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Igor Klep, Dejan Velušček

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The Joly–Becker theorem for $*$–orderings[(*)]

Igor Klep[(1)], Dejan Velušček[(2)]

Abstract. — We prove the $*$–version of the Joly–Becker theorem: a skew field admits a $*$–ordering of level n iff it admits a $*$–ordering of level $n\ell$ for some (resp. all) odd $\ell \in \mathbb{N}$. For skew fields with an imaginary unit and fields stronger results are given: a skew field with imaginary unit that admits a $*$–ordering of higher level also admits a $*$–ordering of level 1. Every field that admits a $*$–ordering of higher level admits a $*$–ordering of level 1 or 2.

Résumé. — Nous démontrons la version involutive du théorème de Joly et Becker : une algèbre à division admet un ordre involutif de niveau n si et seulement si elle admet un ordre involutif de niveau $n\ell$ pour un certain (puis tout) impair $\ell \in \mathbb{N}$. Dans le cas d'une algèbre à division avec une unité imaginaire ou d'un corps commutatif, nous présentons des résultats plus forts : si une algèbre à division avec unité imaginaire admet un ordre involutif de niveau supérieur, elle admet aussi un ordre involutif de niveau 1. Tout corps admettant un ordre involutif de niveau supérieur admet un ordre involutif de niveau 1 ou 2.

An old theorem due to Joly [Jo] states that for a field K the following are equivalent: $-1 \notin \sum K^2$ and $-1 \notin \sum K^{2n}$ for some (resp. all) $n \in \mathbb{N}$. Equivalently, in the terminology of Becker [Be, BHR], a field K admits an ordering of level 1 iff it admits an ordering of level n for some (resp. all)

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(1) Univerza v Ljubljani, Oddelek za matematiko Inštituta za matematiko, fiziko in mehaniko, Jadranska 19, SI–1111 Ljubljana, Slovenia.
igor.klep@fmf.uni-lj.si
(2) University of Ljubljana, Faculty of Mathematics and Physics, Department of Mathematics, Jadranska 19, SI–1000 Ljubljana, Slovenia.
dejan.veluscek@fmf.uni-lj.si

$n \in \mathbb{N}$. This result fails for orderings on skew fields and $*$–orderings on (skew) fields [Ci1, Ci2, KV1], but a variant of it still holds true: a skew field admits an ordering of level n iff it admits an ordering of level $n\ell$ for some (resp. all) odd $\ell \in \mathbb{N}$ [Ci1, KV1]. The aim of this paper is to prove the corresponding result for $*$–orderings on skew fields. Furthermore, this result is improved for skew fields containing a central skew element i satisfying $i^2 = -1$ (an *imaginary unit*): such a skew field admits a $*$–ordering of level 1 iff it admits a $*$–ordering of level n for some (resp. all) $n \in \mathbb{N}$. All this is done in Section 2 after a preparatory Section 1. In the last section we show that a field admits a $*$–ordering of higher level iff it admits a $*$–ordering of level 1 or 2.

1. Introduction and basic definitions

We recall some definitions needed in the sequel. Let D be a skew $*$–field and G a cyclic group. Let $0 \notin G$ and $G^0 := G \cup \{0\}$. As usual, we endow D with two nonassociative multiplications, $[a, b] := ab - ba$ and $\{a, b\} := ab + ba$ for $a, b \in D$. A map $\sigma\colon \operatorname{Sym} D \to G^0$ is a *$*$–signature of level m* if

(S$_1$) $\sigma(-1) = -1$,

(S$_2$) $\sigma\big(\{s, t\}\big) = \sigma(s)\sigma(t)$ for all $s, t \in \operatorname{Sym} D$,

(S$_3$) $\sigma(rsr^*) = \sigma(s)\sigma(rr^*)$ for all $s \in \operatorname{Sym} D$ and $r \in D$,

(S$_4$) $\sigma\big((rr^*)^m\big) = 1$ for all $r \in D^\times$,

(S$_5$) $\sigma^{-1}(1)$ is closed under addition.

