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Special Lagrangian submanifolds in the complex sphere(*)

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Résumé. — We construct a family of Lagrangian submanifolds in the complex sphere which are foliated by $(n-1)$-dimensional spheres. Among them we find those which are special Lagrangian with respect to the Calabi-Yau structure induced by the Stenzel metric.

Abstract. — Nous construisons une famille de sous-variétés lagrangiennes dans la sphère complexe qui sont feuilletées par des sphères de dimension $n-1$. Nous décrivons celles qui sont de plus lagrangiennes spéciales pour la structure de Calabi-Yau induite par la métrique de Stenzel.

Introduction

Special Lagrangian submanifolds of $\mathbb{C}^n$ (or more generally of Calabi-Yau manifolds) may be defined as those submanifolds which are both Lagrangian (an order 1 condition) and minimal (an order 2 condition). Alternatively, they are characterised as those submanifolds which are calibrated by a certain n-form (*cf.* [HL]), so they have the remarkable property of being area minimizing. Their study has received considerable recent attention since connections with string theory were discovered. More particularly, understanding special Lagrangian fibrations (possibly with singularities) of Calabi-Yau manifolds of (complex) dimension 3 is crucial for mirror symmetry (*cf.* [SYZ],[Jo2]). Since the pioneering work of Harvey and Lawson [HL], where this notion was introduced, several authors ([Ha],[Jo1],[CU2]) have discovered many families of special Lagrangian submanifolds in the complex Euclidean space. However very few examples (*cf.* [Br]) of such submanifolds are known in other Calabi-Yau manifolds, where there is also a natural extension of the notion of special Lagrangian submanifold. Perhaps the main reason is that such manifolds are somewhat rare; in particular the existence of compact, Calabi-Yau manifolds is a hard result of S.-T. Yau [Y]

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involving non explicit solutions to a non-linear equation of complex Monge-Ampère type. However there exist some intermediary examples of non-flat, non-compact Calabi-Yau manifolds, perhaps the simplest of which is the complex sphere.

In this paper we describe a family of Lagrangian submanifolds of a complex variety of $\mathbb{C}^{n+1}$ with a $SO(n)$-invariance. In the case of the complex sphere, we obtain among them new examples of special Lagrangian submanifolds, which are a kind of generalization of the Lagrangian catenoid discovered in [HaLa] (*cf.* also [CU1]).

We would like to mention that in [CMU], minimal Lagrangian submanifolds with $SO(n)$-symmetry have been obtained in the complex hyperbolic and complex projective spaces respectively. Though these spaces are not Calabi-Yau manifolds, and so such submanifolds are not minimizing *a priori*, Y.-G. Oh proved in [Oh] that, in the hyperbolic case, they are stable, a necessary condition for being a minimizer; so this situation is not so far from the one of the present article.

The paper is organized as follows:

In Section 1 we give a definition of Calabi-Yau manifolds, we state some basic facts about them and describe the Calabi-Yau structure induced on the complex sphere by the Stenzel metric. In Section 2 we introduce a general class of Lagrangian submanifolds immersed in some complex variety of $\mathbb{C}^{n+1}$. In the particular case of the complex sphere, we then check that they are also Lagrangian for the Calabi-Yau structure. In the last section we find the special Lagrangians within this class and describe them.

The nice argument about $SO(n+1)$-invariance at the end of the next section is due to Dominic Joyce and Mu-Tao Wang suggested to me the Lagrangian Ansatz; it is my pleasure to thank them. I am also grateful to the anonymous referee for numerous improvements to the first version of this paper. The author, as a member of EDGE, Research Training Network HPRN-CT-2000-00101, has been supported by the European Human Potential Programme.

1. The Calabi-Yau structure on the complex sphere

We start with the following definitions — which are not the standard ones — proposed in [Jo1]:

Definition 1.1. — *Let* $n \geqslant 2$. *An* almost Calabi-Yau manifold *is a quadruple* $(\mathcal{M}, J, \omega, \Omega)$ *such that* $(\mathcal{M}, J)$ *is an n-dimensional complex man-*

ifold, ω the symplectic form of a Kähler metric g on $\mathcal{M}$ and Ω a non-vanishing holomorphic $(n,0)$-form.

If in addition

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}(i/2)^n \Omega \wedge \bar{\Omega}, \tag{1.1}$$

$(\mathcal{M}, J, \omega, \Omega)$ *is called a* Calabi-Yau manifold.

