ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

JONATHAN PILA

The density of rational points on a pfaff curve

Tome XVI, nº 3 (2007), p. 635-645.

<http://afst.cedram.org/item?id=AFST_2007_6_16_3_635_0>

Annales de la Faculté des Sciences de Toulouse Vol. XVI, n° 3, 2007
pp. 635–645

The density of rational points on a pfaff curve[(*)]

Jonathan Pila [(1)]

Abstract. — This paper is concerned with the density of rational points on the graph of a non-algebraic pfaffian function.

Résumé. — Cet article est concerné par la densité de points rationnels sur le graphe d'une fonction pfaffienne non-algébrique.

1. Introduction

In two recent papers [8, 9] I have considered the density of rational points on a pfaff curve (see definitions 1.1 and 1.2 below). Here I show that an elaboration of the method of [8] suffices to establish a conjecture stated (and proved under additional assumptions) in [9].

1.1. Definition

Let $H:\mathbb{Q}\to\mathbb{R}$ be the usual height function, $H(a/b)=\max(|a|,b)$ for $a,b\in\mathbb{Z}$ with $b>0$ and $(a,b)=1$. Define $H:\mathbb{Q}^n\to\mathbb{R}$ by $H(\alpha_1,\alpha_2,\ldots,\alpha_n)$ $=\max_{1\leqslant j\leqslant n}(H(\alpha_j))$. For a set $X\subset\mathbb{R}^n$ define $X(\mathbb{Q})=X\cap\mathbb{Q}^n$ and, for $H\geqslant 1$, put

$$X(\mathbb{Q},H)=\{P\in X(\mathbb{Q}):H(P)\leqslant H\}.$$

The *density function* of X is the function

$$N(X,H)=\#X(\mathbb{Q},H).$$

This is not the usual projective height, although this makes no difference to the results here. The class of pfaffian functions was introduced by Khovanskii [5]. The following definition is from [3].

(*) Reçu le 20 octobre 2005, accepté le 9 février 2006
(1) School of Mathematics, University of Bristol, Bristol, BS8 1TW (UK).
j.pila@bristol.ac.uk

1.2. Definition ([3, 2.1])

Let $U \subset \mathbb{R}^n$ be an open domain. A *pfaffian chain* of order $r \geqslant 0$ and degree $\alpha \geqslant 1$ in U is a sequence of real analytic functions $f_1, \dots, f_r$ in U satisfying differential equations

$$df_j = \sum_{i=1}^{n} g_{ij}(\mathbf{x}, f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_j(\mathbf{x}))dx_i$$

for $j = 1, \dots, r$, where $\mathbf{x} = (x_1, \dots, x_n)$ and $g_{ij} \in \mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_r]$ of degree $\leqslant \alpha$. A function f on U is called a *pfaffian function* of order r and degree (α, β) if $f(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \dots, f_r(\mathbf{x}))$, where P is a polynomial of degree at most $\beta \geqslant 1$. In this paper mainly $n = 1$, so $\mathbf{x} = x$.

A *pfaff curve* X is the graph of a pfaffian function f on some connected subset of its domain. The order and degree of X will be taken to be the order and degree of f.

The usual elementary functions $e^x, \log x$ (but not $\sin x$ on all $\mathbb{R}$), algebraic functions, and sums, products and compositions of these are pfaffian functions, such as e.g. $e^{-1/x}, e^{e^x}$, etc: see [5, 3]. Note that, for non-algebraic X, $X(\mathbb{Q})$ can be infinite (e.g. 2^x), or of unknown size (e.g. e^{e^x}).

Suppose X is a pfaff curve that is not semialgebraic. Since the *structure* generated by pfaffian functions is *o-minimal* (see [2, 13]), an estimate of the form

$$N(X, H) \leqslant c(X, \epsilon)H^\epsilon$$

for all positive ϵ (and, with suitable hypotheses, in all dimensions) follows from [10].

I showed in [8] that there is an explicit function $c(r, \alpha, \beta)$ with the following property. Suppose X is a nonalgebraic pfaff curve of order r and degree (α, β). Let $H \geqslant c(r, \alpha, \beta)$. Then

$$N(X, H) \leqslant \exp\left(5\sqrt{\log H}\right).$$

As noted in [6, 7.5], no such quantification of the $c(X, \epsilon)H^\epsilon$ bound can hold for bounded subanalytic sets, and so the estimate cannot be improved for a general o-minimal structure. But much better bounds could be anticipated for sets defined by pfaffian functions, as conjectured in [10].

