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An example of local analytic q -difference equation : Analytic classification^(*)

FRÉDÉRIC MENOUS⁽¹⁾

ABSTRACT. — Using the techniques developed by Jean Ecalle for the study of nonlinear differential equations, we prove that the q -difference equation

$$x\sigma_q y = y + b(y, x)$$

with $(\sigma_q f)(x) = f(qx)$ ($q > 1$) and $b(0, 0) = \partial_y b(0, 0) = 0$ is analytically conjugated to one of the following equations :

$$x\sigma_q y = y \quad \text{ou} \quad x\sigma_q y = y + x$$

RÉSUMÉ. — En utilisant les techniques développées par Jean Ecalle pour l'étude des équations différentielles nonlinéaires, on montre que l'équation aux q -différences

$$x\sigma_q y = y + b(y, x)$$

avec $(\sigma_q f)(x) = f(qx)$ ($q > 1$) et $b(0, 0) = \partial_y b(0, 0) = 0$ est conjuguée analytiquement à l'une des équations suivantes :

$$x\sigma_q y = y \quad \text{ou} \quad x\sigma_q y = y + x$$

1. Introduction

The aim of this paper is to study the formal or analytic conjugacy of q -difference equations, using formal or analytic power series in $\tilde{\mathcal{G}}$ or \mathcal{G} :

$$\begin{aligned} \tilde{\mathcal{G}} &= \{f(z, x) \in \mathbb{C}[[z, x]] \ ; \ \partial_z f(0, 0) \neq 0, \ f(0, 0) = 0\} \\ \mathcal{G} &= \{f(z, x) \in \mathbb{C}\{z, x\} \ ; \ \partial_z f(0, 0) \neq 0, \ f(0, 0) = 0\} \end{aligned} \tag{1.1}$$

$\tilde{\mathcal{G}}$ and \mathcal{G} are groups for the z -composition of series.

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Let $\widetilde{\mathcal{G}}_1$ (resp. \mathcal{G}_1) be the subgroup of elements f of $\widetilde{\mathcal{G}}$ (resp. \mathcal{G}) such that $\partial_z f(0, 0) = 1$. For two series $z + b(z, x)$ and $z + c(z, x)$ of \mathcal{G}_1 , the equation

$$(E_b) \quad x\sigma_q y = y + b(y, x)$$

is formally conjugate to the equation

$$(E_c) \quad x\sigma_q z = z + c(z, x)$$

if there exists $f \in \widetilde{\mathcal{G}}$ such that, if z satisfies (E_c) , then $y = f(z, x)$ satisfies (E_b) . Here $\sigma_q y(x) = y(qx)$, with $q > 1$ (Note that we chose $q > 1$ instead of $q \in \mathbb{C}$, $|q| > 1$ for the sake of simplicity but this wouldn't have change our results.).

The formal conjugacy is an equivalence relation and the equations are *analytically* conjugate if one can find a *conjugating series* $f \in \mathcal{G}$.

It was proven in [2] that, for any $z + b(z, x) \in \mathcal{G}_1$, the equation (E_b) is formally conjugate to

$$(E_0) \quad x\sigma_q z = z$$

and that, if $b(0, x) = 0$, then this conjugacy is analytic. We prove in this paper that,

THEOREM 1.1. — *For any $z + b(z, x) \in \mathcal{G}_1$, there exists $\alpha(b) \in \mathbb{C}$ such that the equation (E_b) is analytically conjugate by an element of \mathcal{G}_1 to the equation*

$$(E_{\alpha(b)}) \quad x\sigma_q z = z + \alpha(b)x$$

Moreover, if $\alpha(b) \neq 0$, (E_b) is analytically conjugate to the equation

$$(E_1) \quad x\sigma_q z = z + x$$

This means that there exists exactly two classes of analytic conjugacy. Using these results, as well as the theory of q -resummation (see [3]), one can then express some local solutions of (E_b) in terms of the solutions of (E_0) or (E_1) .

In section 1, we remind the results of [2] on the formal conjugacy of (E_b) and (E_0) . The main key to this result is that, to any formal (resp. analytic) series f of $\widetilde{\mathcal{G}}_1$ (resp. \mathcal{G}_1), one can bijectively associate a formal (resp. analytic) substitution automorphism $\mathbb{F} \in \mathbb{C}[[z, x, \partial_z]]$ such that

$$\forall \varphi \in \mathbb{C}[[z, x]], \quad \mathbb{F}.\varphi(z, x) = \varphi(f(z, x), x) \tag{1.2}$$

Starting with $z + b(z, x) \in \mathcal{G}_1$, we prove that its associated analytic substitution automorphism \mathbb{D} can be written

$$\mathbb{D} = \text{Id} + \sum_{\eta \in H} x^\sigma \mathbb{D}_\eta \quad (1.3)$$

where H is a semigroup which elements are of type $\eta = \begin{pmatrix} n \\ \sigma \end{pmatrix}$ ($n \in \mathbb{Z}$, $\sigma \in \mathbb{N}$) and

$$\forall \eta \in H, \quad \mathbb{D}_\eta \in \mathbb{C}[z, \partial_z] \quad (1.4)$$

It is then proven that the formal series $w \in \tilde{\mathcal{G}}_1$ that conjugates (E_b) to (E_0) is associated to a formal substitution automorphism \mathbb{W} that can be written :

$$\mathbb{W} = \sum_{s \geq 0} \sum_{(\eta_1, \dots, \eta_s) \in H^s} W^{\eta_1, \dots, \eta_s} \mathbb{D}_{\eta_s} \dots \mathbb{D}_{\eta_1} = \sum W^\bullet \mathbb{D}.$$

where $H^0 = \{\emptyset\}$, $W^\emptyset = 1$ and $\mathbb{D}_\emptyset = \text{Id}$. The collection of weights $\{W^{\eta_1, \dots, \eta_s}\}$ is called a mould W^\bullet . We remind a formula on W^\bullet that proves that its values are in $\mathbb{C}[[x]]$. We recall also that, as W^\bullet satisfies symmetry relations (symmetrelity) : it automatically ensures that \mathbb{W} is a formal substitution automorphism. When $b(0, x) = 0$, \mathbb{W} turns to be an analytic substitution automorphism, or equivalently, $\mathbb{W}.z \in \mathcal{G}_1$. Otherwise, most of the coefficients of W^\bullet are divergent and q -multisummable but, unfortunately, the resummation of all the weights involves an infinite number of critical times in the resummation process (see section 2 below and [4] for details on q -multisummability).

To circumvent this difficulty, we prove in section 3 that the mould W^\bullet can be twisted in a mould U^\bullet whose coefficient are in $\mathbb{C}\{x\}$ (U^\bullet is analytic). Because of the symmetries of U^\bullet the operator

$$\mathbb{U} = \sum U^\bullet \mathbb{D}. \quad (1.5)$$

is an a priori formal substitution automorphism such that $y = u(z, x) = \mathbb{U}.z \in \tilde{\mathcal{G}}_1$ conjugates the equation

$$\begin{aligned} (E_b) \quad & x\sigma_q y = y + b(y, x) \\ (E_c) \quad & x\sigma_q z = z + c(x) = z + \mathbb{V}.z \in z + x\mathbb{C}[[x]] \end{aligned}$$

where \mathbb{V} is also defined by a mould V^\bullet (see section 3).

In fact, \mathbb{U} and \mathbb{V} are analytic substitution automorphisms ($u(z, x) = \mathbb{U}.z \in \mathcal{G}_1$ and $c(x) \in x\mathbb{C}\{x\}$) but the analyticity of U^\bullet does *not* implies the analyticity of \mathbb{U} . Despite nice estimates on U^\bullet (section 4.4), which are

derived from an explicit formula for the coefficients of U^\bullet (section 4.3), there is no hope to prove that the series of analytic functions

$$u(z, x) = \mathbb{U}.z = \sum_{s \geq 0} \sum_{(\eta_1, \dots, \eta_s) \in H^s} (U^{\eta_1, \dots, \eta_s} \mathbb{D}_{\eta_s} \dots \mathbb{D}_{\eta_1}.z) = \sum (U^\bullet \mathbb{D}_{\bullet}.z)$$

is *normally* convergent in a (z, x) -neighborhood of $(0, 0)$.

To prove that \mathbb{U} is analytic, we will need to reorganize the sum defining it, such that $\mathbb{U}.z$ appears as a normally convergent series of analytic functions in a neighborhood of $(0, 0)$. This reorganization of terms is simply the arborification-coarborification defined by J. Ecalle (see [1]). The operator \mathbb{U} can be written

$$\mathbb{U} = \sum (U^{\bullet <} \mathbb{D}_{\bullet <})$$

but here $\bullet <$ means that the sum is over sequences of elements of H equipped with an arborescent order, instead of a total order. Each coefficient of the *arborescent* mould $U^{\bullet <}$ depends on a combination of coefficients of U^\bullet : $U^{\bullet <}$ is obtained by *arborification* of U^\bullet and the operators $\mathbb{D}_{\bullet <}$ are obtained from the operators \mathbb{D}_{\bullet} by a dual (but not unique) operation : the *coarborification* (see section 5).

Using these operations, a formula is obtained for $U^{\bullet <}$ in section 5.2. The estimates derived from this formula prove then that \mathbb{U} is an *analytic* substitution automorphism (section 5.4), as well as \mathbb{V} (section 6.1).

A last but simple remark (section 6.2) leads us to the attempted result, that is to say theorem 1.1.

2. Reminder about previous results and moulds

We resume here the results obtained in [2].

2.1. The operators \mathbb{D}_η

Let us first introduce some notations. We started with the equation

$$x\sigma_q y - y = b(y, x) \tag{2.1}$$

where $\sigma_q y(x) = y(qx)$, with $q > 1$ and $b(y, x) \in \mathbb{C}\{y, x\}$ with the following conditions : $b(0, 0) = 0$ and $(\partial_y b)(0, 0) = 0$.

The function b can be written :

$$b(z, x) = z \sum_{n \geq -1} b_n(x) z^n = z \sum_{n \geq -1} \sum_{\sigma \geq 0} b_{n,\sigma} x^\sigma z^n \quad (b_{-1,0} = 0, b_{0,0} = 0)$$

If we consider the set (2.2)

$$H_0 = \left\{ \eta = \begin{pmatrix} n \\ \sigma \end{pmatrix}, n \geq -1, \sigma \geq 0 \right\} / \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = H_{-1} \cup H_{\geq 0}$$
(2.3)

where $H_{-1} = \{\eta \in H; n = -1\}$ $H_{\geq 0} = \{\eta \in H; n \geq 0\}$

then, one can define the following operators.

$$\forall \eta \in H_0 \quad ; \quad \mathbb{B}_\eta = b_\eta z^{n+1} \partial_z \quad (2.4)$$

where $b_{n,\sigma} = b_\eta$. This kind of operators are useful in the study of nonlinear differential equations but we will need here some slightly different operators which are those described in equations (1.3) and (1.4).

DEFINITION 2.1. — *Let*

$$H = \{\eta_1 + \dots + \eta_s; s \geq 1; \eta_i \in H_0\} \quad (2.5)$$

For $\eta \in H$, the operator \mathbb{D}_η is defined by :

$$\mathbb{D}_\eta = \sum_{s \geq 1} \sum_{\substack{\eta_1 + \dots + \eta_s = \eta \\ \eta_i \in H_0}} \frac{1}{s!} b_{\eta_1} \dots b_{\eta_s} z^{n_1 + \dots + n_s + s} \partial_z^s \quad (2.6)$$

It is important to notice that, for any $\eta = \begin{pmatrix} n \\ \sigma \end{pmatrix} \in H$, the operator \mathbb{D}_η is a finite sum which degree in ∂_z is at most $n + 2\sigma$. Moreover,

$$\forall \eta \in H_0 \quad ; \quad \mathbb{D}_\eta.z = \mathbb{B}_\eta.z \quad (2.7)$$

and,

$$\begin{aligned} \mathbb{B}.z &= \sum_{\eta \in H_0} x^\sigma \mathbb{B}_\eta.z &= b(z, x) \\ \mathbb{D}.z &= \left(\text{Id} + \sum_{\eta \in H} x^\sigma \mathbb{D}_\eta \right).z &= z + b(z, x) \end{aligned} \quad (2.8)$$

In fact, \mathbb{D} is a “convergent” substitution automorphism :

$$\forall f, g \in \mathbb{C}\{z, x\} \quad ; \quad \mathbb{D}(fg) = (\mathbb{D}f)(\mathbb{D}g) \text{ and } (\mathbb{D}.f)(z, x) = f(z + b(z, x), x) \quad (2.9)$$

The notion of convergence will be developed in section 4.4. We introduce also some notations :

DEFINITION 2.2. — Let $\mathbf{H} = \cup_{s \geq 0} H^s$ be the set of sequences of elements of H (H^0 corresponds to the empty sequence \emptyset).

If $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}$ ($s \geq 1$), then

$$l(\boldsymbol{\eta}) = s \quad ; \quad \boldsymbol{\eta} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \quad ; \quad \begin{array}{l} \|\mathbf{n}\| = n_1 + \dots + n_s \\ \|\boldsymbol{\sigma}\| = \sigma_1 + \dots + \sigma_s \end{array} \quad ; \quad \|\boldsymbol{\eta}\| = \begin{pmatrix} \|\mathbf{n}\| \\ \|\boldsymbol{\sigma}\| \end{pmatrix} \quad (2.10)$$

Moreover, if $l(\boldsymbol{\eta}) \geq 1$, $\wedge \boldsymbol{\eta} = (\eta_2, \dots, \eta_s)$ ($\wedge \boldsymbol{\eta} = \emptyset$ if $s = 1$), $\boldsymbol{\eta}_{\text{in}} = \eta_1$ and,

$$\mathbb{D}_{\boldsymbol{\eta}} = \mathbb{D}_{\eta_s} \dots \mathbb{D}_{\eta_1} \quad (2.11)$$

If $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{Z}^s$ ($l(\mathbf{k}) = s$, $s \geq 1$) then, for $1 \leq i \leq s$, $\hat{k}_i = k_i + \dots + k_s$, $\check{k}_i = k_1 + \dots + k_i$, and

$$\begin{aligned} \hat{\mathbf{k}} &= (\hat{k}_1, \dots, \hat{k}_s) \\ \check{\mathbf{k}} &= (\check{k}_1, \dots, \check{k}_s) \end{aligned} \quad (2.12)$$

Finally if \mathbf{k} and \mathbf{l} are two sequences of same length in \mathbb{Z}^s , then

$$\langle \mathbf{k}, \mathbf{l} \rangle = k_1 l_1 + \dots + k_s l_s \quad (2.13)$$

If we refer to [1], the set $\{\mathbb{D}_{\boldsymbol{\eta}}\}_{\boldsymbol{\eta} \in H}$ defines a cosympetrel comould. In fact the operator \mathbb{D} is a first example of automorphism defined by a symmpetrel mould contracted with $\{\mathbb{D}_{\boldsymbol{\eta}}\}_{\boldsymbol{\eta} \in H}$.

2.2. Moulds

For the empty sequence \emptyset we define $\mathbb{D}_{\emptyset} = \text{Id}$. A mould M^\bullet on \mathbf{H} , with values in $\mathbb{C}[[x]]$, is a family $\{M^\boldsymbol{\eta} \in \mathbb{C}[[x]]\}_{\boldsymbol{\eta} \in \mathbf{H}}$.