An important property of a Calabi-Yau manifold is that it is Ricci-flat, *i.e.* its Ricci curvature vanishes. Conversely, on a simply connected open subset a Ricci-flat Kähler metric yields a Calabi-Yau structure. The simplest example of Calabi-Yau manifold is $\mathbb{C}^n$ equipped with its standard structures, that is the symplectic form

$$\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$$

and the $(n,0)$-form

$$\Omega_0 = dz_1 \wedge \ldots \wedge dz_n.$$

Definition 1.2. — *A submanifold L of real dimension n of a Calabi-Yau manifold $\mathcal{M}$ is said to be special Lagrangian if* $\omega|_L \equiv \operatorname{Im}\Omega|_L \equiv 0$.

The first condition says that L is Lagrangian and the second one that L is calibrated with respect to $\pm\operatorname{Re}\Omega$. This implies that a special Lagrangian is always necessarily minimizing (*cf.* [HL]). The same definition may also be stated in the case of an almost Calabi-Yau, however in this case a special Lagrangian is no more minimizing, although it is so for a convenient rescaling of the Kähler metric (*cf.* [Jo2]).

We now describe a class of almost Calabi-Yau manifolds. Given a complex polynomial P, the set

$$Q := \{(z_0, \ldots, z_n), P(z_0) = \sum_{j=1}^{n} z_j^2\}$$

is a complex submanifold of $\mathbb{C}^{n+1}$ which is smooth except if P admits double zeroes. Therefore it inherits from the ambient space both a complex and a Kähler structure. For example, if $P(z) = z$, Q is a paraboloid which is diffeomorphic (and will be identified) to $\mathbb{C}^n$.

On Q we define the holomorphic $(n,0)$-form Ω by the requirement that

$$\Omega \wedge d(-P(z_0) + z_1^2 + \ldots + z_n^2) = dz_0 \wedge \ldots \wedge dz_n.$$

We may give an explicit form on the set $\{P'(z_0) \neq 0\}$:

$$\Omega = \frac{(-1)^{n+1}}{P'(z_0)} dz_1 \wedge \ldots \wedge dz_n.$$

This makes Q an almost Calabi-Yau manifold. If $P(z) = z$, we recover, up to sign, the standard $(n, 0)$-form of $\mathbb{C}^n$. In Section 3.1, we shall recover in this way a classical example of Special Lagrangian in $\mathbb{C}^n$, the *Lagrangian catenoid* (*cf.* [HL],[CU1]).

In the remainder of the section we shall see that in the particular case of the complex sphere, that is when $P(z) = 1 - z^2$, Q admits also a Calabi-Yau structure. This is a consequence of the following result due to Stenzel (*cf.* [St], Theorem 1, p. 152):

Proposition 1.3. — *On the complex sphere* $\{\sum_{j=0}^n z_j^2 = 1\}$, *(with its complex structure inherited from being a smooth complex hypersurface of* $\mathbb{C}^{n+1}$*) there exists a Ricci-flat, Kähler metric* g_{St} *whose corresponding symplectic form is:*

$$\omega_{St} = i\partial\bar{\partial}u(r^2) = i \sum_{j,k=0}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} u(r^2) dz_j \wedge d\bar{z}_k,$$

where $r^2 = \sum_{j=0}^n z_j \bar{z}_j$ *and* u *is some smooth real function. Moreover,* g_{St} *is complete.*

In dimension 2, u takes the explicit form: $u(r^2) = \sqrt{1 + r^2}$, and in higher dimension, it is defined as the solution of some ordinary differential equation (*cf.* [St], pp. 160–161). In [CGLP], the Stenzel metric is described in greater details, in particular the complex sphere is identified to the tangent bundle of the unit sphere. As the complex sphere is simply connected, Stenzel's theorem shows that it is Calabi-Yau. It follows that there exists a holomorphic $(n, 0)$-form which satisties equation (1) and whose real part is a calibration. In fact, the latter is just, up to a multiplicative constant, the $(n, 0)$-form Ω that we have previously introduced:

Lemma 1.4. — *The* $(n, 0)$*-form* Ω *satisfies:*

$$\omega_{St}^n = c\, \Omega \wedge \bar{\Omega},$$

where c *is some real constant. In particular* Ω *is, up to a multiplicative constant, the* $(n, 0)-$*form of the Calabi-Yau structure of the complex sphere induced by the Stenzel metric.*

