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The logarithmic Sobolev constant of some finite Markov chains[(*)]

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Abstract. — The logarithmic Sobolev constant is always bounded above by half the spectral gap. It is natural to ask when this inequality is an equality. We consider this question in the context of reversible Markov chains on small finite state spaces. In particular, we prove that equality holds for simple random walk on the five cycle and we discuss assorted families of chains on three and four points.

Résumé. — La constante de Sobolev logarithmic est toujours inférieure ou égale à la moitié du trou spectral. Il est naturel de se demander dans quels cas l'égalité à lieu. Nous considérons cette question dans le cadre des chaînes de Markov sur un espace fini de petite taille. En particulier, nous montrons l'égalité pour la marche aléatoire simple sur un cycle fini de 5 points et discutons plusieurs familles de chaînes sur 3 et 4 points.

1. Introduction

1.1. Motivation and results

Let (Ω, μ) be a probability space equipped with a Dirichlet form $(\mathcal{E}, \mathcal{D})$. Let $\mathrm{Var}(f)$ denote the variance of f, that is, $\mathrm{Var}(f) = \mu(|f - \mu(f)|^2)$ where

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$\mu(f)$ is the mean (i.e., expectation) of f under μ. The spectral gap λ is defined by the classic variational formula

$$\lambda = \inf\left\{\frac{\mathcal{E}(f,f)}{\operatorname{Var}(f)} : f \in \mathcal{D},\ \operatorname{Var}(f) \neq 0\right\}. \tag{1.1}$$

The logarithmic Sobolev constant α, introduced (implicitly) in the groundbreaking paper of Gross [18], is defined by

$$\alpha = \inf\left\{\frac{\mathcal{E}(f,f)}{\mathcal{L}(f)} : f \in \mathcal{D},\ \mathcal{L}(f) \neq 0\right\} \tag{1.2}$$

where

$$\mathcal{L}(f) = \mu(|f|^2 \log(|f|^2/\mu(|f|^2))) = \operatorname{Ent}(|f|^2/\mu(|f|^2)) \tag{1.3}$$

is the (relative) entropy of the probability measure with density $|f|^2/\mu(|f|^2)$ with respect to μ.

In the most classical example illustrating these definitions, Ω is the real line, μ is the Gaussian measure $d\mu(x) = (2\pi)^{-1/2}e^{-x^2/2}dx$ and $\mathcal{E}(f,f) = \int_{-\infty}^{+\infty}|f'|^2 d\mu$ which is the Dirichlet form of the celebrated Ornstein-Uhlenbeck process. In this case, $\lambda = 1$ is the lowest non-zero eigenvalue of the generator $-d^2/dx^2 + xd/dx$ (diagonalized by the Hermite polynomials) and $\alpha = 1/2$ (attained on any exponential function). See [18, Theorem 4] where it is also proved that $\alpha = 1/2$ is equivalent to Nelson's hypercontractivity [31].

It is a remarkable fact that the constant α captures non-trivial information already in the simplest case where $\Omega = \{0,1\}$ is the symmetric two-point space with $\mu(0) = \mu(1) = 1/2$ and $\mathcal{E}(f,f) = |f(0) - f(1)|^2/2$. Then, $\lambda = 2$ (the minimum is attained on any function such that $f(0) = -f(1)$) and $\alpha = 1$ (the minimum is not attained). If we write $f(0) = 1 + s$, $f(1) = 1 - s$, the fact that $\alpha = 1$ is equivalent to the Calculus inequality

$$\frac{1}{2}\left((1+s)^2\log(1+s)^2 + (1-s)^2\log(1-s)^2 - 2(1+s^2)\log(1+s^2)\right) \leqslant 2s^2 \tag{1.4}$$

which can be proved by taking two derivatives. See [18, p.1068]. An equivalent form of this inequality (via hypercontractivity) appeared first in the work of A. Bonami [9, Lemma 3]. A recent application (via passage to the hypercube, see Theorem 1.3 below) is in [6] where further relevant references can be found.

The constants λ and α are related by the universal inequality stated in the following well-known result.

Theorem 1.1. — *One always has*

$$\alpha \leqslant \lambda/2. \tag{1.5}$$

Moreover, the inequality is strict if the spectral gap λ admits an eigenfunction $\phi \in \mathcal{D}$ such that $\mu(\phi^3) \neq 0$.

The inequality (1.5) was first proved by B. Simon [38] in an equivalent form involving hypercontractivity, and, later, by O. Rothaus [33] in this form. Rothaus' proof consists in testing (1.2) on function of the form $1+\epsilon f$ and performing a Taylor expansion in ϵ. See e.g., [2, 19, 37]. The remark concerning the case of equality is due to Rothaus and follows easily from his proof of (1.5).

Observe that in the two examples discussed above one has

$$\alpha = \lambda/2.$$

Here is a list of examples where this equality holds.

(E1) The Sphere S^n, $n \geqslant 2$, equipped with its natural Riemannian structure has $\lambda = 2\alpha = n$. An important related example is $\Omega = [-1, 1]$ equipped with the measure $d\mu_a(x) = c_a(1-x^2)^{a/2-1}dx$ and the Dirichlet form $\mathcal{E}(f,f) = \int_{-1}^{+1}(1-x^2)|f'(x)|^2 d\mu_a(x)$, $a > 0$. This form is orthonormalized by the ultraspherical polynomials and, for $a = n$, it amounts to projecting the n-sphere on its diameter. For these examples, α was first computed in [30].

(E2) The circle. See [14, 40]. By specializing to functions on the circle such that $f(\theta) = f(-\theta)$, this also gives the interval $[0, 1]$ with Neumann boundary condition.

(E3) Simple random walk on $\mathbb{Z}/k\mathbb{Z}$, $k = 2n$. See [10].

In fact, for the examples in (E1), the equality $\alpha = \lambda/2$ can be obtained by an application of the celebrated Bakry-Émery technique of [4]. For (E2), the equality can be proved using Rothaus' improvement of the Bakry-Émery argument presented in [35]. The finite example (E3) is of a different nature and will be discussed further below. One of the main result of the present paper concerns the case of simple random walk on $\mathbb{Z}/5\mathbb{Z}$ and shows that $\alpha = \lambda/2$ in that case also.

It is now understood that, typically, $\alpha < \lambda/2$ (possibly much smaller). See, e.g., [13, 25, 36]. The first examples that were obtained in this direction are the following.

(E4) The Laguerre polynomials example where $\Omega = (0, \infty)$, $d\mu(x) = e^{-x}dx$ and $\mathcal{E}(f, f) = \int_0^\infty x|f'|^2 e^{-x}dx$. Korzeniowski and Stroock observed in [23] that $\lambda = 1$ and $\alpha = 1/4$. Bakry extended this to part of the Laguerre family (i.e. μ is a Gamma distribution) in the final remarks of [3].

(E5) The asymmetric two point space with

$$\Omega = \{0, 1\},\ \mu(0) = p,\ \mu(1) = q,\ \ \mathcal{E}(f, f) = pq|f(0) - f(1)|^2.$$

Then $\lambda = 1$ and $\alpha = (q - p)(\log q/p)^{-1}$. See [8, 13, 28, 37].

(E6) The one step ergodic chain with Ω finite, μ an arbitrary positive probability measure on Ω and $\mathcal{E}(f, f) = \frac{1}{2}\sum_{x,y} |f(x) - f(y)|^2\mu(x)\mu(y)$. In this case $\lambda = 1$ and $\alpha = (1 - 2\mu_*)(\log(1/\mu_* - 1))^{-1}$, $\mu_* = \min_\Omega \mu(x)$. This generalizes (E5) but in fact the proof is by reduction to (E5). See [8, 13, 37]. A case of special interest is $\Omega = \{0, 1, 2\}$ equipped with the uniform measure which has $\lambda = 1$, $\alpha = (3 \log 2)^{-1}$. By a simple time change argument to get rid of the holding, this gives $\lambda = 3/2$, $\alpha = (2 \log 2)^{-1}$ for simple random walk on $\mathbb{Z}/3\mathbb{Z}$.

(E7) For the natural Riemannian structure on the following objects, λ is known explicitly, α is not, but $\alpha < \lambda/2$: (a) The rank one compact symmetric spaces that are not spheres, in particular, the projective spaces; (b) SU(3), SO(3), and the exceptional simple compact groups G_2, F_4, E_6, E_7, E_8. See [36].

It may be worth emphasizing that, in a sense (e.g., modulo taking direct products), (E4)-(E6) are the only known examples where α is known explicitly and is different from $\lambda/2$. This possibly indicates how difficult it is to compute the constant α. Some of the most natural open problems in this directions are the following.

