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Holomorphic foliations by curves on $\mathbf{P}^3$ with non-isolated singularities(*)

Gilcione Nonato Costa [1]

Abstract. — Let $\mathcal{F}$ be a holomorphic foliation by curves on $\mathbf{P}^3$. We treat the case where the set Sing($\mathcal{F}$) consists of disjoint regular curves and some isolated points outside of them. In this situation, using Baum-Bott's formula and Porteuos'theorem, we determine the number of isolated singularities, counted with multiplicities, in terms of the degree of $\mathcal{F}$, the multiplicity of $\mathcal{F}$ along the curves and the degree and genus of the curves.

Résumé. — Soit $\mathcal{F}$ un feuilletage holomorphe de dimension 1 dans $\mathbf{P}^3$. Nous considérons le cas où l'ensemble Sing($\mathcal{F}$) est formé par des courbes lisses et disjointes et quelques points isolés en dehors de ces courbes. Dans cette situation, en employant la formule de Baum-Bott et le théorème de Porteous, nous déterminons le nombre de singularités isolées, comptées avec multiplicités, en fonction du degré de $\mathcal{F}$, de la multiplicité de $\mathcal{F}$ le long des courbes et du degré et du genre des courbes.

1. Introduction

Throughout this paper $\mathcal{F}$ denotes a holomorphic foliation by curves with non-isolated singularities in a three-dimensional complex manifold M. More precisely, we consider foliations with singular sets consisting of smooth and disjoint curves, possibly with some isolated points. In [8], F. Sancho determines a bound for the number of curves that can appear on Sing($\mathcal{F}$) in terms of the degree of the holomorphic foliation defined on $\mathbf{P}^3$.

Our aim is to describe $\mathcal{F}$ from information obtained by blowing-up M, $\tilde{M} \xrightarrow{\pi} M$, along a regular curve $\mathcal{C} \subset$ Sing($\mathcal{F}$). As in the case of isolated singularities, concepts as dicritical and non-dicritical curve of singularities are

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directly obtained. The algebraic multiplicity of $\mathcal{F}$ along $\mathcal{C}$ and the order of tangency of $\pi^*\mathcal{F}$ on E, the exceptional divisor, will be denoted by $\mathrm{mult}_{\mathcal{C}}(\mathcal{F})$ and $\mathrm{tang}(\pi^*\mathcal{F}, E)$, respectively.

Let $\tilde{\mathcal{F}}$ be the pullback foliation, defined in $\tilde{M}$, obtained from $\mathcal{F}$ via π. The foliation $\mathcal{F}$ will be called *special* along $\mathcal{C}$ if $\tilde{\mathcal{F}}$ has E as an invariant set and contains only isolated singularities on E. As we will see, if $\mathcal{F}$ is special along $\mathcal{C}$ then $\mathrm{mult}_{\mathcal{C}}(\mathcal{F}) = \mathrm{tang}(\pi^*\mathcal{F}, E)$. In case $M = \mathbf{P}^3$ and $\mathrm{Sing}(\mathcal{F})$ consisting of only one curve of singularities, we determine the number of isolated singularities, counted with multiplicities, of $\mathcal{F}$ in $\mathbf{P}^3$. More precisely,

THEOREM 1.1. — *Let $\mathcal{F}$ be a holomorphic foliation by curves on $\mathbf{P}^3$, special along a regular curve $\mathcal{C}$ of genus g and degree d. Suppose that $Sing(\mathcal{F}) = \mathcal{C} \cup \{p_1, \ldots, p_q\}$, disjoint union. Then,*

$$\sum_{j=1}^{q} \mu(\mathcal{F}, p_j) = 1+k+k^2+k^3+(\ell+1)\Big[(2g-2)(\ell^2+\ell+1)+4d\ell^2-d(k-1)(3\ell+1)\Big]$$

where $\mu(\mathcal{F}, p_j))$ is the multiplicity of $\mathcal{F}$ at p_j, $k = degree(\mathcal{F})$ and $\ell = tang(\pi^*\mathcal{F}, E)$.

If we make a small pertubation of $\mathcal{F}$, a regular curve $\mathcal{C} \subset \mathrm{Sing}(\mathcal{F})$ may be destroyed and transformed into isolated singularities. Theorem 1.1 gives the number of isolated singularities, counted with multiplicities, that will appear near $\mathcal{C}$. In fact, this number is $(\ell+1)[(2-2g)(\ell^2+\ell+1)-4d\ell^2+d(k-1)(3\ell+1)]$, because $1+k+k^2+k^3$ is the total number of isolated singularities, counted with multiplicities, after this small pertubation. Therefore, this number may be seen as a Milnor number of $\mathcal{C}$ relative to $\mathcal{F}$.

2. Preliminaries

A foliation by curves (with singularities) $\mathcal{F}$ on a n-dimensional complex manifold M may be defined by a family of holomorphic vector fields $\{X_\alpha\}$ on an open cover $\{U_\alpha\}$ of M, which satisfies $X_\alpha = f_{\alpha\beta} X_\beta$ in $U_\alpha \cap U_\beta$, where $f_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. The singular set of $\mathcal{F}$ is the analytic subvariety defined by

$$\mathrm{Sing}(\mathcal{F}) = \{p \in M | X_\alpha(p) = 0, \text{ for some } \alpha\}.$$

We assume that $\mathrm{cod}(\mathrm{Sing}(\mathcal{F})) \geqslant 2$.

Let z be a coordinate for M near $p \in \mathrm{Sing}(\mathcal{F})$ and let $\mathcal{F}$ be given by a vector field $X(z) = \displaystyle\sum_{i=1}^{n} P_i(z) \frac{\partial}{\partial z_i}$. We have the following objects associated to p:

1. The multiplicity $\mu(\mathcal{F},p)$ of $\mathcal{F}$ at p which is the codimension in the ring $\mathcal{O}_{M,p}$ of the ideal generated by $\{P_i\}_{j=1}^{j=n}$

$$\mu(\mathcal{F},p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{M,p}}{< P_1, \ldots, P_n >}.$$

It is well known that $\mu(\mathcal{F},p)$ is finite if and only if p is an isolated singularity.

2. The algebraic multiplicity of $\mathcal{F}$ at p, which is the degree of the smallest non-zero coefficient in the power series expansion of X. We will say that $\mathcal{F}$ is non-dicritical at p if the terms of smallest degree of X are not a multiple of the radial vector field.

Let us recall the notion of quadratic transformation or blow up of a polydisc along a coordinate plane. Let Δ be a n-dimensional polydisc with holomorphic coordinates $z_1, \ldots, z_n$ and $V \subset \Delta$ be the locus $z_1 = \ldots = z_k = 0$. Let $[l_1, \ldots, l_k]$ be homogeneous coordinates on $\mathbf{P}^{k-1}$, and let

$$\tilde{\Delta} \subset \Delta \times \mathbf{P}^{k-1}$$

be the smooth variety defined by the relations

$$\tilde{\Delta} = \{(z,[l]) \quad | \quad z_i l_j = z_j l_i; \quad 1 \leqslant i,j \leqslant k\}.$$

The projection $\pi : \tilde{\Delta} \to \Delta$ on the first factor is an isomorphism away from V, while the inverse image of a point $z \in V$ is a projective space $\mathbf{P}^{k-1}$. The manifold $\tilde{\Delta}$ together with the map $\pi : \tilde{\Delta} \to \Delta$ is called the blow-up or quadratic transformation of Δ along V. The inverse image $E = \pi^{-1}(V)$ is called the exceptional divisor of the blow-up.

The set $\tilde{\Delta}$ has a natural structure of n-dimensional complex manifold. For each $j \in \{1, 2, \ldots, k\}$ let $U_j = \{[l_1, \ldots, l_k], l_j \neq 0\} \subset \mathbf{P}^{k-1}$ be the standard open cover, then

$$\tilde{U_j} = \{(z,[\varsigma]) \in \tilde{\Delta}; [\varsigma] \in U_j\} \tag{2.1}$$

with holomorphic coordinates $\sigma(\varsigma_1, \ldots, \varsigma_n) = (z_1, \ldots, z_n)$ given by

$$z_i = \begin{cases} \varsigma_i, & \text{for} \quad i = j \quad \text{or} \quad i > k, \\ \varsigma_i \varsigma_j, & \text{for} \quad i = 1, ..., \hat{j}, ..., k. \end{cases}$$

The coordinates $\varsigma \in \mathbf{C}^n$ are affine coordinates on each fiber $\pi^{-1}(p) \cong \mathbf{P}^{k-1}$ of E.

We can generalize this construction. Let $S \subset M$ be a submanifold of dimension $n-k$. Let $\{\phi_\alpha, U_\alpha\}$ be a collection of local charts covering S and

$\phi_\alpha : U_\alpha \to \Delta_\alpha$, where Δ_α is a n-dimensional polydisc. We may suppose that $V_\alpha = \phi_\alpha(X \cap U_\alpha)$ is given by $z_1 = \ldots = z_k = 0$. Let $\pi_\alpha : \tilde{\Delta}_\alpha \to \Delta_\alpha$ be the blow-up of Δ_α along V_α. Then, we have isomorphisms

$$\pi_{\alpha\beta} : \pi_\alpha^{-1}[\phi_\alpha(U_\alpha \cap U_\beta)] \to \pi_\beta^{-1}[\phi_\beta(U_\alpha \cap U_\beta)]$$

and using them, we can patch together the blow-ups $\tilde{\Delta}_{\pi_\alpha}$ to form a manifold $\tilde{\Delta} = \cup_{\pi_{\alpha\beta}} \tilde{\Delta}_\alpha$ with the map $\pi : \tilde{\Delta} \to \cup \tilde{\Delta}_\alpha$.

Finally, since π is an isomorphism away from the exceptional divisor, we can take $\tilde{M} = (M - S) \cup_\pi \tilde{\Delta}$, together with the map $\pi : \tilde{M} \to M$, extending π on $\tilde{\Delta}$ and the identity on $M - S$, is called the blow-up of M along X. The blow-up has the following properties:

1. The *exceptional divisor* E is a fibre bundle over S with fiber $\mathbf{P}^{k-1}$. Indeed, $\pi_E = \pi|_E : E \to S$ is naturally identified with the projectivization $\mathbf{P}(N_{S/M})$ of the normal bundle $N_{S/M}$ of S in M. If M is an algebraic threefold and S a regular compact curve, the exceptional divisor E will be a ruled surface.