Proof. — We first observe that both ω_{St} and Ω are invariant under the natural action of $SO(n+1)$ defined by $z = x + iy \mapsto Az = Ax + iAy, \forall A \in SO(n+1)$. The invariance of ω_{St} follows from the one of its potential $u(r^2)$. In order to show the invariance of Ω, let $v_1, \ldots, v_n$ be n independant tangent vectors to some point z of the complex sphere and let apply the formula

$$\Omega \wedge d(z_0^2 + z_1^2 + \ldots + z_n^2) = dz_0 \wedge \ldots \wedge dz_n,$$

to $(v_1, \ldots, v_n, z)$ and $(Av_1, \ldots, Av_n, Az)$, at the points z and Az respectively. Since the right hand side term is invariant and

$$d(z_0^2 + \ldots + z_n^2)_z(z) = d(z_0^2 + \ldots + z_n^2)_{Az}(Az) = 2,$$

it follows that $\Omega_z(v_1, \ldots, v_n) = \Omega_{Az}(Av_1, \ldots, Av_n)$, which is the required property.

Now let f be the ratio of ω_{St}^n and $\Omega \wedge \bar{\Omega}$. The invariance of both $(2n)$-forms implies that f is constant on the orbits of the action of $SO(n+1)$. But generic orbits have codimension 1 in the complex sphere, so the image of f in $\mathbb{C}$ has at most real dimension 1 locally, which forces f to be constant, since it is holomorphic. Thus the two forms are proportional.

Remark 1.5. — The complex sphere $\{\sum_{j=0}^n z_j^2 = 1\}$ may be canonically identified with the tangent bundle of the real sphere $T\,\mathbb{S}^n$ as follows: to a pair (x, v), where x is some point of $\mathbb{S}^n$ and v a tangent vector to x, we associate the point $\cosh(|v|)x + i\sinh(|v|)|v|^{-1}v$ which belongs to $\{\sum_{j=0}^n z_j^2 = 1\}$ (*cf.* [CGLP]).

2. A Lagrangian Ansatz

Let γ be a curve of the complex plane $\mathbb{C}$. Then, denoting by $x = (x_1, \ldots, x_n)$ a point of the real unit sphere $\mathbb{S}^{n-1}$, the following immersion if Lagrangian with respect to the restriction to Q of the canonical symplectic form of $\mathbb{C}^{n+1}$:

$$\begin{array}{rccl} X: & I \times \mathbb{S}^{n-1} & \to & Q \\ & (s, x) & \mapsto & (\gamma(s), \sqrt{P(\gamma(s))}x_1, \ldots, \sqrt{P(\gamma(s))}x_n). \end{array}$$

Despite the indeterminacy of the complex square root, the image $L = X(I \times \mathbb{S}^{n-1})$ of this immersion is well defined because of the invariance of $\mathbb{S}^{n-1}$ by the antipodal map $x \mapsto -x$. On the other hand it becomes singular when γ is a zero of P. We also observe that L is foliated by round $(n-1)$-dimensional spheres.

From now on we consider only the case $P(\gamma) = 1-\gamma^2$, that is the complex sphere equipped with the Stenzel form. However we shall keep on using the notation P for brevity.

Lemma 2.1. — *L is also Lagrangian with respect to the symplectic form associated to the Stenzel metric.*

Proof. — Let $(z_0, \dots, z_n)$ be coordinates on $\mathbb{C}^{n+1}$. As long as γ does not vanish we can use $(z_1, \dots, z_n)$ as coordinates on L. In particular, we have:

$$\frac{\partial z_0}{\partial z_j} = -\frac{z_j}{z_0}.$$

We compute that

$$\frac{\partial}{\partial z_j} r^2 = \bar{z}_j - \frac{\bar{z}_0}{z_0} z_j,$$

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} r^2 = \delta_{jk} + \frac{z_j \bar{z}_k}{|z_0|^2}.$$

We deduce that

$$\omega_{St} = i \sum_{j,k=1}^{n} a_{jk} dz_j \wedge d\bar{z}_k,$$

where

$$a_{jk} = \left(\delta_{jk} + \frac{z_j \bar{z}_k}{|z_0|^2} \right) u' + 2\mathrm{Re} \left(\bar{z}_j z_k - \frac{\bar{z}_0}{z_0} z_j z_k \right) u''.$$

Hence we can decompose ω_{St} as

$$\omega_{St} = u' \omega_0 + \omega_1,$$

where

$$\omega_0 := i \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$$

and

$$\omega_1 := i \sum_{j,k=1}^{n} \left(\frac{z_j \bar{z}_k}{|z_0|^2} u' + 2\mathrm{Re} \left(\bar{z}_j z_k - \frac{\bar{z}_0}{z_0} z_j z_k \right) u'' \right) dz_j \wedge d\bar{z}_k.$$

