$$\int e^{\frac{2s\delta}{1-\alpha} F} \left(P_t^F (e^F) \right)^{\frac{2}{1-\alpha}} d\nu_F \leqslant \left(\int e^{rF} d\nu_F \right)^{\frac{2s\delta}{r(1-\alpha)}} \left(\int (P_t^F (e^F))^p d\nu_F \right)^{\frac{r(1-\alpha)-2s\delta}{r(1-\alpha)}} ,$$

$$(2.10)$$

if

$$p = \frac{2r}{r(1-\alpha)-2s\delta} \qquad \text{hence} \qquad \alpha = \frac{r(p-2)}{p(2(s-1)+r))} .$$

It remains to check that all these choices are compatible, i.e

$$\frac{r(p-2)}{p(2(s-1)+r))} < \frac{rq}{2s+rq}$$

which is easy.

Plugging (2.9) and (2.10) into (2.7) we obtain

$$\alpha \int f^2 \log(f^2) d\nu_F \leqslant t \mathcal{E}_F(f,f) + 2A , \qquad (2.11)$$

where

$$A = \frac{1}{q} \log \left(C(t) \parallel P_t \parallel_{\frac{2}{q}, \frac{2}{1-\alpha}} \right)$$

$$+ \frac{\alpha}{q} \log \left(\parallel e^F \parallel_{\mathbb{L}^r(\nu_F)} \right) + \frac{1}{s} \log \left(\parallel P_t^F(e^F) \parallel_{\mathbb{L}^p(\nu_F)} \right).$$

For a fixed p we may choose any pair (q, s) with $q < 2$, and the corresponding α yields the result for

$$t \geqslant \frac{a}{2} \log \left(\frac{q(1+\alpha)}{(2-q)(1-\alpha)} \right),$$

according to Gross theorem, if μ satisfies (DLSI) with constants (a, b). $\qquad \square$

Remark 2.6 . — Unfortunately the previous methods cannot furnish the best constants. In particular we cannot get (TLSI) even when μ satisfies (TLSI).

In view of the previous remark it is thus natural to look at the spectral gap properties too. The final result we shall recall is Lemma 2.2 in [1].

THEOREM 2.7. — *Assume that μ is a probability measure satisfying (SGP).Assume that H(F) is in force and $\Gamma(F, F) \in \mathbb{L}^1(\mu)$. Then ν_F satisfies (WSGP).*

One can use Theorems 2.4 (or 2.5) and 2.7 together in order to show that the general Boltzmann measure satisfies (TLSI) provided μ is a Probability measure. Otherwise one has to consider various reference measures μ, as it will be clear in the next sections.

3. The "Well Method"

Our aim in this section is to get sufficient general conditions for (2.4) to hold. To this end we shall slightly modify the "Well Method" of [13], i.e. use the martingale property of the Girsanov density. In the sequel we assume that $F \in D(A)$ satisfies H(F).

The main assumption we shall make is the following, for all x

$$\frac{1}{2}\Gamma(F,F)(x) - AF(x) \geqslant -c > -\infty.\qquad(3.1)$$

It follows that $M_t \leqslant e^{ct} = C(t)$.

Now we define $\lambda(x)$ by the relation,

$$\frac{1}{2}\Gamma(F,F)(x) - AF(x) = \lambda(x)\,F(x).$$

Note that if $F(x) \leqslant 0$, $P_t^F(e^F)(x) \leqslant C(t)$ so that the contribution of the $x's$ with $F(x) \leqslant 0$ belongs to $\mathbb{L}^\infty(\nu_F)$. So we may and will assume that $F(x) > 0$.