2. For any variety $Y \subset M$, we may define the proper transform $\tilde{Y} \subset \tilde{M}$ of $\tilde{Y}$ in the blow-up $\tilde{M}_S$ to be the closure in $\tilde{M}_S$ of the inverse image

$$\pi^{-1}(Y - S) = \pi^{-1}(Y) - E$$

of Y away from the exceptional divisor E. The intersection $\tilde{Y} \cap E \subset \mathbf{P}(N_{S/M})$ corresponds to the image in $N_{S/M}$ of the tangent cones $T_p(Y) \subset T_p(M)$ to Y at points of $Y \cap S$. In particular, for $Y \subset M$ a divisor,

$$\tilde{Y} = \pi^{-1}(Y) - m.E, \tag{2.2}$$

where

$$m = \mathrm{mult}_S(Y)$$

is the multiplicity of Y at a generic point of S.

From (2.2) follows that

$$\mathrm{Pic}(\tilde{M}) = \pi^* \mathrm{Pic}(M) + \mathbf{Z}[E]. \tag{2.3}$$

For additional informations, see [5].

The coholomology of a blow-up. — Let $\rho : F \to S$ be a complex vector bundle with transition functions $\{g_{\alpha\beta}\} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbf{C})$. We write F_p for the fiber over p. The projectivization of F, $\rho_F : \mathbf{P}(F) \to S$, is by definition the fiber bundle whose fiber at a point p in S is the projective space $\mathbf{P}(F_p)$ and whose transition functions $\overline{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{PGL}(r, \mathbf{C})$ are induced from $g_{\alpha\beta}$. Thus a point of $\mathbf{P}(F)$ is a line ℓ_p in the fiber F_p. On $\mathbf{P}(F)$ there are several tautological bundles: the pullback $\pi^{-1}F$, the universal, also called the tautological subbundle T, and the universal quotient bundle Q (See [2]). The cohomology ring $H^*(\mathbf{P}(F))$ is, via the pullback map, $H^*(S) \xrightarrow{\rho_F^*} H^*(\mathbf{P}(F))$ an algebra over the ring $H^*(S)$. A complete description of $H^*(\mathbf{P}(F))$ is given in these terms by the

Proposition 2.1. — *For S any compact oriented C^∞ manifold, $F \to S$ any complex vector bundle of rank r, the cohomology ring $H^*(\mathbf{P}(F))$ is generated, as an $H^*(S)$-algebra, by the Chern class $\zeta = c_1(T)$ of tautological bundle, with the single relation*

$$\zeta^r - \rho_F^* c_1(F)\zeta^{r-1} + \ldots + (-1)^{r-1}\rho_F^* c_{r-1}(F)\zeta + (-1)^r \rho_F^* c_r(F) = 0.$$

Proof. — See [5], page 606. □

Moreover, if $\tilde{M} \to M$ is the blow-up of the manifold M along the submanifold S, $E = \mathbf{P}(N_{S/M})$ the exceptional divisor, then the normal bundle to E in $\tilde{M}$ is just the tautological bundle on $E \cong \mathbf{P}(N_{S/M})$. As a consequence, we see that restriction to E of the cohomology class $e = c_1([E])$ is

$$e|_E = c_1(N_{E/\tilde{M}}) = c_1(T) = \zeta,$$

and correspondingly, with the knowlegde of $H^*(E)$ and the restriction map $H^*(M) \to H^*(S)$, we may compute effectively in the cohomology ring of blow-up $\tilde{M}_S$. We note $c_1(N_{E/\tilde{M}})$ by $[E]$.

Example 2.2. — Let $\tilde{\mathbf{P}}^3 \xrightarrow{\pi} \mathbf{P}^3$ be the blow-up of $\mathbf{P}^3$ along a regular curve $\mathcal{C}$ which has genus g and degree d. From the Proposition 2.1,

$$\pi_E^* c_2(N_{\mathcal{C}/\mathbf{P}^3}) - \pi_E^* c_1(N_{\mathcal{C}/\mathbf{P}^3}) \cdot \zeta + \zeta^2 = 0.$$

As $\displaystyle\int_E \pi_E^* c_2(N_{\mathcal{C}/\mathbf{P}^3}) = \int_{\mathcal{C}} c_2(N_{\mathcal{C}/\mathbf{P}^3}) = 0$, and the restriction of ζ to each fiber of E is just the tautological bundle class of $\mathbf{P}^1$, results that $\int_E \zeta^2 = \int_E \pi_E^* c_1(N_{\mathcal{C}/\mathbf{P}^3}) \cdot \zeta = -\int_{\mathcal{C}} c_1(N_{\mathcal{C}/\mathbf{P}^3})$. From Whitney's formula, we have that

$$\int_E \zeta^2 = \int_{\mathcal{C}} [c_1(T\mathcal{C}) - c_1(T\mathbf{P}^3)] = 2 - 2g - 4d. \tag{2.4}$$

Chern class of a blow-up. — Our objective is to compare $c(T\tilde{M})$ with $\pi^*c(TM)$. Let $i : S \to M$, $j : E \to \tilde{M}$ be the inclusions. We write $N = N_{S/M}$ and $c(M), c(\tilde{M})$ and $c(S)$ for $c(TM)$, $c(T\tilde{M})$ and $c(TS)$ respectively. Then, we have that

THEOREM 2.3 (Porteous). — *With the above notation, and $\zeta = c_1(T)$, we have*

$$c(\tilde{M}) - \pi^*c(M) = j_*(\pi_E^*c(S) \cdot \alpha), \tag{2.5}$$

where

$$\alpha = \frac{1}{\zeta}\sum_{i=0}^{r}\left[1-(1-\zeta)(1+\zeta)^i\right]\pi_E^*c_{r-i}(N).$$

In this expression, the term in brackets is expanded as a polynomial in ζ, and α is the polynomial one obtains after formally dividing by ζ and r is the rank of N.

Proof. — The proof may be found in [7] or [3], page 298. □

Example 2.4. — In order to calculate the Chern class $c(\tilde{M})$ we have to compare the terms of (2.5) with same degree. Equating terms of degree one,

$$c_1(\tilde{M}) - \pi^*c_1(M) = j_*(1-r) = (1-r)[E]. \tag{2.6}$$

For terms of degree two and $r = 2$, then

$$c_2(\tilde{M}) - \pi^*c_2(M) = -j_*\pi_E^*c_1(S) - [E]\cdot[E] = \pi^*i_*[S] - \pi^*c_1(M)\cdot[E], \tag{2.7}$$

where $[S] \in H^4(M)$ is the class of S. The second part of (2.7) may be found in [3], page 114 or in [5], page 609.

For terms of degree three and $r = 2$, as $c_1(M)|_S = c_1(S) + c_1(N)|_E$, we have

$$c_3(\tilde{M}) - \pi^*c_3(M) = -\pi_E^*c_2(N)\cdot[E] - \pi_E^*c_1(M)\cdot[E]^2 + [E]^3. \tag{2.8}$$

Blowing-up curves of singularities of a foliation. — We will assume that M is a 3-dimensional manifold and $\mathcal{C} \subset M$ a regular curve. Let f be a holomorphic complex function on M vanishing along $\mathcal{C}$. By a holomorphic change of coordinates, this curve can be given locally as $z_1 = z_2 = 0$ and f can be written as:

$$f(z) = z_1 f_1(z_1, z_2, z_3) + z_2 f_2(z_1, z_2, z_3). \tag{2.9}$$

If f_1 and f_2 also vanish on the $z_3 - axis$, we can apply (2.9) again to all of them. Thus, the function f can be rewritten as

$$f(z) = z_1^2 f_{2,0}(z_1, z_2, z_3) + z_1 z_2 f_{1,1}(z_1, z_2, z_3) + z_2^2 f_{0,2}(z_1, z_2, z_3).$$

We will repeat this process, until we find some function $f_{i,j}$ which does not vanish on the $z_3 - axis$. Then, the function f will be of the form

$$f(z) = \sum_{i+j=m} z_1^i z_2^j f_{i,j}(z), \tag{2.10}$$

with $f_{i,j}(0,0,z_3) \not\equiv 0$ for some i, j and $z_1^i z_2^j f_{i,j}$ are linearly independent over $\mathbf{C}$.

Definition 2.5. — *The number m in (2.10) will be called the multiplicity of f along $\mathcal{C}$ and will be denoted by $mult_{\mathcal{C}}(f)$.*

Let $\mathcal{F}$ be a holomorphic foliation by curves on M and suppose that $\mathrm{Sing}(\mathcal{F})$ contains regular curves and possibly some isolated points. Assume that $\mathcal{C} \subseteq \mathrm{Sing}(\mathcal{F})$. Then, there exists an open set $U \subset M$ such that $U \cap \mathcal{C} \neq \emptyset$ and the $\mathcal{F}$ is given in U by the vector field

$$X(z) = P(z)\frac{\partial}{\partial z_1} + Q(z)\frac{\partial}{\partial z_2} + R(z)\frac{\partial}{\partial z_3}, \tag{2.11}$$

with P, Q and R vanishing along $\mathcal{C}$. Thus, we can write these functions as

$$\left\{ \begin{array}{lcl} P(z) & = & z_1^m P_0(z) + z_1^{m-1} z_2 P_1(z) + \ldots + z_2^m P_m(z), \\ Q(z) & = & z_1^n Q_0(z) + z_1^{n-1} z_2 Q_1(z) + \ldots + z_2^n Q_n(z), \\ R(z) & = & z_1^p R_0(z) + z_1^{p-1} z_2 R_1(z) + \ldots + z_2^p R_p(z), \end{array} \right. \tag{2.12}$$

with $m = \mathrm{mult}_{\mathcal{C}}(P)$, $\quad n = \mathrm{mult}_{\mathcal{C}}(Q)$ and $p = \mathrm{mult}_{\mathcal{C}}(R)$. By a linear change of variables, we may assume that $m \geqslant n$.