The *positive cone* $P_\sigma := \sigma^{-1}(1) \cup \{0\}$ of the $*$–signature σ of level m is called a *$*$–ordering of level m*. $*$–orderings of higher level were introduced by Cimprič [Ci2]. This notion unifies the notions of orderings of higher level (cf. [Be, BHR] for orderings of higher level of fields, [Cr1, Po1, Po2, KV1] for orderings of higher level of skew fields) and $*$–orderings (cf. [Cr2, Cr3, Ho1, Ho2] for the theory of $*$–orderings of skew fields and [CS, K, Ma1, Ma2] for $*$–orderings of domains). Since $*$–orderings are in general not closed under multiplication, one defines extended $*$–orderings: a pair (W, τ), where $W \subseteq D$ and $\tau\colon W \to G^0$, is an *extended $*$–signature of level m* if

(ES$_1$) $\operatorname{Sym} D \subseteq W$ and $\tau|_{\operatorname{Sym} D}$ is a $*$–signature of level m,

(ES$_2$) if $x \in W$ then $x^* \in W$ and $\tau(x) = \tau(x^*)$,

(ES$_3$) if $x \in W$ and $d \in D$ then $dxd^* \in W$ and $\tau(dxd^*) = \tau(x)\tau(dd^*)$, (ES$_4$) if $x, y \in W$ then $xy \in W$ and $\tau(xy) = \tau(x)\tau(y)$,

(ES$_5$) if $x, y \in W$ and $\tau(x) = \tau(y) = 1$, then $x + y \in W$ and $\tau(x + y) = 1$.

$P_\tau := \tau^{-1}(1) \cup \{0\}$ is called an *extended $*$–ordering of level m*. By [Ci2, Theorem 3.1], every $*$–signature σ of level m of a skew field with imaginary unit extends, that is, there exists an extended $*$–signature (W, τ) of level m such that $\tau|_{\mathrm{Sym}\, D} = \sigma$.

The Artin–Schreier theory for orderings of higher level was developed by Becker [Be, BHR] in the commutative case and by Craven and Powers [Cr1, Po1, Po2] in the noncommutative case. The corresponding theory for $*$–orderings of level 1 is due to Marshall and is nicely presented in [Ma1]; Cimprič extended this to $*$–orderings of higher level of skew fields with imaginary unit in [Ci2]. In order to formulate Cimprič's result, we need to introduce a notion. Let D be a skew field with involution, assume $n \in \mathbb{N}$ is even and let $T_n^s(D, *)$ be the set of all elements of the form $(d_1 d_1^*)^{n/2} \ldots (d_j d_j^*)^{n/2} c_1 \ldots c_k$, where $d_1, \ldots, d_j \in D^\times$, $c_1, \ldots, c_k \in [D^\times, S^\times]$ and $S = \mathrm{Sym}\, D$. $T_n^s(D, *)$ is the proper generalization of n–th powers (resp. n–th permuted powers) in the field (resp. skew field) case. Thus we define the *higher product level* $\mathrm{ps}_n(D, *)$ of a skew $*$–field D as follows: $\mathrm{ps}_n(D, *) = \infty$ if -1 is not a sum of nonzero elements from $T_n^s(D, *)$ and $\mathrm{ps}_n(D, *) = k$ if -1 is a sum of k elements from $T_n^s(D, *)$ but is not a sum of less than k elements from $T_n^s(D, *)$. The higher product level $\mathrm{ps}_n(D, *)$ is the analogue of the higher product level $\mathrm{ps}_n(D)$ as studied in [CV]. We use $\sum T_n^s(D, *)$ to denote the set of all finite sums of elements from $T_n^s(D, *)$.

Theorem 1.1 (cf. Proposition 4.10 in [Ci2]). — *Let D be a skew $*$–field and $n \in \mathbb{N}$ even. Then $\mathrm{ps}_n(D, *) = \infty$ iff D has an extended $*$–ordering of level $n/2$.*

In the sequel we will use some results concerning skew fields and the higher product level $\mathrm{ps}_n(D)$, thus we introduce some additional notation: Let $\prod_n D$ denote the set of permuted products of n–th powers of elements of D and let $\sum^{(b)} B := \{a_1 + \ldots + a_b \mid a_1, \ldots, a_b \in B\}$ for $B \subseteq D$.

2. Higher product levels of skew fields with involution

We start this section by proving that the relation between $\mathrm{ps}_n(D, *)$ and $\mathrm{ps}_{n\ell}(D, *)$ for odd ℓ is similar to the relation between the analogous higher product levels in the noncommutative setting (Theorem 2.4 and Theorem 2.1).