2.2.1. Symmpetrelity

A mould M^\bullet is *symmpetrel* iff $M^\emptyset = 1$ and

$$\forall (\boldsymbol{\eta}, \boldsymbol{\mu}) \in \mathbf{H}^2, \quad \sum_{\boldsymbol{\lambda} \in \text{ctsh}(\boldsymbol{\eta}, \boldsymbol{\mu})} M^\boldsymbol{\lambda} = M^\boldsymbol{\eta} M^\boldsymbol{\mu} \quad (2.14)$$

where the sum is over all the sequence $\boldsymbol{\lambda}$ obtained by contracting shuffling of the sequences $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$: one shuffles the two sequences $(\boldsymbol{\eta}, \boldsymbol{\mu}) \mapsto \boldsymbol{\lambda}^*$ and, eventually, one contracts pairs $(\lambda_i, \mu_j) \mapsto \lambda_i + \mu_j$ that are consecutive in

the sequence λ^* (see [1]). We shall often deal with symmetrel moulds since, if M^\bullet is symmetrel, then the operator

$$\mathbb{M} = \sum_{\eta \in \mathbf{H}} M^\eta \mathbb{D}_\eta = \sum M^\bullet \mathbb{D}_\bullet$$

is a formal substitution automorphism :

$$\forall \varphi \in \mathbb{C}[[z, x]], \quad \mathbb{M}.\varphi(z, x) = \varphi(\mathbb{M}.z, x) = \varphi(m(z, x), x)$$

where $m(z, x) \in \mathbb{C}[[z, x]]$ and $\partial_z m(0, 0) = 1$.

Note that the mould X^\bullet defined by $X^\emptyset = 1$ and

$$\forall \boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}, \quad X^\boldsymbol{\eta} = \begin{cases} x^{\sigma_1} & \text{if } s = 1 \\ 0 & \text{if } s \geq 2 \end{cases} \quad (2.15)$$

is symmetrel and $\mathbb{D} = \sum X^\bullet \mathbb{D}_\bullet$ (see equation 2.8).

2.2.2. Operations on moulds

Product. — Let M^\bullet and N^\bullet two symmetrel moulds. If $\mathbb{M} = \sum M^\bullet \mathbb{D}_\bullet$ and $\mathbb{N} = \sum N^\bullet \mathbb{D}_\bullet$ ($\mathbb{M}.z = m(z, x)$, $\mathbb{N}.z = n(z, x)$), then

$$\begin{aligned} \mathbb{N}.\mathbb{M}.\varphi(z, x) &= \varphi(m(n(z, x), x), x) \\ &= \sum_{\mu \in \mathbf{H}} N^\mu \mathbb{D}_\mu . \sum_{\eta \in \mathbf{H}} M^\eta \mathbb{D}_\eta . \varphi(z, x) \\ \forall \varphi \in \mathbb{C}[[z, x]], &= \sum_{\mu \in \mathbf{H}} \sum_{\eta \in \mathbf{H}} N^\mu M^\eta \mathbb{D}_{\eta\mu} . \varphi(z, x) \\ &= \sum_{\boldsymbol{\eta} \in \mathbf{H}} (M^\bullet \times N^\bullet)^{\boldsymbol{\eta}} \mathbb{D}_{\boldsymbol{\eta}} . \varphi(z, x) \end{aligned}$$

The associative non-commutative product of M^\bullet and N^\bullet is defined by

$$(M^\bullet \times N^\bullet)^\boldsymbol{\eta} = \sum_{\eta^1 \eta^2 = \boldsymbol{\eta}} M^{\eta^1} N^{\eta^2} \quad (2.16)$$

where the sequences $\boldsymbol{\eta}^1$ or $\boldsymbol{\eta}^2$ can be empty. Moreover $M^\bullet \times N^\bullet$ is symmetrel. Note that the symmetrel mould 1^\bullet defined by $1^\emptyset = 1$ and $1^\boldsymbol{\eta} = 0$ if $\boldsymbol{\eta}$ is nonempty is the unity and every symmetrel mould M^\bullet is invertible, of symmetrel inverse $\text{inv}(M)^\bullet$. One can notice that the mould I^\bullet defined by $I^\emptyset = 1$, $I^{\eta_1} = 1$ and $I^{\eta_1, \dots, \eta_s} = 0$ if $s \geq 2$ has a noticeable inverse $H^{\eta_1, \dots, \eta_s} = (-1)^s$.

Composition. — The mould $M^\bullet \circ N^\bullet = P^\bullet$ is defined by $P^\emptyset = 1$ and, if $\eta \in \mathbf{H}$ is nonempty,

$$P^\eta = \sum_{\eta^1 \dots \eta^t = \eta} M^{\|\eta^1\|, \dots, \|\eta^t\|} N^{\eta^1} \dots N^{\eta^t}$$

where the sequences η^i are nonempty and P^\bullet is symmetrel. The composition is associative and

$$(M^\bullet \times N^\bullet) \circ Q^\bullet = (M^\bullet \circ Q^\bullet) \times (N^\bullet \circ Q^\bullet)$$

In fact, for any symmetrel mould M^\bullet , $I^\bullet \circ M^\bullet = M^\bullet$ and

$$\text{inv}(M)^\bullet = H^\bullet \circ M^\bullet \tag{2.17}$$

2.3. Previous results

We remind here the results developed in [2]. We proved that the solution y of the equation

$$x\sigma_q y = y + b(y, x)$$

is formally conjugated to the the solution z of

$$x\sigma_q z = z$$

by a formal substitution automorphism $\mathbb{W} = \sum W^\bullet \mathbb{D}_\bullet$ where W^\bullet is a symmetrel mould on \mathbf{H} with values on $\mathbb{C}[[x]]$. To prove this, one notices that

$$\begin{aligned} y + b(y, x) &= \mathbb{W}z + b(\mathbb{W}.z, x) \\ &= \mathbb{W}.\mathbb{D}z \\ &= \sum (X^\bullet \times W^\bullet) \mathbb{D}_\bullet \end{aligned}$$

and

$$\begin{aligned} x\sigma_q y &= x\sigma_q (\sum W^\bullet \mathbb{D}_\bullet . z) \\ &= \sum s_q(W^\bullet) \mathbb{D}_\bullet . z \end{aligned}$$

where $s_q(W^\eta) = x^{-\|\eta\|} \sigma_q . W^\eta$ with $\eta = \binom{n}{\sigma}$. We shall come back later on this last identity. The mould W^\bullet is thus defined by the identities $W^\emptyset = 1$ and

$$s_q(W^\bullet) = X^\bullet \times W^\bullet \tag{2.18}$$

For a nonempty sequence $\eta = (\eta_1, \dots, \eta_s) \in \mathbf{H}$, this reads

$$x^{-\|\eta\|} \sigma_q . W^\eta = W^\eta + x^{\sigma_1} W^\eta$$

and, solving these equations recursively,

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THEOREM 2.3. — If $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \mathbf{H}$, then the monomial W^η is a formal series with the following expression :

$$W^\eta(x) = \varepsilon_n \sum_{(k_1, \dots, k_s) \in \mathbb{Z}^n} x^{\|\boldsymbol{\sigma}\| + \langle \check{\mathbf{k}}, \mathbf{n} \rangle} q^{-\langle \check{\mathbf{k}}, \boldsymbol{\sigma} \rangle} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s} \quad (2.19)$$

with the notations

- $\varepsilon_n = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$ with $\hat{n}_i = n_i + \dots + n_s$ with $\varepsilon_n = 1$ (resp. $\varepsilon_n = -1$) if $n \geq 0$ (resp. $n < 0$)
- $\mathbb{Z}^n = \mathbb{Z}^{\hat{n}_1} \times \dots \times \mathbb{Z}^{\hat{n}_s}$ with $\mathbb{Z}^n = \mathbb{Z}^{+*}$ (resp. \mathbb{Z}^-) if $n \geq 0$ (resp. $n < 0$)
- $\check{k}_i = k_1 + \dots + k_i$ and $\check{\mathbf{k}} = (\check{k}_1, \dots, \check{k}_s)$.
- $[k + 1]_q = q^{-k(k+1)/2}$

Moreover, the mould W^\bullet is symmetrel. As discussed in [2], there are severe obstacles to apply the techniques of q -resummation developed in [3] and [4] to this mould. It is important to notice that the divergence of W^η is due to possibly negative finishing sequences $\hat{n}_i = n_i + \dots + n_s$: if all the integers \hat{n}_i are non negative, then W^η is convergent.

This last condition is always fulfilled when $b(0, x) = 0$ and then

THEOREM 2.4. — *The equation*

$$(x\sigma_q - 1)y = b(y, x) \quad (2.20)$$

with $b(0, x) = 0$ and $\frac{\partial b}{\partial y}(0, 0) = 0$ is analytically conjugate by $y = \mathbb{W}.z \in \mathcal{G}_1$ to the equation

$$x\sigma_q z = z$$

The ideas that will now be developed are based on the two fundamental remarks

1. The divergence of W^η is due to possibly negative finishing sequences $\hat{n}_i = n_i + \dots + n_s$.
2. Whenever, $b(0, x) \neq 0$, divergence appears but we could try to skip it by analytic conjugacy to an equation which solution contains all the divergence, that is to say of type :

$$x\sigma_q z = z + c(x)$$

3. Formal part : the moulds W^\bullet , J^\bullet and U^\bullet

3.1. Compensating the divergence in W^\bullet

To skip the divergence, one can try to change W^\bullet into a symmetrel mould U^\bullet which values are in $\mathbb{C}\{x\}$. Let

$$\mathbf{H}^+ = \{\eta \in \mathbf{H}; \|\mathbf{n}\| \geq 0\} \quad \mathbf{H}^- = \{\eta \in \mathbf{H}; \|\mathbf{n}\| < 0\} \quad (3.1)$$

For a sequence of length 1, if $\eta_1 \in \mathbf{H}^+$, W^{η_1} is convergent so $U^{\eta_1} = W^{\eta_1}$, otherwise, W^{η_1} is divergent and $U^{\eta_1} = 0$.

Let $(\eta_1, \eta_2) \in \mathbf{H}$.

- If $(\eta_1, \eta_2) \in \mathbf{H}^+$ and $\eta_2 \in \mathbf{H}^+$, W^{η_1, η_2} is convergent so $U^{\eta_1, \eta_2} = W^{\eta_1, \eta_2}$.
- If $(\eta_1, \eta_2) \in \mathbf{H}^+$ and $\eta_2 \in \mathbf{H}^-$ (note that $\eta_1 \in \mathbf{H}^+$), W^{η_1, η_2} is divergent but, because of the symmetrelity,

$$W^{\eta_1} W^{\eta_2} = W^{\eta_1, \eta_2} + W^{\eta_2, \eta_1} + W^{\eta_1 + \eta_2}$$

but $W^{\eta_2, \eta_1} = U^{\eta_2, \eta_1}$ and $W^{\eta_1 + \eta_2} = U^{\eta_1 + \eta_2}$ are convergent thus $U^{\eta_1, \eta_2} = W^{\eta_1, \eta_2} - W^{\eta_1} W^{\eta_2}$ is convergent.

- If $(\eta_1, \eta_2) \in \mathbf{H}^-$, then W^{η_1, η_2} is divergent and $U^{\eta_1, \eta_2} = 0$.

We transformed W^\bullet into a new mould that seems to remain symmetrel. In fact, this suggests the following definition.

DEFINITION 3.1. — *Let*

$$\alpha_+(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} \quad \alpha_-(n) = \begin{cases} 0 & \text{if } n \geq 0 \\ -1 & \text{if } n < 0 \end{cases} \quad (3.2)$$

Then the symmetrel mould J^\bullet on \mathbf{H} is defined as follows : $J^\emptyset = 1$, $J^{\eta_1} = \alpha_+(n_1)$ and, for $s \geq 2$,

$$J^{\eta_1, \dots, \eta_s} = \alpha_+(\hat{n}_1) \alpha_-(\hat{n}_2) \dots \alpha_-(\hat{n}_s) \quad (3.3)$$

The above digression suggest to define

$$U^\bullet = J^\bullet \circ W^\bullet \quad (3.4)$$

3.2. Some results on the mould J^\bullet

As $\alpha_+ = 1 + \alpha_-$,

$$J^\bullet = I^\bullet \times K^\bullet \quad K^\bullet = H^\bullet \times J^\bullet \quad (3.5)$$

where the symmetrel mould K^\bullet is defined by

$$K^{\eta_1, \dots, \eta_s} = \alpha_-(\hat{n}_1) \alpha_-(\hat{n}_2) \dots \alpha_-(\hat{n}_s) \quad (3.6)$$

We give now some identities related to J^\bullet and K^\bullet (whose very simple proofs are left to the reader).

$$J^\bullet \circ K^\bullet = K^\bullet \circ J^\bullet = 1^\bullet \quad (3.7)$$

This implies that

$$\begin{aligned} J^\bullet \circ J^\bullet &= (I^\bullet \times K^\bullet) \circ J^\bullet \\ &= (I^\bullet \circ J^\bullet) \times (K^\bullet \circ J^\bullet) \\ &= J^\bullet \times 1^\bullet \\ &= J^\bullet \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} K^\bullet \circ K^\bullet &= (H^\bullet \times J^\bullet) \circ K^\bullet \\ &= (H^\bullet \circ K^\bullet) \times (J^\bullet \circ K^\bullet) \\ &= \text{inv}(K)^\bullet \times 1^\bullet \\ &= \text{inv}(K)^\bullet \end{aligned} \quad (3.9)$$

We end this section with a useful result :

PROPOSITION 3.2. — *For any moulds A^\bullet and B^\bullet , we have*

$$J^\bullet \circ A^\bullet = J^\bullet \circ (A^\bullet \times (K^\bullet \circ B^\bullet)) \quad (3.10)$$

For the proof, see section 7.1.

3.3. Definition and properties of U^\bullet and V^\bullet

Let :

$$U^\bullet = J^\bullet \circ W^\bullet \quad (3.11)$$

From the definition of J^\bullet one easily deduce that $U^\emptyset = 1$, and, if $\boldsymbol{\eta}$ is a nonempty sequence of \mathbf{H}^- , $U^\boldsymbol{\eta} = 0$.

We remind that, for any mould A^\bullet , $s_q(A^\eta) = x^{-\|\eta\|} \sigma_q \cdot A^\eta$ for a nonempty sequence and $s_q A^\emptyset = A^\emptyset$ if $A^\emptyset = 1$. The mould W^\bullet is defined by the identities $W^\emptyset = 1$ and

$$s_q(W^\bullet) = X^\bullet \times W^\bullet \tag{3.12}$$

If the mould V^\bullet is such that

$$s_q(U^\bullet) \times V^\bullet = X^\bullet \times U^\bullet \tag{3.13}$$

then, as $s_q(U^\bullet)$ is symmetrical (thus invertible), V^\bullet is well defined and symmetrical.