Definition 2.6. — *The multiplicity of $\mathcal{F}$ along $\mathcal{C}$, noted $mult_{\mathcal{C}}(\mathcal{F})$, will be the smallest of the numbers m, n, p.*

Proposition 2.7. — *Let $\mathcal{F}$ be a holomorphic foliation by curves on M with $\mathcal{C} \subseteq Sing(\mathcal{F})$ a regular curve. Then, $mult_{\mathcal{C}}(\mathcal{F})$ is independent of the coordinate system choosen.*

Proof. — Let us suppose that $\mathcal{F}$ is generated in an other coordinate system by the vector field

$$Y(z) = A(w)\frac{\partial}{\partial w_1} + B(w)\frac{\partial}{\partial w_2} + C(w)\frac{\partial}{\partial w_3}$$

with A, B and C vanishing along the w_3-axis. There is a biholomorphism $w = \Phi(z) = (\Phi_1(z), \Phi_2(z), \Phi_3(z))$ such that $X = \Phi^* Y$. Consequently, we have that

$$w_j = z_1\phi_{j1}(z) + z_2\phi_{j2}(z), \text{ for } j = 1, 2. \tag{2.13}$$

In particular,

$$\left[\phi_{11}(z)\phi_{22}(z) - \phi_{12}(z)\phi_{21}(z)\right]\frac{\partial \Phi_3(z)}{\partial z_3}\Bigg|_{z\,=\,(0,0,z_3)} \neq 0.$$

Given that $z_j = w_1\psi_{j1}(w) + w_2\psi_{j2}(w)$ too for $j = 1, 2$, we have that

$$P \circ \Psi(w) = \sum_{i=0}^{m} z_1^{m-i} z_2^{i} P_i(z)\Big|_{z=\Psi(w)} = \sum_{i=0}^{m} w_1^{m-i} w_2^{i} \tilde{P}_i(w), \tag{2.14}$$

with some $\tilde{P}_i(0,0,w_3) \not\equiv 0$. In fact, let us suppose that $\tilde{P}_i(0,0,w_3) \equiv 0$, for all i. From (2.13), if we rewrite the right side of (2.14) in terms of the variable z, we will obtain $P_i(0,0,z_3) \equiv 0$, for $i = 0, \ldots, m$. An absurd, because $\mathrm{mult}_{\mathcal{C}}(P) = m$. From (2.13), follows that

$$Y(w) = \begin{cases} \dot{w}_1 &= [\phi_{11}\circ\Psi(w) + \eta_{11}(w)]P\circ\Psi(w) + [\phi_{21}\circ\Psi(w) \\ &\quad + \eta_{12}(w)]Q\circ\Psi(w) + \eta_{13}(w)R\circ\Psi(w) \\[2ex] \dot{w}_2 &= [\phi_{21}\circ\Psi(w) + \eta_{21}(w)]P\circ\Psi(w) + [\phi_{22}\circ\Psi(w) \\ &\quad + \eta_{22}(w)]Q\circ\Psi(w) + \eta_{23}(w)R\circ\Psi(w) \\[2ex] \dot{w}_3 &= \dfrac{\partial\Phi_3}{\partial z_1}\circ\Psi(w)P\circ\Psi(w) + \dfrac{\partial\Phi_3}{\partial z_2}\circ\Psi(w)Q\circ\Psi(w) \\ &\quad + \dfrac{\partial\Phi_3}{\partial z_3}\circ\Psi(w)R\circ\Psi(w). \end{cases}$$

with $\eta_{ij}(0,0,w_3) \equiv 0$ for all i, j, that is, $\mathrm{mult}_{\mathcal{C}}(\eta_{ij}) \geqslant 1$. As before, $m \geqslant n$, consequently, $\mathrm{mult}_{\mathcal{C}}(\mathcal{F})$ will be n or p. Firstly, we will assume that $p < n$. Because $\partial\Phi_3/\partial z_3 \circ \Psi(0,0,w_3) \neq 0$, the third component of Y has multiplicity equal to p along axis-w_3, while the other components have multipliciy at least $p+1$. Therefore, we have that $\mathrm{mult}_{\mathcal{C}}(Y) = p$.

Now, let us suppose that $n \leqslant p$. The third component of Y has multiplicity at least equal to n along the w_3-axis. Because $\eta_{i3}(w) R \circ \Psi(w)$ has multiplicity at least one, in order to complete the proof, it is enough to verify that one of these functions $M(w) = [\phi_{11} P + \phi_{12} Q] \circ \Psi(w)$ and $N(w) = [\phi_{21} P + \phi_{22} Q] \circ \Psi(w)$ has multiplicity n along $\mathcal{C}$. In fact, as $[\phi_{11}\phi_{22} - \phi_{12}\phi_{21}](0,0,z_3) \neq 0$, we have that

$$P = \frac{M\phi_{22} - N\phi_{12}}{\phi_{11}\phi_{22} - \phi_{21}\phi_{21}} \text{ and } Q = \frac{N\phi_{11} - M\phi_{21}}{\phi_{11}\phi_{22} - \phi_{21}\phi_{21}}.$$

But, if the multiplicity of M and N is greater than n, the same will happen for P and Q. Then, $\operatorname{mult}_{\mathcal{C}}(Y) = n$. □

A bimeromorphic transformation $\phi : N \to M$ is given by a biholomorphism $\Phi|_{N-\Sigma} : N - \Sigma \to M - \Gamma$, which Σ and Γ are analytic subsets. Let $\mathcal{F}$ be as before, on M, with $\mathcal{C} \subset \operatorname{Sing}(\mathcal{F})$ a regular curve. Let us suppose that $\mathcal{C}$ is not contained in Γ. We may define a holomorphic foliation in N called the pullback of $\mathcal{F}$ and denoted by $\mathcal{G} = \Phi^*\mathcal{F}$. This new foliation is also singular along the curve $\mathcal{C}_1 = \Phi^{-1}(\mathcal{C} \setminus \Gamma)$. We will show that $\operatorname{mult}_{\mathcal{C}_1}(\mathcal{G}) = \operatorname{mult}_{\mathcal{C}}(\mathcal{F})$. That is, the multiplicity is a bimeromorphic invariant whenever that $\mathcal{C} \not\subset \Gamma$.

THEOREM 2.8. — *Let $\mathcal{F}$ be a holomorphic foliation by curves on M and $\mathcal{C} \subset Sing(\mathcal{F})$ a regular curve. Consider the bimeromorphism $\Phi : N \to M$ such that $\Phi|_{N-\Sigma} : N - \Sigma \to M - \Gamma$ is a biholomorphism, with $\mathcal{C} \not\subset \Gamma$. Then, $mult_{\mathcal{C}_1}(\mathcal{G}) = mult_{\mathcal{C}}(\mathcal{F})$, where $\mathcal{G} = \Phi^*\mathcal{F}$ and $\mathcal{C}_1 = \Phi^{-1}(\mathcal{C} \setminus \Gamma)$.*

Proof. — Let $\{U_\alpha\}$ be an open cover of M. Shrinking each U_α, if necessary, we may assume that $\mathcal{C} \cap U_\alpha$, non-empty, is given by $z_{\alpha 1} = z_{\alpha 2} = 0$ and $\mathcal{F}$ generated by a holomorphic vector field $X_\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, with P_α, Q_α and R_α as before. If $\mathcal{C} \cap U_\alpha \cap U_\beta \neq \emptyset$ then $X_\alpha = f_{\alpha\beta} X_\beta$, with $f_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. As $\mathcal{C} \not\subset \Gamma$ and $\Phi^{-1}|_{U_\alpha \setminus \Gamma \cap \mathcal{C}} : U_\alpha \setminus \Gamma \cap \mathcal{C} \to \Phi^{-1}(U_\alpha \setminus \Gamma \cap \mathcal{C})$ is a biholomorphism, the vector field Y_α that generates the foliation $\mathcal{G}$ in $\Phi^{-1}(U_\alpha \setminus \Gamma \cap \mathcal{C})$ is analytically conjugated to X_α. As the multiplicity of a foliation along a curve of singularities is independent of coordinate system choosen, X_α and Y_α have the same multiplicity. Given that $X_\alpha = f_{\alpha\beta} X_\beta$, with $f_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, X_α and X_β have the same multiplicity too. Therefore, $\operatorname{mult}_{\mathcal{C}_1}(\mathcal{G}) = \operatorname{mult}_{\mathcal{C}}(\mathcal{F})$. □

Now, we blow-up M along $\mathcal{C}$ and describe the behavior of $\mathcal{F}$ under this transformation. Let $\mathcal{F}$ generated by vector a vector field as in (2.11). In an open set in $\tilde{U}_1$, as in (2.1), we have

$$\sigma(\varsigma) = (\varsigma_1, \varsigma_1\varsigma_2, \varsigma_3) = (z_1, z_2, z_3).$$

Then, given that $z_1 = \varsigma_1$ and $z_2 = \varsigma_1\varsigma_2$, we have that

$$\dot{\varsigma}_1 = \sum_{i=0}^{m} (\varsigma_1)^{m-i}(\varsigma_1\varsigma_2)^i P_i(\varsigma_1, \varsigma_1\varsigma_2, \varsigma_3) = \varsigma_1^m \sum_{i=0}^{m} \varsigma_2^i P_i(\varsigma_1, \varsigma_1\varsigma_2, \varsigma_3).$$

But, $P_i(\varsigma_1, \varsigma_1\varsigma_2, \varsigma_3) = P_i(0, 0, \varsigma_3) + \varsigma_1 \tilde{P}_i(\varsigma_1, \varsigma_2, \varsigma_3) = p_i(\varsigma_3) + \varsigma_1 \tilde{P}_i(\varsigma)$. Thus, we obtain that

$$\dot{\varsigma}_1 = \varsigma_1^m \left[\sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) + \varsigma_1 P_1(\varsigma) \right].$$

with $P_1(\varsigma) = \sum_{i=0}^{m} \varsigma_2^i \tilde{P}_i(\varsigma)$. In the same way, we obtain that

$$\dot{\varsigma}_3 = \varsigma_1^p \left[\sum_{i=0}^{p} \varsigma_2^i r_i(\varsigma_3) + \varsigma_1 R_1(\varsigma) \right].$$