(Q1) Decide whether or not, on any flat torus of dimension $n \geqslant 2$, $\alpha = \lambda/2$. If not, compute α. A flat torus is the quotient of Euclidean space by a (cocompact) lattice and the spectral gap can be computed in terms of the lattice. See [7].

(Q2) Show that for simple random walk on $\mathbb{Z}/n\mathbb{Z}$, n odd, $n \neq 3$, one has $\alpha = \lambda/2$.

(Q3) Among all ergodic chains on the three-point space $\Omega = \{0, 1, 2\}$, which have $\alpha = \lambda/2$?

This paper is devoted to partial results concerning (Q2) and (Q3). We will show that $\alpha = \lambda/2$ for simple random walk on $\mathbb{Z}/5\mathbb{Z}$ and for some Markov chains on three-point and four-point spaces. We also believe our results give some insights on the difficulties that arise in computing or estimating the logarithmic Sobolev constant α.

1.2. Hypercontractivity, products and projections

Let us recall two of the main basic properties of the logarithmic Sobolev constant α. The first gives the equivalent formulation in terms of hypercontractivity. The second concerns taking products.

THEOREM 1.2 (Gross [18]). — *Let (Ω, μ) and $(\mathcal{E}, \mathcal{D})$ be as above. Let H_t, $t > 0$, be the associated Markov semigroup acting on $L^2(\Omega, \mu)$. The logarithmic Sobolev constant α at* (1.2) *is also the largest of all real β such that $\|H_t\|_{p\to q} \leqslant 1$ for all t, p, q satisfying $t \in (0, \infty)$, $1 < p \leqslant q < \infty$ and $e^{4\beta t} \geqslant \frac{q-1}{p-1}$.*

Hypercontractivity (for the Gaussian measure) first appeared in the work of Nelson. We refer the reader to [19] for a historical perspective. Observe that the spectral gap λ defined at (1.1) admits a similar (much simpler) characterization as the largest real β such that $\|H_t - \mu\|_{2\to 2} \leqslant e^{-\beta t}$, for all $t > 0$.

Suppose now that we are given n Dirichlet forms $(\mathcal{E}_i, \mathcal{D}_i)$ on probability spaces (Ω_i, μ_i). For any sequence $w = (w_1, \ldots, w_n)$ of positive weights, we can form the Dirichlet form

$$\mathcal{E}_w(f, f) = \sum_1^n w_i \widetilde{\mathcal{E}}_i(f, f)$$

on $\Omega = \bigotimes_1^n \Omega_i$ equipped with the measure $\mu = \bigotimes_1^n \mu_i$ where

$$\widetilde{\mathcal{E}}_i(f, f) = \int_{\Omega^i} \mathcal{E}_i(f^i_{x^i}, f^i_{x^i}) d\mu^i(x^i)$$

with $\Omega^i = \bigotimes_{j\neq i} \Omega_j$, $\mu^i = \bigotimes_{j\neq i} \mu_j$, x^i is the $(n-1)$-tuple where the i-th coordinate of $x = (x_1, \ldots, x_n)$ has been omitted, and $f^i_{x^i} : \Omega_i \mapsto \mathbb{R}$ is the function defined by $f^i_{x^i}(x_i) = f(x)$. We omit the description of the domain. The associated semigroup is the commutative product of the semigroups acting on the individual factors (with time scale adjusted to the corresponding weight).

Theorem 1.3 (Faris, Segal, See [19, Theorem 2.3]). — *Referring to the notation introduced above the logarithmic Sobolev constant* α *and the spectral gap* λ *of the form* $\mathcal{E}_w$ *are given by*

$$\alpha = \min\{w_i\alpha_i : i = 1, \dots n\}, \quad \lambda = \min\{w_i\lambda_i : i = 1, \dots n\}$$

where α_i, λ_i *are, respectively, the logarithmic Sobolev constant and the spectral gap of the i-th factor* $(\Omega_i, \mu_i, (\mathcal{E}_i, \mathcal{D}_i))$.

This theorem is the single most important source of examples for which the logarithmic Sobolev constant is known. For instance, consider the hypercube $\Omega = \{0,1\}^d$ equipped with the uniform measure and the Dirichlet form

$$\mathcal{E}(f,f) = \frac{1}{2d}\sum_x \sum_1^d |f(x) - f(x+e_i)|^2\mu(x)$$

where e_i denotes the binary vector with a single 1 in position i and addition is mod 2. This is the product of d symmetric two point chains and thus Theorem 1.3 yields $2\alpha = \lambda = 2/d$. See [6] and the references therein for problems where this example is relevant. With the help of the central limit theorem, the tensorization of the two-point space above leads to the sharp logarithmic Sobolev constant for the Gauss measure on the real line. See [18, Theorem 4]. This shows that computing the logarithmic Sobolev constants of "small" examples is not an entirely futile exercise. For $d = 2$, this is also a simple random walk on $\mathbb{Z}/4\mathbb{Z}$.

Another simple but useful technique that belongs to the folklore of the subject involves collapsing to a smaller state space.

Theorem 1.4. — *Let* $(\Omega, \mu, (\mathcal{E}, \mathcal{D}))$ *and* $(\widetilde{\Omega}, \widetilde{\mu}, (\widetilde{\mathcal{E}}, \widetilde{\mathcal{D}}))$ *be two Dirichlet spaces as above. Assume that there is a map* $p : \widetilde{\Omega} \mapsto \Omega$ *such that for any* $f \in \mathcal{D}$ *we have*

$$\widetilde{f} = f \circ p \in \widetilde{\mathcal{D}} \text{ and } \widetilde{\mathcal{E}}(\widetilde{f}, \widetilde{f}) = \mathcal{E}(f,f).$$

Assume further that μ *is the pushforward of* $\widetilde{\mu}$ *under* p, *i.e.,* $\widetilde{\mu}(\widetilde{f}) = \mu(f)$ *for any measurable non-negative* f *on* Ω. *Let* $\widetilde{\lambda}, \widetilde{\alpha}$, *be the spectral gap and logarithmic Sobolev constant on* $\widetilde{\Omega}$. *Then*

$$\alpha \geqslant \widetilde{\alpha}, \quad \lambda \geqslant \widetilde{\lambda}.$$

In particular, if $\widetilde{\alpha} = \widetilde{\lambda}/2$ *and* $\widetilde{\lambda} = \lambda$ *then* $\alpha = \lambda/2$.

This result is useful both for finding examples with $\alpha = \lambda/2$ and examples with $\alpha < \lambda/2$. The reason is that it is often easy to decide whether or not $\widetilde{\lambda} = \lambda$ since it simply involves finding an eigenfunction associated to $\widetilde{\lambda}$ on $\widetilde{\Omega}$ that can be projected on Ω. If no such eigenfunctions exist, then λ will often be significantly larger than $\widetilde{\lambda}$ because it must come from a higher part of the spectrum on $\widetilde{\Omega}$. Here are explicit known examples.

(E8) Let $\Omega = \{0, 1, \ldots, d\}$ with

$$\mu(k) = 2^{-d}\binom{d}{k}, \;\; \mathcal{E}(f,f) = (1/d)\sum_{k=1}^{d} |f(k) - f(k-1)|^2 k\mu(k)$$

This birth and death chain with binomial stationary distribution corresponds to following the number of 1 on the hypercube $\widetilde{\Omega} = \{0,1\}^d$. If $|x|$ denotes the number of 1 in $x \in \widetilde{\Omega}$, then $|x| - d/2$ is an eigenfunction with eigenvalue $\widetilde{\lambda} = 2/d$ which obviously "lives" on Ω. Thus Theorem 1.4 gives $\alpha = \lambda/2 = 1/d$.

(E9) The n dimensional real projective space P^n is the quotient of the sphere S^n by the antipodal map $x \mapsto -x$. All the eigenfunctions associated to the spectral gap $\widetilde{\lambda} = n$ on the sphere are odd and thus, cannot be projected on P^n. Indeed, the spectral gap on P^n is $\lambda = 2(n+1)$, coming from the second non-zero eigenvalue on the sphere. The logarithmic Sobolev constant of the projective space is not known but satisfies $\alpha < \lambda/2$. Moreover, it is proved in [36] that as n tends to infinity, λ/α tends to 4. This means that, asymptotically as the dimension goes to infinity, the logarithmic Sobolev constant of the real projective space and the sphere are the same.

We now treat in more details two applications of these techniques that are not in the literature. Consider the following questions.