Since J^\bullet is a constant mould (with values in \mathbb{C}),

$$\begin{aligned} s_q(U^\bullet) &= s_q(J^\bullet \circ W^\bullet) \\ &= J^\bullet \circ s_q(W^\bullet) \\ &= J^\bullet \circ (X^\bullet \times W^\bullet) \\ &= J^\bullet \circ (X^\bullet \times W^\bullet \times (K^\bullet \circ W^\bullet)) \\ &= J^\bullet \circ (X^\bullet \times ((I^\bullet \times K^\bullet) \circ W^\bullet)) \\ &= J^\bullet \circ (X^\bullet \times (J^\bullet \circ W^\bullet)) \\ &= J^\bullet \circ (X^\bullet \times U^\bullet) \end{aligned} \tag{3.14}$$

thus V^\bullet is defined by

$$(J^\bullet \circ (X^\bullet \times U^\bullet)) \times V^\bullet = X^\bullet \times U^\bullet$$

but

$$\begin{aligned} (J^\bullet \circ (X^\bullet \times U^\bullet)) &= ((I^\bullet \times K^\bullet) \circ (X^\bullet \times U^\bullet)) \\ &= X^\bullet \times U^\bullet \times (K^\bullet \circ (X^\bullet \times U^\bullet)) \end{aligned}$$

thus

$$(K^\bullet \circ (X^\bullet \times U^\bullet)) \times V^\bullet = 1^\bullet$$

and, since the inverse of $K^\bullet \circ (X^\bullet \times U^\bullet)$ is $(\text{inv}(K^\bullet)) \circ (X^\bullet \times U^\bullet)$,

$$V^\bullet = (\text{inv}(K^\bullet)) \circ (X^\bullet \times U^\bullet) \tag{3.15}$$

but if $L^\bullet = \text{inv}(K^\bullet)$ then $L^\emptyset = 1$, $L^{\eta_1} = -\alpha_-(\eta_1)$ and, for $(\eta_1, \dots, \eta_s) \in \mathbf{H}$ ($s \geq 2$):

$$L^{\eta_1, \dots, \eta_s} = (-1)^s \alpha_+(\check{\eta}_1) \dots \alpha_+(\check{\eta}_{s-1}) \alpha_-(\check{\eta}_s) \tag{3.16}$$

so it is clear that, if η is a nonempty sequence in \mathbf{H}^+ , then $V^\eta = 0$.

3.4. Back to the equations

We have two moulds U^\bullet and V^\bullet on \mathbf{H} , with values in $\mathbb{C}[[x]]$, such that U^\bullet (resp. V^\bullet) vanishes on \mathbf{H}^- (resp. \mathbf{H}^+). Let us consider the analytic equation

$$x\sigma_q y = y + b(y, x)$$

and the a priori formal equation

$$x\sigma_q z = \mathbb{V}.z = \left(\sum V^\bullet \mathbb{D}_\bullet \right).z$$

where \mathbb{V} is a formal automorphism. We will prove that these equation are conjugate by the a priori formal automorphism

$$\mathbb{U} = \sum U^\bullet \mathbb{D}_\bullet. \tag{3.17}$$

If $y = \mathbb{U}.z$, then,

$$\begin{aligned} x\sigma_q y &= x\sigma_q(\mathbb{U}.z) \\ &= \sum_{\boldsymbol{\eta} \in \mathbf{H}} x\sigma_q(U^\eta \mathbb{D}_\eta.z) \\ &= \sum_{\boldsymbol{\eta} \in \mathbf{H}} x\sigma_q(U^\eta) \sigma_q(\mathbb{D}_\eta.z) \end{aligned}$$

but $\mathbb{D}_\emptyset.z = z$, and, for a nonempty sequence $\boldsymbol{\eta}$, there exists a constant β_η such that $\mathbb{D}_\eta.z = \beta_\eta z^{1+\|\boldsymbol{\eta}\|}$:

$$\begin{aligned} \sigma_q(\mathbb{D}_\eta.z) &= \sigma_q(\beta_\eta z^{1+\|\boldsymbol{\eta}\|}) \\ &= \beta_\eta \sigma_q(z^{1+\|\boldsymbol{\eta}\|}) \\ &= x^{-1-\|\boldsymbol{\eta}\|} \beta_\eta (x\sigma_q z)^{1+\|\boldsymbol{\eta}\|} \\ &= x^{-1-\|\boldsymbol{\eta}\|} \beta_\eta (\mathbb{V}.z)^{1+\|\boldsymbol{\eta}\|} \\ &= x^{-1-\|\boldsymbol{\eta}\|} \beta_\eta \mathbb{V}.(z^{1+\|\boldsymbol{\eta}\|}) \\ &= x^{-1-\|\boldsymbol{\eta}\|} \mathbb{V}.(\beta_\eta z^{1+\|\boldsymbol{\eta}\|}) \\ &= x^{-1-\|\boldsymbol{\eta}\|} \mathbb{V}.\mathbb{D}_\eta.z \end{aligned}$$

this identity still holds for the empty sequence with $\|\mathbf{n}\| = 0$. One can deduce that,

$$\begin{aligned}
 x\sigma_q y &= \sum_{\eta \in \mathbf{H}} x\sigma_q(U^\eta)\sigma_q(\mathbb{D}_\eta.z) \\
 &= \sum_{\eta \in \mathbf{H}} x\sigma_q(U^\eta)x^{-1-\|\mathbf{n}\|}\mathbb{V}.\mathbb{D}_\eta.z \\
 &= \mathbb{V}.\sum_{\eta \in \mathbf{H}} s_q(U^\eta)\mathbb{D}_\eta.z \\
 &= \left(\sum V^\bullet \mathbb{D}_\bullet\right).\left(\sum s_q(U^\bullet)\mathbb{D}_\bullet\right).z \\
 &= \left(\sum (s_q(U^\bullet) \times V^\bullet)\mathbb{D}_\bullet\right).z
 \end{aligned} \tag{3.18}$$

But $s_q(U^\bullet) \times V^\bullet = X^\bullet \times U^\bullet$ thus

$$\begin{aligned}
 x\sigma_q y &= \left(\sum (X^\bullet \times U^\bullet)\mathbb{D}_\bullet\right).z \\
 &= \mathbb{U}.\left(\sum X^\bullet \mathbb{D}_\bullet\right).z \\
 &= \mathbb{U}.(z + b(z, x)) \\
 &= \mathbb{U}.z + b(\mathbb{U}.z, x) \\
 &= y + b(y, x)
 \end{aligned} \tag{3.19}$$

Finally, $\mathbb{D}_\emptyset.z = z$, and, for a nonempty sequence η , there exists a constant β_η such that $\mathbb{D}_\eta.z = \beta_\eta z^{1+\|\mathbf{n}\|}$ but since \mathbb{D}_η is a differential operator in z , if $\|\mathbf{n}\| < -1$, then $\beta_\eta = 0$. As V^\bullet vanishes on \mathbf{H}^+ , it is clear that

$$\mathbb{V}.z = z + c(x) \in z + x\mathbb{C}[[x]] \tag{3.20}$$

In conclusion

THEOREM 3.3. — *Let us consider the equation*

$$(x\sigma_q - 1)y = b(y, x) \tag{3.21}$$

with $b(0, 0) = 0$ and $\frac{\partial b}{\partial y}(0, 0) = 0$. This equation is conjugate by the a priori formal automorphism \mathbb{U} :

$$y = \mathbb{U}.z = \left(\sum U^\bullet \mathbb{D}_\bullet\right).z \in \mathbb{C}[[z, x]] \tag{3.22}$$

to the a priori formal equation

$$x\sigma_q z = \mathbb{V}.z = \left(\sum V^\bullet \mathbb{D}_\bullet\right).z \in z + x\mathbb{C}[[x]] \tag{3.23}$$

It remains to prove that $\mathbb{U}.z \in \mathbb{C}\{z, x\}$ and $\mathbb{V}.z \in z + x\mathbb{C}\{x\}$. We will first prove that the values of U^\bullet and V^\bullet are in $\mathbb{C}\{x\}$ (i.e. U^\bullet and V^\bullet are analytic) then we will use the arborification process to prove that \mathbb{U} and \mathbb{V} are *convergent* automorphisms, namely, $\mathbb{U}.z \in \mathbb{C}\{z, x\}$ and $\mathbb{V}.z \in z + x\mathbb{C}\{x\}$.

4. Analyticity of U^\bullet and V^\bullet

4.1. The mould U^\bullet

We prove that

$$\forall \boldsymbol{\eta} \in \mathbf{H}, \quad U^\boldsymbol{\eta} \in x^{\|\boldsymbol{\sigma}\|} \mathbb{C}\{x\} \quad (4.1)$$

We already know that $\forall \boldsymbol{\eta} \in \mathbf{H}, \quad U^\boldsymbol{\eta} \in \mathbb{C}[[x]]$. Let us remind (see [2]) that if $n > 0$ and $g \in \mathbb{C}\{x\}$, then the equation

$$(x^{-n} \sigma_q - 1)f = g$$

has a unique solution in $\mathbb{C}\{x\}$ of greater valuation than g and if $g \in x\mathbb{C}\{x\}$, then the equation

$$(\sigma_q - 1)f = g$$

has a unique solution in $x\mathbb{C}\{x\}$ of same valuation.

Let us prove 4.1 by induction on the length of the sequences in \mathbf{H} . The result is obvious for the empty sequence ($U^\emptyset = 1$). We remind that :

$$s_q(U^\bullet) = J^\bullet \circ (X^\bullet \times U^\bullet)$$

For a sequence $\boldsymbol{\eta} \in \mathbf{H}$ of length $l(\boldsymbol{\eta}) = 1$ ($\boldsymbol{\eta} = \eta_1$), if $\eta_1 \in \mathbf{H}^-$ then $U^{\eta_1} = 0$ and, otherwise,

$$\begin{aligned} s_q(U^{\eta_1}) &= J^{\eta_1}(x^{\sigma_1} + U^{\eta_1}) \\ x^{-n_1} \sigma_q U^{\eta_1} &= x^{\sigma_1} + U^{\eta_1} \end{aligned}$$

note that $n_1 \geq 0$, $x^{\sigma_1} \in \mathbb{C}\{x\}$ and if $n_1 = 0$ then, by definition of \mathbf{H} , $\sigma_1 \geq 1$ and $x^{\sigma_1} \in x\mathbb{C}\{x\}$. So, in any case,

$$U^{\eta_1} \in x^{\sigma_1} \mathbb{C}\{x\}.$$

Let $s \geq 2$ and suppose that

$$\forall \boldsymbol{\eta} \in \mathbf{H}; l(\boldsymbol{\eta}) < s \quad U^\boldsymbol{\eta} \in x^{\|\boldsymbol{\sigma}\|} \mathbb{C}\{x\}$$

and $\boldsymbol{\eta} \in \mathbf{H}$ such that $l(\boldsymbol{\eta}) = s$ ($\boldsymbol{\eta} = \eta_1, \dots, \eta_s$). If $\boldsymbol{\eta} \in \mathbf{H}^-$ then $U^\boldsymbol{\eta} = 0$, otherwise

$$\begin{aligned}
 s_q(U^\boldsymbol{\eta}) &= (J^\bullet \circ (X^\bullet \times U^\bullet))^\boldsymbol{\eta} \\
 x^{-\|\mathbf{n}\|} \sigma_q(U^\boldsymbol{\eta}) &= J^{\|\boldsymbol{\eta}\|} (U^\boldsymbol{\eta} + x^{\sigma_1} U^\boldsymbol{\eta}) \\
 &\quad + \sum_{\substack{\eta^1 \dots \eta^t = \boldsymbol{\eta} \\ t \geq 2}} J^{\|\boldsymbol{\eta}^1\|, \dots, \|\boldsymbol{\eta}^t\|} (X^\bullet \times U^\bullet)^{\eta^1} \dots (X^\bullet \times U^\bullet)^{\eta^t} \\
 &= U^\boldsymbol{\eta} + x^{\sigma_1} U^\boldsymbol{\eta} \\
 &\quad + \sum_{\substack{\eta^1 \dots \eta^t = \boldsymbol{\eta} \\ t \geq 2}} J^{\|\boldsymbol{\eta}^1\|, \dots, \|\boldsymbol{\eta}^t\|} (X^\bullet \times U^\bullet)^{\eta^1} \dots (X^\bullet \times U^\bullet)^{\eta^t} \\
 &= U^\boldsymbol{\eta} + P^\boldsymbol{\eta}
 \end{aligned}$$

By induction, it is clear the $P^\boldsymbol{\eta}$ depends on values of U^\bullet for sequences of length smaller than s , thus, $P^\boldsymbol{\eta} \in x^{\|\boldsymbol{\sigma}\|} \mathbb{C}\{x\}$. If $\|\mathbf{n}\| > 0$, then $U^\boldsymbol{\eta} \in x^{\|\boldsymbol{\sigma}\|} \mathbb{C}\{x\}$, if $\|\mathbf{n}\| = 0$, by construction of \mathbf{H} , $\|\boldsymbol{\sigma}\| > 0$ and, once again $U^\boldsymbol{\eta} \in x^{\|\boldsymbol{\sigma}\|} \mathbb{C}\{x\}$.

This ends the proof by induction.

4.2. The mould V^\bullet

There is not much to say to prove that,

$$\forall \boldsymbol{\eta} \in \mathbf{H}, \quad V^\boldsymbol{\eta} \in x^{\|\boldsymbol{\sigma}\|} \mathbb{C}\{x\} \tag{4.2}$$

since this is already true for U^\bullet and

$$V^\bullet = (\text{inv}(K^\bullet)) \circ (X^\bullet \times U^\bullet)$$

We end this section by giving an interesting formula for U^\bullet which is similar to the one given in theorem 2.3.

4.3. A formula for U^\bullet

THEOREM 4.1. — *If $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \mathbf{H}$, then the monomial $U^\boldsymbol{\eta}$ is a convergent series with the following expression :*

$$U^\boldsymbol{\eta}(x) = \varepsilon_{\mathbf{n}} \sum_{(k_1, \dots, k_s) \in \mathbb{Z}^{\mathbf{n}} \cap P^{\mathbf{n}}} x^{\|\boldsymbol{\sigma}\| + \langle \check{\mathbf{k}}, \mathbf{n} \rangle} q^{-\langle \check{\mathbf{k}}, \boldsymbol{\sigma} \rangle} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s} \tag{4.3}$$

with the notations

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- $\varepsilon_n = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$ with $\hat{n}_i = n_i + \dots + n_s$ with $\varepsilon_n = 1$ (resp. $\varepsilon_n = -1$) if $n \geq 0$ (resp. $n < 0$)
- $\mathbb{Z}^n = \mathbb{Z}^{\hat{n}_1} \times \dots \times \mathbb{Z}^{\hat{n}_s}$ with $\mathbb{Z}^n = \mathbb{Z}^{+*}$ (resp. \mathbb{Z}^-) if $n \geq 0$ (resp. $n < 0$)
- $P^n = \{(k_1, \dots, k_s) \in \mathbb{Z}^s; \check{k}_i > 0 \text{ if } \varepsilon_{\hat{n}_i} = -1\}$
- $\check{k}_i = k_1 + \dots + k_i$ and $\check{\mathbf{k}} = (\check{k}_1, \dots, \check{k}_s)$.
- $[k+1]_q = q^{-k(k+1)/2}$

Note that if $\|\mathbf{n}\| < 0$, then $\varepsilon_{\hat{n}_1} = \varepsilon_{\|\mathbf{n}\|} = -1$ thus $k_1 \in \mathbb{Z}^-$ but, in P^n , $k_1 > 0$, thus $\mathbb{Z}^n \cap P^n = \emptyset$ and we recover that $U^\eta = 0$. We shall focus on sequences in \mathbf{H}^+ . The complete proof can be found in section 7.2. We just give an example here to catch the idea.