THEOREM 2.1 (Compare Theorem 8 in [Ci1] or Corollary 18 in [KV1]) *Let D be a skew $*$–field and $n \in \mathbb{N}$ even. If $\mathrm{ps}_n(D,*) < \infty$ then $\mathrm{ps}_{n\ell}(D,*) < \infty$ for every odd ℓ.*

LEMMA 2.2. — *For every odd ℓ and every $n = 2^k$ there exists $b_{n\ell}$ such that for every skew $*$–field D, every $s,t \in \mathrm{Sym}D$ and every $i = 0,\ldots,n-1$ there exist $v_{i,0},\ldots,v_{i,n-1} \in \sum^{(b_{n\ell})} T^s_{n\ell}(D,*)$ such that $(s+t)^{i\ell} = v_{i,0}s^{i\ell} + v_{i,1}s^{(i-1)\ell}t^{\ell} + \ldots + v_{i,i-1}s^{\ell}t^{(i-1)\ell} + v_{i,i}t^{i\ell} + v_{i,i+1}s^{(n-1)\ell}t^{(i+1)\ell} + \ldots + v_{i,n-1}s^{(i+1)\ell}t^{(n-1)\ell}$.*

Proof. — Take any symmetric elements $s,t \in D$. By [CV, Lemma 4.2], for every odd ℓ and every $n = 2^k$ there exists $a_{n\ell}$ such that for every skew field D, every $x,y \in D$ and every $i = 0,\ldots,n-1$ there exist elements $u_{i,0},\ldots,u_{i,n-1} \in \sum^{(a_{n\ell})}\prod_{n\ell} D$ satisfying $(x+y)^{i\ell} = u_{i,0}x^{i\ell}+u_{i,1}x^{(i-1)\ell}y^{\ell}+\ldots+u_{i,i-1}x^{\ell}y^{(i-1)\ell}+u_{i,i}y^{i\ell}+u_{i,i+1}x^{(n-1)\ell}y^{(i+1)\ell}+\ldots+u_{i,n-1}x^{(i+1)\ell}y^{(n-1)\ell}$.

Set $x = 2$ and $y = \{s^{-1},t\}$. On the left–hand side we get $\left(2+\{s^{-1},t\}\right)^{i\ell} = \{s+t,s^{-1}\}^{i\ell} = c_i(s+t)^{i\ell}s^{-i\ell}$, where c_i is a sum of $2^{i\ell}$ products of commutators. On the right–hand side write $\{s^{-1},t\}^{j\ell} = c_{i,j}s^{i\ell-j\ell}t^{j\ell}s^{-i\ell}$, where $c_{i,j}$ is a sum of $2^{j\ell}$ products of commutators for every $j = 0,\ldots,n-1$. Since $2 \in Z(D)$ it follows from the proof of [CV, Lemma 4.2] that $u_{i,j} \in \sum^{(a_{n\ell})}\mathbb{Q}(s^{-1}t+ts^{-1})^{n\ell}$. Hence,

$$c_i(s+t)^{i\ell} = u_{i,0}c_{i,0}s^{i\ell} + u_{i,1}c_{i,1}s^{(i-1)\ell}t^{\ell} + \ldots + +u_{i,i}c_{i,i}t^{i\ell} +$$

$$+u_{i,i+1}c_{i,i+1}s^{-l}t^{(i+1)\ell} + \ldots + u_{i,n-1}c_{i,n-1}s^{(i+1-n)\ell}t^{(n-1)\ell}.$$

Write $v_{i,j} = c_i^{-1}u_{i,j}c_{i,j}$ for $j = 0,\ldots,i$, $v_{i,j} = c_i^{-1}u_{i,j}c_{i,j}s^{-n\ell}$ for $j = i+1,\ldots,n$ and $b_{n\ell} = 2^{2n\ell}a_{n\ell}$. It follows that $v_{i,0},\ldots,v_{i,n-1} \in \sum^{(b_{n\ell})} T^s_{n\ell}(D,*)$ and $(s+t)^{i\ell} = v_{i,0}s^{i\ell} + v_{i,1}s^{(i-1)\ell}t^{\ell} + \ldots + v_{i,i-1}s^{\ell}t^{(i-1)\ell} + v_{i,i}t^{i\ell} + v_{i,i+1}s^{(n-1)\ell}t^{(i+1)\ell} + \ldots + v_{i,n-1}s^{(i+1)\ell}t^{(n-1)\ell}$. □

Remark 2.3. — From [CV, Lemma 2.1] we get

$$a_{n\ell} = G(n)\max\left\{25\binom{n\ell+7}{5}, (n\ell+1)u(4,n\ell)\right\},$$

where $u(t,2^m s) = G(2^m s)L(s,t)L(2s,t+1)\cdots L(2^{m-1}s,t+1)$ for $t,m \in \mathbb{N}$ and odd $s \in \mathbb{N}$. Here the function L is defined by $L(s,t) := \binom{t+2s-1}{t-1}$ and $G(m)$ stands for the m–th Waring number, i.e., the least number $k \in \mathbb{N}$, such that every positive integer is a sum of k m–th powers of positive integers.