For $0 < \varepsilon < 1$ define the stopping time τ_x as

$$\tau_x = \inf\{s > 0,\ (\frac{1}{2}\Gamma(F,F) - AF)(X_s) \leqslant \varepsilon\,\lambda(x)\,F(x)\ \text{or}\ F(X_s) \leqslant \varepsilon\,F(x)\}.\qquad(3.2)$$

First we assume that $(\frac{1}{2}\Gamma(F,F) - AF)(x) > 0$. In this case $\tau_x > 0$ $\mathbb{P}_x$ a.s. Introducing the previous stopping time we get

$$\mathbb{E}^{\mathbb{P}_x}[M_t] = \mathbb{E}^{\mathbb{P}_x}[M_t\,\mathbb{1}_{t<\tau_x}] + \mathbb{E}^{\mathbb{P}_x}[M_t\,\mathbb{1}_{\tau_x\leqslant t}] = A + B,$$

with

$$A = \mathbb{E}^{\mathbb{P}_x}[M_t\,\mathbb{1}_{t<\tau_x}] \leqslant \exp-\big(\varepsilon\,t\,\lambda(x)\,F(x)\big),\qquad(3.3)$$

and

$$
\begin{aligned}
B &= \mathbb{E}^{\mathbb{P}_x}[M_t\,\mathbb{1}_{\tau_x\leqslant t}] &(3.4)\\
&\leqslant e^{ct}\,\mathbb{E}^{\mathbb{P}_x}[\exp\Big(\int_0^t \big(AF - \frac{1}{2}\Gamma(F,F) + c\big)(X_s)\,ds\Big)\,\mathbb{1}_{\tau_x\leqslant t}]\\
&\leqslant e^{ct}\,\mathbb{E}^{\mathbb{P}_x}[\exp\Big(\int_0^{\tau_x} \big(AF - \frac{1}{2}\Gamma(F,F) + c\big)(X_s)\,ds\Big)\,\mathbb{1}_{\tau_x\leqslant t}]\\
&\leqslant e^{ct}\,\mathbb{E}^{\mathbb{P}_x}[\exp\Big(\int_0^{\tau_x} \big(AF - \frac{1}{2}\Gamma(F,F)\big)(X_s)\,ds\Big)\,\mathbb{1}_{\tau_x\leqslant t}]\\
&= e^{ct}\,\mathbb{E}^{\mathbb{P}_x}[M_{\tau_x}\,\mathbb{1}_{\tau_x\leqslant t}].
\end{aligned}
$$

But $e^{-F(X_s)}\,M_s$ is a $\mathbb{L}^2$ (thanks to 3.1) $\mathbb{P}_x$ martingale. Hence, according to Doob's Optional Stopping Theorem

$$\mathbb{E}^{\mathbb{P}_x}[e^{-F(X_{\tau_x})}\,M_{\tau_x}\,\mathbb{1}_{\tau_x\leqslant t}] \leqslant \mathbb{E}^{\mathbb{P}_x}[e^{-F(X_{t\wedge\tau_x})}\,M_{t\wedge\tau_x}] = e^{-F(x)}.\qquad(3.5)$$

According to (3.2),

$$e^{-F(X_{\tau_x})} \geqslant e^{-\varepsilon F(x)},$$

so that thanks to (3.5),

$$\mathbb{E}^{\mathbb{P}_x}[M_{\tau_x} \mathbb{1}_{\tau_x \leqslant t}] \leqslant e^{-(1-\varepsilon)F(x)}.$$

Using this estimate in (3.4) and using (3.3) we finally obtain

$$\mathbb{E}^{\mathbb{P}_x}[M_t] \leqslant e^{-\varepsilon t \lambda(x) F(x)} + e^{ct} e^{-(1-\varepsilon)F(x)}. \tag{3.6}$$

Finally if $(\frac{1}{2} \Gamma(F,F) - AF)(x) < 0$ we certainly have

$$\mathbb{E}^{\mathbb{P}_x}[M_t] \leqslant e^{ct} e^{-\varepsilon t \lambda(x) F(x)},$$

since in this case $\lambda(x) < 0$ while we assume $F(x) > 0$.