Finally, from $z_2 = \varsigma_1\varsigma_2$, we have that $\dot{z}_2 = \dot{\varsigma}_1\varsigma_2 + \varsigma_1\dot{\varsigma}_2$. Then

$$\varsigma_1^n \left[\sum_{i=0}^{n} \varsigma_2^i q_i(\varsigma_3) + \varsigma_1 \tilde{Q}_1(\varsigma) \right] = \varsigma_2\varsigma_1^m \left[\sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) + \varsigma_1 P_1(\varsigma) \right] + \varsigma_1\dot{\varsigma}_2,$$

thus we obtain

$$\dot{\varsigma}_2 = \varsigma_1^{n-1} \left[\sum_{i=0}^{n} \varsigma_2^i q_i(\varsigma_3) - \varsigma_1^{m-n}\varsigma_2 \sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) + \varsigma_1(\tilde{Q}(\varsigma) - \varsigma_1^{m-n}\varsigma_2 P_1(\varsigma)) \right].$$

The following are equations for $\pi^*(\mathcal{F})$

$$\left\{ \begin{array}{lcl} \dot{\varsigma}_1 & = & \varsigma_1^m \left[\displaystyle\sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) + \varsigma_1 P_1(\varsigma) \right] \\ \dot{\varsigma}_2 & = & \varsigma_1^{n-1} \left[\displaystyle\sum_{i=0}^{n} \varsigma_2^i q_i(\varsigma_3) - \varsigma_1^{m-n}\varsigma_2 \sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) + \varsigma_1 Q_1(\varsigma) \right] \\ \dot{\varsigma}_3 & = & \varsigma_1^p \left[\displaystyle\sum_{i=0}^{p} \varsigma_2^i r_i(\varsigma_3) + \varsigma_1 R_1(\varsigma) \right] \end{array} \right. \tag{2.15}$$

with $Q_1(\varsigma) = \tilde{Q}(\varsigma) - \varsigma_1^{m-n}\varsigma_2 P_1(\varsigma)$. Now, all points of E given by $\varsigma_1 = 0$ are singularities of $\pi^*(\mathcal{F})$. We have some ways of desingularizing it, according to the possible values of m, n and p. And if $n = m$ we must verify whether $\sum_{i=0}^{n} \varsigma_2^i (q_i(\varsigma_3) - \varsigma_2 p_i(\varsigma_3))$ is identically zero or not. Thus, we may divide it in two cases, dicritical or non-dicrital curves of singularities, according to fact that the exceptional divisor is, or is not, invariant by the induced foliation $\tilde{\mathcal{F}}$.

(a) Non-dicritical curve of singularities.

(i) If $p+1=n<m-1$ or $p+1=n=m$ and $\sum_{i=0}^{n} \varsigma_2^i[q_i(\varsigma_3)-\varsigma_2 p_i(\varsigma_3)]$ is not identically zero. Dividing (2.15) by ς_1^p we get

$$\left\{\begin{array}{lcl} \dot{\varsigma}_1 & = & \varsigma_1^{m-p}\left[\sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3)+\varsigma_1 P_1(\varsigma)\right] \\ \dot{\varsigma}_2 & = & \sum_{i=0}^{n} \varsigma_2^i q_i(\varsigma_3)-\varsigma_1^{m-n}\varsigma_2 \sum_{i=0}^{m} \varsigma_2 p_i(\varsigma_3))+\varsigma_1 Q_1(\varsigma) \\ \dot{\varsigma}_3 & = & \sum_{i=0}^{m-1} \varsigma_2^i r_i(\varsigma_3)+\varsigma_1 R_1(\varsigma) \end{array}\right. \tag{2.16}$$

The expression in the other coordinate system (after dividing by ς_2^p) fits with (2.16) to define a foliation $\tilde{\mathcal{F}}$ in $\tilde{U}_1$ having the exceptional divisor as an invariant set. More precisely, the singularities on E are given by the roots of

$$\sum_{i=0}^{m} \varsigma_2^i[q_i(\varsigma_3)-\varsigma_2 p_i(\varsigma_3)]=0 \quad \text{and} \quad \sum_{i=0}^{p} \varsigma_2^i r_i(\varsigma_3)=0$$

if $n=m$ or

$$\sum_{i=0}^{m} \varsigma_2^i q_i(\varsigma_3)=0 \qquad \text{and} \qquad \sum_{i=0}^{p} \varsigma_2^i r_i(\varsigma_3)=0$$

if $n<m$, E is an invariant set of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ and $\pi^*(\mathcal{F})$ coincide outside E.

(ii) If $p+1<n \leqslant m$, dividing (2.15) by ς_1^p, we get

$$\left\{\begin{array}{lcl} \dot{\varsigma}_1 & = & \varsigma_1^{m-p}\left[\sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3)+\varsigma_1 P_1(\varsigma)\right] \\ \dot{\varsigma}_2 & = & \varsigma_1^{l}\left[\sum_{i=0}^{n} \varsigma_2^i q_i(\varsigma_3)-\varsigma_2\varsigma_1^{m-n} \sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3)+\varsigma_1 Q_1(\varsigma)\right] \\ \dot{\varsigma}_3 & = & \sum_{i=0}^{p} \varsigma_2^i r_i(\varsigma_3)+\varsigma_1 R_1(\varsigma) \end{array}\right. \tag{2.17}$$

with $l \geqslant 1$. In this situation, the exceptional divisor is also invariant by the foliation, but the restriction of the foliation to it is given by $\varsigma_2=\beta$, β a constant.

(iii) If $n \leqslant p < m$ or $n < m \leqslant p$ or $n = m \leqslant p$ and $\sum_{i=0}^{m} \varsigma_2^i [q_i(\varsigma_3) - \varsigma_2 p_i(\varsigma_3)]$ is not identically zero. Dividing (2.15) by ς_1^{n-1}, we get

$$\begin{cases} \dot{\varsigma}_1 &= \varsigma_1^{m-n+1} \left[\sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) + \varsigma_1 P_1(\varsigma) \right] \\ \dot{\varsigma}_2 &= \sum_{i=0}^{n} \varsigma_2^i q_i(\varsigma_3) - \varsigma_1^{m-n} \varsigma_2 \sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) + \varsigma_1 Q_1(\varsigma) \\ \dot{\varsigma}_3 &= \varsigma_1^l \sum_{i=0}^{n} \varsigma_2^i r_i(\varsigma_3) + \varsigma_1 R_1(\varsigma) \end{cases} \tag{2.18}$$

with $l \geqslant 1$. The exceptional divisor is invariant by the foliation $\tilde{\mathcal{F}}$, but now the restriction of this foliation to it is given by $\varsigma_3 = \beta$, β a constant.

Remark. — If $\mathcal{F}$ is special along a regular curve then this condition (i) must be satisfied, because in the other two cases, new curves of singularities will appear on E.

(b) Dicritical curve of singularities:

(i) If $p = n = m$ and $\sum_{i=0}^{m} \varsigma_2^i [q_i(\varsigma_3) - \varsigma_2 p_i(\varsigma_3)]$ is identically zero . Dividing (2.15) by ς_1^m we get

$$\begin{cases} \dot{\varsigma}_1 &= \sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) + \varsigma_1 P_1(\varsigma) \\ \dot{\varsigma}_2 &= Q_1(\varsigma_1, \varsigma_2, \varsigma_3) \\ \dot{\varsigma}_3 &= \sum_{i=0}^{m} \varsigma_2^i r_i(\varsigma_3) + \varsigma_1 R_1(\varsigma) \end{cases} \tag{2.19}$$

Combining this with the corresponding expression in the other coordinate systems, we get defining equations for a foliation $\tilde{\mathcal{F}}$ which coincides with $\pi^*(\mathcal{F})$ outside E but this time the exceptional divisor is no longer invariant. The foliation $\tilde{\mathcal{F}}$ is transverse to E except at the hypersurface locally given by $\sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) = 0$, which may or may not consist of singularities of $\tilde{\mathcal{F}}$.

(ii) If $n = m < p$ and $\sum_{i=0}^{n} \varsigma_2 [q_i(\varsigma_3) - \varsigma_2 p_i(\varsigma_3)]$ is identically zero. Dividing

(2.15) by ς_1^m, we get

$$\begin{cases} \dot{\varsigma}_1 &= \sum_{i=0}^{m} \varsigma_2^i p_i(\varsigma_3) + \varsigma_1 P_1(\varsigma) \\ \dot{\varsigma}_2 &= Q_1(\varsigma_1, \varsigma_2, \varsigma_3) \\ \dot{\varsigma}_3 &= \varsigma_1^l \big[\sum_{i=0}^{m} \varsigma_2^i r_i(\varsigma_3) + \varsigma_1 R_1(\varsigma) \big] \end{cases} \tag{2.20}$$

where $l \geqslant 1$. The exceptional divisor is not invariant by the foliation, but, on it, the third component of the vector field vanishes.

From (2.15) we have the following definition:

Definition 2.9. — *The order of tangency of $\pi^*\mathcal{F}$, denoted by $tang(\pi^*\mathcal{F}, E)$, is*

$$tang(\pi^*(\mathcal{F}), E) = \begin{cases} min\{m, n-1, p\}, & \textit{if } \mathcal{C} \textit{ is non dicritical} \\ min\{m, n, p\}, & \textit{if } \mathcal{C} \textit{ is dicritical} \end{cases} \tag{2.21}$$

Observe that if $\mathcal{F}$ is special along $\mathcal{C}$ then $\mathrm{mult}_{\mathcal{C}}(\mathcal{F}) = \mathrm{tang}(\pi^*\mathcal{F}, E)$.