For a sequence $(k_1, \dots, k_s) \in \mathbb{Z}^s$, let $l_i = \check{k}_i$, thus $k_1 = l_1$ and, for i greater than 1, $k_i = l_i - l_{i-1}$. For a sequence $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+$ and $\mathbf{l} = (l_1, \dots, l_s) \in \mathbb{Z}^s$ ($s \geq 1$), let

$$T_l^\eta = x^{|\sigma| + \langle \mathbf{l}, \mathbf{n} \rangle} q^{-\langle \mathbf{l}, \sigma \rangle} [l_1 + 1]_q^{\eta_1} \dots [l_s + 1]_q^{\eta_s} = \prod_{i=1}^s T_{l_i}^{\eta_i} \quad (4.4)$$

With this notation, the formula in theorem 2.3 becomes

$$W^\eta(x) = \varepsilon_n \sum_{(l_1, l_2 - l_1, \dots, l_s - l_{s-1}) \in \mathbb{Z}^n} T_l^\eta$$

using the function $\rho_1(n) = \begin{cases} 1 & \text{if } n > 0 \\ 0 & \text{if } n \leq 0 \end{cases}$ and $\rho_{-1} = 1 - \rho_1$, we get

$$W^\eta(x) = \varepsilon_n \sum_{\mathbf{l} \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1) \rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1) \dots \rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) T_l^\eta = \sum_{\mathbf{l} \in \mathbb{Z}^s} w_l^\eta T_l^\eta \quad (4.5)$$

and we have to prove that

$$\begin{aligned} U^\eta(x) &= \varepsilon_n \sum_{\mathbf{l} \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1) \rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1) \dots \rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) T_l^\eta \prod_{\substack{1 \leq i \leq s \\ \varepsilon_{\hat{n}_i} = -1}} \rho_1(l_i) \\ &= \sum_{\mathbf{l} \in \mathbb{Z}^s} \tilde{u}_l^\eta T_l^\eta \end{aligned} \quad (4.6)$$

Let $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) \in \mathbf{H}^+$ such that $\hat{n}_1 \geq 0$, $\hat{n}_2 < 0$, $\hat{n}_3 < 0$, then,

$$\begin{aligned}
 \tilde{u}_l^\eta &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2)\rho_1(l_3) \\
 &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2)(1 - \rho_{-1}(l_3)) \\
 &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2)\rho_{-1}(l_3) \\
 &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_1(l_2)\rho_{-1}(l_3) \\
 &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)(1 - \rho_{-1}(l_2)) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)(1 - \rho_{-1}(l_2))\rho_{-1}(l_3) \\
 &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_{-1}(l_2) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3) \\
 &\quad + \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_2)\rho_{-1}(l_3) \\
 &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2) - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3 - l_2) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3) + \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3)
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 U^\eta &= (J^\bullet \circ W^\bullet)^\eta \\
 &= W^{\eta_1, \eta_2, \eta_3} - W^{\eta_1, \eta_2} W^{\eta_3} - W^{\eta_1} W^{\eta_2, \eta_3} + W^{\eta_1} W^{\eta_2} W^{\eta_3} \\
 &= \sum_{l \in \mathbb{Z}^s} u_l^\eta T_l^\eta
 \end{aligned}$$

It is clear that $n_1 \geq 0$, $n_1 + n_2 \geq 0$.

- If $n_2 < 0$, then

$$\begin{aligned}
 u_l^\eta &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2) - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3 - l_2) + \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3) \\
 &= \tilde{u}_l^\eta
 \end{aligned}$$

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- If $n_2 \geq 0$, then

$$\begin{aligned}
 u_l^\eta &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2) + \rho_1(l_1)\rho_1(l_2 - l_1)\rho_{-1}(l_3) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3 - l_2) - \rho_1(l_1)\rho_1(l_2)\rho_{-1}(l_3) \\
 &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2) + \rho_1(l_1)(1 - \rho_{-1}(l_2 - l_1))\rho_{-1}(l_3) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3 - l_2) - \rho_1(l_1)(1 - \rho_{-1}(l_2))\rho_{-1}(l_3) \\
 &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2) - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3) \\
 &\quad - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3 - l_2) + \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3) \\
 &= \tilde{u}_l^\eta
 \end{aligned}$$

and that proves that the formula holds in this case.

4.4. Estimates for U^\bullet and “divergence” of $\sum U^\bullet \mathbb{D}_\bullet$.

Using the above formula for U^\bullet , one gets

PROPOSITION 4.2. — *Let $0 \leq |x| \leq \varepsilon \leq q^{-1/2}$. For $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+$,*

$$|U^\boldsymbol{\eta}(x)| \leq C_q^s |x|^{\|\mathbf{n}\| + \|\boldsymbol{\sigma}\|} q^{-(\|\mathbf{n}\| + \|\boldsymbol{\sigma}\|)} \quad (4.7)$$

with $C_q = \frac{1}{1 - q^{-1/2}}$.

For the proof see section 7.3.

Unfortunately, this is not sufficient to assume that $\mathbb{U} = \sum U^\bullet \mathbb{D}_\bullet$ is “convergent” : in order to study the convergence of the automorphism $\mathbb{U} = \sum U^\bullet \mathbb{D}_\bullet$, it is sufficient to prove that $\mathbb{U}.z = \sum U^\bullet \mathbb{D}_\bullet.z$ is in $\mathbb{C}\{x, z\}$. We can’t establish directly the convergence of the series of power series because, for a sequence $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+$ the only estimates (in a neighborhood of $(z, x) = (0, 0)$) we got are

$$|U^\boldsymbol{\eta}(x)| \leq C_q^s |x|^{\|\mathbf{n}\| + \|\boldsymbol{\sigma}\|} q^{-(\|\mathbf{n}\| + \|\boldsymbol{\sigma}\|)}$$

whereas (see [1]) we can’t hope to improve lower estimate such as

$$|\mathbb{D}_{\boldsymbol{\eta}}.z| \geq s! C^{s + \|\boldsymbol{\sigma}\| + \|\mathbf{n}\|} |z|^{1 + \|\mathbf{n}\|} \quad (4.8)$$

Note that this does not prove that $\mathbb{U}.z$ is formal but that its expansion $\sum U^\bullet \mathbb{D}_\bullet.z$ is not normally convergent. In fact, we prove in the next section that \mathbb{U} is analytic, using a different expansion obtained by arborification, and that appears to be normally convergent.

5. Analytic part : arborification

5.1. Reminder on arborification

5.1.1. Contracting arborification

We follow the definitions of J. Ecalle [1]. Let us consider an additive semigroup H . The set \mathbf{H} is the set of sequences on H , where a sequence is a totally ordered sequence of elements of H , with possible repetitions.

An *arborescent sequence* on H is a sequence $\eta^< = (\eta_1, \dots, \eta_s)^< \in \mathbf{H}^<$ of elements of H with an arborescent order on the indices $\{1, \dots, s\}$: each $i \in \{1, \dots, s\}$ possess at most one predecessor i_- . We note $\eta^< = \eta'^< \oplus \eta''^<$ the disjoint union of $\eta'^<$ and $\eta''^<$, the partial orders of $\eta'^<$ and $\eta''^<$ being preserved and the elements $\eta'^<$ are not comparable with those of $\eta''^<$. \emptyset is the empty sequence. A sequence $\eta^<$ is irreducible if it is not a disjoint union of smaller nontrivial sequences ; that is to say that it has exactly one least element.

We remind here that a mould $A^\bullet = \{A^\eta\}$ on H with values in a commutative algebra is a family of elements A^η indexed by the sequences $\eta \in \mathbf{H}$ of H . For example, U^\bullet is a mould on H with values in $\mathbb{C}\{x\}$. Moreover, this mould is symmetrel : $U^\emptyset = 1$ and, for any pair (η', η'') , we get

$$U^{\eta'} U^{\eta''} = \sum \text{ctsh} \left(\begin{array}{c} \eta', \eta'' \\ \eta \end{array} \right) U^\eta \quad (5.1)$$

Where $\text{ctsh} \left(\begin{array}{c} \eta', \eta'' \\ \eta \end{array} \right)$ is the number of ways to get η by contracting shuffling of η' and η'' .

We also remind that an arborescent mould $A^{\bullet<} = \{A^{\eta^<}\}$ on H with values in a commutative algebra is a family of elements $A^{\eta^<}$ indexed by the arborescent sequences $\eta^< \in \mathbf{H}^<$ of H . Such an arborescent mould $A^{\bullet<}$ is separative if :

$$A^\emptyset = 1 \quad \text{and} \quad \forall \eta'^<, \eta''^<, \quad A^{\eta'^< \oplus \eta''^<} = A^{\eta'^<} A^{\eta''^<} \quad (5.2)$$

We get such arborescent separative moulds by *contracting arborification* of symmetrel moulds. This operation is defined as follows.

Let $\eta^< = (\eta_1, \dots, \eta_s)^<$ be an arborescent sequence and $\eta' = (\eta'_1, \dots, \eta'_{s'})$ a totally ordered sequence. Let $\text{cont} \left(\begin{array}{c} \eta^< \\ \eta' \end{array} \right)$ be the number of monotonic

contractions of $\eta^<$ on η' , that is to say the number of surjections σ from $\{1, \dots, s\}$ into $\{1, \dots, s'\}$ such that :

$$(i_1 < i_2 \text{ in } \eta^<) \implies (\sigma(i_1) < \sigma(i_2) \text{ in } \eta') \quad (5.3)$$

$$\forall j \in \{1, \dots, s'\} \quad ; \quad \eta'_j = \sum_{\sigma(i)=j} \eta_i \quad (5.4)$$

The relation

$$A^{\eta^<} = \sum \text{cont} \left(\begin{array}{c} \eta^< \\ \eta' \end{array} \right) A^{\eta'} \quad (5.5)$$

defines a homomorphism from the algebra of moulds into the algebra of arborescent moulds. Moreover, the contracting arborification of a symmetrel mould is separative. One can also notice that, if η is a totally ordered sequence and $\eta^<$ is that arborescent sequence with the same order (total), then $A^{\eta^<} = A^\eta$.

5.1.2. Product and composition

We give here some formulas for the arborification of a product and of a composition of moulds.

Product. — Let M^\bullet and N^\bullet two moulds on H , then

$$P^{\bullet^<} = (M^\bullet \times N^\bullet)^< = M^{\bullet^<} \times N^{\bullet^<} \iff P^{\eta^<} = \sum M^{\eta'^<} N^{\eta''^<} \quad (5.6)$$

with a sum extended to all the monotonic partitions of $\eta^<$ in $\eta'^<$ and $\eta''^<$, that is to say all the partitions such that no element in $\eta'^<$ is greater than any element of $\eta''^<$. The order of $\eta'^<$ and $\eta''^<$ is of course inherited from the one of $\eta^<$. For example, if $\eta^< = \eta_1 \bullet (\eta_2 \oplus \eta_3)$ (η_2 and η_3 are not comparable but have η_1 as a common predecessor), then

$$P^{\eta^<} = M^{\eta^<} N^\emptyset + M^\emptyset N^{\eta^<} + M^{\eta_1} N^{\eta_2 \oplus \eta_3} + M^{\eta_1 \bullet \eta_2} N^{\eta_3} + M^{\eta_1 \bullet \eta_3} N^{\eta_2}$$

The arborescent separative moulds are stable by multiplication and the multiplicative inverse $N^{\bullet^<}$ of a separative arborescent mould $M^{\bullet^<}$ can be computed with the formula :

$$N^{\eta^<} = \sum (-1)^s M^{\eta^{1^<}} \dots M^{\eta^{s^<}} \quad (5.7)$$

where the sum is extended to the monotonic partitions $\eta^{1^<}, \dots, \eta^{s^<}$ of $\eta^<$, with no empty sequence $\eta^{i^<}$, counting separately partitions which differs only by the order of the $\eta^{i^<}$.

Composition. — Let M^\bullet and N^\bullet two moulds on H , then

$$P^{\bullet <} = (M^\bullet \circ N^\bullet)^{<} \iff P^{\eta^{<}} = \sum M^{\|\eta^{1 <}\|, \dots, \|\eta^{s <}\|} N^{\eta^{1 <}} \dots N^{\eta^{s <}} \quad (5.8)$$

where the sum is over the same set as in the formula for the multiplicative inverse and $\|\eta^{<}\| = \eta_1 + \dots + \eta_s$ if $\eta^{<} = (\eta_1, \dots, \eta_s)$.

5.2. The arborified mould $U^{\bullet <}$

We will first change some notations. Let $\eta^{<} = (\eta_1, \dots, \eta_s)^{<}$ be an arborescent sequence of length s and of sum $\|\eta^{<}\| = \eta_1 + \dots + \eta_s$. We redefine the partial sums :

$$\begin{aligned} \tilde{n}_i &= \sum_{j \leq i} n_j \\ \hat{n}_i &= \sum_{j \geq i} n_j \end{aligned} \quad (5.9)$$

where the orders \leq and \geq are now relative to the partial order on $\{1, \dots, s\}$. We have the following theorem :

THEOREM 5.1. — *If $\eta^{<} = (\eta_1, \dots, \eta_s)^{<} = \left(\begin{smallmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{smallmatrix} \right)^{<} = \left(\begin{smallmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{smallmatrix} \right)^{<} \in \mathbf{H}^{<}$, then the monomial $W^{\eta^{<}}$ is a formal series and :*

$$W^{\eta^{<}}(x) = \varepsilon_{\mathbf{n}^{<}} \sum_{(k_1, \dots, k_s)^{<} \in \mathbb{Z}^{\mathbf{n}^{<}}} x^{\|\boldsymbol{\sigma}\| + \langle \check{\mathbf{k}}, \mathbf{n} \rangle} q^{-\langle \check{\mathbf{k}}, \boldsymbol{\sigma} \rangle} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s} \quad (5.10)$$

with the following rules

- The sequences $\mathbf{n}^{<}$ and $(k_1, \dots, k_s)^{<}$ inherit the partial order of $\eta^{<}$.
- $\varepsilon_{\mathbf{n}} = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$ with $\hat{n}_i = \sum_{j \geq i} n_j$ and $\varepsilon_n = 1$ (resp. $\varepsilon_n = -1$) if $n \geq 0$ (resp. $n < 0$).
- $\mathbb{Z}^{\mathbf{n}^{<}} = (\mathbb{Z}^{\hat{n}_1} \times \dots \times \mathbb{Z}^{\hat{n}_s})^{<}$ and $\mathbb{Z}^n = \mathbb{Z}^{+*}$ (resp. \mathbb{Z}^-) if $n \geq 0$ (resp. $n < 0$).
- $\check{k}_i = \sum_{j \leq i} k_j$.