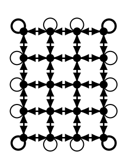


Figure 1. — The box R_b with its Dirichlet form structure, $b = (b_1, b_2) = (4, 5)$
All edges have weight 1/4 except the corner loops which have weight 1/2
The stationary measure is uniform

(Q4) Fix an integer vector $b = (b_1, \ldots, b_d)$, $1 \leqslant b_1 \leqslant \ldots \leqslant b_d$. In $\mathbb{Z}^d$ with basis $\{e_1, \ldots, e_d\}$, consider the rectangular box $R_b = \{x \in \mathbb{Z}^d :$

$x_i \in \{1, \dots, b_i\}$, $1 \leqslant i \leqslant d\}$. Let μ be the uniform distribution on R_b and

$$\mathcal{E}(f,f) = \frac{1}{4d} \sum_{x \in R_b} \sum_{i=1}^{d} \sum_{u \in \{\pm e_i\}} |f(x) - f(x+u)|^2 \mu(x) \tag{1.6}$$

with the convention that $x + u = x$ if $x \in R_b$ and $x + u \notin R_b$. This is the Dirichlet form of the simple random walk on $\mathbb{Z}^d$, forced to stay in R_b. See Figure 1. It is well-known and easy to check that $\lambda = \frac{1}{d}(1 - \cos \pi / b_d)$. What is α?

(Q5) Fix an integer n. Let $p = (p_1, \dots, p_n)$ be a probability vector on $\Omega = \{1, \dots, n\}$. Define the relative entropy and Fisher information of p by

$$\mathrm{Ent}(p) = \log n + \sum_1^n p_i \log p_i, \quad J(p) = 2 \sum_2^n |\sqrt{p_i} - \sqrt{p_{i-1}}|^2.$$

Can one control the (relative) entropy by the Fisher information and what is the best inequality? In classical terms, the Fisher information can be defined in a number of different ways, one of which is $J(f) = 4 \int |\nabla \sqrt{f}|^2 d\mu$. Note that, in discrete cases where derivatives are replaced by differences, the various definitions are not equivalent anymore.

The two questions above are essentially the same. By Theorem 1.3, (Q4) reduces to finding the logarithmic Sobolev constant of $\{1, \dots, n\}$ equipped with the uniform measure and the Dirichlet form

$$\mathcal{E}(f,f) = \frac{1}{2n} \sum_2^n |f(k) - f(k-1)|^2.$$

This is the Dirichlet form of the simple random walk on an n-point stick with loops at the ends. It is easily seen that (Q5) amounts to the same question because $\mathrm{Ent}(p) = \mathcal{L}(\sqrt{np})$ and $J(p) = 4\mathcal{E}(\sqrt{np}, \sqrt{np})$. Finding α for an n-point stick is not an easy problem. However, the n-point stick $\Omega = \{0, \dots, n-1\}$ (note the slight change of notation) can be obtained by collapsing (in the sense of Theorem 1.4) a $2n$-cycle via the identification of x with $2n - x - 1$. See Figure 2.

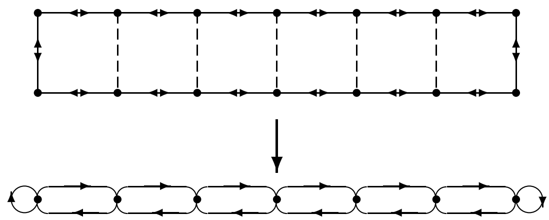


Figure 2. — The $2n$ cycle collapses to the n-stick with loops at the ends, $n = 7$. All edges have weight 1/2.

On the $2n$-cycle $\widetilde{\Omega}$, we have $\widetilde{\lambda} = 1 - \cos(\pi/n)$ with eigenfunctions $e^{\pm \pi i x/n}$. This two dimensional eigenspace contains the function $f(x) = \cos(\frac{\pi}{n}(x + \frac{1}{2}))$ which has the property that $f(x) = f(2n - x - 1)$ and thus passes to the quotient Ω. It is proved in [10] that

$$2\widetilde{\alpha} = \widetilde{\lambda} = 1 - \cos(\pi/n).$$

Thus, by Theorem 1.4,

$$2\alpha = \lambda = 1 - \cos(\pi/n).$$

This provides the answers to questions (Q4)-(Q5).

THEOREM 1.5. — *For a d-dimensional rectangular box*

$$R_b = \{x \in \mathbb{Z}^d : x_i \in \{1, \dots, b_i\},\ 1 \leqslant i \leqslant d\}$$

with $b = (b_1, \dots, b_d)$, $1 \leqslant b_1 \leqslant \dots \leqslant b_d$, equipped with the uniform probability measure and the Dirichlet form defined at (1.6), *we have*

$$\alpha = \frac{1}{2d}\left(1 - \cos\frac{\pi}{b_d}\right).$$

THEOREM 1.6. — *For any probability vector $p = (p_1, \dots, p_n)$, we have*

$$\log n + \sum_1^n p_i \log p_i \leqslant (1 - \cos(\pi/n))^{-1} \left(\sum_2^n |\sqrt{p_i} - \sqrt{p_{i-1}}|^2\right).$$

that is, $\mathrm{Ent}(p) \leqslant \frac{1}{2}(1 - \cos(\pi/n))^{-1} J(p)$. *This inequality is best possible, saturated by $f_\epsilon(x) = 1 + \epsilon\cos(\frac{\pi}{n}(x + 1/2))$ as ϵ tends to* 0.

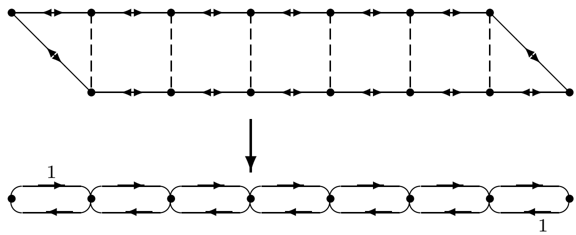


Figure 3. — The $n+1$-stick with reflecting barriers, $n=7$.
All edges have weight 1/2 except those marked which have weight 1.

Let us mention that the even cycle can also be collapsed onto the $(n+1)$-stick $\{0,\dots,n\}$ with reflecting barriers (identify x with $2n-x$ modulo $2n$). See Figure 3. In this case, the eigenfunction $x\mapsto\cos(\pi x/n)$ passes to the quotient. Hence for the $n+1$-stick equipped with simple random walk with reflecting boundary condition and stationary measure μ given by

$$\mu(x)=\begin{cases}1/(2n) & \text{if } x\in\{0,n\}\\ 1/n & \text{if } x\in\{1,\dots,n-1\},\end{cases}$$

we have $2\alpha=\lambda=1-\cos(\pi/n)$.

We end the introduction by considering two types of collapses of the even cycle generalizing those from Figures 1-2. Let $n>1$ be an integer, $s=(s_1,...,s_n)\in\{0,1\}^n$ and set $\Omega_s=\bigcup_i\{x_{i,s_i},x_{i,-s_i}\}$, where $x_{i_1,j_1}=x_{i_2,j_2}$ if and only if $i_1=i_2$ and $j_1=j_2$. In words, Ω_s is made of two copies of $\{1,\dots,n\}$ (i.e., $+$ and the $-$ copies) with certain elements in the two copies being identified (i.e., when $s_i=0$). Let $p_s:\mathbb{Z}_{2n}\to\Omega_s$ be the projection defined by

$$\forall 1\leqslant i\leqslant n,\quad p_s(i)=x_{i,s_i},\ p_s(2n-i+1)=x_{i,-s_i}.$$

Let K_s be the Markov chain defined by

$$\forall x,y\in\Omega_s,\quad K_s(x,y)=\frac{1}{|p_s^{-1}(\{x\})|}\sum_{\substack{p_s(z)=x\\ p_s(w)=y}}K(z,w), \tag{1.7}$$

where K is the transition matrix of the simple random walk on the $2n$ cycle. Thus, starting from $x_{i,j}$, we first choose a direction, to the right ($x_{i+1,\cdot}$) or to the left ($x_{i-1,\cdot}$), with equal probability, and then, independently and uniformly, move to a neighboring state in that direction. By convention, the left neighbors of $x_{1,1}$, $x_{1,-1}$ and $x_{1,0}$ are respectively $x_{1,-1}$, $x_{1,1}$ and $x_{1,0}$. Similarly, the right neighbors of $x_{n,1}$, $x_{n,-1}$ and $x_{n,0}$ are respectively $x_{n,-1}$, $x_{n,1}$ and $x_{n,0}$. See Figure 4. When $s=(0,0,0,0,0,0,0)$, the projection p_s is the same as that in Figure 2.