3. Special foliations

In this section, unless said otherwise, $\mathcal{F}$ will be a holomorphic foliation by curves on $\mathbf{P}^3$, special along the compact, smooth and disjoint curves $\mathcal{C}_j$ for $j = 1, \ldots, r$. We write

$$\mathrm{Sing}(\mathcal{F}) = \cup_{j=1}^{r} \mathcal{C}_j \cup \{p_1, \ldots, p_q\}, \tag{3.1}$$

where p_j are isolated points. Our objective is to calculate $n_{\mathcal{F}} = \sum_{j=1}^{q} \mu(\mathcal{F}, p_j)$, the number of isolated singularities, counted with multiplicities, of $\mathcal{F}$. We assume that $r = 1$, that is, $\mathrm{Sing}(\mathcal{F})$ has only one one-dimensional component, noted $\mathcal{C}$. The case where $r > 1$ will follow without difficulty.

In order to reach this goal, we blow-up $\mathbf{P}^3$ along $\mathcal{C}$. In this manner, we will obtain a foliation $\tilde{\mathcal{F}}$ on $\tilde{\mathbf{P}}^3$ which has only isolated singularities as well as the exceptional divisor E as an invariant set. Thus, using Baum-Bott's formula and Porteous'theorem we can calculate the number $n_{\mathcal{F}}$ which is a difference between the total number of singularities of $\tilde{\mathcal{F}}$ in $\tilde{P}^3$ and in E because the blow-up is an isomorphism away from the E.

In order to use the Baum-Bott's formula, we must calculate the Chern class of tangent bundle of the foliation $T_{\tilde{\mathcal{F}}}$. From [1], it follows that

$$T_{\tilde{\mathcal{F}}} \cong \pi^*(T_{\mathcal{F}}) \otimes [E]^{\ell}.$$

Therefore, in order to know $T_{\tilde{\mathcal{F}}}$ is enough to calculate the number ℓ. With this notation, we have that

$$c_1(T_{\tilde{\mathcal{F}}}) = \pi^* c_1(T_{\mathcal{F}}) + \ell[E], \tag{3.2}$$

where $\ell = \mathrm{tang}(\pi^*\mathcal{F}, E)$.

Theorem 3.1. — *Let $\mathcal{F}$ be a holomorphic foliation by curves on $\mathbf{P}^3$, special along some regular curve $\mathcal{C}$ of genus g and degree d. Consider $\tilde{\mathbf{P}}^3 \xrightarrow{\pi} \mathbf{P}^3$ the blow-up centered at $\mathcal{C}$ with E the exceptional divisor. Then*

$$\sum_{q \in Sing(\mathcal{F}_1)} \mu(\mathcal{F}_1, q) = (2-2g)(\ell^2 + 2\ell + 2) + 2d(\ell+1)(k - 2\ell - 1),$$

where $\mathcal{F}_1 = \tilde{\mathcal{F}}|_E$, $k = degree(\mathcal{F})$ and $\ell = tang(\pi^*\mathcal{F}, E)$.

Proof. — By Baum-Bott's formula, we have that

$$\sum_{q \in \mathrm{Sing}(\mathcal{F}_1)} \mu(\mathcal{F}_1, q) = \int_E c_2(TE \otimes T^*_{\tilde{\mathcal{F}}}),$$

with

$$c_2(TE \otimes T^*_{\tilde{\mathcal{F}}}) = c_2(TE) + c_1(TE) \cdot c_1(T^*_{\tilde{\mathcal{F}}}) + c_1^2(T^*_{\tilde{\mathcal{F}}}).$$

From Whitney and (2.6), it follows that

$$c_1(TE) = (c_1(\tilde{\mathbf{P}}^3) - [E])\big|_E = (\pi^* c_1(\mathbf{P}^3) - 2[E])\big|_E.$$

As $c_1(T^*_{\tilde{\mathcal{F}}}) = \pi^* c_1(T^*_{\mathcal{F}}) - \ell[E]$, $\int_E \pi^* c_1(\mathbf{P}^3) \cdot \pi^* c_1(T^*_{\mathcal{F}}) = \int_E \pi^* c_1^2(T^*_{\mathcal{F}}) = 0$ and $\int_E \pi^*[H] \cdot [E] = -\int_{\mathcal{C}} [H] = -d$, from the example 2.2, it follows that

$$\begin{aligned}
\int_E c_2(TE \otimes T^*_{\tilde{\mathcal{F}}}) &= \int_E \Big[c_2(TE) - [\ell \pi^* c_1(\mathbf{P}^3) + 2(1+\ell)\pi^* c_1(T^*_{\mathcal{F}})] \cdot [E] \\
&\qquad + (2\ell + \ell^2)[E]^2\Big] \\
&= 2(2-2g) + \int_{\mathcal{C}} [\ell c_1(\mathbf{P}^3) + 2(\ell+1) c_1(T^*_{\mathcal{F}})] \\
&\qquad + (2\ell + \ell^2) \int_E [E]^2.
\end{aligned}$$

Therefore,

$$\sum_{q \in \mathrm{Sing}(\mathcal{F}_1)} \mu(\mathcal{F}_1, q) = 2(2-2g)+4\ell d+2(1+\ell)(k-1)d+(2\ell+\ell^2)(2-2g-4d).$$

Regrouping, we obtain the theorem. $\square$

Example 3.2. — Let $\mathcal{F}_k$ be a holomorphic foliation by curves on $\mathbf{P}^3$ with degree$(\mathcal{F}_k) = k \geqslant 2$, induced on the affine open set $V_3 = \{[\xi_0 : \xi_1 : \xi_2 : \xi_3] \in \mathbf{P}^3 | \xi_3 \neq 0\}$ by the vector field

$$X_k(z) = \begin{cases} \dot{z}_1 &= a_0 z_1^k + a_1 z_1^{k-1} z_2 + \ldots + a_{k-1} z_1 z_2^{k-1} + a_k z_2^k \\ \dot{z}_2 &= b_0 z_1^k + b_1 z_1^{k-1} z_2 + \ldots + b_{k-1} z_1 z_2^{k-1} + b_k z_2^k \\ \dot{z}_3 &= z_1^{k-1} R_0(z) + z_1^{k-2} z_2 R_1(z) \ldots + z_2^{k-1} R_{k-1}(z), \end{cases} \tag{3.3}$$

with $z_1 = \xi_0/\xi_3, z_2 = \xi_1/\xi_3, z_3 = \xi_2/\xi_3$, $\sum_{i=0}^{k} a_i z_1^{k-i} z_2^i$ and $\sum_{i=0}^{k} b_i z_1^{k-i} z_2^i$ linearly independent over $\mathbf{C}$ and $R_i(z) = \alpha_i + \beta_i z_1 + \gamma_i z_2 + \delta_i z_3$ for $i = 0, \ldots, k-1$.

The curve defined by $\xi_0 = \xi_1 = 0$ is a curve of singularities of $\mathcal{F}_k$. We blow-up $\mathbf{P}^3$ along this curve. In the open set $\tilde{U}_1$ with coordinates $\varsigma \in \mathbf{C}^3$, we have the relations

$$\sigma_1(\varsigma_1, \varsigma_2, \varsigma_3) = (\varsigma_1, \varsigma_1\varsigma_2, \varsigma_3) = (z_1, z_2, z_3).$$

Because $m = n = p+1 = k$ we have that $\ell = \mathrm{tang}(\pi^*\mathcal{F}, E) = k-1$. In this way, the foliation $\tilde{\mathcal{F}}_k$ induced by $\mathcal{F}_k$ via π is generated in $\tilde{V}_3$ by the vector field

$$\tilde{X}_k(z) = \begin{cases} \dot{\varsigma}_1 &= \varsigma_1(a_0 + a_1\varsigma_2 + \ldots + a_k\varsigma_2^k) \\ \dot{\varsigma}_2 &= b_0 + b_1\varsigma_2 + \ldots + b_k\varsigma_2^k - \varsigma_2(a_0 + a_1\varsigma_2 + \ldots + a_k\varsigma_2^k) \\ \dot{\varsigma}_3 &= \alpha_0 + \alpha_1\varsigma_2 + \ldots + \alpha_{k-1}\varsigma_2^{k-1} + \varsigma_3(\delta_0 + \delta_1\varsigma_2 + \ldots \\ & \quad +\delta_{k-1}\varsigma_2^{k-1}) + \varsigma_1 R(\varsigma) \end{cases} \tag{3.4}$$

for some polynomial R. It is not hard to see that on the affine open set, $\varsigma_3 \in \mathbf{C}$, the foliation $\tilde{\mathcal{F}}_k$, when restricted on the exceptional divisor, has $k+1$ singularities, counted with multiplicities. But, at fiber the $\pi^{-1}([0:0:1:0])$ the foliation $\tilde{\mathcal{F}}_k$ has $k+1$ additional singularities. Therefore, $\tilde{\mathcal{F}}_k$ has $2k+2$ singularities on E.

Theorem 3.3. — *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbf{P}^3$, special along a regular curve $\mathcal{C}$ of genus g and degree d. Moreover, suppose that $\mathcal{C}$ is the unique one-dimensional irreducible component of Sing($\mathcal{F}$). Consider $\tilde{\mathbf{P}}^3 \xrightarrow{\pi} \mathbf{P}^3$, the blow-up centered at $\mathcal{C}$ and $\tilde{\mathcal{F}}$ the foliation induced by $\mathcal{F}$ via π. Then,*

$$\sum_{q \in Sing(\tilde{\mathcal{F}})} \mu(\tilde{\mathcal{F}}, q) = 1 + k + k^2 + k^3 - d(k-1)(3\ell^2 + 2\ell - 1) - (2-2g)(\ell^3 + \ell^2 - 1) + 4\ell d(\ell^2 - 1),$$

where degree($\mathcal{F}$) $= k$ and $\ell = tang(\pi^*\mathcal{F}, E)$.