Theorem 2.4. — *For every $n = 2^k$ and every odd $\ell \in \mathbb{N}$ there exists $d_{n\ell}$ such that $\mathrm{ps}_{n\ell}(D,*) \leqslant d_{n\ell}^{\mathrm{ps}_n(D,*)}$ for every skew field D.*

Proof. — Let D be a skew $*$–field, $t = \mathrm{ps}_n(D,*)$ and $p_1, \ldots, p_t \in T_n^s(D,*)$ such that $-1 = p_1 + \ldots + p_t$. Writing $s_i = p_i + p_i^*$ for $i = 1, \ldots, t$, we get $-1 = 1 + s_1 + \ldots + s_t$. Pick $r \in \mathbb{N}$ such that $2^{r-1} \leqslant t < 2^r$ and write $s_0 = 1$ and $s_{t+1} = \ldots = s_{2^r-1} = 0$. It follows that $-1 = (s_0 + s_1 + \ldots + s_t + s_{t+1} + \ldots + s_{2^r-1})^\ell$.

For every $i = 0, \ldots, n-1$ there exists the smallest number $f_{i,r}$ such that

$$(s_0 + \ldots + s_{2^r-1})^{i\ell} \in \sum^{(f_{i,r})} T_{n\ell}^s(D,*) \tag{†}$$

for every choice of $p_0, \ldots, p_{2^r-1} \in T_n^s(D,*)$: since we can use Lemma 2.2 on the expression (†) recursively, it is enough to show that for every $p \in T_n^s(D,*)$ we have $(p+p^*)^{j\ell} \in \sum^{(f_{j,0})} T_{n\ell}^s(D,*)$ for some $f_{j,0} \in \mathbb{N}$. Observe that p can be written in the form $p = (dd^*)^{n/2}c$ for some $d \in D$ and some product of commutators $c \in \prod[D^\times, \mathrm{Sym}(D)]$. Hence,

$$\begin{aligned}(p+p^*)^{j\ell} &= \big((dd^*)^{n/2}c + c^*(dd^*)^{n/2}\big)^{j\ell} = \big((dd^*)^{n/2}c + (dd^*)^{n/2}\widetilde{c}\big)^{j\ell} \\ &= \big((dd^*)^j\big)^{\ell n/2}(c+\widetilde{c})^{j\ell}c',\end{aligned}$$

where $\widetilde{c}, c' \in \prod[D^\times, \mathrm{Sym}(D)]$. Clearly, $(dd^*)^j = d'(d')^*$ for some $d' \in D$ and the expansion of $(c+\widetilde{c})^{j\ell}$ gives a sum of $2^{j\ell}$ terms from $T_{n\ell}^s(D,*)$. We conclude that there exists a $f_{j,0} \in \mathbb{N}$ such that $s_0^{j\ell} = (p_0 + p_0^*)^{j\ell} \in \sum^{(f_{j,0})} T_{n\ell}^s(D,*)$.

Note that $f_{0,r} = 1$ for every r and from Lemma 2.2 it follows that

$$\begin{aligned}f_{i,r} \leqslant\ & b_{n\ell}\big(f_{i,r-1}f_{0,r-1} + f_{i-1,r-1}f_{1,r-1} + \ldots + \\ &+ f_{0,r-1}f_{i,r-1} + f_{n-1,r-1}f_{i+1,r-1} + \ldots + f_{i+1,r-1}f_{n-1,r-1}\big).\end{aligned}$$

Writing $F_r := \max_i f_{i,r}$ we get $F_0 \leqslant 2^{(n-1)\ell}$ and $F_r \leqslant nb_{n\ell}F_{r-1}^2$. Hence,

$$\mathrm{ps}_{n\ell}(D,*) \leqslant F_r \leqslant (nb_{n\ell})^{2^r-1}\big(2^{(n-1)\ell}\big)^{2^r} \leqslant (nb_{n\ell}2^{(n-1)\ell})^{2\mathrm{ps}_n(D,*)}.$$

Setting $d_{n\ell} := (nb_{n\ell}2^{(n-1)\ell})^2$ finishes the proof. □

Proof of Theorem 2.1. — Observe that $\infty > \mathrm{ps}_{2^k q}(D,*) \geqslant \mathrm{ps}_{2^k}(D,*)$ for every $k, q \in \mathbb{N}$ with q odd. Thus if $n = 2^k q$ and $\ell \in \mathbb{N}$ is odd, then $\mathrm{ps}_{n\ell}(D,*) \leqslant d_{2^k q\ell}^{\mathrm{ps}_{2^k}(D,*)} < \infty$ by Theorem 2.4. This concludes the proof. □

Corollary 2.5. — *For $n \in \mathbb{N}$ the following assertions are equivalent :*

(i) $-1 \notin \sum T^s_{2n}(D,*)$.