We have thus obtained choosing first $\varepsilon = r/p$,

THEOREM 3.1. — *Assume that H(F) and (3.1) are fulfilled. Assume in addition that there exists some $0 < r$ such that $e^F \in \mathbb{L}^r(\nu_F)$. Then $e^F \mathbb{E}^{\mathbb{P}_x}[M_t] \in \mathbb{L}^p(\nu_F)$ as soon as*

$$\int e^{(p-2)F} e^{-(rt/p)(\frac{1}{2}\Gamma(F,F) - AF)} d\mu < +\infty.$$

In particular ν_F satisfies (DLSI) as soon as

$$\int e^{\beta F} e^{-\lambda(\frac{1}{2}\Gamma(F,F) - AF)} d\mu < +\infty,$$

for some $\beta > 0$ and some $\lambda > 0$. Furthermore if the previous holds for all pair (β, λ) of positive real numbers, then P_t^F is immediately hyperbounded (i.e. P_t^F is bounded from $\mathbb{L}^2(\nu_F)$ in $\mathbb{L}^p(\nu_F)$ for all $t > 0$ and all $p > 2$).

Remark 3.2 . — This result extends previous ones obtained by Davies [8] (especially Theorem 4.7.1 therein) in the ultracontractive context, by Rosen [19] in the hyperbounded context (based on deep Sobolev inequalities available in $\mathbb{R}^N$) or by Kusuoka and Stroock [15]. In addition it is an "almost" necessary condition too, in the sense of the next result.

THEOREM 3.3. — *Assume that H(F) holds and that there exists some $1 < r$ such that $e^F \in \mathbb{L}^r(\nu_F)$. A necessary condition for ν_F to satisfy (DLSI) is*

$$\int e^{\beta F(x)} e^{-\lambda\left(\frac{1}{2}\Gamma(F,F) - AF\right)(x)} \mathbb{P}_x^{2+\beta}(\tau_x > \lambda/2(2+\beta))\, d\mu < +\infty,$$

for some $\beta > 0$ and some $\lambda > 0$, where τ_x is the stopping time defined by

$$\tau_x = \inf\left\{s \geqslant 0 \text{ s.t.} \left(\frac{1}{2}\Gamma(F,F) - AF\right)(X_s) \geqslant 2\lambda(x)\, F(x)\right\},$$

$\lambda(x)$ being defined as

$$\frac{1}{2}\Gamma(F,F)(x) - AF(x) = \lambda(x)\, F(x).$$

Proof. — It is enough to remark that $\tau_x = 0$ if $\lambda(x) \leqslant 0$ and then write for $\lambda(x) > 0$

$$\mathbb{E}_x[M_t] \geqslant \mathbb{E}_x[M_t \,\mathbb{1}_{t<\tau_x}]$$
$$\geqslant e^{-2t\lambda(x)F(x)}\, \mathbb{E}_x[\mathbb{1}_{t<\tau_x}],$$

and then to apply the necessary part of Theorem 2.5. □

4. A direct approach for the sufficient condition and others consequences

In the previous two sections we used the hyperbounded point of view. As suggested by an anonymous referee, Theorem 3.1 can be directly obtained by using logarithmic Sobolev inequalities.

Indeed assume that

$$\int f^2 \log f^2\, d\mu \leqslant C_1 \int \Gamma(f,f)\, d\mu + C_2, \tag{4.1}$$

for all nice f such that $\int f^2\, d\mu = 1$. Take $f = e^{-F} g$ for some g such that $\int g^2\, d\nu_F = 1$. Thanks to the chain rule, i.e.

$$\int \varphi'(f)\, Af + \frac{1}{2}\varphi''(f)\,\Gamma(f,f)\, d\mu = 0$$

it is easy to see that (4.1) can be rewritten

$$\int g^2 \log g^2 \, d\nu_F \quad \leqslant \quad C_1 \int \Gamma(g,g) \, d\nu_F \tag{4.2}$$

$$+ \quad \int g^2 \left(2C_1 \big(AF - \frac{1}{2}\Gamma(F,F)\big) + 2F \right) d\nu_F + C_2 \, .$$