It is straightforward to see that L is Lagrangian with respect to ω_0, so it remains to show that it is also the case for ω_1. This will be a consequence of the fact that the two following 2-forms vanish on L:

$$\sum_{j,k=1}^{n} z_j \bar{z}_k dz_j \wedge d\bar{z}_k, \qquad \sum_{j,k=1}^{n} z_j z_k dz_j \wedge d\bar{z}_k.$$

We shall denote $v^\alpha = (v_1^\alpha, \dots, v_n^\alpha)$, $1 \leqslant \alpha \leqslant n-1$, a basis of tangent vectors to $\mathbb{S}^{n-1}$ at $x = (x_1, \dots, x_n)$.

This yields a basis of the tangent space of L at $X(s,x) = (\gamma, \sqrt{P(\gamma)}x_1, \dots, \sqrt{P(\gamma)}x_n)$:

$$X_s = (\dot\gamma, \frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}}x_1, \dots, \frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}}x_n),$$

$$X_* v^\alpha = (0, \sqrt{P(\gamma)}v_1^\alpha, \dots, \sqrt{P(\gamma)}v_n^\alpha),$$

where the dot '$\dot{}$' denotes the derivation with respect to the variable s.

We then compute:

$$\left(\sum_{j,k=1}^n z_j \bar z_k dz_j \wedge d\bar z_k\right)(X_s, X_* v^\alpha)$$

$$= \sum_{j,k=1}^n |P(\gamma)| x_j x_k \begin{vmatrix} \frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}} x_j & \sqrt{P(\gamma)} v_j^\alpha \\ \overline{\frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}}} x_k & \overline{\sqrt{P(\gamma)}} v_k^\alpha \end{vmatrix}$$

$$= \sum_{j,k=1}^n |P(\gamma)| x_j x_k \left(\frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}} \overline{\sqrt{P(\gamma)}} x_j v_k^\alpha - \overline{\frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}}} \sqrt{P(\gamma)} x_k v_j^\alpha \right)$$

$$= |P(\gamma)| \frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}} \overline{\sqrt{P(\gamma)}} \sum_{j=1}^n x_j^2 \sum_{k=1}^n x_k v_k^\alpha - |P(\gamma)| \overline{\frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}}} \sqrt{P(\gamma)} \sum_{k=1}^n x_k^2 \sum_{j=1}^n x_j v_j^\alpha.$$

The latter vanishes because tangent vectors to the real sphere at some point are orthogonal at this point.

On the other hand we have:

$$\left(\sum_{j,k=1}^n z_j \bar z_k dz_j \wedge d\bar z_k\right)(X_* v^\alpha, X_* v^\beta)$$

$$= \sum_{j,k=1}^n |P(\gamma)| x_j x_k \begin{vmatrix} \sqrt{P(\gamma)} v_j^\alpha & \sqrt{P(\gamma)} v_j^\beta \\ \overline{\sqrt{P(\gamma)}} v_k^\alpha & \overline{\sqrt{P(\gamma)}} v_k^\beta \end{vmatrix}$$

$$= \sum_{j,k=1}^{n} |P(\gamma)| x_j x_k \left(|P(\gamma)| v_j^\alpha v_k^\beta - |P(\gamma)| v_k^\alpha v_j^\beta \right) = 0.$$

The computations showing that $\sum_{j,k} z_j z_k dz_j \wedge d\bar{z}_k$ vanishes on L are analogous and left to the Reader.

3. Special Lagrangian submanifolds

In this section, we shall consider the two cases in which there is a Calabi-Yau structure and look for those submanifolds within the class defined in the previous section which are calibrated by $\pm\mathrm{Re}\,\Omega$, that is those on which $\mathrm{Im}\,\Omega$ vanishes.