Proof. — By Baum-Bott's formula, we have that

$$\sum_{q \in \mathrm{Sing}(\tilde{\mathcal{F}})} \mu(\tilde{\mathcal{F}}, q) = \int_{\tilde{\mathbf{P}}^3} c_3(T\tilde{\mathbf{P}}^3 \otimes T^*_{\tilde{\mathcal{F}}}),$$

with

$$c_3(T\tilde{P}^3 \otimes T^*_{\tilde{\mathcal{F}}}) = c_3(T\tilde{\mathbf{P}}^3) + c_2(T\tilde{\mathbf{P}}^3)c_1(T^*_{\tilde{\mathcal{F}}}) + c_1(T\tilde{\mathbf{P}}^3)c_1^2(T^*_{\tilde{\mathcal{F}}}) + c_1^3(T^*_{\tilde{\mathcal{F}}}).$$

Let us calculate separately each term of the above expression. Writing $c_i(\mathbf{P}^3)$ for $c_i(T\mathbf{P}^3)$, from (2.8) we obtain that

$$\int_{\tilde{\mathbf{P}}^3} c_3(T\tilde{\mathbf{P}}^3) = \int_{\tilde{\mathbf{P}}^3} \Big[\pi^* c_3(\mathbf{P}^3) - \pi^* c_2(N)\cdot[E] - \pi^* c_1(\mathbf{P}^3)\cdot[E]^2 + [E]^3\Big],$$

where $N = N_{\mathcal{C}/\mathbf{P}^3}$ is the normal bundle of $\mathcal{C}$ in $\mathbf{P}^3$. Therefore,

$$\int_{\tilde{\mathbf{P}}^3} c_3(T\tilde{\mathbf{P}}^3) = \int_{\mathbf{P}^3} c_3(\mathbf{P}^3) + \int_E \Big[-\pi^* c_2(N) - \pi^* c_1(\mathbf{P}^3)\cdot[E] + [E]^2\Big],$$

because $[E]$ is Poincaré dual of E in $\tilde{\mathbf{P}}^3$. As $\int_E \pi^* c_2(N) = \int_{\mathcal{C}} c_2(N) = 0$ and $\int_E [E]^2 = 2 - 2g - 4d$, example (2.2), follows that

$$\int_{\tilde{\mathbf{P}}^3} c_3(T\tilde{\mathbf{P}}^3) = 4 + 4d + 2 - 2g - 4d = 4 + (2-2g). \tag{3.5}$$

From (2.7) and (3.2) we obtain that

$$c_2(T\tilde{\mathbf{P}}^3)c_1(T^*_{\tilde{\mathcal{F}}}) = \Big[\pi^* c_2(\mathbf{P}^3) + \pi^*[\mathcal{C}] - \pi^* c_1(\mathbf{P}^3)\cdot[E]\Big]\Big[\pi^* c_1(T^*_{\mathcal{F}}) - \ell[E]\Big].$$

As in the previous calculation,

$$\int_{\tilde{\mathbf{P}}^3} c_2(T\tilde{\mathbf{P}}^3)c_1(T^*_{\tilde{\mathcal{F}}}) = \int_{\mathbf{P}^3} c_2(\mathbf{P}^3)c_1(T^*_{\mathcal{F}}) + \int_{\mathcal{C}} c_1(T^*_{\mathcal{F}}) - \ell \int_{\mathcal{C}} c_1(\mathbf{P}^3).$$

Therefore, we conclude that

$$\int_{\tilde{\mathbf{P}}^3} c_2(T\tilde{\mathbf{P}}^3)c_1(T^*_{\tilde{\mathcal{F}}}) = 6(k-1) + (k-1)d - 4\ell d. \tag{3.6}$$

From (2.6) and (3.2) follows that

$$c_1(T\tilde{\mathbf{P}}^3)c_1^2(T^*_{\tilde{\mathcal{F}}}) = \Big[\pi^* c_1(\mathbf{P}^3) - [E]\Big]\Big[\pi^* c_1^2(T^*_{\mathcal{F}}) - 2\ell\pi^* c_1(T^*_{\mathcal{F}}) \cdot [E] + \ell^2 [E]^2\Big].$$

In the same way,

$$\int_{\tilde{\mathbf{P}}^3} c_1(T\tilde{\mathbf{P}}^3)c_1^2(T^*_{\tilde{\mathcal{F}}}) = \int_{\mathbf{P}^3} c_1(\mathbf{P}^3)c_1(T^*_{\mathcal{F}}) - \int_{\mathcal{C}} [\ell^2 c_1(\mathbf{P}^3) + 2\ell c_1(T^*_{\mathcal{F}})] - \ell^2 \int_E [E]^2.$$

Thus, we obtain that

$$\int_{\tilde{\mathbf{P}}^3} c_1(\tilde{\mathbf{P}}^3)c_1^2(T^*_{\tilde{\mathcal{F}}}) = 4(k-1)^2 - \ell^2(2-2g) - 2\ell(k-1)d. \tag{3.7}$$

As $\int_E \pi^* c_1^2(T_{\tilde{\mathcal{F}}}) \cdot [E] = 0$, from (3.2), we have that

$$\int_{\tilde{\mathbf{P}}^3} c_1^3(T^*_{\tilde{\mathcal{F}}}) = \int_{\mathbf{P}^3} c_1^3(T^*_{\mathcal{F}}) - 3\ell^2 \int_{\mathcal{C}} c_1(T^*_{\mathcal{F}}) - \ell^3 \int_E [E]^2.$$

Finally,

$$\int_{\tilde{\mathbf{P}}^3} c_1^3(T^*_{\tilde{\mathcal{F}}}) = (k-1)^3 - 3\ell^2(k-1)d - \ell^3(2-2g-4d). \tag{3.8}$$

With the equations (3.5), (3.6), (3.7) and (3.8) added and regrouped, we conclude the proof of the theorem. □

As a direct consequence of the Theorems 3.1 and 3.3 we can effectively calculate $n_{\mathcal{F}}$, that is, the proof of the Theorem 1.1.

Example 3.4. — Let $\mathcal{F}_k$ as in the example (3.2). The foliation $\mathcal{F}_k$ has no singularity in $V_3 = \{[\xi_j] \in \mathbf{P}^3 | \xi_3 \neq 0\}$ moreover $\mathcal{C} \cap V_3$, which $\mathcal{C}$ is given by $\xi_0 = \xi_1 = 0$.

Let $H_3 = \mathbf{P}^3 \setminus V_3$ be the infinity hyperplane. This hyperplane is isomorphic to $\mathbf{P}^2$ as well as is invariant by $\mathcal{F}_k$. As degree$(\mathcal{F}_k\big|_{H_3}) = k$ too, the number of isolated singularities, counted with multiplicities, of $\mathcal{F}_k$ on H_3 is $1 + k + k^2$. Given that the singularity $q = [0 : 0 : 1 : 0] \in \mathcal{C}$ has Milnor number $\mu(\mathcal{F}_k\big|_{H_3}, q) = k^2$, $\mathcal{F}_k$ has $k+1$ singularities isolated on $\mathbf{P}^3$, counted with multiplicities.

The Theorem 1.1 may be generalized for special foliation along disjoint curves.

Theorem 3.5. — *Let $\mathcal{F}_0$ be a holomorphic foliation by curves on $\mathbf{P}^3$ with degree k. Suppose that $\mathcal{C}_i^0 \subset Sing(\mathcal{F})$ are regular and disjoint curves with genus g_i and degree d_i for $i = 1, \ldots, r$. If $\mathcal{F}_0$ is special along each curve $\mathcal{C}_i$ then its number of isolated singularities, counted the multiplicities, will be*

$$\sum_{i=0}^{3} k^i + \sum_{i=1}^{r} (\ell_i + 1)\Big[(2g_i - 2)(\ell_i^2 + \ell_i + 1) + 4d_i\ell_i^2 - d_i(k-1)(3\ell_i + 1)\Big]$$

where $\ell_i = mult_{\mathcal{C}_i^0}(\mathcal{F}_0)$.

Proof. — Let $M_0 = \mathbf{P}^3$ and $\{\pi_i\}$ be a sequence of blow-up $\pi_i : M_i \to M_{i-1}$ centered at $\mathcal{C}_i^{i-1}$ which $\mathcal{C}_j^i = \pi_i^{-1}(\mathcal{C}_j^{i-1})$ for $j = i+1, \ldots, r$ and $E_i = \pi_i^{-1}(\mathcal{C}_i^{i-1})$ be the exceptional divisor of each blow-up. Apply successively the example (2.4), we obtain the Chern class of $c_j(TM_r)$. In the same way, we obtain $c_1(T_{\mathcal{F}_r})$. We can assume that $E_i \cdot E_j = 0$ if $i \neq j$ because the curves $\mathcal{C}_j$ are disjoint. Using Baum-Bott's formula, the proof follows like in Theorem 3.3. $\square$

We show that $n_{\mathcal{F}} = \displaystyle\sum_{j=1}^{q} \mu(\mathcal{F}, p_j) > 0$ when Sing$(\mathcal{F})$ has a unique regular curve $\mathcal{C}$ which is also a complete intersection of surfaces. Let f_1, f_2 be two polynomials defined an affine open set of $\mathbf{P}^3$ such that $\mathcal{C} = f_1^{-1}(0) \cap f_2^{-1}(0)$ with $d_j = $ degree(f_j) for $j = 1, 2$. Therefore, the degree of $\mathcal{C}$ is $d = d_1d_2$ while its genus is $g = 1 + d_1d_2(d_1 + d_2 - 4)/2$, see [6]. As $\mathcal{C}$ is a regular curve, we have $df_1 \wedge df_2 \neq 0$ along $\mathcal{C}$. Thus, given an open set U such that $U \cap \mathcal{C} \neq \emptyset$, we may assume that $\dfrac{\partial f_1}{\partial z_1}\dfrac{\partial f_2}{\partial z_2} - \dfrac{\partial f_1}{\partial z_2}\dfrac{\partial f_2}{\partial z_1} \neq 0$ for $z \in U$. Let $F : U \to V \subset \mathbf{C}^3$, defined by $F(z) = (f_1(z), f_2(z), z_3)$, be local biholomorphism and $G = (g_1(w), g_2(w), w_3)$ its inverse biholomorphism. Notice the image of $\mathcal{C}$ by F is the w_3-axis. Consider $\mathcal{F}$ described by a vector field X.