(i') *There exists an extended $*$–ordering of level n of D.*

(ii) $-1 \notin \sum T^s_{2n\ell}(D,*)$ *for some odd ℓ.*

(ii') *There exists an extended $*$–ordering of D of level $n\ell$ for some odd ℓ.*

(iii) $-1 \notin \sum T^s_{2n\ell}(D,*)$ *for all odd ℓ.*

(iii') *There exists an extended $*$–ordering of D of level $n\ell$ for all odd ℓ.*

For the rest of this section D will denote a *skew $*$–field with an imaginary unit i* and σ will be a $*$–signature of level m of D. The set $A(\sigma) := \{d \in D \mid \exists r \in \mathbb{Q}_{>0} : \ \sigma(r \pm dd^*) = 1\}$ is an invariant valuation $*$–subring of D with maximal $*$–ideal $I(\sigma) := \{d \in D \mid \forall r \in \mathbb{Q}_{>0} : \ \sigma(r \pm dd^*) = 1\}$, see [Ci2, Theorem 2.1]. Let $k(\sigma)$ denote the residue skew $*$–field $A(\sigma)/I(\sigma)$. σ induces an archimedean $*$–ordering of level 1 of $k(\sigma)$ and thus $k(\sigma)$ is a $*$–ordered skew subfield of $\mathbb{H}$ by [Ho1]. The valuation corresponding to $A(\sigma)$ will be denoted by $v_\sigma : D \to \Gamma_\sigma \cup \{\infty\}$ and $\wp_\sigma : D \to k(\sigma) \cup \{\infty\}$ will denote the corresponding place. By [Ci2, Proposition 3.2], v_σ is *compatible* with P_σ, i.e., for all $x, y \in P_\sigma$ we have $v_\sigma(x+y) = \min\{v_\sigma(x), v_\sigma(y)\}$. Furthermore, by [Ci2, Proposition 3.7], v_σ is *quasi–commutative for symmetric elements*, i.e., for every $a, b \in \operatorname{Sym} D^\times$ we have

$$v(ab - ba) > v(ab) = v(ba). \qquad (*)$$

By [Ho2, 4.1] or [Cr3, Theorem 2.3], this implies that $(*)$ holds for all $a \in \operatorname{Sym} D^\times$ and $b \in D^\times$. We will prove that v_σ is actually *quasi–commutative*, i.e., $(*)$ holds for all $a, b \in D^\times$. This will be used to deduce a strong version of Corollary 2.5 for skew fields with imaginary unit.

Proposition 2.6. — *Every $a \in D$ decomposes uniquely as $a = a_1 + i\, a_2$ for symmetric a_1, a_2 and $v_\sigma(a) = \min\{v_\sigma(a_1), v_\sigma(a_2)\}$.*

Proof. — As D admits a $*$–signature, it is of characteristic 0. Thus we may write $x = \frac{x+x^*}{2} + i\,\frac{x-x^*}{2i}$. This proves the first assertion. For $a = a_1 + i\, a_2$ with a_j symmetric, we have $v_\sigma(a) \geqslant \min\{v_\sigma(a_1), v_\sigma(a_2)\}$. If $v_\sigma(a_1) \neq v_\sigma(a_2)$, then $v_\sigma(a) = \min\{v_\sigma(a_1), v_\sigma(a_2)\}$. Thus, we may assume $v_\sigma(a_1) = v_\sigma(a_2)$.

As v_σ is quasi–commutative for symmetric elements, $v_\sigma\big(\{s,t\}\big) = v_\sigma(s) + v_\sigma(t)$ for all $s, t \in \operatorname{Sym} D$. This result also holds if only one of s, t is symmetric by the result of Holland mentioned above. Let $s \in \operatorname{Sym} D$ satisfy

$v_\sigma(s) = -v_\sigma(a_1)$. Clearly, $\{a, s\} = \{a_1, s\} + i\,\{a_2, s\}$. In other words, we may assume $v_\sigma(a_1) = v_\sigma(a_2) = 0$.

We claim that $v_\sigma(a) = 0$. Note that $aa^* = a_1^2 + a_2^2 + i\,[a_2, a_1]$. Obviously, $v_\sigma\big([a_2, a_1]\big) > v_\sigma(a_2 a_1) = 0$. So it suffices to prove that $v_\sigma(a_1^2 + a_2^2) = 0$. Once we have shown this, we get $v_\sigma(aa^*) = 0$ and hence (as v_σ is a $*$–valuation) $v_\sigma(a) = 0$.