3.1. Case of the Euclidean complex space $P(z) = z$

As we have already observed, when $P(z) = z$ in the construction made in Section 1, the manifold Q may be identified with $\mathbb{C}^n$. Therefore we consider the stantard holomorphic $(n, 0)$-form of $\mathbb{C}^n$ given in coordinates by

$$\Omega_0 = dz_1 \wedge \ldots \wedge dz_n.$$

Putting $\alpha = \sqrt{\gamma}$, the equation for L so be special Lagrangian is $\mathrm{Im}\,(\dot{\alpha}\alpha^{n-1}) = 0$, which is easy to solve: $\mathrm{Im}\,(\alpha^n) = C$, where C is some real constant. This implies the existence of two types of solutions to the special Lagrangian equation:

- If $C = 0$, α is the union of n lines passing through the origin $\{\arg z = k\pi/n \mod \pi\}$, $1 \leqslant k \leqslant n-1$, and the corresponding special Lagrangians are just Lagrangian n-spaces;
- If $C \neq 0$, the curve is asymptotic to two of the lines described above. All the curves are congruent (and so the corresponding Lagrangian as well). We recognize the Lagrangian catenoid which was first identified in [HL] and characterised in [CU1] as the only special Lagrangian of $\mathbb{C}^n$ which is foliated by $(n-1)$-dimensional round spheres.

3.2. Case of the complex sphere $P(z) = 1 - z^2$

Before we start the computations, we observe that we already know a special Lagrangian in the complex sphere: if a Calabi-Yau has a "real structure" $z \mapsto \bar{z}$, *i.e.* an anti-holomorphic involution which is in some sense compatible (see [Br] and [Au], pp. 62–64), then the set of real points $\{z = \bar{z}\}$, if non empty, is a special Lagrangian submanifold. Here, the set of real points is the real sphere embedded in Q, so the latter will be a trivial solution to our problem.

We now compute the holomorphic form Ω on L:

$$
\begin{aligned}
&\Omega(X_s, X_* v^1, \ldots, X_* v^{n-1}) \\
&\quad = \frac{(-1)^{n+1}}{P'(\gamma)} \begin{vmatrix} \frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}} x_1 & \sqrt{P(\gamma)} v_1^1 & \cdots & \sqrt{P(\gamma)} v_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\dot\gamma P'(\gamma)}{2\sqrt{P(\gamma)}} x_n & \sqrt{P(\gamma)} v_n^1 & \cdots & \sqrt{P(\gamma)} v_n^{n-1} \end{vmatrix} \\
&\quad = \frac{(-1)^{n+1}\dot\gamma}{2\sqrt{P(\gamma)}} \sqrt{P(\gamma)}^{\,n-1} \begin{vmatrix} x_1 & v_1^1 & \cdots & v_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & v_n^1 & \cdots & v_n^{n-1} \end{vmatrix} \\
&\quad = \dot\gamma \sqrt{1-\gamma^2}^{\,n-2} C_{\mathbb{R}},
\end{aligned}
$$

where $C_{\mathbb{R}}$ is some real, non-vanishing constant.

Thus the differential equation for L to be special Lagrangian is

$$\mathrm{Im}\left(\dot\gamma \sqrt{1-\gamma^2}^{\,n-2}\right) = 0.$$

This equation has two singular points ± 1 and is regular elsewhere. Moreover, there is always a simple solution to this equation, the real segment $[-1, 1]$. This corresponds to the standard embedding of the real sphere. We shall analyse separately the even and odd cases.

3.2.1. Even case

In this case, the equation is polynomial: $\mathrm{Im}\left(\dot\gamma(1-\gamma^2)^{n/2-1}\right) = 0$. It is easy to integrate and we get $\mathrm{Im}\,(Q(\gamma)) = C$, where Q is some polynomial such that $Q'(z) = (1-z^2)^{n/2-1}$. The degree of Q is $n-1$, so the integral curves are algebraic curves of the same degree. In order to have a better description of the solutions, we make an asymptotic analysis of the equation when $|\gamma| \to \infty$ and when $\gamma \sim \pm 1$.

As $|\gamma|$ tends to infinity, we have

$$\mathrm{Im}\left(\dot\gamma(1-\gamma^2)^{n/2-1}\right) \sim -\mathrm{Im}\,(\dot\gamma\gamma^{n-2}),$$

so asymptotically, the phase portrait of the equation looks like the one of the flat case (however not of the same dimension). In particular all integral curves are asymptotic to the half lines $\{\arg(z) = k\pi/(n-1)\}$, $1 \leqslant k \leqslant 2n-2$.

We now write $\gamma = 1 + w$ and we see that as w tends to 0,

$$\mathrm{Im}\left(\dot\gamma(1-\gamma^2)^{n/2-1}\right) = \mathrm{Im}\left(\dot w(-2w-w^2)^{n/2-1}\right) \sim \mathrm{Im}\left(\dot w(-2w)^{n/2-1}\right)$$

It follows that next to the singular point 1, the integral lines look like the level sets of

$$\mathrm{Im}\,(-1/n(-2w)^{n/2}).$$

In particular, the level set of 0 is the union of n branches tangent at 1 to the half-lines $\{\arg(z-1) = 2k\pi/n + \pi\}$, $1 \leqslant k \leqslant n$, one of them being of course the real segment $[-1, 1]$.