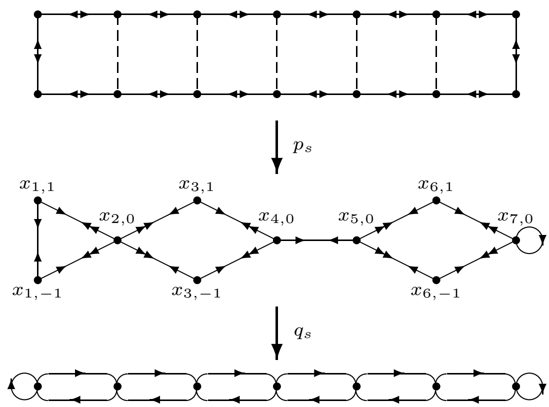


Figure 4. — The $2n$ cycle collapses to Ω_s through the projection p_s with $n = 7$ and $s = (1, 0, 1, 0, 0, 1, 0)$.
All single arrows have weight 1/2 and double arrows have weight 1/4

For $n > 1$ and $s \in \{0,1\}^n$, the Markov kernel defined in (1.7) has stationary distribution

$$\pi_s(x) = \frac{|p_s^{-1}(\{x\})|}{2n}, \quad x \in \Omega_s.$$

Let $\mathcal{E}$ and $\mathcal{E}_s$ be the Dirichlet forms associated to K and K_s and π be the uniform probability measure on $\mathbb{Z}_{2n}$. In a few computations, we obtain $\mathcal{E}(f \circ p_s, f \circ p_s) = \mathcal{E}_s(f, f)$ and $\pi(f \circ p_s) = \pi_s(f)$ for all $f \in \mathbb{R}^{|\Omega_s|}$. As a consequence of Theorem 1.4, we have the following result.

Theorem 1.7. — *For $n \geqslant 2$ and $s = (s_1, ..., s_n) \in \{0,1\}^n$, let Ω_s be as above and K_s be the Markov kernel on Ω_s defined at (1.7). Then the spectral gap λ_s and the logarithmic Sobolev constant α_s satisfy $2\alpha_s = \lambda_s \equiv 1 - \cos\frac{\pi}{n}$.*

Proof. — In order to apply Theorem 1.4, we need to investigate whether K and K_s have the same spectral gap. Consider another projection map $q_s : \Omega_s \to \mathbb{Z}_n$ defined by

$$q_s(x_{i,s_i}) = q_s(x_{i,-s_i}) = i, \quad 1 \leqslant i \leqslant n.$$

See Figure 4. Let K' be the simple random walk on the n-stick with loops at the ends and π', $\mathcal{E}'$ and λ' be its stationary distribution, associated Dirichlet form and spectral gap. By the discussion after Figure 2, we know that $\lambda' = 1 - \cos\frac{\pi}{n}$. It is also an easy exercise that $\mathcal{E}_s(f \circ q_s, f \circ q_s) = \mathcal{E}'(f, f)$ and $\pi_s(f \circ q_s) = \pi'(f)$ for any function f. This implies $\lambda' \leqslant \lambda_s \leqslant \lambda'$ and hence, by Theorem 1.4, $2\alpha_s = \lambda_s = 1 - \cos\frac{\pi}{n}$. □

Note that further examples are obtained by a similar construction based on Figure 3.

2. The Euler-Lagrange equation

In this section, the state space Ω is a finite set and the Dirichlet form $\mathcal{E}$ has the form

$$\mathcal{E}(f,f) = \frac{1}{2} \sum_{x,y \in \Omega} |f(x) - f(y)|^2 K(x,y)\mu(x)$$

where $K(x,y)$ is a Markov kernel with reversible measure μ, i.e., $K(x,y) \geqslant 0$, $\sum_y K(x,y) = 1$ and $\mu(x)K(x,y)$ is symmetric. In this case, the spectral gap λ is the smallest non-zero eigenvalue of the operator $I - K$ acting on $L^2(\Omega,\mu)$ ($Kf = \sum_y K(\cdot,y)f(y)$ and I denotes the identity). Of course there is an associated eigenfunction ϕ satisfying $(I-K)\phi = \lambda\phi$.

THEOREM 2.1. — *Referring to the reversible finite Markov chain setting introduced above, let λ, α denote the spectral gap and logarithmic Sobolev constant.*

(i) If ψ is a minimizer for α, i.e.,

$$\alpha = \frac{\mathcal{E}(\psi,\psi)}{\mathcal{L}(\psi)}.$$

then ψ is solution of the Euler-Lagrange equation

$$(I-K)\psi = 2\alpha\psi \log(\psi/\|\psi\|_2). \tag{2.1}$$

(ii) For any $\beta > 0$, any non-constant solution ϕ of the equation

$$(I-K)\phi = 2\beta\phi \log(\phi/\|\phi\|_2) \tag{2.2}$$

satisfy $\beta = \mathcal{E}(\phi,\phi)/\mathcal{L}(\phi)$. In particular, for $\beta \in (0,\alpha)$, (2.2) has no non-constant solutions.

(iii) If $\alpha < \lambda/2$, then α admits a positive non-constant minimizer.

This result is obvious from the perspective of Calculus of variation, and only the last sentence (existence of minimizers) needs attention in more general settings where capacity is not obvious. The idea to use the Euler-Lagrange equation was first emphasized in the work of Rothaus [32, 33, 34] in the (more difficult) context of diffusion on manifolds. It was used in [30] to compute the logarithmic Sobolev inequality in Example (E1) of the introduction (the ultraspherical polynomials). In the context of finite Markov chains, it appears in [8, 10, 11, 12, 13, 37]. Theorem 2.1 will be one of the main tools we use to treat specific examples below. Here we illustrate it with the case of the asymmetric two-point space.

Theorem 2.2 ([13, Theorem A.2]). — *Fix $p, q \in (0,1)$, $p+q=1$. For the two-point space $\Omega = \{0,1\}$ equipped with the chain*

$$K(0,0) = K(1,0) = q,\ K(0,1) = K(1,1) = p,\ \pi(0) = q,\ \pi(1) = p. \quad (2.3)$$

we have $\lambda = 1$ and $\alpha = 1/2$ if $p = q = 1/2$ and

$$\alpha = \frac{p-q}{\log(p/q)} \quad \textit{if } p \neq q.$$

Proof. — That $\lambda = 1$ is a very easy exercise. We prove the statement concerning α using Theorem 2.1. Setting $\psi(0) = b, \psi(1) = a$ and normalizing by $qb^2 + pa^2 = 1$, we look for triplets (α, a, b) of positive numbers that are solutions of (2.1), that is,

$$\begin{cases} p(b-a) & = 2\alpha b \log b \\ q(a-b) & = 2\alpha a \log a \\ pa^2 + qb^2 & = 1. \end{cases}$$

Luckily, α can be eliminated by using the first two equations. This yields the system

$$\begin{cases} pa\log a + qb \log b & = 0 \\ p(a^2-1) + q(b^2-1) & = 0. \end{cases}$$

Setting aside the solution $a = b = 1$, we can assume $a, b \in (0,1) \cup (1, +\infty)$ and write this system as

$$\begin{cases} pa\log a + qb \log b & = 0 \\ \frac{a-a^{-1}}{\log a} = \frac{b-b^{-1}}{\log b}. \end{cases}$$

Calculus shows that the function $x \mapsto (x - x^{-1})/\log x$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. As it obviously satisfies $f(x) = f(1/x)$, it follows that the second equation can only be satisfied if $b = 1/a$. Reporting in the first equation yields $pa - q/a = 0$, that is, $a = \sqrt{q/p}$. It follows that the solutions of our original system are the triplets $(\alpha, 1, 1)$ (α arbitrary) and, when $p \neq q$,

$$\left(\frac{p-q}{\log(p/q)}, \sqrt{q/p}, \sqrt{p/q}\right)$$

As $\frac{p-q}{\log(p/q)} < 1/2$ when $p \neq q$, we conclude from Theorem 2.1 that the logarithmic Sobolev constant of the asymmetric two-point space at (2.3) is

$$\alpha = \frac{p-q}{\log(p/q)},\ p \neq q$$

and that, in the symmetric case $p = q = 1/2$, we have $2\alpha = \lambda = 1$. □

Remark 2.3. — The proof of Theorem 2.2 given above is outlined without details in [8]. It is much simpler than the two different proofs given in [13, 37]. Here, we have been careful to treat both the symmetric and the asymmetric cases at once. In fact, the proof in [37] is incorrect (it can however be corrected with additional pain but without changing the main ideas). On the one hand, in the case $p = q = 1/2$, the proof above consists in showing that no non-constant minimizers exist, leading to the conclusion that $\alpha = \lambda/2$. This is the main line of reasoning that will be used in this work to treat other examples. On the other hand, in the case $p \neq q$, we were able to find a unique normalized non-constant solution of (2.1) with $\alpha < \lambda/2$ leading to the explicit computation of α. To the best of our knowledge, this is the only case with $\alpha < \lambda/2$ where α has been computed by solving (2.1). Our study of other small examples indicates that such a computation is typically extremely difficult.