Introducing some $0 < \varepsilon < 1$, we write the second integral in the right hand side

$$\varepsilon \int g^2 \frac{1}{\varepsilon} H \, d\nu_F \, ,$$

and use Young's inequality in order to get

$$(1-\varepsilon) \int g^2 \log g^2 \, d\nu_F \tag{4.3}$$

$$\leqslant \ C_1 \int \Gamma(g,g) \, d\nu_F + \varepsilon \, e^{-1} \int e^{\frac{2C_1}{\varepsilon} \left(AF - \frac{1}{2}\Gamma(F,F) \right) + 2(\frac{1}{\varepsilon} - 1) F} \, d\mu + C_2 \, ,$$

and we recover Theorem 3.1 since we may choose ε arbitrarily close to 1 and independently C_1 arbitrarily large. Actually in Theorem 3.1, since (3.1) is fulfilled, we may choose any $\lambda' > \lambda$. The only difference here is that we do not need to assume (3.1), but in contrast, we have to assume that λ is large enough.

The above proof is given with less details than the previous martingale proof. Actually both are short and elementary. The main advantage of the martingale point of view is to indicate how get a necessary condition.

However it is interesting at this point to compare our condition for (DLSI) and known results on (SGP) obtained by Gong and Wu [10] for Feynman-Kac semigroups.

The unitary transform $U : \mathbb{L}^2(E, d\mu) \to \mathbb{L}^2(E, d\nu_F)$ defined by $U(f) = e^F f$ satisfies

$$\int \Gamma(U(f), U(g)) \, d\nu_F \ = \ \int \big(\Gamma(f,g) + V_F \, fg \big) \, d\mu$$

where $V_F = \Gamma(F,F) - 2\,AF$. The latter Dirichlet form is the one associated with the Schrödinger operator $H_F = A + V_F$. Since U is unitary the spectrum of H_F on $\mathbb{L}^2(d\mu)$ and the one of $-A_F$ on $\mathbb{L}^2(\nu_F)$ coincide. Hence the existence of a spectral gap for ν_F follows from Corollary 6 in [10], namely

PROPOSITION 4.1. — *Let μ be a probability measure satisfying (TLSI) (i.e. (4.1) with $C_2 = 0$) and assume that H(F) holds. If*

$$\int e^{(2\,C_1+\varepsilon)(\frac{1}{2}\Gamma(F,F)\,-\,AF)^-}\,d\mu \;<\; +\infty$$

for some $\varepsilon > 0$ then ν_F satisfies (SGP). This result holds in particular when (3.1) is satisfied.

It follows in particular that, provided F is bounded below, the condition in Proposition 4.1 is implied by the condition in Theorem 3.1 without assuming (3.1), but assuming that $\lambda > 2C_1$.

COROLLARY 4.2. — *If μ satisfies (TLSI) (or equivalently P_t is hypercontractive) and H(F) holds, then*

$$\int e^{\beta F}\,e^{-\lambda\,(\frac{1}{2}\,\Gamma(F,F)\,-\,AF)}\,d\mu \;<\; +\infty\,,$$

for some $\beta > 0$ and $\lambda > 0$ is a sufficient condition for ν_F to satisfy (TLSI) provided in addition

(1) *either $\frac{1}{2}\Gamma(F,F) - AF$ is bounded from below and $e^F \in \mathbb{L}^r(\nu_F)$ for some $r > 0$,*

(2) *or F is bounded below and $\lambda > 2C_1$ where C_1 is the optimal constant in (TLSI) for μ.*

The interested reader will find a stronger statement (Theorem 5) in [10], but with less tractable hypotheses.

5. Examples: $\mathbb{R}^N$ valued Boltzmann measures

In this section we shall deal with the $\mathbb{R}^N$ valued case, i.e. $E = \mathbb{R}^N$, dx is Lebesgue measure, $A = \frac{1}{2}\Delta$ is one half of the Laplace operator and ∇ is the usual gradient operator. $\mathbb{P}_x$ is thus the law of the Brownian motion starting at x, whose associated semigroup P_t is dx symmetric and ultracontractive with $\|\,P_t\,\|_{2,+\infty} = (4\pi\,t)^{-\frac{N}{4}}$. $\mathbb{D}$ is the algebra generated by the usual set of test functions and the constants.