Since the phase portrait is invariant by the reflection $z \mapsto -\bar z$, the situation at the point -1 is symmetric with respect to that of the point 1.

3.2.2. Odd case

When n is odd, there is also a first integral which takes the following form:

$$\mathrm{Im}\left(\gamma\sqrt{1-\gamma^2}R(\gamma) + A\arcsin(\gamma)\right),$$

where R is an even polynomial of degree $n-3$ and A a real constant. Although this function is not well-defined on the whole complex plane, it turns out that all its level curves are globally defined.

As $|\gamma|$ tends to infinity we have:

$$\mathrm{Im}\left(\dot\gamma\sqrt{1-\gamma^2}^{\,n-2}\right) = \mathrm{Im}\left(\dot\gamma\sqrt{1+(i\gamma)^2}^{\,n-2}\right) \sim \mathrm{Im}\,(\dot\gamma(i\gamma)^{n-2})$$

$$= \pm\mathrm{Im}\,(i\dot\gamma\gamma^{n-2})$$

Again, the phase portrait looks like the flat case at infinity, up to a rotation of angle $\pm\pi/2(n-1)$. In particular, all curves are asymptotic to the half lines $\{\arg(z) = k\pi/(n-1) + \pi/2(n-1)\}$, $1 \leqslant k \leqslant 2n-2$.

The asymptotic analysis at the singular points ± 1 is the same than in the even case.

3.2.3. Concluding remarks

From the preceding observations, we can describe the general picture of the phase portrait:

- there is a singular curve which is the union the real segment $[-1,1]$ and of $2n-2$ branches, one half of them starting from 1 and the other half from -1, making between them an angle of $2\pi/n$ and going to infinity where they are asymptotic to the half lines $\{\arg(z) = k\pi/(n-1)\}$, $1 \leqslant k \leqslant 2n-2$ when n is even and to $\{\arg(z) = k\pi/(n-1)+\pi/2(n-1)\}$, $1 \leqslant k \leqslant 2n-2$ when n is odd; when n is even, two of these branches are the horizontal half-lines $\{\pm s, s \in [1,\infty)\}$;
- the other curves are smooth and have two ends asymptotic to two successive branches described above.

Remark 3.1. — In the case of dimension 2, the integral lines are simply the horizontal lines.

We deduce the existence of the following special Lagrangians in the complex sphere equipped with the Stenzel metric:

- The standard embedding of the real sphere;
- $2n-2$ "exceptional" special Lagrangians homeomorphic to $\mathbb{R}^n$; one half of them touches the real sphere at the north pole $(1,0,\dots,0)$ and the other half at the south pole $(-1,0\dots,0)$;
- A one-parameter smooth family of special Lagrangians homeomorphic to $\mathbb{R}\times\mathbb{S}^{n-1}$. Each of them has two ends asymptotic to two "exceptional" Lagrangians.

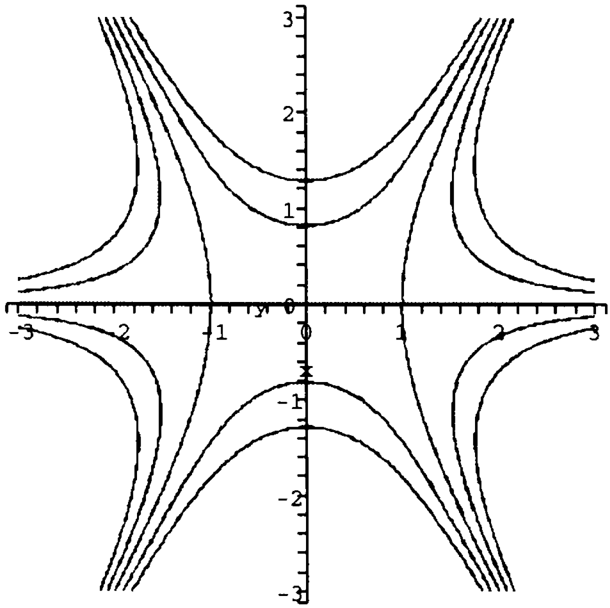

Figure 1. — Phase portrait, $n = 4$

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