The following corollary deals with all Markov kernels on the two-point space and is an immediate application of Theorem 2.2. The proof is omitted.

Corollary 2.4. — *Let K be a Markov kernel on the two-point space $\Omega = \{0, 1\}$ defined by*

$$K(0,0) = p_1,\ K(0,1) = q_1,\ K(1,0) = q_2,\ K(1,1) = p_2,$$

where $p_1 + q_1 = p_2 + q_2 = 1$. Assume that $q_1 q_2 \neq 0$. Then $\lambda = q_1 + q_2$ and $\alpha = q_1$ if $q_1 = q_2$, whereas

$$\alpha = \frac{q_2 - q_1}{\log q_2 - \log q_1} \quad \text{if} \quad q_2 \neq q_1.$$

We end this paragraph by recording two elementary lemmas that will be useful in showing that the Euler-Lagrange equation (2.1) has no non-constant solutions in some specific cases.

Lemma 2.5. — *Consider the continuous function $u : [0, \infty) \to \mathbb{R}$ defined by*

$$u(s) = \begin{cases} 0 & \text{if } s = 0 \\ s \log s & \text{if } s \in (0, \infty). \end{cases} \tag{2.4}$$

The function u has the following properties:

$$\forall\, t \in [0,\infty),\quad u(t) \geqslant t-1. \tag{2.5}$$

$$\forall\, s,t \in [0,\infty) \text{ with } s \leqslant t,\quad u(t)-u(s) \leqslant (t-s)(1+\log((s+t)/2)). \tag{2.6}$$

$$\forall\, s,t \in [0,\infty) \text{ with } s \leqslant t \text{ and } s+t \leqslant 2,\quad u(t)-u(s) \leqslant t-s. \tag{2.7}$$

$$\forall\, s,t \in [1,\infty) \text{ with } s \leqslant t,\quad u(t)-u(s) \geqslant t-s. \tag{2.8}$$

Proof. — The function $s \mapsto s\log s - s + 1$ has derivative $s \mapsto \log s$ on $(0,\infty)$. Hence it attains its minimum at $s=1$. As the value at $s=1$ is 0, (2.5) follows.

To prove (2.6)-(2.7), fix $s \geqslant 0$ and set, for $t \geqslant s$,

$$\begin{aligned} g(t) &= u(t)-u(s)-(t-s)u'((t+s)/2) \\ &= t\log t - s\log s - (t-s)(1+\log((t+s)/2)). \end{aligned}$$

Compute the derivatives

$$g'(t) = \log\left(\frac{2t}{t+s}\right) - \frac{t-s}{t+s},\quad g''(t) = \frac{s(s-t)}{t(t+s)^2}.$$

It follows that g is non-increasing on $[s,\infty)$. Hence $g(t) \leqslant g(s) = 0$ on $[s,\infty)$, that is,

$$u(t)-u(s) \leqslant (t-s)(1+\log((t+s)/2)).$$

The inequality (2.7) obviously follows when $s+t \leqslant 2$.

Finally, (2.8) follows from the Mean Value Theorem applied to the function u since $u' \geqslant 1$ on $[1,\infty)$. □

LEMMA 2.6. — *Consider the function $v : [0,\infty)^2 \to \mathbb{R}$ defined by*

$$v(\beta,t) = \begin{cases} 0 & \text{if } t=0 \\ t-\beta t\log t & \text{if } t>0. \end{cases}$$

Fix $\beta > 0$. For $s \in [0, \beta e^{1/\beta-1})$, let $0 \leqslant t_1(s) < t_2(s)$ be the two reals such that $v(\beta,t_1(s)) = v(\beta,t_2(s)) = s$. Then:

(i) $t_1(s)t_2(s) < e^{2/\beta-2}$ for all $s \in [0,\beta e^{1/\beta-1})$.

(ii) The map $s \mapsto t_1(s)+t_2(s)$ is strictly decreasing on $[0,\beta e^{1/\beta-1})$.

In particular, for $0 \leqslant s < \beta e^{1/\beta-1}$,

$$t_1(s) + t_2(s) > 2e^{1/\beta-1}, \quad t_1(s)^2 + t_2(s)^2 > 2e^{2/\beta-2}. \tag{2.9}$$

Proof. — For fixed $\beta > 0$, we write $v(t)$ as a shorthand of $v(\beta, t)$. Note that $v(t)$ is a concave function attaining its maximum at $t = e^{1/\beta-1}$ with value $\beta e^{1/\beta-1}$. This ensures that $t_1(s)$ and $t_2(s)$ are well defined. By the concavity of v, (i) is equivalent to $v(e^{2/\beta-2}/t_1(s)) < s$, or

$$v(e^{2/\beta-2}/t) < v(t), \quad \forall t \in (0, e^{1/\beta-1}).$$

For $t > 0$, let f be the difference of both sides, that is,

$$f(t) = v(e^{2/\beta-2}/t) - v(t) = \frac{e^{2/\beta-2}}{t}(-1 + 2\beta + \beta \log t) - t(1 - \beta \log t).$$

A simple computation gives

$$f'(t) = \frac{1 - \beta - \beta \log t}{t^2}(e^{2/\beta-2} - t^2) > 0, \quad \forall t \in (0, e^{1/\beta-1}).$$

Hence $f(t) < f(e^{1/\beta-1}) = 0$.

To prove the monotonicity of $g(s) = t_1(s) + t_2(s)$, we consider two intervals $A = (0, e^{1/\beta-1})$, $B = (e^{1/\beta-1}, e^{1/\beta})$ and the restrictions $v|_A$, $v|_B$ of v on them. It is obvious that $t_1 \circ v|_A = I_A$ and $t_2 \circ v|_B = I_B$, where I_D is the identity map on D. By the inverse function theorem, $t_1(\cdot)$ and $t_2(\cdot)$ are differentiable on $(0, \beta e^{1/\beta-1})$ with derivatives

$$t_1'(s) = \frac{1}{v'(t_1(s))} = \frac{1}{1 - \beta - \beta \log t_1(s)} > 0,$$

and

$$t_2'(s) = \frac{1}{v'(t_2(s))} = \frac{1}{1 - \beta - \beta \log t_2(s)} < 0.$$

Putting both identities together and then applying part (i) gives

$$\forall s \in (0, \beta e^{1/\beta-1}), \quad g'(s) = \frac{2 - 2\beta - \beta \log(t_1(s)t_2(s))}{v'(t_1(s))v'(t_2(s))} < 0.$$

Hence, g is strictly decreasing.

The first inequality in (2.9) is obtained by applying part (ii) and observing that

$$\lim_{s \to \beta e^{1/\beta-1}} t_1(s) = \lim_{s \to \beta e^{1/\beta-1}} t_2(s) = e^{1/\beta-1}.$$

The second inequality in (2.9) follows from the first one and part (i). □

3. The five cycle

This section is devoted to the study of the five cycle $\mathbb{Z}/5\mathbb{Z}$ equipped with the uniform probability measure $\pi(x) = 1/5$ and the Markov kernel $K(x, y) = 1/2$ if $|x-y| = 1$ modulo 5. The Dirichlet form can be written as

$$\mathcal{E}(f, f) = \frac{1}{10} \sum_{x \in \mathbb{Z}/5\mathbb{Z}} |f(x) - f(x+1)|^2 \tag{3.1}$$

where addition is understood modulo 5. We refer to this chain as the simple random walk on the 5 cycle. The spectral gap is

$$\lambda = 1 - \cos(2\pi/5).$$

The 5 cycle can be projected to the three point space $\{0, 1, 2\}$ by identifying x with $5-x$ (modulo 5). The corresponding chain is the simple random walk on the 3 stick with a loop at one end. It has kernel $K(0, 1) = 1$, $K(1, 0) = K(1, 2) = K(2, 1) = K(2, 2) = 1/2$, and stationary measure $\pi(0) = 1/5$, $\pi(1) = \pi(2) = 2/5$. See Figure 5.