The canonical place $\wp_\sigma$ maps $\operatorname{Sym} D$ into $\mathbb{R} \cup \{\infty\}$. Hence $\wp_\sigma(a_1), \wp_\sigma(a_2) \in \mathbb{R}^\times$, so $\wp_\sigma(a_1^2), \wp_\sigma(a_2^2) \in \mathbb{R}_{>0}$. This implies $\wp_\sigma(a_1^2 + a_2^2) \in \mathbb{R}_{>0}$. In particular, $v_\sigma(a_1^2 + a_2^2) = 0$, as desired. $\square$

Theorem 2.7. — *v_σ is quasi–commutative and $k(\sigma)$ is a $*$–ordered subfield of $\mathbb{C}$.*

Proof. — Let $a, b \in D^\times$ be arbitrary. Write $a = a_1 + i\,a_2$ and $b = b_1 + i\,b_2$ for symmetric a_1, a_2, b_1, b_2. Then

$$\begin{aligned} ab - ba &= (a_1 + i\,a_2)(b_1 + i\,b_2) - (b_1 + i\,b_2)(a_1 + i\,a_2) = \\ &= (a_1 b_1 - b_1 a_1) + (b_2 a_2 - a_2 b_2) + i\,(a_1 b_2 - b_2 a_1) + i\,(a_2 b_1 - b_1 a_2). \end{aligned}$$

Hence

$$\begin{aligned} v_\sigma(ab - ba) \quad &\geqslant \min\big\{v_\sigma(a_1 b_1 - b_1 a_1), v_\sigma(b_2 a_2 - a_2 b_2), \\ &\qquad v_\sigma(a_1 b_2 - b_2 a_1), v_\sigma(a_2 b_1 - b_1 a_2)\big\}. \end{aligned}$$

We now use the fact that v_σ is quasi–commutative for symmetric elements to get

$$\begin{aligned} v_\sigma(ab - ba) \quad &> \min\big\{v_\sigma(a_1 b_1), v_\sigma(a_2 b_2), v_\sigma(a_1 b_2), v_\sigma(a_2 b_1)\big\} = \\ &= \min\{v_\sigma(a_1), v_\sigma(a_2)\} + \min\{v_\sigma(b_1), v_\sigma(b_2)\}. \end{aligned}$$

By Proposition 2.6, the right–hand side of the last equation equals $v_\sigma(a) + v_\sigma(b) = v_\sigma(ab)$. Hence $v_\sigma(ab - ba) > v_\sigma(ab)$, as required. Now $k(\sigma)$ is commutative subfield of $\mathbb{H}$ and thus a $*$–ordered subfield of $\mathbb{C}$. $\square$

Corollary 2.8. — *The following statements are equivalent :*

(i) *D admits a $*$–signature of level* 1.

(ii) *D admits a $*$–signature of level n for some $n \in \mathbb{N}$.*

(iii) *D admits a $*$–signature of level n for all $n \in \mathbb{N}$.*

Proof. — Implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are clear. It remains to prove (ii) $\Rightarrow$ (i). For this let χ be a $*$–signature of level n of D. Then v_χ is a quasi–commutative valuation and $k(\chi)$ is a $*$–ordered subfield of $\mathbb{C}$ by Theorem 2.7. Hence by [Cr2, Theorem 3.4], there is a bijection between all $*$–signatures of level 1 of D compatible with v_χ and $\{*$–orderings of $k(\chi)\} \times \mathrm{Hom}\big(\Gamma_\chi/2\Gamma_\chi, \{-1,1\}\big)$. In particular, there exists a $*$–signature of level 1 of D. $\square$

By [Ci2, Theorem 3.1], every $*$–signature of D extends to an extended $*$–signature of D. Hence:

Corollary 2.9. — *The following statements are equivalent :*

(i) $\mathrm{ps}_2(D,*) < \infty$.

(ii) $\mathrm{ps}_{2n}(D,*) < \infty$ *for some* $n \in \mathbb{N}$.

(iii) $\mathrm{ps}_{2n}(D,*) < \infty$ *for all* $n \in \mathbb{N}$.

3. Fields with $*$–orderings of higher level

In this section we focus on *fields*. We first investigate the existence of valuations compatible with $*$–orderings of higher level of fields. Cimprič [Ci2, Theorem 2.1] showed that for a skew field with imaginary unit $A(\sigma)$ is an invariant $*$–valuation ring. For (skew) fields without an imaginary unit the existence of a compatible valuation was left open.