Let $Y = F_*(X)(w)$ be the push-forward of X,

$$Y = P(w)\frac{\partial}{\partial w_1} + Q(w)\frac{\partial}{\partial w_2} + R(w)\frac{\partial}{\partial w_3},$$

which P, Q, and R are given as in (2.12). Given that $w_j = f_j(z)$, we obtain after the normalization by the factor $\dfrac{\partial f_1}{\partial z_1}\dfrac{\partial f_2}{\partial z_2} - \dfrac{\partial f_1}{\partial z_2}\dfrac{\partial f_2}{\partial z_1}$ that

$$X(z) = \left\{ \begin{array}{rcl} \dot{z}_1 & = & \dfrac{\partial f_2}{\partial z_2}\displaystyle\sum_{i=0}^{m} f_1^{m-i}(z) f_2^i(z) P_i \circ F(z) \\ & & -\dfrac{\partial f_1}{\partial z_2}\displaystyle\sum_{i=0}^{n} f_1^{n-i}(z) f_2^i(z) Q_i \circ F(z) \\ & & +\Big(\dfrac{\partial f_1}{\partial z_2}\dfrac{\partial f_2}{\partial z_3} - \dfrac{\partial f_1}{\partial z_3}\dfrac{\partial f_2}{\partial z_2}\Big)\displaystyle\sum_{i=0}^{p} f_1^{p-i}(z) f_2^i(z) R_i \circ F(z) \\ \dot{z}_2 & = & -\dfrac{\partial f_2}{\partial z_1}\displaystyle\sum_{i=0}^{m} f_1^{m-i}(z) f_2^i(z) P_i \circ F(z) \\ & & +\dfrac{\partial f_1}{\partial z_1}\displaystyle\sum_{i=0}^{n} f_1^{n-i}(z) f_2^i(z) Q_i \circ F(z) \\ & & -\Big(\dfrac{\partial f_1}{\partial z_1}\dfrac{\partial f_2}{\partial z_3} - \dfrac{\partial f_1}{\partial z_3}\dfrac{\partial f_2}{\partial z_1}\Big)\displaystyle\sum_{i=0}^{p} f_1^{p-i}(z) f_2^i(z) R_i \circ F(z) \\ \dot{z}_3 & = & \Big(\dfrac{\partial f_1}{\partial z_1}\dfrac{\partial f_2}{\partial z_2} - \dfrac{\partial f_1}{\partial z_2}\dfrac{\partial f_2}{\partial z_1}\Big)\displaystyle\sum_{i=0}^{p} f_1^{p-i}(z) f_2^i(z) R_i \circ F(z). \end{array} \right. \tag{3.9}$$

Lemma 3.6. — *Let $\mathcal{F}$ be a special foliation along $\mathcal{C} \subset \mathbf{P}^3$, a curve given by the complete intersection of surfaces $f_1^{-1}(0)$ and $f_2^{-1}(0)$, with $d_j = degree(f_j)$ for $j = 1, 2$. Then*

$$k = \mathit{degree}(\mathcal{F}) \geqslant \begin{cases} \ell + 1, & \text{if } d_2 = 1 \\ (\ell+1)d_2 + d_1 - 2, & \text{if } d_2 \geqslant 2 \end{cases}$$

which $d_2 \geqslant d_1$ and $\ell = mult_{\mathcal{C}}(\mathcal{F})$.

Proof. — Let us suppose by absurd that exists a special foliation $\mathcal{F}$ along $\mathcal{C}$ such that $k < (\ell+1)d_2 + d_1 - 2$ with $d_2 \geqslant 2$. As $\mathcal{F}$ is special along $\mathcal{C}$, we have that $p = n - 1 = \ell$ in (3.9).

Let f_{j,d_j} be the homogeneous terms of f_j with degree d_j for $j = 1, 2$. Given that $\mathcal{C}$ is the complete intersection of surfaces, the degree of $df_1 \wedge df_2$ is $d_1 + d_2 - 2$. In fact, if the three terms of $df_1 \wedge df_2$ have degree smaller than

d_1+d_2-2 then we will have that $f_{1,d_1}=\lambda f_{2,d_2}$, for a some constant λ. But, it is an absurd. By the same reason, degree$(\frac{\partial f_j}{\partial z_1})=d_j-1$ or degree$(\frac{\partial f_j}{\partial z_2})=d_j-1$, for $j=1,2$.

If $P_{\ell+1}\not\equiv 0$ or $Q_{\ell+1}\not\equiv 0$, the degree of the first or the second component of (3.9) will be at least $(\ell+1)d_2+d_1-1$. Consequently, we must have $P_{\ell+1}\equiv Q_{\ell+1}\equiv 0$ and $R_\ell\not\equiv 0$ at most a constant because $\mathrm{cod}_{\mathbf{C}}\mathrm{Sing}(\mathcal{F})\geqslant 2$.

In this way, the degree of each component of (3.9) is, at least, $\ell d_2+d_1+d_2-2=(\ell+1)d_2+d_1-2$. In order to exists a special foliation along $\mathcal{C}$ with $k<(\ell+1)d_2+d_1-2$, the infinity hyperplane must be non-invariant by $\mathcal{F}$. As the homogeneous term of $\displaystyle\sum_{j=0}^{p} f_1^{p-j}f_2^jR_j\circ F(z)$ of degree $(\ell+1)d_2+d_1-2$ is not divisible by f_{1,d_1} because $R_\ell\not\equiv 0$, the homogeneous term of

$$z_1\left[\frac{\partial f_1}{\partial z_1}\frac{\partial f_2}{\partial z_2}-\frac{\partial f_1}{\partial z_2}\frac{\partial f_2}{\partial z_1}\right]-z_3\left[\frac{\partial f_1}{\partial z_2}\frac{\partial f_2}{\partial z_3}-\frac{\partial f_1}{\partial z_3}\frac{\partial f_2}{\partial z_2}\right]$$

with degree $(\ell+1)d_2+d_1-2$ must have f_{1,d_1} as factor. That is,

$$d_1f_{1,d_1}\frac{\partial f_{2,d_2}}{\partial z_2}-d_2f_{2,d_2}\frac{\partial f_{1,d_1}}{\partial z_2}$$

must be divisible by f_{1,d_1}. An absurd, because $\mathcal{C}$ is a complete intersection.

From (2.12) it is not hard to see that $k\geqslant(\ell+1)$ if $d_2=1$. $\square$

Theorem 3.7. — *Let $\mathcal{F}$ be a special foliation along $\mathcal{C}\subset\mathbf{P}^3$, with $\mathcal{C}$ a complete intersection and the unique one-dimensional component of Sing$(\mathcal{F})$. Then $\mathcal{F}$ has isolated singularities.*

Proof. — Let $\mathcal{C}$ be as in the Lemma 3.6. As d and g was calculated in terms of d_1 and d_2, for $k=(\ell+1)d_2+d_1-2$, we have that

$$\begin{aligned}n_{\mathcal{F}} \;\geqslant\; & d_2(\ell+1)\Big\{(d_2-1)(d_2-2)+(d_1-1)\Big[3(d_1+d_2)-7\Big]+(d_2-d_1)\\ & +\ell(d_2-d_1)\Big[2(d_2+d_1)-5\Big]+\ell^2(d_2-d_1)^2\Big\}.\end{aligned}$$

Then, $n_{\mathcal{F}}\geqslant 0$ for $d_2\geqslant d_1\geqslant 1$ with the equality only if $d_2=d_1=1$. But, if $d_2=1$ there is the sharp bound for k, that is, $k\geqslant(\ell+1)$. With the same procedure above, $n_{\mathcal{F}}=\ell+2$ if $k=(\ell+1)$ and $d_1=d_2=1$. In this way, $n_{\mathcal{F}}>0$ when k assumes its minimal value.

Assuming that k is a continuous variable, the partial derivative of $n_{\mathcal{F}}$ with respect to k is

$$n'_{\mathcal{F}} = 1 + 2k + 3k^2 - d(\ell+1)(3\ell+1).$$

As $k \geqslant (\ell+1)d_2 + d_1 - 2$, we have that

$$n'_{\mathcal{F}} > (d_1-1)^2 + 2(d_1-2)^2 + d_2(\ell+1)\big[3\ell(d_2-d_1) + 5d_1 + 3d_2 - 10\big].$$

If $d_2 \geqslant 2$ then $n'_{\mathcal{F}} > 0$ because we will have that $5d_1 + 3d_2 \geqslant 11$. But, if $d_2 = 1$ then $n'_{\mathcal{F}} \geqslant 1 + 4(\ell+1) > 0$ because $k \geqslant (\ell+1)$. Therefore, $n_{\mathcal{F}} > 0$. □

4. Holomorphic foliations in ruled surfaces

A special foliation $\mathcal{F}$ along $\mathcal{C}$ gives a foliation with isolated singularites on E and in case $\mathcal{F}$ is dicritical but not special new curves of singularities will appear. Two questions arise: given a foliaton $\mathcal{F}_1$ on E with isolated singularities, is there a condition on $\mathcal{F}_1$ to be the restriction of $\tilde{\mathcal{F}}$ on E where $\tilde{\mathcal{F}}$ is the foliation induced foliation from some holomorphic foliation $\mathcal{F}$ of $\mathbf{P}^3$? How many curves of singularities will appear on E if $\mathcal{F}$ is not special? We shall give the answer to these questions with the determination of the Chern class of the holomorphic tangent bundle $T_{\mathcal{F}_1}$. Firstly, we describe the results on ruled surfaces that will be needed later.