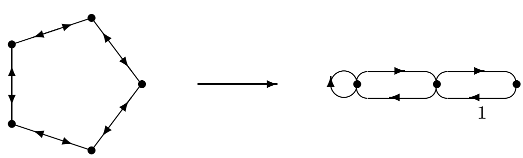


Figure 5. — The 5 cycle collapses to the 3-point stick with a loop at one end. All edges have weight 1/2 except marked otherwise. In both diagrams the right most point is 0

THEOREM 3.1. — *The spectral gap and logarithmic Sobolev constant of the simple random walk on the* 5 *cycle satisfy*

$$\alpha = \frac{\lambda}{2} = \frac{1}{2}\left(1 - \cos\frac{2\pi}{5}\right).$$

THEOREM 3.2. — *The spectral gap and logarithmic Sobolev constant of the simple random walk on the* 3*-point stick with a loop at one end satisfy*

$$\alpha = \frac{\lambda}{2} = \frac{1}{2}\left(1 - \cos\frac{2\pi}{5}\right).$$

Observe that $\cos(2\pi\cdot/5)$ is an eigenvector of the transition kernel of the simple random walk on $\mathbb{Z}/(5\mathbb{Z})$ and the associated eigenvalue is $\cos(2\pi/5)$.

Then, by Theorem 1.4, Theorem 3.2 is a corollary of Theorem 3.1. However, the proof below proceeds differently. We will first show that the logarithmic constants of the 5 cycle and the 3-point stick with a loop at one end are equal. Then, we will show that $\alpha = \lambda/2$ for the 3-point stick, proving both Theorem 3.1 and Theorem 3.2 at the same time.

In what follows, we will always consider a positive function ψ on the 5 cycle normalized by $\|\psi\|_2 = 1$ and which is a potential non-constant solution of the Euler-Lagrange equation (2.2) for a given $\beta > 0$. In this case, (2.2) reads

$$\forall\, x \in \mathbb{Z}/5\mathbb{Z}, \quad 2\psi(x) - (\psi(x+1) + \psi(x-1)) = 4\beta u(\psi(x)) \tag{3.2}$$

with u as in (2.4). It will be convenient to label the value of ψ around the cycle as indicated in Figure 6.

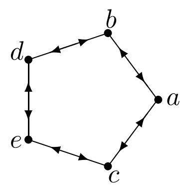


Figure 6. — The values of ψ around the 5 cycle

This notation is justified by the following lemma. Because we use the letter e as one of the values of ψ, we will use the notation exp for the exponential function.

Lemma 3.3. — *Assume that ψ is a non-constant function such that*

$$\frac{\mathcal{E}(\psi,\psi)}{\mathcal{L}(\psi)} = \inf\left\{\frac{\mathcal{E}(f,f)}{\mathcal{L}(f)} : \mathcal{L}(f) \neq 0\right\}.$$

Let the values taken by ψ be (a,b,c,d,e) as indicated in Figure 6 with $a = \|\psi\|_\infty$ and $b \geqslant c$. Then we must have

$$a \geqslant b \geqslant c \geqslant d \geqslant e.$$

Proof. — Observe that there is no loss of generality in fixing the position where the maximum is taken. Without loss of generality, we can also assume that $\|\psi\|_2 = 1$. Observe that

$$\frac{\mathcal{E}(\psi,\psi)}{\mathcal{L}(\psi)} = \frac{(a-b)^2 + (a-c)^2 + (b-d)^2 + (c-e)^2 + (d-e)^2}{a^2\log a^2 + b^2\log b^2 + c^2\log c^2 + d^2\log d^2 + e^2\log e^2}. \tag{3.3}$$

Hence swapping the different values of ψ does not change the denominator. Suppose that the smallest value taken by ψ is $c < e$. Then we must have $a \geqslant b \geqslant d \geqslant e > c$ because the following inequality holds

$$\sum_{i=1}^{n-1}(x_i - x_{i+1})^2 < \sum_{i=1}^{n-1}(x_{\sigma(i)} - x_{\sigma(i+1)})^2$$

for any real numbers $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n-1} \leqslant x_n$ and for all $\sigma \in S_n$ satisfying $\sigma(1) = 1$, $\sigma(n) = n$ and $x_{\sigma(i)} \neq x_i$ for some $1 < i < n$. However a direct computation shows that, in this situation,

$$(a-c)^2 + (d-e)^2 \geqslant (a-e)^2 + (d-c)^2$$

with equality if and only if $a = b = d$ which, by (2.1) would imply that ψ is constant. It follows that swapping the positions of e and c decreases the quotient at (3.3), a contradiction. Thus we can assume that the smallest value taken by ψ is either e or d. As $b \geqslant c$, it follows immediately that we must have $d \geqslant e$. Hence the smallest value taken by ψ is e and we must have $a \geqslant b \geqslant d \geqslant e$ and $a \geqslant c \geqslant e$. Assume that $c < d$. By inspection, we then have

$$(a-c)^2 + (b-d)^2 \geqslant (b-c)^2 + (a-d)^2$$

with equality if and only if $a = b$. By (2.1), $a = b$ implies $c = d$ which is not possible. Hence, swapping the positions of c and d decrease the quotient at (3.3), a contradiction. It follows that

$$a \geqslant b \geqslant c \geqslant d \geqslant e$$

as desired. □

The equations in the following lemma correspond to the Euler-Lagrange equations (2.2) for a minimizer on the 5 cycle using the notation of Figure 6.

Lemma 3.4. — *Let u be the function defined at (2.4). Let (a,b,c,d,e) be such that $a \geqslant b \geqslant c \geqslant d \geqslant e > 0$. Assume that*

$$a^2 + b^2 + c^2 + d^2 + e^2 = 5 \tag{3.4}$$

and that the equations

$$2a - (b+c) = 4\beta u(a) \tag{3.5}$$

$$2b - (a+d) = 4\beta u(b) \tag{3.6}$$

$$2c - (a+e) = 4\beta u(c) \tag{3.7}$$

$$2d - (b+e) = 4\beta u(d) \tag{3.8}$$

$$2e - (c+d) = 4\beta u(e) \tag{3.9}$$

are satisfied for some β, $0 \leqslant \beta < \frac{1}{2}(1 - \cos\frac{2\pi}{5})$. Then $d \leqslant 1$.

Proof. — The proof will produce a number of additional conditions on (a, b, c, d, e). Namely, we claim that under the hypotheses of Lemma 3.4, we must have

$$b \geqslant 1 \text{ with equality only if } a = b = c = d = e = 1 \tag{3.10}$$

and

$$b + d < 2 < a + e < 2c \quad \text{if } c > 1. \tag{3.11}$$

Let us prove that (3.10) and (3.11) imply that $d \leqslant 1$. Indeed, on the one hand, if $c \leqslant 1$ then $d \leqslant c \leqslant 1$. On the other hand, if $c > 1$ then (3.11) gives $b+d < 2$ and (3.10) implies $d < 1$. Thus we are left with the task of proving (3.10) and (3.11).

One of the key to the proof given below is to recognize that $\frac{1}{2}(1-\cos\frac{2\pi}{5})$ is the smallest root of the polynomial

$$g(t) = 16t^2 - 20t + 5 = (2 - 4t)(3 - 4t) - 1. \tag{3.12}$$

Since the constant β in Lemma 3.4 satisfies $\beta < \frac{1}{2}(1-\cos\frac{2\pi}{5})$, we must have

$$g(\beta) > 0. \tag{3.13}$$

To prove (3.10), assume that $b \leqslant 1$. Observe that (3.6) and (2.5) give $2(b - 1) - (a + d - 2) = 2b - (a + d) \geqslant 4\beta(b - 1)$, that is,

$$(2 - 4\beta)(b - 1) \geqslant (a + d - 2).$$

Note that the hypothesis $\beta \in [0, \frac{1}{2}(1 - \cos\frac{2\pi}{5}))$ implies that $\beta < 1/2$. Hence we must have $a + d \leqslant 2$. This also implies $c + e \leqslant 2$. Subtract (3.8) from (3.5) and apply (2.7) — which is justified since $a + d \leqslant 2$ — to obtain

$$2(a - d) - (c - e) \leqslant 4\beta(a - d)$$

or, equivalently,

$$(2 - 4\beta)(a - d) - (c - e) \leqslant 0. \tag{3.14}$$

Similarly, since $c + e \leqslant 2$, subtracting (3.9) from (3.7) and applying (2.7) produces

$$(3 - 4\beta)(c - e) - (a - d) \leqslant 0. \tag{3.15}$$

Multiplying (3.14) by $(3 - 4\beta)$ and adding (3.15) yields

$$g(\beta)(a - d) \leqslant 0.$$

As $g(\beta) > 0$ this implies $a = b = c = d$. Using (3.5) we must have $a = b = d = c = 1$. By (3.4), we must have also $e = 1$. Thus $b > 1$ or $a = b = c = d = e = 1$.