(TLSI) can thus be written

$$\int f^2 \log \Big(\frac{f^2}{\| f \|^2_{\mathbb{L}^2(\nu_F)}} \Big) \, e^{-2F} \, dx \leqslant a \int |\nabla f|^2 \, e^{-2F} \, dx \, .$$

Note that Lebesgue measure satisfies a family of logarithmic Sobolev inequalities i.e. for all $\eta > 0$ and all f belonging to $\mathbb{L}^1(dx) \cap \mathbb{L}^\infty(dx)$ such that $\int f^2 \, dx = 1$

$$\int f^2 \log f^2 \, dx \leqslant 2\eta \int |\nabla f|^2 dx + \frac{N}{2} \log \Big(\frac{1}{4\pi\eta} \Big) \, ,$$

see e.g. [8] Theorem 2.2.3.

In the sequel we will consider functions F that are of class C^2 and according to Proposition 2.2 we shall then (if necessary) add to F some bounded perturbation. Furthermore in this particular finite dimensional situation we may replace H(F) by the following Lyapounov control:

there exists some ψ such that $\psi(x) \to +\infty$ as $|x| \to +\infty$, (5.1)

and $\Delta\psi(x) - (\nabla F . \nabla\psi)(x) \leqslant K < +\infty$ for all x .

In order to complete the picture, we have to describe some sufficient conditions allowing to tight the logarithmic Sobolev inequality.

One is given by Theorem 2.7. Indeed if $\nu_U(dx) = e^{-2U(x)}dx$ is another Boltzmann measure satisfying (SGP) and

$$\int |\nabla U|^2 \, d\nu_U \, < \, +\infty$$

a sufficient condition for ν_F to satisfy (WSGP) is

$$\int |\nabla F|^2 \, d\nu_U \, < \, +\infty,$$

since $d\nu_F = e^{-2(F-U)}d\nu_U$. It is thus not difficult to guess that (WSGP) holds for *any* Boltzmann measure (such that F is smooth). This result is actually true and shown (using another route) in [18] Theorem 3.1 and Remark (1) following this Theorem. Hence

PROPOSITION 5.1. — *For a Boltzmann measure* ν_F *with* $F \in C^2$, *(WSGP) is satisfied. Consequently (DLSI) and (TLSI) are equivalent.*

It is nevertheless interesting, at least for counter examples to know some sufficient conditions for the usual (SGP). If $N = 1$ a necessary and sufficient condition was obtained by Muckenhoupt (see [3] chapter 6). We recall below a tractable version due to Malrieu and Roberto of this result as well as its N dimensional counterpart

PROPOSITION 5.2. — *Let F of C^2 class.*

(1) *(see [3] Theorem 6.4.3)   If $N = 1$, $|F'(x)| > 0$ for $|x|$ large enough and $\frac{F''(x)}{|F'(x)|^2}$ goes to 0 as $|x|$ goes to ∞, then ν_F satisfies (SGP) if and only if*

$$\liminf_{|x| \to +\infty} |F'(x)|^2 = C > 0.$$

(2) *(see e.g. [14] Proposition 3.7)   For any N, if*

$$\liminf_{|x| \to +\infty} (|\nabla F|^2 - \Delta F) = C > 0,$$

then (SGP) holds for ν_F.

Now if we want to use Proposition 4.1 we may choose $d\mu = (1/Z_\rho) e^{-2\rho|x|^2} dx$ which is known to satisfy (TLSI) with constant $C_1 = 1/2\rho$, and is associated to the generator

$$A_\rho = \frac{1}{2} \Delta - 2\rho x.\nabla.$$

We thus have to look at

$$\frac{1}{2} |\nabla(F - \rho|x|^2)|^2 - A_\rho(F - \rho|x|^2) = \frac{1}{2} \left(|\nabla F|^2 - \Delta F\right) - 2\rho^2|x|^2 + \rho N.$$
$$(5.2)$$

Note thus that we cannot recover 5.2(2).