1.3. Theorem

Let $X \subset \mathbb{R}^2$ be a pfaff curve, and suppose that X is not semialgebraic. There are constants $c(r, \alpha, \beta), \gamma(r) > 0$ such that (for $H \geqslant e$)

$$N(X, H) \leqslant c(\log H)^{\gamma}.$$

Indeed, if X is the graph of a pfaffian function f of order r and degree (α, β) on an interval $I \subset \mathbb{R}$ then the above holds with $\gamma = 5(r+2)$ and suitable $c(r, \alpha, \beta)$.

In fact the result may be strengthened (with suitable γ) to apply to a *plane pfaffian curve* $X \subset \mathbb{R}^2$ defined as the set of zeros of a pfaffian function $F(x, y)$, where F is defined e.g. on $U = I \times J$ where $I, J \subset \mathbb{R}$ are open intervals. Such X may contain semialgebraic subsets of positive dimension, which must be excluded: see 1.4 and 1.5 below. This extension is sketched after the proof of 1.3 in §4. I thank the referee for suggesting that such an extension be considered.

Theorem 1.3 affirms a conjecture made in [9, 1.3]. That conjecture was an extrapolation of part of the one-dimensional case of a conjecture in [10, 1.5]. It is natural to frame the following generalization.

1.4. Definition ([10, §1; 7, §1])

Let $X \subset \mathbb{R}^n$. The *algebraic part* of X, denoted X^{alg}, is the union of all connected semialgebraic subsets of X of positive dimension. The *transcendental part* of X is the complement $X - X^{\mathrm{alg}}$.

1.5. Conjecture

Let $\mathbb{R}_{\mathrm{Pfaff}}$ be the structure generated by pfaffian sets ([13, §0]). Let X be definable in $\mathbb{R}_{\mathrm{Pfaff}}$. Then there exist constants $c(X), \gamma(X)$ such that (for $H \geqslant e$)

$$N(X - X^{\mathrm{alg}}, H) \leqslant c(\log H)^{\gamma}.$$

In [9] I obtained the conclusion of Theorem 1.3 under an additional hypothesis on the curve X and further conjectured that in fact this additional hypothesis always holds: This conjecture remains of interest as it might yield a better dependence of γ on r, and may moreover be more susceptible of extension to higher dimensions.

2. Preliminaries

2.1. Definition

Let I be an interval (which may be closed, open or half-open; bounded or unbounded), $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $L > 0$ and $f : I \to \mathbb{R}$ a function with k continuous derivatives on I. Set $T_{L,0}(f) = 1$ and, for positive k,

$$T_{L,k}(f) = \max_{1 \leqslant i \leqslant k} \left(1, \sup_{x \in I} \left(\frac{|f^{(i)}(x)| L^{i-1}}{i!}\right)^{1/i}\right).$$

(so possibly $T_{L,k}(f) = \infty$ if a derivative of order $i, 1 \leqslant i \leqslant k$, is unbounded, and then the conclusion of the following proposition is empty.) Set further

$$\tau_{L,k} = \left(\prod_{i=0}^{k-1} T_{L,i}(f)^i\right)^{2/(k(k-1))}.$$

2.2. Proposition

Let $d \geqslant 1, D = (d+1)(d+2)/2, H \geqslant 1, L \geqslant 1/H^3$. *Let* I *be an interval of length* $\ell(I) \leqslant L$. *Let* f *be a function possessing* $D-1$ *continuous derivatives on* I, *with* $|f'| \leqslant 1$ *and with graph* X. *Then* $X(\mathbb{Q}, H)$ *is contained in the union of at most*

$$6\, T_{L,D-1}(f) L^{8/(3(d+3))} H^{8/(d+3)}$$

real algebraic curves of degree $\leqslant d$.

Proof. — This is [7, Corollary 2.5]. □

It is shown in [9] that the conclusion holds with $\tau_{L,D}$ in place of $T_{L,D-1}$. This is an improvement if the derivatives of f grow super-geometrically, but is not required here.

3. Non-oscillating functions

The following elementary lemma is a trivial variant of [1, Lemma 7]. For related, sharper formulations and relations to theory of analytic functions see Pólya [11], the references therein and commentary (in the collected papers).

3.1. Proposition

Let $k \in \mathbb{N}, L > 0, T \geqslant 1$ and let I be an interval with $\ell(I) \leqslant L$. Suppose $g : I \to \mathbb{R}$ has k continuous derivatives on I. Suppose that $|g'| \leqslant 1$ throughout I and that

(a) $|g^{(i)}(x)| \leqslant i!T^i L^{1-i}$, all $1 \leqslant i \leqslant k-1, t \in I$, and

(b) $|g^{(k)}(x)| \geqslant k!T^k L^{1-k}$ all $t \in I$.