Example 3.1 (Compare Example 2.2 in [Ci2]). — Consider the field $F := \mathbb{Q}(\sqrt{2})(X)$ with the involution given by $(p(X) + q(X)\sqrt{2})^* = p(X) - q(X)\sqrt{2}$ for $p, q \in \mathbb{Q}(X)$. Clearly, $\mathrm{Sym}F = \mathbb{Q}(X)$, so every $*$–ordering of higher level induces an ordering of higher level of $\mathbb{Q}(X)$.

Every nonzero $q \in F$ can be written as $q(X) = r(X)(X^2-2)^k$ for $r \in F$ with $r(\sqrt{2}) \neq 0$. Then $\sigma(q) := \mathrm{sign}(r(\sqrt{2}))$ defines a $*$–signature of level 2 and induces an ordering of level 1 of $\mathbb{Q}(X)$. There is only one non–trivial valuation of $\mathbb{Q}(X)$ compatible with this ordering. It is discrete and given by $v(r(X)(X^2-2)^k) = k$. We claim that v cannot be extended to a $*$–valuation of F. Assume otherwise and let u denote an extension. Since $2u(X-\sqrt{2}) = u\big((X-\sqrt{2})(X-\sqrt{2})^*\big) = v(X^2-2) = 1$, we have $\wp_u(X) = \wp_u(\sqrt{2})$. But $\wp_u(X)$ is symmetric and $\wp_u(\sqrt{2})$ is skew. This contradicts the fact that u is a $*$–valuation.

As observed in the last example, if σ is a $*$–signature of a field F, then $\sigma|_{\mathrm{Sym}F}$ is a signature of higher level. Hence there is a natural valuation v of

Sym$F$ compatible with $\sigma|_{\mathrm{Sym}F}$. If $F = \mathrm{Sym}F$, then v is a $*$–valuation compatible with σ. Otherwise $[F : \mathrm{Sym}F] = 2$ and there is a skew element $\zeta \in F$ and $\{1, \zeta\}$ is a Sym$F$–basis of F. Let us investigate under which conditions v extends to a $*$–valuation u of F (which is automatically compatible with σ). Set $\gamma := v(\zeta^2)$.

Since v is a valuation of Sym$F$, it extends to a valuation of F by Chevalley's extension theorem (see e.g. [E, §9]). Let $\mathcal{E}$ denote the set of all extensions. For each $u \in \mathcal{E}$ we define the *ramification index* of u as $e_u := [\Gamma_u : \Gamma_v]$ and the *inertia degree* of u as $f_u := [k_u : k_v]$. Then by the fundamental inequality [E, §17]

$$[F : \mathrm{Sym}F] \geqslant \sum_{u \in \mathcal{E}} e_u f_u.$$

Thus there are at most two extensions of v. As $F|\mathrm{Sym}F$ is a normal field extension, we can use [E, (14.1)]. If $u \in \mathcal{E}$, then u^* defined as $u^*(x) := u(x^*)$ for $x \in F$ is also in $\mathcal{E}$. This shows that v extends to a $*$–valuation of F iff the extension is unique. And in this case the extension cannot be immediate. Furthermore, if such an extension u exists, it will satisfy

$$u(a_1 + a_2\zeta) = \min\{v(a_1), v(a_2) + u(\zeta)\}$$

for $a_1, a_2 \in \mathrm{Sym}F$ (see e.g. [KV2]).

Assume that $\gamma \notin 2\Gamma_v$. In this case set $\Gamma_u := \Gamma_v(\gamma/2)$ and extend the ordering. Every $a \in F^\times$ can be written uniquely as $a = a_s + a_k\zeta$ with a_s, a_k symmetric. The extension u of v to F is unique and given by $u(a) := \min\{v(a_s), v(a_k) + \gamma/2\}$. By the above, u is a $*$–valuation.

Now assume $\gamma \in 2\Gamma_v$ and let $s \in \mathrm{Sym}\,F$ satisfy $\gamma = v(s^2)$. By replacing ζ by ζs^{-1}, we may assume $v(\zeta^2) = 0$. We distinguish two cases. If $\wp_v(\zeta^2) \in k_v^2$, then let $\bar{P}$ denote the ordering of k_v induced by the $*$–ordering of F. Assume v extends to a $*$–valuation u of F. Then the automorphism of k_u induced by the $*$–conjugation with ζ leaves $\bar{P}$ invariant. But $\overline{\zeta P \zeta^*} \subseteq -k_v^2 \bar{P} \subseteq -\bar{P}$, a contradiction. For the final case, assume that $\wp_v(\zeta^2) \notin k_v^2$. If there are two extensions of v, they are necessarily immediate by the fundamental inequality. Let u denote one of the extensions. Then $\wp_u(\zeta) = \wp_v(s) \in k_v$ for some $s \in \mathrm{Sym}\,S$. On the other hand, $\wp_v(\zeta^2) = \wp_u(\zeta^2) = \wp_v(s)^2 \in k_v^2$, a contradiction. It follows that the extension in this case is unique and is a $*$-valuation.