According to Proposition 5.1 and the previous sections we know that a sufficient condition for (TLSI) to hold is the integral condition in Theorem 3.1 (assuming in addition one of conditions (1) and (2) in Corollary 4.2), while a necessary one is given in Theorem 3.2. Up to our knowledge, except the bounded perturbation recalled in Proposition 2.2, three others family of sufficient conditions have been given for ν_F:

- the renowned Bakry-Emery criterion saying that (TLSI) holds as soon as F is uniformly convex, i.e. $Hess(F) \geqslant K\,Id$ for some $K > 0$,

- Wang's results (see [21] Theorem 1.1 for this final version) saying that provided $Hess(F) \geqslant -K\,Id$ for some $K \geqslant 0$, a sufficient condition is

$$\int e^{\varepsilon |x|^2}\, d\nu_F \leqslant +\infty$$

  for some $\varepsilon > K$,

- the beautiful Bobkov-Götze criterion for $N = 1$, and its weak version due to Malrieu and Roberto (see [3] Theorem 6.4.3) saying that if $|F'(x)| > 0$ for $|x|$ large enough and $\frac{F''(x)}{|F'(x)|^2}$ goes to 0 as $|x|$ goes to ∞, then ν_F satisfies (TLSI) if and only if there exists some A such that

$$\frac{F}{|F'|^2} + \frac{\log |F'|}{|F'|^2}$$

  is bounded on $\{|x| \geqslant A\}$.

It is not difficult to see that our results contain Malrieu-Roberto result.

It is also easy to see that if $Hess(F)(x) \geqslant \rho\,Id$ for some positive ρ and all x, then

$$|\nabla F|^2(x) \geqslant 2\rho F(x) - C\,,$$

for some constant C. Hence if F is uniformly convex and such that

$$|\Delta F|(x) \leqslant (1 - \varepsilon)\,|\nabla F|^2(x) + c(F)\,,$$

for some $\varepsilon > 0$, all x, and some constant $c(F)$, we recover the result by Bakry-Emery. The same holds for Wang's result if the perturbed $F + \frac{1}{2}(K + \varepsilon)|x|^2$ is a nice uniformly convex function as before.

Unfortunately, it is not difficult to build uniformly convex functions such that

$$\limsup_{|x|\to+\infty} \left(\frac{\Delta F}{|\nabla F|^2}\right) = +\infty\,.$$

Actually the counter examples built by Wang are such that the previous property holds.

Remark 5.3 . — Assume that $\lim_{|x|\to+\infty} F(x) = +\infty$.

Applying Theorem 3.1 we see that P_t^F is hypercontractive in particular as soon as

$$|\nabla F|^2(x) - \Delta F(x) \geqslant \eta\, F(x) - c\,,$$

for some constant c and some $\eta > 0$.

As we remarked in Theorem 3.1, we can get conditions for *immediate* hypercontractivity, for instance P_t^F will be *immediately* hypercontractive as soon as

$$|\nabla F|^2(x) - \Delta F(x) \geqslant G(F(x)),$$

for some function G such that

$$\lim_{y \to +\infty} \frac{G(y)}{y} = +\infty.$$

One can also see from [13] that a condition like

$$\int^{+\infty} \frac{y}{G(y)\, g'(g^{-1}(y))}\, dy < +\infty \text{ for some } g \text{ satisfying } \int^{+\infty} e^{-g(y)}\, dy < +\infty,$$

for the function G we have introduced above, implies that P_t^F is *ultracontractive*. This result with $G(y) = y^\theta$ for some $\theta > 1$ (take then $g(y) = e^y$) is contained in [8] Theorem 4.7.1.

As shown in [5] the same control but with $0 < \theta < 1$ yields a weaker form of hypercontractivity.

As we discussed before, if our results can only be partly compared (at least easily) with existing ones in the bounded below curvature case (i.e. when the Hessian is bounded from below), they allow to look at interesting examples in the unbounded curvature case. We shall below discuss such a family of examples. But first we recall a basic estimate for the Brownian motion that allows us to give a precise meaning to the necessary condition stated in Theorem 3.3.