Then $\ell(I) \leqslant 2L/T$.

Proof. — Let $a, b \in I$. By Taylor's formula, for a suitable intermediate point ξ,

$$g(b) - g(a) = \sum_{i=1}^{k-1} \frac{g^{(i)}(a)}{i!}(b-a)^i + \frac{g^{(k)}(\xi)}{k!}(b-a)^k.$$

Therefore

$$L\left(\frac{(b-a)T}{L}\right)^k \leqslant (b-a)^k T^k L^{1-k} \leqslant \sum_{i=1}^{k-1}(b-a)^i T^i L^{1-i} + L \leqslant L \sum_{i=0}^{k-1}\left(\frac{(b-a)T}{L}\right)^i.$$

Thus, if $q = (b-a)T/L$, then $q^k \leqslant \sum_{i=0}^{k-1} q^i$, whence $q \leqslant 2$, completing the proof. □

The following proposition contains the new feature of this paper. It is a more careful version of the recursion argument [8, 2.1].

3.2. Proposition

Let $d \geqslant 1, D = (d+1)(d+2)/2, H \geqslant e, L > 1/H^2$ and I an interval of length $\ell(I) \leqslant L$. Let $f : I \to \mathbb{R}$ have D continuous derivatives, with $|f'| \leqslant 1$ and $f^{(j)}$ either non-vanishing in the interior of I or identically zero for $j = 1, 2, \ldots, D$. Let X be the graph of f. Then $X(\mathbb{Q}, H)$ is contained in at most

$$66\, D \log(eLH^2) \left(LH^3\right)^{8/(3(d+3))}$$

real algebraic curves of degree $\leqslant d$.

Proof. — Under the hypotheses I is a finite interval. Let a, b, with $a < b$ be its boundary points, which may or may not belong to I. If J

is a subinterval of I, and $X|_J$ is the graph of the restriction of f to J, write $G(f,J)$ for the minimal number of algebraic curves of degree $\leqslant d$ required to contain $X|_J(\mathbb{Q},H)$.

Let $T \geqslant 2D$.

By the hypotheses, any equation of the from $|f^{(\kappa)}(x)| = K$, where $0 \leqslant \kappa \leqslant D-1$ and $K \in \mathbb{R}$ has at most one solution $x \in I$, unless it is satisfied indentically. Thus the equation $|f^{(2)}(x)| = 2TL^{-1}$ has at most one solution unless it is satisfied identically. In the case that there is a unique solution $x = c$, it follows from the monotonicity of $|f^{(2)}|$ that $|f^{(2)}| \geqslant 2TL^{-1}$ on either (a,c) or (c,b) and by 3.1, this interval has length at most $2L/T$. On the remaining interval (c,b) or (a,c), the inequality $|f^{(2)}| \leqslant 2TL^{-1}$ holds.

Continue to split I at those points (if they exist) where $|f^{(\kappa)}(x)| = \kappa! T^{\kappa} L^{1-\kappa}$, for $\kappa = 3, \ldots, D-1$. This yields an interval $I_0 = (s,t)$, possibly empty, in which $|f^{(\kappa)}(x)| \leqslant \kappa! T^{\kappa} L^{1-\kappa}$ for all $\kappa = 1, 2, \ldots D-1$, while the remaining intervals $J_1^L = (a,s)$ and $J_1^R = (t,b)$ (which may also be empty) comprise at most D subintervals each of length $\leqslant 2L/T$, and hence have length $\leqslant 2DL/T$.

The bounds for f and its derivatives on I_0 imply that

$$T_{L,D-1}(f) \leqslant T$$

on I_0 and hence, by 2.2,

$$G(f, I_0) \leqslant 6TL^{8/(3(d+3))} H^{8/(d+3)}.$$

Put $\lambda = 2D/T$, so that $\lambda \leqslant 1$ by the hypotheses. Then

$$G(f,I) \leqslant 6TL^{8/(3(d+3))} H^{8/(d+3)} + G(f, J_1^L) + G(f, J_1^R)$$

where $\ell(J_1^L), \ell(J_2^L) \leqslant \lambda L$.