Proposition 3.2. — *Let F be a field that admits a $*$–ordering of higher level. Then F admits a $*$–ordering of level* 1 *or* 2.

Proof. — Let σ denote a $*$–signature of higher level of F. Then $\sigma|_{\mathrm{Sym}F}$ is a signature of higher level. This implies (since $\mathrm{Sym}F$ is a field) that $\mathrm{Sym}F$ is formally real, i.e., admits an ordering of level 1, by the classical Joly–Becker theorem. Fix an ordering P of $\mathrm{Sym}F$. We claim that P is a $*$–ordering of level 1 or 2.

For every $a \in F$, $aa^* \in \mathrm{Sym}F$. If all these products are in P, then P is a $*$–ordering of level 1. Otherwise $(aa^*)^2 \in P$ for all $a \in F$ and so P is a $*$–ordering of level 2 of F. □

Corollaries 2.8 and 2.9 suggest that the upper bound of $\mathrm{ps}_{2n}(D,*)$, $n \in \mathbb{N}$, can be expressed in terms of $\mathrm{ps}_2(D,*)$. We are able to give an explicit upper bound in the case where D is commutative (Theorem 3.3), but not for general skew $*$–fields with an imaginary unit. We present the commutative case.

For the rest of the section assume that F is a $*$–field of characteristic 0 containing an imaginary unit i. In this case the higher product level $\mathrm{ps}_{2n}(F,*)$ coincides with the *higher hermitian level* $\mathrm{s}_{2n}(F,*)$, which is the smallest $t \in \mathbb{N}$ such that $-1 = (a_1a_1^*)^n + \ldots + (a_ta_t^*)^n$ and equals infinity if -1 is not a sum of hermitian $2n$–th powers. We will give an upper bound of $\mathrm{s}_{2n}(F,*)$ expressed in terms of $\mathrm{s}_2(F,*)$.

The proof of the next theorem will be omitted since it is quite straightforward generalization of Joly's proof of [Jo, Theoreme 6.16], i.e., we use the following result of Hilbert from 1909: For every $r, n \in \mathbb{N}$ there exist $\lambda_i \in \mathbb{Q}_{>0}$ and $c_{ij} \in \mathbb{Z}$ ($1 \leqslant i \leqslant L(r,n)$, $1 \leqslant j \leqslant n$) such that

$$(x_1^2 + \ldots + x_n^2)^r = \sum_{i=1}^{L(r,n)} \lambda_i (c_{i1}x_1 + \ldots + c_{in}x_n)^{2r}.$$

Observe that by replacing x_i's with symmetric elements of F the expression $c_{i1}x_1 + \ldots + c_{in}x_n$ on the right–hand side of the equation becomes a symmetric element of F.

THEOREM 3.3. — *Let $t := \mathrm{s}_2(F,*) < \infty$ and let $k, \ell \in \mathbb{N}$ with ℓ odd. Then*

$$\mathrm{s}_{2^k\ell}(F,*) \leqslant G(2^k\ell)L(\ell,2t)L(2\ell,2t+1)\cdots L(2^{k-1}\ell,2t+1).$$

Proposition 3.2 shows that in the commutative case the absence of the imaginary unit seems to affect only the connection between $\mathrm{s}_2(F,*)$ and $\mathrm{s}_{4k}(F,*)$ for $k \in \mathbb{N}$.

COROLLARY 3.4. — *For a $*$–field K of characteristic* 0 *the following statements are equivalent :*

(i) $\mathrm{s}_4(K,*) < \infty$.

(ii) $\mathrm{s}_{4n}(K,*) < \infty$ *for some* $n \in \mathbb{N}$.

(iii) $\mathrm{s}_{2n}(K,*) < \infty$ *for all* $n \in \mathbb{N}$.

Remark 3.5. — By replacing $s_2(K,*)$ with $s_4(K,*)/2$ in Theorem 3.3 we get an explicit upper bound for $\mathrm{s}_{2^k\ell}(K,*)$ expressed in terms of $s_4(K,*)$.

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