LEMMA 5.4. — *For a standard Brownian motion B_s on $\mathbb{R}^N$, there exists a constant θ_N such that*

$$\mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} |B_s| < A\right) \geqslant e^{-\theta_N \frac{t}{A^2}}.$$

Example 5.5 . — Let us consider on $\mathbb{R}^+$ the potential $F_\beta(x) = x^2 + \beta\, x\, \sin(x)$ extended by symmetry to the full real line. We shall only look at its behaviour near $+\infty$.

The derivatives are given by $F_\beta'(x) = (2 + \beta\, \cos(x))\, x + \beta\, \sin(x)$ and $F_\beta''(x) = -\beta\, x\, \sin(x) + 2(1 + \beta\, \cos(x))$. Hence $-\infty = \liminf_{x \to +\infty} F''(x)$.

For $|\beta| < 2$ we may apply Malrieu-Roberto result (or Theorem 3.1) and show that (TLSI) holds.

For $|\beta| \geqslant 2$ the hypotheses of Theorem 3.1 are no more satisfied. Indeed

$$F'^2(x) - F''(x) = (2 + \beta \cos(x))^2 x^2 + (4 + 2\beta \cos(x) - \beta \sin(x))x + h(x)$$

where h is bounded, can be very negative for the x's such that $2 + \beta \cos(x) = 0$.

We shall discuss below the case $\beta = -2$ in details. Instead of using Theorem 3.3 we shall directly study $P_t^F(e^F)$ for $F = F_{-2}$.

Introduce $x_k = 2k\pi$. Then for k large enough one can find ε small enough and some constant c such that

$$\text{for all } y \text{ such that } 1/2\,k^{-\frac{1}{2}} \leqslant y - x_k \leqslant 3/2\,k^{-\frac{1}{2}} \text{ it holds} \qquad (5.3)$$
$$F''(y) \geqslant (1-\varepsilon)k^{\frac{1}{2}} \quad \text{and} \quad |F'(y)| \leqslant c\,.$$

Introduce the stopping times $\tau_k = \inf\left\{s \geqslant 0,\, |X_s - y| \geqslant \frac{1}{4}\,k^{-\frac{1}{2}}\right\}$. Then according to (5.3), for $3/4\,k^{-\frac{1}{2}} \leqslant y - x_k \leqslant 5/4\,k^{-\frac{1}{2}}$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}_y}(M_t) &\geqslant \mathbb{E}^{\mathbb{P}_y}(M_t\,\mathbb{1}_{t < \tau_k}) \\
&\geqslant e^{\frac{t}{2}\left((1-\varepsilon)k^{\frac{1}{2}} - c^2\right)}\,\mathbb{P}_y(t < \tau_k) \\
&\geqslant e^{\frac{t}{2}\left((1-\varepsilon)k^{\frac{1}{2}} - c^2\right)}\,e^{-4\theta tk}
\end{aligned}$$

for the constant θ appearing in Lemma 5.4. It follows

$$\int^{+\infty} e^{(q-2)F}\left(\mathbb{E}^{\mathbb{P}_x}[M_t]\right)^q dx \geqslant \frac{1}{2}\sum_k k^{-\frac{1}{2}}\,e^{4\pi^2\,(q-2-\varepsilon)\,k^2}\,e^{-4q\theta tk} = +\infty\,.$$

Hence (DLSI) does not hold.

For $\beta = 2$ the discussion is similar, while for $|\beta| > 2$ it is a little bit different. Indeed (again with $\beta < 0$) this time if $F'(x_k) = 0$, on $2k^{-\frac{3}{4}} \leqslant y - x_k \leqslant k^{-\frac{3}{4}}$ we have $F''(y) \geqslant (1 - \varepsilon)k$ while $|F'(x)| \leqslant c\,k^{\frac{1}{4}}$. Hence we can prove as before that $\mathbb{E}^{\mathbb{P}_y}(M_t) \geqslant C\,e^{-c'\theta t\,k^{\frac{3}{2}}}$ for some constants C and c' and conclude again that (DLSI) does not hold.

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