Now repeat the subdivision process for each of J_1^L, J_1^R with λL in place of L and the same T. Since $\lambda \leqslant 1$, the new subdivision values $\kappa! T^{\kappa} (\lambda L)^{1-\kappa}$ exceed the previous ones for each κ; the subinterval on which $|f^{(\kappa)}(x)| \geqslant \kappa! T^{\kappa} (\lambda L)^{1-\kappa}$, if non-empty, must have the form (a,u) for J_1^L, or (v,b) for J_1^R. This process yields two subintervals I_1^L, I_1^R on which $|f^{(\kappa)}(x)| \leqslant \kappa! T^{\kappa} (\lambda L)^{1-\kappa}$ for all κ, and two subintervals $J_2^L = (a,u), J_2^R = (v,b)$ of length at most $\lambda^2 L$ so that now (provided $\lambda L \geqslant 1/H^3$)

$$G(f,I) \leqslant 6TL^{8/(3(d+3))} H^{8/(d+3)} + 2.6T(\lambda L)^{8/(3(d+3))} H^{8/(d+3)}$$
$$+ G(f, J_2^L) + G(f, J_2^R).$$

Continuing in this way yields, after n iterations, provided $\lambda^{n-1}L \geqslant 1/H^3$, and putting $\sigma = 8/(3(d+3))$,

$$G(f,I) \leqslant 6TL^\sigma H^{8/(d+3)}\Big(1 + 2\lambda^\sigma + \ldots + 2\lambda^{(n-1)\sigma}\Big) + G(f,J_n^L) + G(f,J_n^R)$$

where $\ell(J_n^L), \ell(J_n^R) \leqslant \lambda^n L$. Since $\lambda \leqslant 1$, $1 + 2\lambda^\sigma + \ldots + 2\lambda^{(n-1)\sigma} \leqslant 2n-1$ so that, provided $\lambda^n LH^3 \geqslant 1$,

$$G(f,I) \leqslant 6\,(2n-1)\,TL^\sigma H^{8/(d+3)} + G(f,J_n^L) + G(f,J_n^R).$$

Take n so that

$$\frac{\lambda}{LH^2} \leqslant \lambda^n < \frac{1}{LH^2}.$$

Then J_n^L, J_n^R, having length $< 1/H^2$, contain at most one rational point of height $\leqslant H$, so that $G(f,J_n^L) + G(f,J_n^R) \leqslant 2$, while

$$n \leqslant \log(LH^2/\lambda)/\log(1/\lambda).$$

Thus taking $\lambda = 1/e$, i.e. $T = 2eD$,

$$\begin{aligned} G(f,I) &\leqslant 12eD\Big(2\log(eLH^2) - 1\Big)L^\sigma H^{8/(d+3)} + 2 \\ &\leqslant 66\,D\,\log(eLH^2)\,\big(LH^3\big)^{8/(3(d+3))} \end{aligned}$$

as required. $\square$

3.3. Corollary

Under the conditions of 3.2, if also $L \leqslant 2H$ and $H \geqslant e$ then $X(\mathbb{Q},H)$ is contained in at most

$$660\,D\,H^{32/(3(d+3))}\,\log H$$

algebraic curves of degree $\leqslant d$. $\square$

Proof. — Observe that $\log(eLH^2) \leqslant \log(2e) + 3\log H \leqslant 5\log H$, and $(LH^3)^{8/(3(d+3))} \leqslant 2H^{32/(3(d+3))}$. $\square$

4. Proof of theorem 1.3

If f is a pfaffian function, then its derivatives are also pfaffian, and the number of zeros of a derivative (if it is not identically zero) may be bounded uniformly in the order and degree of f, and the order of derivative.

The intersection multiplicity of the graph X of a pfaffian function and an algebraic curve is (if non-degenerate) also explicitly bounded.

The following explicit bounds are drawn from [3]. With these bounds and Corollary 3.3, the proof of 1.3 is easily concluded.

4.1. Proposition

Let $f_1, \ldots, f_r$ be a pfaffian chain of order $r \geqslant 1$ and degree α on an open interval $I \subset \mathbb{R}$, and f a pfaffian function on I having this chain and degree (α, β).

(a) Let $k \in \mathbb{N}$. Then $f^{(k)}$ is a pfaffian function with the same chain as f (so of order r) and degree $\big(\alpha, \beta + k(\alpha - 1)\big)$.

(b) If f is not identically zero, it has at most $2^{r(r-1)/2+1}\beta(\alpha+\beta)^r$ zeros.

Suppose further that f is non-algebraic.