This result was given in [2]. The formula for W^\bullet was derived from the fact that

$$s_q(W^\bullet) = X^\bullet \times W^\bullet$$

An example of local analytic q -difference equation : Analytic classification

As X^\bullet and W^\bullet are symmetrel the arborescent moulds $X^{\bullet<}$ and $W^{\bullet<}$ are separative. Moreover, for an irreducible sequence $\eta^< = (\eta_1, \dots, \eta_s)^< = \eta_1 \bullet \eta'^< \in \mathbf{H}^<$, it is easy to check that

$$X^{\eta^<} = \begin{cases} x^{\sigma_1} & \text{if } s = 1 \\ 0 & \text{otherwise} \end{cases}$$

and then

$$s_q(W^{\eta^<}) = W^{\eta^<} + x^{\sigma_1} W^{\eta'^<}$$

It is then easy to check that the formula works. From this one can deduce a formula for $U^{\bullet<}$:

THEOREM 5.2. — *If $\eta^< = (\eta_1, \dots, \eta_s)^< = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix}^< = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix}^< \in \mathbf{H}^<$, then the monomial $U^{\eta^<}$ is a convergent series with the following expression :*

$$U^{\eta^<}(x) = \varepsilon_{\mathbf{n}^<} \sum_{(k_1, \dots, k_s)^< \in \mathbb{Z}^{\mathbf{n}^<} \cap P^{\mathbf{n}^<}} x^{\|\boldsymbol{\sigma}\| + \langle \check{k}, \mathbf{n} \rangle} q^{-\langle \check{k}, \boldsymbol{\sigma} \rangle} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s} \quad (5.11)$$

with the notations

- The sequences $\mathbf{n}^<$ and $(k_1, \dots, k_s)^<$ inherit the partial order of $\eta^<$.
- $\varepsilon_{\mathbf{n}} = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$ with $\hat{n}_i = \sum_{j \geq i} n_j$ with $\varepsilon_n = 1$ (resp. $\varepsilon_n = -1$) if $n \geq 0$ (resp. $n < 0$)
- $\mathbb{Z}^{\mathbf{n}^<} = \mathbb{Z}^{\hat{n}_1} \times \dots \times \mathbb{Z}^{\hat{n}_s}$ with $\mathbb{Z}^n = \mathbb{Z}^{+*}$ (resp. \mathbb{Z}^-) if $n \geq 0$ (resp. $n < 0$)
- $P^{\mathbf{n}^<} = \{(k_1, \dots, k_s)^< \in \mathbb{Z}^s; \check{k}_i > 0 \text{ if } \varepsilon_{\hat{n}_i} = -1\}$
- $\check{k}_i = \sum_{j \leq i} k_j$.
- $[k + 1]_q = q^{-k(k+1)/2}$

The proof is essentially the same as in theorem 4.1, using the formula linking arborification to composition.

One can notice that, for an irreducible sequence $\eta^<$, if $\|\mathbf{n}^<\| < 0$ then $U^{\eta^<} = 0$ and as $U^{\bullet<}$ is separative, for a sequence $\eta^< = \eta^{1<} \oplus \dots \oplus \eta^{s<}$

($\eta^{i<}$ irreducible), if there exist i_0 such that $\|\mathbf{n}^{i_0<}\| < 0$ then $U^{\eta^{i_0<}} = 0$. It means that we can restrict ourselves to arborescent sequences

$$\eta^{i<} \in \mathbf{H}^{+i<} = \{\eta^{1i<} \oplus \dots \oplus \eta^{s_i i<} \ ; \ \eta^{i_0 i<} \text{ irreducible and } \|\mathbf{n}^{i_0 i<}\| \geq 0\} \quad (5.12)$$

Using this theorem, it is easy to obtain, as in proposition 4.2, that

PROPOSITION 5.3. — *Let $0 \leq |x| \leq q^{-1/2}$. For $\eta^{i<} = (\eta_1, \dots, \eta_s)^{i<} \in \mathbf{H}^{+i<}$,*

$$|U^{\eta^{i<}}(x)| \leq C_q^s |x|^{\|\mathbf{n}^{i<}\| + \|\sigma^{i<}\|} q^{-\|\mathbf{n}^{i<}\| - \|\sigma^{i<}\|} \quad (5.13)$$

5.3. Coarborification

THEOREM 5.4. — *There exists a unique arborescent comould $\mathbb{D}_{\bullet i<}$ with the three following properties :*

i. $\mathbb{D}_{\bullet i<}$ is coseparative : $\mathbb{D}_{\emptyset} = 1$ and

$$\text{col}(\mathbb{D}_{\eta^{i<}}) = \sum \mathbb{D}_{\eta'^{i<}} \otimes \mathbb{D}_{\eta''^{i<}} \quad (\eta'^{i<} \oplus \eta''^{i<} = \eta^{i<}) \quad (5.14)$$

with a sum extended to the arborescent sequences $\eta'^{i<}, \eta''^{i<}$ (even the empty sequences) which disjoint union is $\eta^{i<}$.

ii. If $\text{deg}(\eta^{i<}) = d$, $\mathbb{D}_{\eta^{i<}}$ is a differential operator of degree d in ∂_z : if the sequence $\eta^{i<}$ has exactly d minimal elements and thus :

$$\eta^{i<} = \eta^{1i<} \oplus \dots \oplus \eta^{d i<} \quad (\text{with } \eta^{i_0 i<} \text{ irreducible and } \neq \emptyset) \quad (5.15)$$

the operator $\mathbb{D}_{\eta^{i<}}$ can be written :

$$\mathbb{D}_{\eta^{i<}} = b(z) \partial_z^d \quad (5.16)$$

iii. If $\eta^{i<} = \eta_1 \cdot \eta^{*i<}$ ($\eta^{i<}$ has a least element η_1 followed by an arborescent sequence $\eta^{*i<}$) we get :

$$\mathbb{D}_{\eta^{i<}} \cdot u = \mathbb{D}_{\eta^{*i<}} \cdot \mathbb{D}_{\eta_1} \cdot z \quad (5.17)$$

Moreover, as \mathbb{D}_{\bullet} is cosympetrel

$$\mathbb{D}_{\eta'} = \sum \text{cont} \left(\begin{array}{c} \eta^{i<} \\ \eta' \end{array} \right) \mathbb{D}_{\eta^{i<}} \quad (5.18)$$

These results were proven by Jean Ecalle (see [1]). Note that $\mathbb{D}_\emptyset = 1$ and

$$\mathbb{D}_{\eta_1^<} = (\mathbb{D}_{\eta_1} \cdot z) \partial_z = \begin{cases} \mathbb{B}_{\eta_1} & \text{if } \eta_1 \in H_0 \\ 0 & \text{otherwise} \end{cases} \quad (5.19)$$

and if the length of $\eta^<$ is greater than two :

- Either $\eta^<$ is irreducible : $\eta^< = \eta_1 \cdot \eta^{*<}$ and of degree $d = 1$. Thus :

$$\mathbb{D}_{\eta^<} = (\mathbb{D}_{\eta^{*<}} \cdot \mathbb{D}_{\eta_1} \cdot z) \partial_z \quad (5.20)$$

- Either $\eta^<$ is reducible of degree $d \geq 2$ and $\eta^< = \eta^{1<} \oplus \dots \oplus \eta^{d<}$ (with $\eta^{i<}$ irreducible and $\neq \emptyset$), in this case :

$$\mathbb{D}_{\eta^<} = \frac{1}{d_1! \dots d_s!} (\mathbb{D}_{\eta^{1<}} \cdot z) \dots (\mathbb{D}_{\eta^{d<}} \cdot z) \partial_z^d \quad (5.21)$$

where d_1, \dots, d_s are the numbers of identical arborescent sequences $\eta^{j<}$ in the decomposition into irreducible sequences, of course $\sum d_i = d$.

One can also notice that if a sequence $\eta^<$ is irreducible of sum $\|\eta^<\| \notin H_0$, then $\mathbb{D}_{\eta^<} = 0$. This property remains valid if $\eta^<$ has at least a monotonic partition $\eta^{1<}, \dots, \eta^{s<}$ with an irreducible part $\eta^{i<}$ such that $\|\eta^{i<}\| \notin H_0$. It means that we can restrict ourselves to the arborescent sequences $(\eta_1, \dots, \eta_s)^<$ such that, for $1 \leq i \leq s$, $\hat{\eta}_i = \sum_{j \geq i} \eta_j \in H_0$. We note $\mathbf{H}_0^<$ this set of sequences. For details, see [1].

Finally, because of equation 5.18,

$$\mathbb{U} = \sum U^\bullet \mathbb{D}_\bullet = \sum U^{\bullet<} \mathbb{D}_{\bullet<} \quad (5.22)$$

5.4. Analyticity of \mathbb{U}

As U^\bullet is symmetrel, \mathbb{U} is a *formal* substitution automorphism :

$$\forall \varphi \in \mathbb{C}[[z, x]], \quad \mathbb{U} \cdot \varphi(z, x) = \varphi(u(z, x), x)$$

with $u \in \mathbb{C}[[z, x]]$, $u(0, 0) = 0$ and $\partial_z u(0, 0) = 1$. \mathbb{U} is analytic iff $u \in \mathbb{C}\{z, x\}$. We will prove now that the series of analytic functions

$$u(z, x) = \mathbb{U} \cdot z = \sum_{\eta^< \in \mathbf{H}^<} U^{\eta^<} \mathbb{D}_{\eta^<} \cdot z$$

is normally convergent in a neighborhood of $(0, 0)$. First of all, we remind that, because of proposition 5.3, if $0 \leq |x| \leq q^{-1/2}$ and $\eta^< = (\eta_1, \dots, \eta_s)^< \in \mathbf{H}^<$,

$$|U^{\eta^<}(x)| \leq C_q^s |x|^{\|\sigma^<\|} \tag{5.23}$$

In section 2.1, starting with

$$b(z, x) = z \sum_{\eta \in H_0} b_\eta x^\sigma z^\eta \in \mathbb{C}\{z, x\}$$

we built on $H = \{\eta_1 + \dots + \eta_s; s \geq 1; \eta_i \in H_0\}$ the operators \mathbb{D}_η is defined by :

$$\mathbb{D}_\eta = \sum_{s \geq 1} \sum_{\substack{\eta_1 + \dots + \eta_s = \eta \\ \eta_i \in H_0}} \frac{1}{s!} b_{\eta_1} \dots b_{\eta_s} z^{n_1 + \dots + n_s + s} \partial_z^s \tag{5.24}$$

Similarly, if

$$\bar{b}(t, x) = t \sum_{\eta \in H_0} |b_\eta| x^\sigma t^\eta \in \mathbb{C}\{t, x\}$$

one can build the operators on H :

$$\bar{\mathbb{D}}_\eta = \sum_{s \geq 1} \sum_{\substack{\eta_1 + \dots + \eta_s = \eta \\ \eta_i \in H_0}} \frac{1}{s!} |b_{\eta_1}| \dots |b_{\eta_s}| t^{n_1 + \dots + n_s + s} \partial_t^s$$

It is clear that, if $|z| \leq t$, then,

$$\forall \eta \in \mathbf{H}, \quad |\mathbb{D}_\eta.z| \leq \bar{\mathbb{D}}_\eta.t$$

and, because of the chosen rule of coarborification, if, once again, $|z| \leq t$, then,

$$\forall \eta^< \in \mathbf{H}^<, \quad |\mathbb{D}_{\eta^<}.z| \leq \bar{\mathbb{D}}_{\eta^<}.t \tag{5.25}$$

So $u(z, x) = \mathbb{U}.z = \sum_{\eta^< \in \mathbf{H}^<} U^{\eta^<} \mathbb{D}_{\eta^<}.z$ is a series of analytic functions such

that, for $|x| \leq q^{-1/2}$ and $|z| \leq t$,

$$\forall \eta^< \in \mathbf{H}^<, \quad |U^{\eta^<} \mathbb{D}_{\eta^<}.z| \leq C_q^{s(\eta^<)} |x|^{\|\sigma^<\|} \bar{\mathbb{D}}_{\eta^<}.t \tag{5.26}$$

Let

$$\bar{u}(t, |x|) = \sum_{\eta^< \in \mathbf{H}^<} C_q^{s(\eta^<)} |x|^{\|\sigma^<\|} \bar{\mathbb{D}}_{\eta^<}.t = \bar{\mathbb{U}}.t = \sum C^{\bullet} \bar{\mathbb{D}}_{\bullet}.t \quad (5.27)$$

where $s(\eta^<) = s$ if $\eta^< = (\eta_1, \dots, \eta_s)^<$ and $C^{\emptyset} = 1$ ($\bar{\mathbb{D}}_{\emptyset} = \text{Id}$). If \bar{u} is convergent for $|x|$ and t small enough, then $u(z, x) = \mathbb{U}.z = \sum_{\eta^< \in \mathbf{H}^<} U^{\eta^<} \mathbb{D}_{\eta^<}.z$ is normally convergent in a neighborhood of $(0, 0)$, thus $u \in \mathbb{C}\{z, x\}$ and \mathbb{U} is an analytic substitution automorphism.

We have

where $\bar{\mathbb{U}}_0 = \text{Id}$, and $\bar{\mathbb{U}} = \bar{\mathbb{U}}_0 + \bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2 + \bar{\mathbb{U}}_3$

$$\begin{aligned} \bar{\mathbb{U}}_1 &= \sum_{\substack{\eta^< \in \mathbf{H}^< \\ \eta^< \text{ irreducible} \\ s(\eta^<) = 1}} C^{\eta^<} \bar{\mathbb{D}}_{\eta^<} = \sum_{\eta^< \in \mathbf{H}_1^<} C^{\eta^<} \bar{\mathbb{D}}_{\eta^<} \\ \bar{\mathbb{U}}_2 &= \sum_{\substack{\eta^< \in \mathbf{H}^< \\ \eta^< \text{ irreducible} \\ s(\eta^<) \geq 2}} C^{\eta^<} \bar{\mathbb{D}}_{\eta^<} = \sum_{\eta^< \in \mathbf{H}_2^<} C^{\eta^<} \bar{\mathbb{D}}_{\eta^<} \\ \bar{\mathbb{U}}_3 &= \sum_{\substack{\eta^< \in \mathbf{H}^< \\ \eta^< \text{ not irreducible}}} C^{\eta^<} \bar{\mathbb{D}}_{\eta^<} = \sum_{\eta^< \in \mathbf{H}_3^<} C^{\eta^<} \bar{\mathbb{D}}_{\eta^<} \end{aligned} \quad (5.28)$$

Because of the coarborification rule (applied to $\bar{\mathbb{D}}_{\bullet}$), we get the following identities

$$\bar{\mathbb{U}}_1 = C_q \sum_{\eta \in H_0} |b_{\eta}| |x|^{\sigma} t^{n+1} \partial_t \quad (5.29)$$

and $\bar{\mathbb{U}}_1.t = C_q b(t, |x|) = c(t, |x|)$. Using the rule 5.20, we get,

$$\begin{aligned} \bar{\mathbb{U}}_2 &= \sum_{\substack{\eta^< \in \mathbf{H}^< \\ \eta^< \text{ irreducible} \\ s(\eta^<) \geq 2}} C^{\eta^<} \bar{\mathbb{D}}_{\eta^<} \\ &= \sum_{\eta^< \in \mathbf{H}_2^<} C^{\eta^<} \bar{\mathbb{D}}_{\eta^<} \\ &= \sum_{\substack{\eta^< = \eta_1 \bullet \eta^{*<} \\ \eta_1 \in \mathbf{H}_1^< \\ \eta^{*<} \in \mathbf{H}_1^< \cup \mathbf{H}_2^< \cup \mathbf{H}_3^<}} C^{\eta^<} C^{\eta_1} (\bar{\mathbb{D}}_{\eta^{*<}}. \bar{\mathbb{D}}_{\eta_1}.t) \partial_t \\ &= ((\bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2 + \bar{\mathbb{U}}_3). \bar{\mathbb{U}}_1.t) \partial_t \end{aligned} \quad (5.30)$$

and finally, using the rule 5.21,

$$\begin{aligned}
 \bar{\mathbb{U}}_3 &= \sum_{\substack{\eta^< \in \mathbf{H}^< \\ \eta^< \text{ not irreducible}}} C^{\eta^<} \bar{\mathbb{D}}_{\eta^<} \\
 &= \sum_{\substack{\eta^< = \eta^{1^<} \oplus \dots \oplus \eta^{d^<} \\ d \geq 2 \ ; \ \eta^{i^<} \in \mathbf{H}_1^< \cup \mathbf{H}_2^<}} \frac{1}{d!} \frac{d!}{d_1! \dots d_s!} (\bar{\mathbb{D}}_{\eta^{1^<}}.t) \dots (\bar{\mathbb{D}}_{\eta^{d^<}}.t) \partial_t^d \\
 &= \sum_{d \geq 2} \frac{1}{d!} ((\bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2).t)^d \partial_t^d
 \end{aligned} \tag{5.31}$$