We now prove (3.11). Assume $c > 1$. Observe that

$$a + b + c + d + e \leqslant \sqrt{5(a^2 + b^2 + c^2 + d^2 + e^2)} = 5.$$

Thus $c > 1$ implies $d+e < 2$. Also, by (3.7), $a+e < 2c$. Subtract (3.6) from (3.5) and apply (2.8) — this is possible since $a \geqslant b \geqslant c > 1$ — to obtain

$$(3 - 4\beta)(a - b) - (c - d) \geqslant 0. \tag{3.16}$$

Similarly, subtract (3.9) from (3.8) and apply (2.7) — this is justified because $d + e < 2$ — to obtain

$$(3 - 4\beta)(e - d) - (c - b) \geqslant 0. \tag{3.17}$$

Adding up (3.16) and (3.17) yields

$$(2 - 4\beta)(a + e - b - d) \geqslant 2c - a - e = 4\beta u(c) > 0.$$

Hence

$$a + e \geqslant b + d. \tag{3.18}$$

We now claim that $a + e > 2$. Indeed, assume that $a + e \leqslant 2$. Then, by (3.18), we also have $b+d \leqslant 2$. Subtracting (3.9) from (3.5), (3.8) from (3.6), and using (2.7) —which is justified since $a + e$ and $b + d$ are not greater than 2— we obtain

$$(2 - 4\beta)(a - e) - (b - d) \leqslant 0, \tag{3.19}$$

$$(3 - 4\beta)(b - d) - (a - e) \leqslant 0. \tag{3.20}$$

Multiplying (3.19) by $(3-4\beta)$ and subtracting (3.20) yields $g(\beta)(a-e) \leqslant 0$ which implies $a = b = d = c = e = 1$, a contradiction since we assume that $c > 1$. Thus we must have $a+e > 2$. As $a+b+c+d+e \leqslant 5$, $a+e > 2$ and $c > 1$ implies $b+d < 2$ as desired. □

The next Lemma is one of the crucial step in the proof of Theorem 3.1.

Lemma 3.5. — *Referring to the notation and hypotheses of Lemma 3.4, we must have $b = c$, $d = e$, and $a \in [1, 1.42)$.*

Before we prove this lemma we rephrase its conclusion in different terms.

Lemma 3.6. — *The logarithmic Sobolev constants of the 5 cycle and the 3-point stick with a loop at one end are equal. Call it α. If $\alpha < \frac{1}{2}(1-\cos\frac{2\pi}{5})$ then any non-constant positive normalized solution of the corresponding Euler-Lagrange equation (2.1) on the 3-point stick with a loop at one end is monotone, attains its maximum a at the loopless end of the stick and $a \in (1, 1.42)$.*

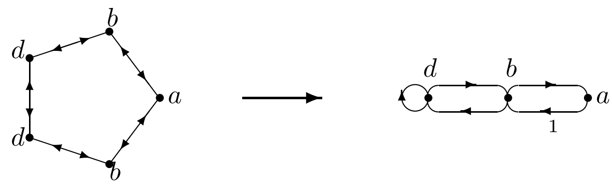


Figure 7. — The minimizers on the 5 cycle and the associated 3-point stick

Proof. — Let λ, α be the spectral gap and logarithmic Sobolev constant of the 5 cycle. By Theorem 2.1, either $\alpha = \lambda/2$ and then, by Theorem 1.4, α is also the logarithmic Sobolev constant of the 3-point stick with a loop at one end, or there exists a positive non-constant minimizer ψ satisfying (2.1) on the 5 cycle. By Lemma 3.3, we can assume that the values (a, b, c, d, e) of ψ as presented on Figure 6 satisfy $a \geqslant b \geqslant c \geqslant d \geqslant e$. Applying Lemma 3.4 and Lemma 3.5 with $\beta = \alpha < \lambda/2 = \frac{1}{2}(1 - \cos 2\pi/5)$, we conclude that the minimizer ψ is symmetric, that is, satisfy $b = c, d = e$. Hence ψ projects on the 3-point stick with a loop at one end. See Figure 7. It follows that α is also the logarithmic Sobolev constant of that chain. The other statement in the lemma follows from lifting a (potential) minimizer from the 3-point stick to the 5 cycle. By Lemma 3.3 and Lemma 3.5, the minimizer on the 3 point stick must be monotone with its maximum a at the loopless end with $a \in (1, 1.42)$. $\square$

Proof of Lemma 3.5. — Let $\lambda = 1 - \cos \frac{2\pi}{5}$. Suppose we have $b = c$ and $d = e$. Then, by (3.10) and (3.11), we must have $b \geqslant 1$ and $d < 1$ which implies, by (3.4), $a \in (1, \sqrt{3})$. If $a \geqslant 1.42$, then $b = a - 2\beta a \log a \geqslant a - (1 - \cos \frac{2\pi}{5}) a \log a \geqslant 1.0759$. Similarly, if $d < 1/\sqrt{2}$, then (3.8) implies $b = d - 4\beta d \log d \leqslant d - 2(1 - \cos \frac{2\pi}{5}) d \log d \leqslant 1.0458$. Thus, if $a \geqslant 1.42$, then $d \geqslant 1/\sqrt{2}$ and $a^2 + 2b^2 + 2d^2 \geqslant 1.42^2 + 2 + 1 > 5$, a contradiction. Hence a must be in $[1, 1.42)$.

We now prove that $b = c$ and $d = e$. Set $v(0) = 0$ and

$$v(s) = 2s - 4\beta s \log s, \quad s > 0.$$

We will need the following elementary facts about v.

(v1) $v'(s) = 2 - 4\beta - 4\beta \log s$, $v''(s) = -4\beta/s$, $v'''(s) = 4\beta/s^2$. In particular, $v''' > 0$, $v'' < 0$ and v' decreasing on $(0, \infty)$.

(v2) v is increasing on $(0, \exp(-1 + 1/2\beta))$, decreasing on $(\exp(-1 + 1/2\beta), \infty)$.

(v3) $s \mapsto v(s) - s$ as a unique maximum $4\beta \exp(-1 + 1/4\beta)$.

(v4) $v'(v(s) - s) > 0$ when $s \in (0, \exp(1/4\beta))$. Moreover, $\exp(1/4\beta) > 2$.

The first three assertions are straightforward. By (v2) and (v3), to prove that $v'(v(s) - s) > 0$ when $s \in (0, \exp(1/4\beta))$ it suffices to check that $4\beta \exp(-1 + 1/4\beta) \leqslant \exp(-1 + 1/2\beta)$, that is, $4\beta \log 4\beta \leqslant 1$. This is true because $2\lambda \log 2\lambda < 1/2$ and $\beta \leqslant \lambda/2$. The last inequality can also be used to check that $\exp(1/4\beta) > 2$.

Observe that the equations (3.5)–(3.9) can be written in a neat form using the function v. For instance, (3.5) and (3.9) read $v(a) = b + c$ and $v(e) = c + d$, respectively. Now, using (3.7) and (3.9) in that form, we obtain

$$a = v(v(e) - d) - e.$$

Similarly, (3.6) and (3.8) yields

$$a = v(v(d) - e) - d.$$

Thus, we must have

$$v(v(e) - d) - v(v(d) - e) = e - d.$$

Set

$$J = [0, d] \cap \{s : v(s) \geqslant d\}.$$

As $d \leqslant 1$ (by Lemma 3.4), this is an interval containing d and contained in $[0, 1]$. By (3.9), J also contains the value e of ψ. On J, consider the function

$$w(s) = v(v(s) - d) - v(v(d) - s).$$

The idea of the proof is to show that the only solution in J of the equation $w(s) = s - d$ is d. As e must satisfy this equation, it then follows that $e = d$. By (3.8) and (3.9) this also implies that $b = c$.

Thus we are left with the task of proving that $w(s) = s - d$ implies $s = d$. This will follow if we can show:

(1) w' is decreasing on J;

(2) $w'(d) > 1$.