(c) Let $P(x, y)$ be a polynomial of degree d. Then the number of zeros of $P(x, f(x)) = 0$ in I is at most

$$2^{r(r-1)/2+1}\, d\beta \left(\alpha + d\beta\right)^r.$$

(d) Let $J \subset I$ be an open interval on which $f' \neq 0$ and $k \geqslant 1$. Then on $f(J)$ there is an inverse function g of f. Then g is not algebraic and the number of zeros of $g^{(k)}$ on $f(J)$ is at most

$$2^{r(r-1)/2+1}\,(k-1)(\beta + k(\alpha - 1))\Big(\alpha + (k-1)(\beta + k(\alpha-1))\Big)^r.$$

Proof. — Part (a) is by [3, 2.5].

Part (b) follows from [3, 3.3], which states in particular that the set of zeros of a pfaffian function f of order r and degree (α, β) on an interval I has at most $2^{r(r-1)/2+1}\beta(\alpha+\beta)^r$ connected components.

Part (c). Since $P(x, f(x))$ is a pfaffian function of order $r \geqslant 1$ and degree $(\alpha, d\beta)$, the conclusion follows from (b).

Part (d). By differentiating the relation $g(f(x)) = x$ and simple induction, for $k \geqslant 1$,

$$g^{(k)}(y) = \frac{Q_k(f^{(1)}, f^{(2)}, \ldots, f^{(k)})}{(f'(x))^{2k-1}}$$

where $Q_k(z_1, z_2, \ldots, z_k)$ is a polynomial of degree $\gamma_k = k-1$. Since $f^{(j)}$ are pfaffian functions with the same chain, the function $Q_k(f^{(1)}, f^{(2)}, \ldots, f^{(k)})$ is a pfaffian function of order r and degree $(\alpha, \gamma_k(\beta + k(\alpha-1)))$. The statement now follows from (b). $\square$

4.2. Proof of 1.3

Suppose f is defined on an interval I. Divide I into at most

$$2 \cdot 2^{r(r-1)/2+1}(\beta+\alpha-1)(\alpha+\beta+\alpha-1)^r + 1 \leqslant 2^{2+r(r-1)/2}\big(2\alpha+\beta\big)^{r+1}$$

subintervals on which $f' \leqslant -1$, $-1 \leqslant f' \leqslant 1$ or $f' \geqslant 1$, and then divide further into subintervals on which the inverse g of f has nonvanishing derivatives up to order D in the first and third case, or f has nonvanishing derivatives up to order D in the second case. For $k \leqslant D$, the number of zeros of $f^{(k)}$ or $g^{(k)}$ on an interval is, by 4.1 (b) or (c), at most $c_0(r, \alpha, \beta)D^{2r+2}$ for some explicit function $c_0(r, \alpha, \beta)$. The total number of intervals is therefore at most

$$c_1(r, \alpha, \beta)D^{2r+3}$$

for some explicit function $c_1(r, \alpha, \beta)$.

Intersecting with the interval $[-H, H]$ of the appropriate axis (which contains all points of height $\leqslant H$), the relevant intervals are of length $\leqslant 2H$. By 3.3, in each such interval the points of $X(\mathbb{Q}, H)$ lie on at most

$$660\, D\, H^{32/(3(d+3))} \log H$$

real algebraic curves of degree $\leqslant d$. The number of points in the intersection of X with a curve of degree d is at most

$$2^{r(r-1)/2+1}\, d\beta\, (\alpha + d\beta)^r = c_2(r, \alpha, \beta) d^{r+1}.$$

Combining these estimates yields

$$N(X, H) \leqslant c_3(r, \alpha, \beta)\, d^{5r+9}\, H^{32/(3(d+3))} \log H.$$

Taking $d = [\log H]$, where $[.]$ is the integer part, completes the proof. $\square$

Suppose that $F(x, y)$ is a pfaffian function of order r and degree (α, β) defined on $U = I \times J$ where $I, J \subset \mathbb{R}$ are open intervals. I sketch how to extend the conclusion of Theorem 1.3 to the (transcendental part of the) zero set $X \subset U$ of F.

The set X consists of at most $c_4(r,\alpha,\beta)$ isolated points and at most $c_5(r,\alpha,\beta)$ graphs $y = f(x)$ or $x = g(y)$ of real analytic functions f, g defined on open intervals and satisfying $F(x, f(x)) = 0, F(g(y), y) = 0$, with $F_y(x, f(x)), F_x(g(y), y) \neq 0$ (respectively), and with further derivatives f', g' bounded in absolute value by 1. It thus suffices to consider X to be such a graph, which may be assumed to be non-algebraic.