But

$$\begin{aligned}
 \bar{u}(t, |x|) &= \bar{\mathbb{U}}.t \\
 &= (\bar{\mathbb{U}}_0 + \bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2 + \bar{\mathbb{U}}_3).t \\
 &= t + \bar{\mathbb{U}}_1.t + \bar{\mathbb{U}}_2.t \\
 &= t + \bar{\mathbb{U}}_1.t + (\bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2 + \bar{\mathbb{U}}_3).\bar{\mathbb{U}}_1.t \\
 &= t + \bar{\mathbb{U}}_1.t + \sum_{d \geq 1} \frac{1}{d!} ((\bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2).t)^d \partial_t^d (\bar{\mathbb{U}}_1.t) \\
 &= t + \sum_{d \geq 0} \frac{1}{d!} ((\bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2).t)^d \partial_t^d (c(t, |x|)) \\
 &= t + \sum_{d \geq 0} \frac{1}{d!} (\bar{u}(t, |x|) - t)^d \partial_t^d (c(t, |x|)) \\
 &= t + c(\bar{u}(t, |x|), |x|)
 \end{aligned}$$

But, as $c \in \mathbb{R}^+\{t, |x|\}$ and $c(0, 0) = \partial_t c(0, 0) = 0$, using the implicit function theorem and majorant series, it can easily be proved that $\bar{u} \in \mathbb{R}^+\{t, |x|\}$. Thus,

THEOREM 5.5. — *There exists $\alpha > 0$ such that the series of analytic functions*

$$u(z, x) = \mathbb{U}.z = \sum_{\eta^< \in \mathbf{H}^<} U^{\eta^<} \mathbb{D}_{\eta^<}.z$$

is normally convergent in $V_\alpha = \{|x| \leq \alpha, |z| \leq \alpha\}$ which means that \mathbb{U} is an analytic substitution automorphism : $u \in \mathbb{C}\{z, x\}$.

6. Analytic classification

We first prove that \mathbb{V} is also an analytic substitution automorphism.

6.1. Analyticity of \mathbb{V}

Let $\mathbb{U}_q = \sum s_q(U^\bullet)\mathbb{D}_\bullet = \sum$

$$\mathbb{U}_q = \sum s_q(U^\bullet)\mathbb{D}_\bullet = \sum_{\eta \in \mathbf{H}} x^{-\|\eta\|} U^\eta(qx)\mathbb{D}_\eta$$

The operator \mathbb{U}_q is a formal substitution automorphism, but, after arborification,

$$\forall \eta^< \in \mathbf{H}^<, \quad s_q(U^\bullet)^{\eta^<}(x) = \begin{cases} 0 & \text{if } \eta^< \in \mathbf{H}^</\mathbf{H}^{+<} \\ x^{-\|\eta^<\|} U^{\eta^<}(qx) & \text{if } \eta^< \in \mathbf{H}^{+<} \end{cases}$$

thus, for $|x| \leq q^{-3/2}$, using proposition 5.3,

$$\forall \eta^< \in \mathbf{H}^<, \quad |s_q(U^\bullet)^{\eta^<}(x)| \leq C_q^s |x|^{\|\sigma^<\|} \quad (6.1)$$

this proves once again that \mathbb{U}_q is an analytic substitution automorphism, such that

$$u_q(z, x) = \mathbb{U}_q.z \in \mathbb{C}\{z, x\} \quad (6.2)$$

and,

$$\forall \varphi \in \mathbb{C}\{z, x\}, \quad \mathbb{U}_q.\varphi(z, x) = \varphi(u_q(z, x), x)$$

Finally (see section 3.4) $\mathbb{V} = \sum V^\bullet\mathbb{D}_\bullet$ is a formal substitution such that

$$v(z, x) = \mathbb{V}.z \in z + x\mathbb{C}[[x]]$$

and

$$\mathbb{V}.\mathbb{U}_q = \mathbb{U}.\mathbb{D} \quad (6.3)$$

where $\mathbb{D} = \sum X^\bullet\mathbb{D}_\bullet$ is an analytic substitution automorphism such that

$$\forall \varphi \in \mathbb{C}\{z, x\}, \quad \mathbb{D}.\varphi(z, x) = \varphi(z + b(z, x), x)$$

but, as \mathbb{U}_q is analytic, its inverse substitution automorphism \mathbb{U}_q^{-1} is also analytic and $\mathbb{V} = \mathbb{U}.\mathbb{D}.\mathbb{U}_q^{-1}$ is analytic :

$$v(z, x) = \mathbb{V}.z \in z + x\mathbb{C}\{x\}$$

We can resume this in the following theorem

THEOREM 6.1. — *Let us consider the equation*

$$(x\sigma_q - 1)y = b(y, x) \tag{6.4}$$

with $b(0, 0) = 0$ and $\frac{\partial b}{\partial y}(0, 0) = 0$. This equation is conjugate by the analytic automorphism \mathbb{U} :

$$y = \mathbb{U}.z = \left(\sum U \bullet \mathbb{D} \bullet \right).z = u(z, x) \in \mathbb{C}\{z, x\} \tag{6.5}$$

to the equation

$$x\sigma_q z = \mathbb{V}.z = v(z, x) \in z + x\mathbb{C}\{x\} \tag{6.6}$$

To prove theorem 1.1, it remains to prove that the equation above is analytically conjugate to

$$x\sigma_q z = z + \alpha x \tag{6.7}$$

where $\alpha \in \mathbb{C}$ depends on v .

6.2. The equation $x\sigma_q z = \mathbb{V}.z = v(z, x) \in z + x\mathbb{C}\{x\}$

We have

$$v(z, x) = z + \sum_{n \geq 1} v_n x^n \in z + x\mathbb{C}\{x\}$$

If

$$\alpha = \sum_{n \geq 1} v_n q^{-n(n-1)/2} \in \mathbb{C} \tag{6.8}$$

the equation

$$x\sigma_q(f) = v(f, x) - \alpha x = f + (v_1 - \alpha)x + \sum_{n \geq 2} v_n x^n \in z + x\mathbb{C}\{x\} \tag{6.9}$$

possess an analytic solution : If $f(x) = \sum_{n \geq 1} f_n x^n$,

$$\begin{aligned} x\sigma_q f - f &= \sum_{n \geq 2} f_{n-1} q^{n-1} x^n - \sum_{n \geq 1} f_n x^n \\ &= (v_1 - \alpha)x + \sum_{n \geq 2} v_n x^n \\ &= - \left(\sum_{k \geq 2} v_k q^{-k(k-1)/2} \right) x + \sum_{n \geq 2} v_n x^n \end{aligned}$$

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thus $f_1 = \sum_{k \geq 2} v_k q^{-k(k-1)/2}$ and

$$\forall n \geq 2, \quad f_n = f_{n-1} q^{n-1} - v_n$$

but,

$$\forall n \geq 1, \quad f_n = \sum_{k \geq n+1} v_k q^{-(k+n-1)(k-n)/2} = \sum_{k \geq n+1} v_k [k]_q [n]_q^{-1}$$

and $f \in \mathbb{C}\{x\}$: there exists $C > 0$ such that

$$\forall n \geq 1, \quad |v_n| \leq C^n$$

thus, for $n \geq 1$,

$$\begin{aligned} |f_n| &\leq \sum_{k \geq n+1} C^k q^{-(k+n-1)(k-n)/2} \\ &\leq C^n \sum_{k \geq n+1} C^{k-n} q^{-(k+n-1)(k-n)/2} \\ &\leq C^n \sum_{k \geq 1} C^k q^{-(k+2n-1)k/2} \\ &\leq C^n \sum_{k \geq 1} C^k q^{-(k-1)k/2} \\ &\leq M_C C^n \end{aligned} \tag{6.10}$$

But if $\tilde{z} = z - f$, then

$$\begin{aligned} x\sigma_q(\tilde{z}) &= x\sigma_q(z) - x\sigma_q(f) \\ &= z + \sum_{n \geq 1} v_n x^n - f - (v_1 - \alpha)x - \sum_{n \geq 2} v_n x^n \\ &= \tilde{z} + \alpha x \end{aligned}$$

This proves finally theorem 1.1 :

Let us consider the equation

$$(x\sigma_q - 1)y = b(y, x) \tag{6.11}$$

with $b(0,0) = 0$ and $\frac{\partial b}{\partial y}(0,0) = 0$. There exists $\alpha(b) \in \mathbb{C}$, such that this equation is analytically conjugate to the equation

$$x\sigma_q z = z + \alpha(b)x \tag{6.12}$$

and, if $\alpha(b) \neq 0$, this last equation is analytically conjugate to

$$x\sigma_q z_0 = z_0 + x \tag{6.13}$$

(Take $z = \alpha(b)z_0$).

7. Proofs

7.1. Proof of proposition 3.2

PROPOSITION 3.2. — *For any moulds A^\bullet and B^\bullet , we have*

$$J^\bullet \circ A^\bullet = J^\bullet \circ (A^\bullet \times (K^\bullet \circ B^\bullet)) \quad (7.1)$$

Let $C^\bullet = K^\bullet \circ B^\bullet$ and $D^\bullet = A^\bullet \times (K^\bullet \circ B^\bullet) = A^\bullet \times C^\bullet$ and let $\eta \in \mathbf{H}$ nonempty. We can restrict ourselves to the case $\eta \in \mathbf{H}^+$ because, otherwise, the equation becomes $0 = 0$. We remind that :

$$(J^\bullet \circ D^\bullet)^\eta = \sum_{\eta^1 \dots \eta^t = \eta} J^{\|\eta^1\|, \dots, \|\eta^t\|} D^{\eta^1} \dots D^{\eta^t}$$

One can expand the products $D^{\eta^1} \dots D^{\eta^t}$ such that we only get 5 type of monomials :

1. The monomials $A^{\alpha^1} \dots A^{\alpha^r}$ where $\alpha^1 \dots \alpha^r = \eta$,
2. The monomials $A^{\alpha^1} \dots A^{\alpha^r} C^\beta$ where $\alpha\beta = \eta$ and $\alpha^1 \dots \alpha^r = \alpha$,
3. The monomials $A^{\alpha^1} \dots A^{\alpha^r} C^\beta D^{\gamma^1} \dots D^{\gamma^s}$ where $\alpha\beta\gamma = \eta$ and $\alpha^1 \dots \alpha^r = \alpha$ and $\gamma^1 \dots \gamma^s = \gamma$,
4. The monomials $C^{\beta^1} \dots C^{\beta^r} D^{\gamma^1} \dots D^{\gamma^s}$ where $\beta\gamma = \eta$ and $\beta^1 \dots \beta^r = \beta$ and $\gamma^1 \dots \gamma^s = \gamma$,
5. The monomials $C^{\beta^1} \dots C^{\beta^r}$ where $\beta^1 \dots \beta^r = \eta$.

Let us now take look at the contribution to each type of monomial of the expression $(J^\bullet \circ D^\bullet)^\eta$.

1. The first type gives :

$$\sum_{\alpha^1 \dots \alpha^r = \eta} J^{\|\alpha^1\|, \dots, \|\alpha^r\|} A^{\alpha^1} \dots A^{\alpha^r} = (J^\bullet \circ A^\bullet)^\eta$$

2. When $\alpha\beta = \eta$ and $\alpha^1 \dots \alpha^r = \alpha$, the monomial $A^{\alpha^1} \dots A^{\alpha^r} C^\beta$ has, as a coefficient :

$$J^{\|\alpha^1\|, \dots, \|\alpha^r\|, \|\beta\|} + J^{\|\alpha^1\|, \dots, \|\alpha^r\| + \|\beta\|} = J^{\|\alpha^1\|, \dots, \|\alpha^r\| + \|\beta\|} (1 + \alpha_-(\|\beta\|))$$

but, either $\|\beta\| < 0$ and it vanishes, either $\|\beta\| \geq 0$ and then

$$C^\beta = (K^\bullet \circ B^\bullet)^\beta = 0 \text{ (see the definition of } K^\bullet\text{).}$$

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3. When $\alpha\beta\gamma = \eta$ and $\alpha^1 \dots \alpha^r = \alpha$ and $\gamma^1 \dots \gamma^s = \gamma$, the monomial $A^{\alpha^1} \dots A^{\alpha^r} C^\beta D^{\gamma^1} \dots D^{\gamma^s}$ has, as a coefficient :

$$J^{\|\alpha^1\|, \dots, \|\alpha^r\|, \|\beta\|, \|\gamma^1\|, \dots, \|\gamma^s\|} + J^{\|\alpha^1\|, \dots, \|\alpha^r\| + \|\beta\|, \|\gamma^1\|, \dots, \|\gamma^s\|}$$

for the same reason, if $\|\beta\| + \|\gamma^1\| + \dots + \|\gamma^s\| < 0$, this term vanishes and if $\|\beta\| + \|\gamma^1\| + \dots + \|\gamma^s\| \geq 0$, this term still vanishes if $\|\gamma^1\| + \dots + \|\gamma^s\| \geq 0$ (definition of J^\bullet) and otherwise $\|\gamma^1\| + \dots + \|\gamma^s\| < 0$, which means that $\|\beta\| \geq 0$ and then $C^\beta = (K^\bullet \circ B^\bullet)^\beta = 0$.

4. When $\beta\gamma = \eta$ and $\beta^1 \dots \beta^r = \beta$ and $\gamma^1 \dots \gamma^s = \gamma$, the monomial $C^{\beta^1} \dots C^{\beta^r} D^{\gamma^1} \dots D^{\gamma^s}$ has, as a factor :

$$J^{\|\beta^1\|, \dots, \|\beta^r\|, \|\gamma^1\|, \dots, \|\gamma^s\|}$$

this vanishes if $\|\beta^1\| + \dots + \|\beta^r\| + \|\gamma^1\| + \dots + \|\gamma^s\| < 0$ or $\|\gamma^1\| + \dots + \|\gamma^s\| \geq 0$, otherwise $\|\gamma^1\| + \dots + \|\gamma^s\| < 0$ and $\|\beta^1\| + \dots + \|\beta^r\| + \|\gamma^1\| + \dots + \|\gamma^s\| \geq 0$ thus $\|\beta^1\| + \dots + \|\beta^r\| \geq 0$ which means that at least one of the $\|\beta^i\|$ is non negative and so $C^{\beta^1} \dots C^{\beta^r}$ vanishes.