Definition 4.1. — *A ruled surface S is a connected compact complex surface with a holomorphic map $\Psi : S \to \mathcal{C}$ to a regular complex curve $\mathcal{C}$ giving S the structure of a holomorphic $\mathbf{P}^1$-bundle over $\mathcal{C}$.*

The map Ψ induces on the level of cohomology an isomorphism $\Psi^* : H^1(\mathcal{C}, \mathbf{Z}) \cong \mathbf{Z}^{2g} \to H^1(S, \mathbf{Z})$, where g is the genus of $\mathcal{C}$, and an injection $\Psi^* : H^2(\mathcal{C}, \mathbf{Z}) \cong \mathbf{Z} \to H^2(S, \mathbf{Z})$ sending the fundamental class of $\mathcal{C}$ to the Poincaré dual of a fiber of the ruling Ψ, $f = [\Psi^{-1}(b)]^*$. If $\sigma : \mathcal{C} \to S$ denotes a holomorphic section of Ψ and f' denotes the Poincaré dual of $\sigma(\mathcal{C})$, then f and f' form a basis of $H^2(S, \mathbf{Z})$ satisfying $f \cdot f = 0$ and $f \cdot f' = 1$. We shall carry out computations in $H^2(S, \mathbf{Z})$ by expanding its elements in terms of f and $h = f' - \frac{1}{2}(f' \cdot f')f$, using that $f \cdot h = 1$ and $h \cdot h = 0$. Then, if L is a line bundle, there are $a, b \in \mathbf{Z}$ such that $c_1(L) = af + bh$ which $c_1(L)$ is the first Chern class.

Let TS be the tangent bundle of S and $\tau \hookrightarrow TS$ be the sub-line bundle defined as the kernel of the Jacobian of Ψ,

$$0 \longrightarrow \tau \longrightarrow TS \xrightarrow{D\Psi} \Psi^*(T\mathcal{C}) = N \longrightarrow 0, \tag{4.1}$$

where N is the normal bundle to the ruling.

Lemma 4.2. — *The Chern classes of τ and N are*

$$c_1(\tau) = 2h \text{ and } c_1(N) = (2-2g)f$$

where g is the genus of $\mathcal{C}$.

Proof. — See [4]. $\square$

Definition 4.3. — *A holomorphic foliation by curves in the connected complex surface S is a nonidentically zero holomorphic bundle map $X : L \to TS$ from the line bundle L to the tangent bundle of S.*

Proposition 4.4. — *Let $\mathcal{F}$ be a holomorphic foliation by curves on the ruled surface S with isolated singularities and let $af + bh$ be the first Chern class of $T_{\mathcal{F}}$. Then,*

(i) $\displaystyle\sum_{p \in Sing(\mathcal{F})} \mu(\mathcal{F}, p) = 2(a+g-1)(b-1) + (2-2g)$,

(ii) $\displaystyle\sum_{p \in Sing(\mathcal{F})} BB(\mathcal{F}, p) = 2(a+2g-2)(b-2)$, *where $BB(\mathcal{F}, p)$ is the Baum-Bott index of $\mathcal{F}$ at p.*

Proof. — See [9]. $\square$

Proposition 4.5. — *Let $\tilde{\mathbf{P}}^3 \xrightarrow{\pi} \mathbf{P}^3$ be the blow-up of $\mathbf{P}^3$ along a regular curve $\mathcal{C}$ of genus g and degree d. Consider a holomorphic foliation by curves $\mathcal{F}$ such that $\mathcal{C} \subset Sing(\mathcal{F})$ is non-dicritical, not necessarily special, with $\tilde{\mathcal{F}}$ and E as before. Then*

$$c_1(T_{\mathcal{F}_1}) = -\big[d(k-2\ell-1) + \ell(1-g)\big]f - \ell h,$$

where $\mathcal{F}_1 = \tilde{\mathcal{F}}|_E$, $k = degree(\mathcal{F})$ and $\ell = tang(\pi^*\mathcal{F}, E)$.

Proof. — From (3.2), we have that $c_1(T_{\tilde{\mathcal{F}}}) = \pi^* c_1(T_{\mathcal{F}}) + \ell[E]$. Let us suppose that $c_1(T_{\mathcal{F}_1}) = af + bh$. Then

$$\begin{aligned} \int_E c_1^2(T_{\tilde{\mathcal{F}}}) &= \int_E \big[\pi^* c_1^2(T_{\mathcal{F}}) + 2\ell\pi^* c_1(T_{\mathcal{F}}) \cdot [E] + \ell^2 [E]^2\big] \\ &= 2\ell(k-1)d + \ell^2(2-2g-4d). \end{aligned}$$

By other side, $\displaystyle\int_E c_1^2(T_{\tilde{\mathcal{F}}}) = c_1^2(T_{\mathcal{F}_1}) = 2ab$.

In the same way, we obtain that

$$\begin{aligned}\int_E c_1(T_{\tilde{\mathcal{F}}})c_1(TE) &= \int_E \left[\pi^* c_1(T_{\mathcal{F}}) + \ell[E]\right]\left[\pi^* c_1(\mathbf{P}^3) - 2[E]\right] \\ &= 2(1-k)d - 4\ell d - 2\ell(2-2g-4d)\end{aligned}$$

On the other hand,

$$\begin{aligned}\textstyle\int_E c_1(T_{\tilde{\mathcal{F}}})c_1(TE) &= c_1(T_{\mathcal{F}_1})\cdot c_1(S) \\ &= 2a + (2-2g)b.\end{aligned}$$

From these equations, we obtain a linear system. Solving it for a and b, the proposition is then proved. □

With the determination of the Chern class of $T_{\mathcal{F}_1}$ we can see that the parameters a and b are related with the genus and the degree of the curve of singularities as well as the degree of the foliation and the order of tangency $\operatorname{tang}(\pi^*\mathcal{F}, E)$. Therefore, there is a restriction for a foliation on E to be given by $\tilde{\mathcal{F}}|_E$.

Theorem 4.6. — *Let $\mathcal{F}$ be a special foliation along $\mathcal{C} \subset \mathbf{P}^3$ where $\mathcal{C}$ is the complete intersection, with $\tilde{\mathbf{P}}^3$, $\tilde{\mathcal{F}}$ and E as before. Then the foliation $\tilde{\mathcal{F}}$ has singularities on E.*

Proof. — Let us suppose by absurd that $\mathcal{F}_1 = \tilde{\mathcal{F}}|_E$ is non-singular. From item (ii) of the proposition 4.4, we must have that

$$2(a+2g-2)(b-2) = 0.$$

As $b = -\ell < 0$, the unique possibility is $a = 2-2g$. From item (i) of the same proposition 4.4,

$$2(a+g-1)(b-1) + (2-2g) = (2-2g)b = 0.$$

Therefore, necessarily $g = 1$.

From the Theorem 3.1, since $g = 1$, we obtain $2d(\ell+1)(k-2\ell-1) = 0$. In order to exist a foliation $\mathcal{F}$ such that $\mathcal{F}_1$ is non-singular, we must have that $k = 2\ell+1$. As $\mathcal{C} = f_1^{-1}(0) \cap f_2^{-1}(0)$ with $d_j = \operatorname{degree}(f_j)$ and $d_1 \leqslant d_2$ and from the Lemma 3.6, we obtain

$$k = 2\ell + 1 \geqslant (\ell+1)d_2 + d_1 - 2 \Leftrightarrow \ell(2-d_2) + 3 - d_1 - d_2 \geqslant 0.$$

We have two possible cases for this inequality, that is, $d_1 = d_2 = 1$ or $d_1 = 1$ and $d_2 = 2$. But, in both cases, we have that $g = 0$. An absurd, because $g = 1$. □

Let us consider $\mathcal{F}$ and $\mathcal{C} \subset \mathrm{Sing}(\mathcal{F})$ as before, but $\mathcal{F}$ non-dicritical and non-special along $\mathcal{C}$. Thus, we will assume locally that $\mathcal{F}$ is given by a vector field $X(z)$ as in (2.11) with $p+1 \neq n \leqslant m$. The foliation induced $\tilde{\mathcal{F}}$ when restricted to the exceptional divisor E is either tangent or normal to a fiber $\pi^{-1}(q) \cong \mathbf{P}^1$, $q \in \mathcal{C}$, as was observed by equations (2.17) and (2.18). But, in both cases, new curves of singularities will appear on E. The number of these new curves is determined in the next result.

Theorem 4.7. — *Let $\tilde{\mathbf{P}}^3 \xrightarrow{\pi} \mathbf{P}^3$ be the blow-up of $\mathbf{P}^3$ along a regular curve $\mathcal{C}$ of genus g and degree d. Consider a holomorphic foliation by curves $\mathcal{F}$, with degree k, non-special along $\mathcal{C}$, with $p+1 \neq n \leqslant m$ as given above. The number of curves of singularities in the exceptional divisor, counted the multiplicities, is*

$$2+\ell$$

in case $\mathcal{F}_1 = \tilde{\mathcal{F}}|_E$ be tangent to the fiber $\pi^{-1}(q) \cong \mathbf{P}^1$, $q \in \mathcal{C}$ and

$$d(k-2\ell-1)+(\ell+2)(1-g)$$

in case $\mathcal{F}_1$ be normal to the fiber $\pi^{-1}(q) \cong \mathbf{P}^1$, $q \in \mathcal{C}$ with $\ell = tang(\pi^*\mathcal{F}, E)$.

Proof. — Firstly, let us suppose $\mathcal{F}_1$ be tangent to the fiber $\pi^{-1}(q)$, $q \in \mathcal{C}$, as in (2.18). The number of singularities in each fiber is given by

$$\begin{aligned}\int_\tau c_1(\tau \otimes T^*_{\mathcal{F}_1}) &= \int_\tau [2h-af-bh] = [(2-b)h-af]\cdot f \\ &= 2-b.\end{aligned}$$

As $\mathcal{F}$ is analytical and $b=-\ell$ we conclude that there are $2+\ell$ curves of singularities on E.

Let us suppose that $\mathcal{F}_1$ is normal to the fiber $\pi^{-1}(q)$, $q \in \mathcal{C}$, as in (2.17). In the same way, the number of singularities in each fiber is given by

$$\begin{aligned}\int_N c_1(N \otimes T^*_{\mathcal{F}_1}) &= \int_N [(2-2g)f-af-bh] \\ &= [(2-2g-a)f-bh]\cdot h \\ &= (2-2g-a).\end{aligned}$$

As $a = -d(k-2\ell-1)-\ell(1-g)$ and by the same reason of the previous case we conclude that there are $2-2g-a = d(k-2\ell-1)+(\ell+2)(1-g)$ curves of singularities on E. □

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