To proceed with the proof following the proof of 1.3, we need only show that the number of zeros of $f^{(k)}$ is suitably bounded (i.e. by a polynomial function of k), and that the number of zeros of an equation $P(x, f(x)) = 0$ is suitably bounded (i.e. polynomially in the degree of P). The zeros of $P(x, f(x))$ are isolated and contained in the common zeros of $F(x, y) = 0, P(x, y) = 0$. The number of connected components of this set is at most $c_6(r,\alpha,\beta)d^{2r+2}$ by [3, 3.3].

By differentiating the relation $F(x, f(x)) = 0$ we may write

$$f^{(k)} = \frac{H_k}{F_y(x, f(x))^{a_k}}$$

where H_k is a polynomial in partial derivatives of F. If H_k consists of terms of the form $\phi_1\phi_2 \dots \phi_m$, where ϕ_i is a partial derivative of F of order δ_i, we will say that the *weight* of this term is $\sum \delta_i$, and the *weight* h_k of H_k is the maximum weight of its terms. A straightforward induction (very similar to the one in [1, Lemma 5]) shows that $a_k = 2k - 1, h_k = 3k - 2$. The zeros of $f^{(k)}$ are isolated, since f is non-algebraic. They are contained in the common zero set of $F = 0, H_k = 0$. The number of connected components of this set is at most $c_7(r,\alpha,\beta)k^{2r+2}$, again by [3, 3.3].

4.3. Final remarks

1. I know of no example in which $N(X, H)$ grows faster than $\log H$; For $X : y = 2^x$, clearly $N(X, H) >> \log H$.

2. The curves $y = x^\mu, \mu \in \mathbb{R}, x > 0$ are pfaffian (with $r = 2$) and non-algebraic provided $\mu \notin \mathbb{Q}$. Thus theorem 1.3 directly implies a very weak form of the "six exponentials" theorem ([12]).

3. Theorem 1.3 holds for curves $X : y = f(x)$ for which f admits appropriate control over the zeros of derivatives (i.e. the number of zeros of $f^{(k)}$ grows polynomially with k) and over the number of solutions of $P(x, f(x)) = 0$ (i.e. a bound that depends only on the degree of P and is polynomial d). For examples that do not lie in any o-minimal structure see [4].

Bibliography

[1] Bombieri (E) and Pila (J.). — The number of integral points on arcs and ovals, Duke Math. J. 59, p. 337-357 (1989).

[2] van den Dries (L.). — Tame topology and *o*-minimal structures, LMS Lecture Note Series 248, CUP, Cambridge, (1998).

[3] Gabrielov (A.) and Vorobjov (N.). — Complexity of computations with pfaffian and noetherian functions, in Normal Forms, Bifurcations and Finiteness problems in Differential Equations, Kluwer, (2004).

[4] Gwozdziewicz (J.), Kurdyka (K.), Parusinski (A.). — On the number of solutions of an algebraic equation on the curve $y = e^x + \sin x, x > 0$, and a consequence for o-minimal structures, Proc. Amer. Math. Soc. 127, p. 1057-1064 (1999).

[5] Khovanskii (A. G.). — Fewnomials, Translations of Mathematical Monographs 88, AMS, Providence, (1991).

[6] Pila (J.). — Integer points on the dilation of a subanalytic surface, Quart. J. Math. 55, p. 207-223 (2004).

[7] Pila (J.). — Rational points on a subanalytic surface, Ann. Inst. Fourier 55, p. 1501-1516 (2005).

[8] Pila (J.). — Note on the rational points of a pfaff curve, Proc. Edin. Math. Soc., 49 (2006), 391-397.

[9] Pila (J.). — Mild parameterization and the rational points of a pfaff curve, Commentari Mathematici Universitatis Sancti Pauli, 55 (2006), 1-8.

[10] Pila (J.) and Wilkie (A. J.). — The rational points of a definable set, Duke Math. J., 133 (2006), 591-616.

[11] Pólya (G.). — On the zeros of the derivative of a function and its analytic character, Bull. Amer. Math. Soc. 49, 178-191 (1943). Also Collected Papers: Volume II, MIT Press, Cambridge Mass., p. 394-407 (1974).

[12] Waldschmidt (M.). — Diophantine approximation on linear algebraic groups, Grund. Math. Wissen. 326, Springer, Berlin, (2000).

[13] Wilkie (A. J.). — A theorem of the complement and some new o-minimal structures, Selecta Math. (N. S.) 5, p. 397-421 (1999).