5. The fifth type gives 0 for the same reason.

In conclusion, the only nonzero contribution is $(J^\bullet \circ A^\bullet)^\eta$ and we proved that

$$J^\bullet \circ A^\bullet = J^\bullet \circ (A^\bullet \times (K^\bullet \circ B^\bullet))$$

7.2. Proof of theorem 4.1

THEOREM 4.1. — *If $\eta = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} = \begin{pmatrix} \mathbf{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \mathbf{H}$, then the monomial U^η is a convergent series with the following expression :*

$$U^\eta(x) = \varepsilon_n \sum_{(k_1, \dots, k_s) \in \mathbb{Z}^n \cap P^n} x^{\|\boldsymbol{\sigma}\| + \langle \check{\mathbf{k}}, \boldsymbol{\sigma} \rangle} q^{-\langle \check{\mathbf{k}}, \boldsymbol{\sigma} \rangle} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s} \quad (7.2)$$

with the notations

- $\varepsilon_n = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$ with $\hat{n}_i = n_i + \dots + n_s$ with $\varepsilon_n = 1$ (resp. $\varepsilon_n = -1$) if $n \geq 0$ (resp. $n < 0$)
- $\mathbb{Z}^n = \mathbb{Z}^{\hat{n}_1} \times \dots \times \mathbb{Z}^{\hat{n}_s}$ with $\mathbb{Z}^n = \mathbb{Z}^{+*}$ (resp. \mathbb{Z}^-) if $n \geq 0$ (resp. $n < 0$)
- $P^n = \{(k_1, \dots, k_s) \in \mathbb{Z}^s; k_i > 0 \text{ if } \varepsilon_{\hat{n}_i} = -1\}$
- $\check{k}_i = k_1 + \dots + k_i$ and $\check{\mathbf{k}} = (\check{k}_1, \dots, \check{k}_s)$.

- $[k + 1]_q = q^{-k(k+1)/2}$

Proof. — To prove this theorem we will introduce some notations. For a sequence $(k_1, \dots, k_s) \in \mathbb{Z}^s$, let $l_i = \check{k}_i$, thus $k_1 = l_1$ and, for $i \geq 2$, $k_i = l_i - l_{i-1}$. For a sequence $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+$ and $\mathbf{l} = (l_1, \dots, l_s) \in \mathbb{Z}^s$ ($s \geq 1$), let

$$T_{\mathbf{l}}^{\boldsymbol{\eta}} = x^{\|\boldsymbol{\sigma}\| + \langle \mathbf{l}, \boldsymbol{n} \rangle} q^{-\langle \mathbf{l}, \boldsymbol{\sigma} \rangle} [l_1 + 1]_q^{\eta_1} \dots [l_s + 1]_q^{\eta_s} = \prod_{i=1}^s T_{l_i}^{\eta_i} \quad (7.3)$$

With this notation, the formula in theorem 2.3 becomes

$$W^{\boldsymbol{\eta}}(x) = \varepsilon_{\mathbf{n}} \sum_{(l_1, l_2 - l_1, \dots, l_s - l_{s-1}) \in \mathbb{Z}^{\mathbf{n}}} T_{\mathbf{l}}^{\boldsymbol{\eta}}$$

using the function $\rho_1(n) = \begin{cases} 1 & \text{if } n > 0 \\ 0 & \text{if } n \leq 0 \end{cases}$ and $\rho_{-1} = 1 - \rho_1$, we get

$$W^{\boldsymbol{\eta}}(x) = \varepsilon_{\mathbf{n}} \sum_{\mathbf{l} \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1) \rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1) \dots \rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) T_{\mathbf{l}}^{\boldsymbol{\eta}} = \sum_{\mathbf{l} \in \mathbb{Z}^s} w_{\mathbf{l}}^{\boldsymbol{\eta}} T_{\mathbf{l}}^{\boldsymbol{\eta}} \quad (7.4)$$

and we have to prove that

$$\begin{aligned} U^{\boldsymbol{\eta}}(x) &= \varepsilon_{\mathbf{n}} \sum_{\mathbf{l} \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1) \rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1) \dots \rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) T_{\mathbf{l}}^{\boldsymbol{\eta}} \prod_{\substack{1 \leq i \leq s \\ \varepsilon_{\hat{n}_i} = -1}} \rho_1(l_i) \\ &= \sum_{\mathbf{l} \in \mathbb{Z}^s} \tilde{w}_{\mathbf{l}}^{\boldsymbol{\eta}} T_{\mathbf{l}}^{\boldsymbol{\eta}} = \sum_{\mathbf{l} \in \mathbb{Z}^s} w_{\mathbf{l}}^{\boldsymbol{\eta}} \left(\prod_{\substack{1 \leq i \leq s \\ \varepsilon_{\hat{n}_i} = -1}} \rho_1(l_i) \right) T_{\mathbf{l}}^{\boldsymbol{\eta}} \end{aligned} \quad (7.5)$$

Let $\tilde{U}^{\boldsymbol{\eta}}$ be the second term of this equation. It is clear that the identities hold if $\hat{n}_1 < 0$ because $(J^{\bullet} \circ W^{\bullet})^{\boldsymbol{\eta}} = 0$ and in $U^{\boldsymbol{\eta}}$ we have the factor $\rho_{-1}(l_1) \rho_1(l_1) = 0$. Let us suppose now that, in the sequel, $\hat{n}_1 \geq 0$. If $I = \{1 \leq i \leq s \ ; \ \varepsilon_{\hat{n}_i} = -1\} = \emptyset$ then, once again, we obtain the identity because

$$\prod_{\substack{1 \leq i \leq s \\ \varepsilon_{\hat{n}_i} = -1}} \rho_1(l_i) = \prod_{i \in \emptyset} \rho_1(l_i) = 1 \quad \text{and} \quad (J \circ W)^{\boldsymbol{\eta}} = W^{\boldsymbol{\eta}}$$

We need some notations to prove the identity. For $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_t) \in \mathbf{H}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_t) \in \{-1, +1\}^t$ and $\mathbf{m} = (m_1, \dots, m_t) \in \mathbb{Z}^t$, then

$$w_{\mathbf{m}}^{\boldsymbol{\gamma}, \boldsymbol{\alpha}} = \left(\prod \alpha_i \right) \rho_{\alpha_1}(m_1) \rho_{\alpha_2}(m_2 - m_1) \dots \rho_{\alpha_t}(m_t - m_{t-1})$$

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and if $\varepsilon(\gamma) = (\varepsilon_{\hat{\rho}_1}, \dots, \varepsilon_{\hat{\rho}_t})$ with $\gamma = \begin{pmatrix} p_1, \dots, p_t \\ \sigma_1, \dots, \sigma_t \end{pmatrix}$, then

$$w_m^\gamma = w_m^{\gamma, \varepsilon(\gamma)}$$

If $\eta = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+$ ($s \geq 1$) then

$$U^\eta(x) = \sum_{\eta^1 \dots \eta^t = \eta} J^{\|\eta^1\|, \dots, \|\eta^s\|} W^\eta(x) \dots W^{\eta^t(x)}$$

If $I = \{1 \leq i \leq s \ ; \ \varepsilon_{\hat{\rho}_i} = -1\} = \{i_1 < \dots < i_q\}$ (note that $1 \notin I$), because of the definition of \mathbf{J}^\bullet , we get

$$U^\eta(x) = \sum_{K=\{k_1 < \dots < k_j\} \subset I} (-1)^{|K|} W^{\eta_{k_0}^{k_1}} \dots W^{\eta_{k_j}^{k_{j+1}}}$$

where $j = |K|$ is the cardinal of K , $k_0 = 1$, $k_{j+1} = s + 1$ and

$\eta_{k_i}^{k_{i+1}} = (\eta_{k_i}, \dots, \eta_{k_{i+1}-1})$. Using formula 7.4, we get

$$\begin{aligned} U^\eta(x) &= \sum_{l \in \mathbb{Z}^s} \sum_{K \subset I} (-1)^{|K|} w_{l_{k_0}^{k_1}}^{\eta_{k_0}^{k_1}} \dots w_{l_{k_j}^{k_{j+1}}}^{\eta_{k_j}^{k_{j+1}}} T_l^\eta \\ &= \sum_{l \in \mathbb{Z}^s} \sum_{K \subset I} (-1)^{|K|} w_{l_{k_0}^{k_1}}^{\eta_{k_0}^{k_1}, \varepsilon(\eta_{k_0}^{k_1})} \dots w_{l_{k_j}^{k_{j+1}}}^{\eta_{k_j}^{k_{j+1}}, \varepsilon(\eta_{k_j}^{k_{j+1}})} T_l^\eta \end{aligned}$$

For $0 \leq r \leq q$, let $I_r = \{i_1 < \dots < i_r\}$ ($I_q = I$ and $I_0 = \emptyset$). If

$$\tilde{U}_r^\eta = \sum_{l \in \mathbb{Z}^s} w_l^\eta \left(\prod_{i \in I_r} \rho_1(l_i) \right) T_l^\eta$$

then $\tilde{U}_0^\eta = W^\eta$ and $\tilde{U}_q^\eta = \tilde{U}^\eta$. For $1 \leq r \leq q$, we get

$$\begin{aligned} \tilde{U}_r^\eta &= \sum_{l \in \mathbb{Z}^s} w_l^\eta \left(\prod_{i \in I_r} \rho_1(l_i) \right) T_l^\eta \\ &= \sum_{l \in \mathbb{Z}^s} w_l^\eta (1 - \rho_{-1}(l_{i_r})) \left(\prod_{i \in I_{r-1}} \rho_1(l_i) \right) T_l^\eta \\ &= \tilde{U}_{r-1}^\eta - \sum_{l \in \mathbb{Z}^s} w_l^\eta \rho_{-1}(l_{i_r}) \left(\prod_{i \in I_{r-1}} \rho_1(l_i) \right) T_l^\eta \end{aligned}$$

One can notice that, in the sum, $l_{i_{r-1}} > 0$ (take $i_{r-1} = i_0 = 1$ if $r = 1$) and $(l_{i_{r-1}+1} - l_{i_{r-1}}), \dots, (l_{i_{r-1}} - l_{i_{r-2}})$ must be positive, thus $l_{i_{r-1}} > 0$. As l_{i_r} is non positive, this automatically ensures that $(l_{i_r} - l_{i_{r-1}})$ is non positive, thus we get the simplification

$$\begin{aligned}
 Q &= w_l^\eta \rho_{-1}(l_{i_r}) \left(\prod_{i \in I_{r-1}} \rho_1(l_i) \right) \\
 &= -\varepsilon_n \prod_{i < i_r} \rho_{\varepsilon_{\hat{n}_i}}(l_i - l_{i-1}) \prod_{i > i_r} \rho_{\varepsilon_{\hat{n}_i}}(l_i - l_{i-1}) \prod_{i \in I_{r-1}} \rho_1(l_i) \\
 &\quad \times \rho_{-1}(l_{i_r} - l_{i_{r-1}}) \rho_{-1}(l_{i_r}) \\
 &= -\varepsilon_n \prod_{i < i_r} \rho_{\varepsilon_{\hat{n}_i}}(l_i - l_{i-1}) \rho_{-1}(l_{i_r}) \prod_{i > i_r} \rho_{\varepsilon_{\hat{n}_i}}(l_i - l_{i-1}) \prod_{i \in I_{r-1}} \rho_1(l_i) \\
 &= w_l^{\eta_{i_0}^{i_r}, \varepsilon(\eta)_{i_0}^{i_r}} \left(\prod_{i \in I_{r-1}} \rho_1(l_i) \right) w_{l_{i_r}^{i_{q+1}}}^{\eta_{i_r}^{i_{q+1}}, \varepsilon(\eta)_{i_r}^{i_{q+1}}}
 \end{aligned}$$

Where $i_0 = 1$ and $i_{q+1} = s$. This gives

$$\tilde{U}_r^\eta = \tilde{U}_{r-1}^\eta - \sum_{l \in \mathbb{Z}^s} w_{l_{i_0}^{i_r}, \varepsilon_{i_0}^{i_r}} \left(\prod_{i \in I_{r-1}} \rho_1(l_i) \right) w_{l_{i_r}^{i_{q+1}}}^{\eta_{i_r}^{i_{q+1}}, \varepsilon(\eta)_{i_r}^{i_{q+1}}} T_l^\eta$$

Using this formula recursively, we get

$$\tilde{U}^\eta(x) = \sum_{l \in \mathbb{Z}^s} \sum_{K \subset I} (-1)^{|K|} w_{l_{k_0}^{k_1}}^{\eta_{k_0}^{k_1}, \varepsilon(\eta)_{k_0}^{k_1}} \dots w_{l_{k_j}^{k_{j+1}}}^{\eta_{k_j}^{k_{j+1}}, \varepsilon(\eta)_{k_j}^{k_{j+1}}} T_l^\eta$$

which must be compared to

$$U^\eta(x) = \sum_{l \in \mathbb{Z}^s} \sum_{K \subset I} (-1)^{|K|} w_{l_{k_0}^{k_1}}^{\eta_{k_0}^{k_1}, \varepsilon(\eta_{k_0}^{k_1})} \dots w_{l_{k_j}^{k_{j+1}}}^{\eta_{k_j}^{k_{j+1}}, \varepsilon(\eta_{k_j}^{k_{j+1}})} T_l^\eta$$

For a given $K = \{k_1 < \dots < k_j\} \subset I$, it is clear that

$$\varepsilon(\eta)_{k_j}^{k_{j+1}} = \varepsilon(\eta_{k_j}^{k_{j+1}}) = \varepsilon_{\hat{n}_{k_j}}, \dots, \varepsilon_{\hat{n}_s} \quad (k_{j+1} = s)$$

but generally speaking, we can have

$$\varepsilon(\eta)_{k_{j_0}}^{k_{j_0+1}} \neq \varepsilon(\eta_{k_{j_0}}^{k_{j_0+1}})$$

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for $0 \leq j_0 < j$. Nonetheless, we will prove that $\tilde{U}^\eta = U^\eta$. Let $1 \leq i \leq s$ and $i_r \in I$ (or $r = q + 1$ and $i_{q+1} = s + 1$) such that $i < i_r$. Let also

$$\mathcal{K}_{i,i_r} = \{K = \{k_1 < \dots < k_j\} \subset I \quad ; \exists 0 \leq j_0 \leq j; \quad k_{j_0} \leq i < k_{j_0+1} = i_r\}$$

For the partitions $K \in \mathcal{K}_{i,i_r}$, the i^{th} sign of $\tilde{\alpha}_K = \varepsilon(\eta)_{k_0}^{k_1} \dots \varepsilon(\eta)_{k_j}^{k_{j+1}} = \varepsilon(\eta)$ is $\varepsilon_{n_i+\dots+n_s}$ whereas, in $\alpha_K = \varepsilon(\eta)_{k_0}^{k_1} \dots \varepsilon(\eta)_{k_j}^{k_{j+1}}$ is $\varepsilon_{n_i+\dots+n_{i_r-1}}$. If $r = q + 1$ these signs are equal. Otherwise, if $\varepsilon_{n_i+\dots+n_s} = +1$, since $\varepsilon_{n_i+\dots+n_s} = -1$, it implies that $\varepsilon_{n_i+\dots+n_{i_r-1}} = +1$ so that the sign at i^{th} position in $\tilde{\alpha}_K$ and α_K is the same (+1). But, if ever $\varepsilon_{n_i+\dots+n_s} = -1$, we can't conclude : $\varepsilon_{n_i+\dots+n_{i_r-1}}$ could be +1 or -1. The main fact to prove is that in \tilde{U}^η , we can, if necessary, change $\varepsilon_{n_i+\dots+n_s} = -1$ into $\varepsilon_{n_i+\dots+n_{i_r-1}} = +1$, for the terms corresponding to $K \in \mathcal{K}_{i,i_r}$, without changing the value of \tilde{U}^η . It is clear that, iterating this result proves finally that $\tilde{U}^\eta = U^\eta$. So, consider the case when $\varepsilon_{n_i+\dots+n_s} = -1$ and $\varepsilon_{n_i+\dots+n_{i_r-1}} = +1$ that can occur iff $r \leq q$ and $i = i_t \in I$ ($1 \leq t < r$). The terms where we want to change this sign are

$$\tilde{U}_{i,i_r}^\eta = \sum_{l \in \mathbb{Z}^s} \sum_{K \subset \mathcal{K}_{i,i_r}} (-1)^{|K|} w_{k_0}^{\eta_{k_0}^{k_1}, \varepsilon(\eta)_{k_0}^{k_1}} \dots w_{k_j}^{\eta_{k_j}^{k_{j+1}}, \varepsilon(\eta)_{k_j}^{k_{j+1}}} T_l^\eta$$

Once again, let $I_{t-1} = \{i_1 < \dots < i_{t-1}\}$ and ${}_r I = \{i_r < \dots < i_q\}$. It is clear that $\mathcal{K}_{i_t,i_r} = \mathcal{K}_{i_t,i_r}^1 \cup \mathcal{K}_{i_t,i_r}^2$ ($\mathcal{K}_{i_t,i_r}^1 \cap \mathcal{K}_{i_t,i_r}^2 = \emptyset$) where

$$\mathcal{K}_{i_t,i_r}^1 = \left\{ K = K^1 \cup K^2 ; \quad \begin{array}{l} K^1 = \{k_1^1 < \dots < k_{j_1}^1\} \subset I_{t-1} \\ K^2 = \{i_r = k_1^2 < \dots < k_{j_2}^2\} \subset {}_r I \end{array} \right\}$$

$$\mathcal{K}_{i_t,i_r}^2 = \{K = K' \cup \{i_t\} \quad ; \quad K' \in \mathcal{K}_{i_t,i_r}^1\}$$

thus, if

$$w_{K^1} = w_{k_0}^{\eta_{k_0}^{k_1}, \varepsilon(\eta)_{k_0}^{k_1}} \dots w_{k_{j_1-1}}^{\eta_{k_{j_1-1}}^{k_{j_1}^1}, \varepsilon(\eta)_{k_{j_1-1}}^{k_{j_1}^1}}$$

and

$$w_{K^2} = w_{k_1}^{\eta_{k_1}^{k_2}, \varepsilon(\eta)_{k_1}^{k_2}} \dots w_{k_{j_2}^s}^{\eta_{k_{j_2}^s}^s, \varepsilon(\eta)_{k_{j_2}^s}^s}$$

then

$$\begin{aligned} \tilde{U}_{i,i_r}^\eta &= \sum_{l \in \mathbb{Z}^s} (-1)^{|K^1|+|K^2|} w_{K^1} w_{l_{k_{j_1}^1}^{i_r}, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} w_{K^2} T_l^\eta \\ &\quad K^1 = \{k_1^1 < \dots < k_{j_1}^1\} \subset I_{t-1} \\ &\quad K^2 = \{i_r = k_1^2 < \dots < k_{j_2}^2\} \subset_r I \\ &+ \sum_{l \in \mathbb{Z}^s} (-1)^{|K^1|+|K^2|+1} w_{K^1} w_{l_{k_{j_1}^1}^{i_t}, \varepsilon(\eta)_{k_{j_1}^1}^{i_t}} w_{l_{i_t}^{i_r}, \varepsilon(\eta)_{i_t}^{i_r}} w_{K^2} T_l^\eta \\ &\quad K^1 = \{k_1^1 < \dots < k_{j_1}^1\} \subset I_{t-1} \\ &\quad K^2 = \{i_r = k_1^2 < \dots < k_{j_2}^2\} \subset_r I \end{aligned}$$

If $y_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} = \prod_{k_{j_1}^1 \leq i \leq i_r-1, i \neq i_t} \varepsilon_{\hat{n}_i} \rho_{\varepsilon_{\hat{n}_i}}(l_i - l_{i-1})$ (with the convention here $l_{k_{j_1}^1-1} = 0$), then

$$\begin{aligned} w_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} &= y_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} \varepsilon_{\hat{n}_{i_t}} \rho_{\varepsilon_{\hat{n}_{i_t}}}(l_{i_t} - l_{i_t-1}) \\ &= -\rho_{-1}(l_{i_t} - l_{i_t-1}) y_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} \\ &= \rho_1(l_{i_t} - l_{i_t-1}) y_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} - y_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} \end{aligned}$$

and

$$\begin{aligned} w_{k_{j_1}^1}^{i_t, \varepsilon(\eta)_{k_{j_1}^1}^{i_t}} w_{i_t}^{i_r, \varepsilon(\eta)_{i_t}^{i_r}} &= y_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} \varepsilon_{\hat{n}_{i_t}} \rho_{\varepsilon_{\hat{n}_{i_t}}}(l_{i_t}) \\ &= -\rho_{-1}(l_{i_t}) y_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} \\ &= \rho_1(l_{i_t}) y_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} - y_{k_{j_1}^1}^{i_r, \varepsilon(\eta)_{k_{j_1}^1}^{i_r}} \end{aligned}$$

thus, using this in the expression \tilde{U}_{i,i_r}^η , we changed $\varepsilon_{\hat{n}_{i_t}} = -1$ into $\varepsilon_{n_{i_t}+\dots+n_{i_r}} = +1$ without changing the value of \tilde{U}_{i,i_r}^η and thus of \tilde{U}^η . This ends the proof of the theorem. \square

7.3. Proof of Proposition 4.2

PROPOSITION 4.2. — Let $0 \leq |x| \leq \varepsilon \leq q^{-1/2}$. For $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+$,

$$|U^\boldsymbol{\eta}(x)| \leq C_q^s |x|^{\|\boldsymbol{\eta}\| + \|\boldsymbol{\sigma}\|} q^{-(\|\boldsymbol{\eta}\| + \|\boldsymbol{\sigma}\|)} \quad (7.6)$$

with $C_q = \frac{1}{1 - q^{-1/2}}$.

Proof. — We remind that, for $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+$

$$T_l^\boldsymbol{\eta}(x) = x^{\|\boldsymbol{\sigma}\| + \langle l, \boldsymbol{n} \rangle} q^{-\langle l, \boldsymbol{\sigma} \rangle} [l_1 + 1]_q^{\eta_1} \dots [l_s + 1]_q^{\eta_s} = \prod_{i=1}^s T_{l_i}^{\eta_i}(x)$$

and

$$U^\boldsymbol{\eta}(x) = \varepsilon_{\mathbf{n}} \sum_{l \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1) \rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1) \dots \rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) T_l^\boldsymbol{\eta}(x) \prod_{\substack{1 \leq i \leq s \\ \varepsilon_{\hat{n}_i} = -1}} \rho_1(l_i)$$

thus

$$|U^\boldsymbol{\eta}(x)| \leq \sum_{l \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1) \rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1) \dots \rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) T_l^\boldsymbol{\eta}(|x|) \prod_{\substack{1 \leq i \leq s \\ \varepsilon_{\hat{n}_i} = -1}} \rho_1(l_i)$$

Let

$$R_l^\boldsymbol{\eta} = \rho_{\varepsilon_{\hat{n}_1}}(l_1) \rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1) \dots \rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) \left(\prod_{\substack{1 \leq i \leq s \\ \varepsilon_{\hat{n}_i} = -1}} \rho_1(l_i) \right)$$

and $L^\boldsymbol{\eta} = \{l \in \mathbb{Z}^s \ ; \ R_l^\boldsymbol{\eta} = 1\} \subset \mathbb{Z}^s$. First of all, one can notice that

$$L^\boldsymbol{\eta} \subset (\mathbb{N}^*)^s \quad (7.7)$$

because, for $1 \leq i \leq s$, if $\varepsilon_{\hat{n}_i} = -1$, then the factor $\rho_1(l_i)$ in $R_l^\boldsymbol{\eta}$ ensures that $l_i > 0$ and if $\varepsilon_{\hat{n}_i} = +1$, then

- if $i = 1$ then $\rho_1(l_1) = 1$ and $l_1 > 0$,
- if $i > 1$ and all the signs $\varepsilon_{\hat{n}_j}$ ($j \leq i$) are $+1$, then $l_1 > 0, l_2 > l_1, \dots, l_i > l_{i-1}$ thus $l_i > 0$,
- otherwise $i > 1$ and there exist $i_0 = \max\{i; \varepsilon_{\hat{n}_{i_0}} = -1\}$. Thanks to the factor $\rho_1(l_{i_0})$, once again, $l_{i_0} > 0, l_{i_0+1} > l_{i_0}, \dots, l_i > l_{i-1}$ and $l_i > 0$.

Based on a similar discussion, we get that

$$\forall \boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+, \quad n(\boldsymbol{\eta}) = \min_{\mathbf{l} \in L^\boldsymbol{\eta}} (n_1 l_1 + \dots + n_s l_s) \geq (n_1 + \dots + n_s) \geq 0 \quad (7.8)$$

and

$$\forall \boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+, \quad \begin{aligned} p_q(\boldsymbol{\eta}) &= \max_{\mathbf{l} \in L^\boldsymbol{\eta}} ([l_1 + 1]_q^{n_1} \dots [l_s + 1]_q^{n_s}) \\ &\leq [1]_q^{n_1 + \dots + n_s} = q^{-\|\mathbf{n}\|} \end{aligned} \quad (7.9)$$

These inequalities can be proved by induction on the length s of the sequence $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbf{H}^+$. These results are obvious for $\boldsymbol{\eta} = (\eta_1) \in \mathbf{H}^+$ since $l_1 > 0$. Let $s \geq 2$, we remind that $L^\boldsymbol{\eta} \subset (\mathbb{N}^*)^s$ thus,

- If $\varepsilon_{\hat{n}_s} = +1$, then $n_s \geq 0$ and $l_s > l_{s-1} > 0$. It implies that $n_s l_s \geq n_s l_{s-1}$ and $[l_s + 1]_q^{n_s} \leq [l_{s-1} + 1]_q^{n_s}$ ($l_s(l_s + 1) \geq l_{s-1}(l_{s-1} + 1)$) so

$$\begin{aligned} n((\eta_1, \dots, \eta_s)) &\geq n((\eta_1, \dots, \eta_{s-1} + \eta_s)) \\ p_q((\eta_1, \dots, \eta_s)) &\leq p_q((\eta_1, \dots, \eta_{s-1} + \eta_s)) \end{aligned} \quad (7.10)$$

- If $\varepsilon_{\hat{n}_s} = -1$, then $n_s < 0$ and $0 < l_s \leq l_{s-1}$. It implies that $n_s l_s \geq n_s l_{s-1}$ and $[l_s + 1]_q^{n_s} \leq [l_{s-1} + 1]_q^{n_s}$ ($l_s(l_s + 1) \leq l_{s-1}(l_{s-1} + 1)$) so, once again

$$\begin{aligned} n((\eta_1, \dots, \eta_s)) &\geq n((\eta_1, \dots, \eta_{s-1} + \eta_s)) \\ p_q((\eta_1, \dots, \eta_s)) &\leq p_q((\eta_1, \dots, \eta_{s-1} + \eta_s)) \end{aligned}$$

Since $(\eta_1, \dots, \eta_{s-1} + \eta_s) \in \mathbf{H}^+$, the induction is then trivial. This result implies that, for $\mathbf{l} \in L^\boldsymbol{\eta}$,

$$\begin{aligned} T_l^\boldsymbol{\eta}(|x|) &\leq q^{-\|\mathbf{n}\|} |x|^{\|\boldsymbol{\sigma}\|} |x|^{l_1 n_1 + \dots + l_s n_s} q^{-l_1 \sigma_1 - \dots - l_s \sigma_s} \\ &\leq |x|^{\|\boldsymbol{\sigma}\| + \|\mathbf{n}\|} q^{-\|\boldsymbol{\sigma}\| - \|\mathbf{n}\|} |x|^{l_1 n_1 + \dots + l_s n_s - \|\mathbf{n}\|} \prod_{i=1}^s q^{-\sigma_i (l_i - 1)} \\ &\leq |x|^{\|\boldsymbol{\sigma}\| + \|\mathbf{n}\|} q^{-\|\boldsymbol{\sigma}\| - \|\mathbf{n}\|} \prod_{i=1}^s |x|^{n_i (l_i - 1)} q^{-\sigma_i (l_i - 1)} \end{aligned}$$

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but, if $|x| \leq q^{-1/2}$, since $l_1 n_1 + \dots + l_s n_s - \|\mathbf{n}\| \geq 0$,

$$\begin{aligned} T_l^\eta(|x|) &\leq |x|^{\|\sigma\| + \|\mathbf{n}\|} q^{-\|\sigma\| - \|\mathbf{n}\|} (q^{-1/2})^{l_1 n_1 + \dots + l_s n_s - \|\mathbf{n}\|} \prod_{i=1}^s q^{-\sigma_i (l_i - 1)} \\ &\leq |x|^{\|\sigma\| + \|\mathbf{n}\|} q^{-\|\sigma\| - \|\mathbf{n}\|} \prod_{i=1}^s q^{-1/2 (n_i + 2\sigma_i) (l_i - 1)} \end{aligned}$$

and, as for $\eta \in \mathbf{H}^+$ and $l \in L^\eta \in (\mathbb{N}^*)^s$, $n_i + 2\sigma_i \geq 1$,

$$T_l^\eta(|x|) \leq |x|^{\|\sigma\| + \|\mathbf{n}\|} q^{-\|\sigma\| - \|\mathbf{n}\|} \prod_{i=1}^s q^{-1/2 (l_i - 1)}$$

and, finally

$$\begin{aligned} |U^\eta(|x|)| &= \sum_{l \in L^\eta} T_l^\eta(|x|) \\ &\leq |x|^{\|\sigma\| + \|\mathbf{n}\|} q^{-\|\sigma\| - \|\mathbf{n}\|} \sum_{l \in L^\eta} \prod_{i=1}^s q^{-1/2 (l_i - 1)} \\ &\leq |x|^{\|\sigma\| + \|\mathbf{n}\|} q^{-\|\sigma\| - \|\mathbf{n}\|} \sum_{l \in (\mathbb{N}^*)^s} \prod_{i=1}^s q^{-1/2 (l_i - 1)} \\ &\leq |x|^{\|\sigma\| + \|\mathbf{n}\|} q^{-\|\sigma\| - \|\mathbf{n}\|} C_q^s \quad \square \end{aligned}$$

8. Conclusion

We shall prove in a forthcoming paper that this result still holds for systems of nonlinear q -difference equations (with some restrictions due to resonance and small divisors). This means that q -difference nonlinear equations are almost always conjugated to linear q -difference equations: This situation is totally different from the one encountered in the case of singular irregular equations.

We did not say much about the solutions of the equation

$$(x\sigma_q - 1)y = b(y, x)$$

but, since it is analytically conjugated to an equation

$$x\sigma_q z = z + \alpha(b)x$$

it would be sufficient to study the “small” solutions of such an equation, which can be obtained by q -resummation of the formal solution :

$$\tilde{z}(x) = -\alpha(b) \sum_{n \geq 1} q^{-n(n-1)/2} x^n$$

See [3] for details. One should also deduce some results on the Stokes phenomenon.

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