To this end, we compute 3 derivatives of w.

$$\begin{aligned} w'(s) &= v'(v(s)-d)\times v'(s)+v'(v(d)-s) \\ w''(s) &= v'(v(s)-d)\times v''(s)+v''(v(s)-d)\times v'(s)^2-v''(v(d)-s) \\ w'''(s) &= v'(v(s)-d)\times v'''(s)+v''(v(s)-d)\times v'(s)\times v''(s) \\ &\quad +v''(v(s)-d)\times 2v'(s)\times v''(s)+v'''(v(s)-d)\times v'(s)^3 \\ &\quad +v'''(v(d)-s). \end{aligned}$$

Note that, by (v2), v' is positive on $J\subset[0,1]$. Moreover, by (v1), v'' is negative and v''' is positive on J. Observe further that $v(s)-d\leqslant v(d)-d$ and thus, by (v4),

$$v'(v(s)-d)\geqslant v'(v(d)-d)>0.$$

It now follows that $w'''\geqslant 0$ on J. Hence, to show that w' is decreasing on J, it suffices to show that $w''(d)<0$. Set

$$V(d)=w''(d)=v'(v(d)-d)\times v''(d)+v''(v(d)-d)\times v'(d)^2-v''(v(d)-d),$$

and consider V as a function of $d\in[0,1]$. An elementary, straightforward but tedious computation shows that

$$\begin{aligned} V'(s) &= [v'(s)-1]^2\times v'''(v(s)-s)\times[v'(s)+1] \\ &\quad +v''(v(s)-s)\times v''(s)\times[3v'(s)-1]+v'''(s)\times v'(v(s)-s). \end{aligned}$$

Again, on $[0,1]$, v' is positive, v'' negative, v''' positive and $v'(v(s)-s)>0$. So the only terms whose sign is unknown is $3v'(s)-1$. However, on $[0,1]$, $v'(s)\geqslant v'(1)=2-4\beta>2\cos 2\pi/5$ and thus one can check that $3v'(s)-1$ is positive. It follows that V is increasing on $[0,1]$. With g defined at (3.12), we have

$$\begin{aligned} V(1) &= -4\beta(2-4\beta)-4\beta(2-4\beta)^2+4\beta \\ &= -4\beta[(2-4\beta)(3-4\beta)-1]=-4\beta g(\beta)<0. \end{aligned}$$

Hence, for any $d\in[0,1]$, $w''(d)=V(d)<V(1)<0$. We have proved that the function w' is decreasing on J (see item (1) above).

We now need to show that $w'(d)>1$ for any $d\in[0,1]$. Set

$$W(d)=w'(d)=v'(v(d)-d)\times[1+v'(d)]$$

and consider W as a function on $[0,1]$. Observe that

$$W(1)=(2-4\beta)(3-4\beta)=1+g(\beta)>1.$$

We have

$$W'(s) = v''(v(s) - s)[v'(s)^2 - 1] + v''(s) \times v'(v(s) - s).$$

If $v'(s) \geqslant 1$, we have $W'(s) < 0$. If $v'(s) < 1$, we use the facts that $v(s) - s \geqslant s$ on $[0,1]$ and v'' is increasing to obtain

$$W'(s) \leqslant v''(s)[v'(s)^2 - 1 + v'(v(s) - s)].$$

Let $W_1(s) = v'(s)^2 - 1 + v'(v(s) - s)$. Then $W_1(1) = g(\beta) > 0$ and

$$\begin{aligned} W_1'(s) &= v''(v(s) - s) \times [v'(s) - 1] + 2v'(s)v''(s) \\ &\leqslant v''(s)[3v'(s) - 1] < 0 \end{aligned}$$

because we already checked that $3v'(s) - 1 > 0$ and $v''(s) < 0$ on $[0,1]$. Hence, $W_1 > 0$ and $W' \leqslant 0$ on $[0,1]$. As $W(1) > 1$, it follows that $W > 1$ on $[0,1]$, that is, for any $d \in (0,1]$, $w'(d) > 1$ as desired (see item (2) above). □

The following lemma says that, for $\beta < \frac{1}{2}(1 - \cos 2\pi/5)$, the corresponding equation (2.2) on the 3-point stick has no non-constant solutions with the properties stated in Lemma 3.6. By Lemma 3.6, this finishes the proof of both Theorem 3.1 and Theorem 3.2: the logarithmic Sobolev constant of both the 5 cycle and the 3-point stick with a loop at one end must be equal to $\frac{1}{2}(1 - \cos 2\pi/5)$.

LEMMA 3.7. — *Fix $\beta < \lambda/2$ with $\lambda = 1 - \cos 2\pi/5$. If (a, b, d) is such that $a \geqslant b \geqslant d > 0$, $a \in [1, 1.42)$ and satisfies*

$$a^2 + 2b^2 + 2d^2 = 5 \tag{3.21}$$

and

$$\begin{aligned} 2\beta a \log a &= a - b &(3.22)\\ 4\beta b \log b &= 2b - a - d &(3.23)\\ 4\beta d \log d &= d - b &(3.24) \end{aligned}$$

then we must have $a = b = d = 1$.

Proof. — For $s, \eta > 0$, set

$$v(s, \eta) = s - 2\eta s \log s.$$

Note that

$$\forall\, (s, \eta) \in [1, 1.42) \times [0, \lambda/2], \;\; 1 \leqslant v(s, \eta) \leqslant s. \tag{3.25}$$

Using the equations (3.22)–(3.24), for any $\beta < \lambda/2$, we can compute b and d in terms of a. Namely,

$$b = v(a,\beta), \quad d = 2v(v(a,\beta),\beta) - a.$$

For $(s,\eta) \in [1, 1.42) \times [0, \lambda/2]$, set

$$w(s,\eta) = s^2 + 2v(s,\eta)^2 + 2(2v(v(s,\eta),\eta) - s)^2 - 5.$$

Note that, given that a, b, d satisfy (3.22)–(3.24), (3.21) can be equivalently stated as

$$w(a,\beta) = 0.$$

Hence, the Lemma will be proved if we can show that

$$s \in [1, 1.42) \text{ and } w(s,\beta) = 0 \text{ implies } s = 1. \tag{3.26}$$

The key to the proof is in the following two observations.

(1) For each $s \in [1, 1.42)$, the function

$$\eta \mapsto w(s,\eta)$$

is a non-increasing function of $\eta \in (0, \lambda/2)$.

(2) The function $s \mapsto w(s, \lambda/2)$ is strictly positive on $(1, 1.42)$.

Indeed, if we can prove (1) and (2) above then, obviously, (3.26) follows.

In order to prove item (1) above, we compute

$$\frac{\partial w}{\partial \eta}(s,\eta) = 4v(s,\eta)\frac{\partial v}{\partial \eta}(s,\eta)$$
$$+8\left(2v(v(s,\eta),\eta) - s\right)\left(\frac{\partial v}{\partial \eta}(v(s,\eta),\eta) + \frac{\partial v}{\partial s}(v(s,\eta),\eta)\frac{\partial v}{\partial \eta}(s,\eta)\right).$$

For $(s,\eta) \in [1, 1.42) \times [0, \lambda/2]$, we have

$$v(s,\eta) \geqslant 1, \quad 2v(v(s,\eta),\eta) - s \geqslant 0.$$

As $\frac{\partial v}{\partial \eta}(s,\eta) = -2s\log s$, and $s \geqslant 1, v(s,\eta) \geqslant 1$, one has

$$\frac{\partial v}{\partial \eta}(s,\eta) \leqslant 0, \quad \frac{\partial v}{\partial \eta}(v(s,\eta),\eta) \leqslant 0.$$

Thus, to see that $\frac{\partial w}{\partial \eta} \leqslant 0$, it suffices to check that $\frac{\partial v}{\partial s}(v(s,\eta),\eta) \geqslant 0$ for $(s,\eta) \in [1, 1.42) \times [0, \lambda/2]$. Now, $\frac{\partial v}{\partial s}$ is a decreasing function of s and one can check that it is positive on $[1, 1.42)$ for each $\eta \in [0, \lambda/2]$. Moreover, $v(s, \eta) \leqslant s$ for $s \geqslant 1$. Hence we have

$$\frac{\partial v}{\partial s}(v(s,\eta),\eta) \geqslant \frac{\partial v}{\partial s}(s,\eta) \geqslant 0$$

for $(s, \eta) \in [1, 1, 42) \times [0, \lambda/2]$. This proves item (1) above.

We are left to prove item (2), that is, $w(s, \lambda/2) > 0$ for $s \in (1, 1.42)$. To simplify notation, we set

$$w(s) = w(s, \lambda/2), \quad v(s) = v(s, \lambda/2).$$

The first and second derivatives of w are

$$\begin{aligned} w'(s) &= 2s + 4v(s)v'(s) + 4(2v(v(s)) - s)(2v'(v(s))v'(s) - 1) \\ w''(s) &= 2 + 4v'(s)^2 + 4v(s)v''(s) + 4(2v'(v(s))v'(s) - 1)^2 \\ &\quad +4(2v(v(s)) - s)\left(2v''(s)v'(v(s)) + 2v''(v(s))v'(s)^2\right). \end{aligned}$$

When evaluated at $s = 1$ both are zero. Indeed, using the polynomial g introduced at (3.12),

$$w'(1) = 4(1 - 2(\lambda/2))(3 - 4(\lambda/2)) - 2 = 2g(\lambda/2) = 0,$$

and

$$\begin{aligned} w''(1) &= 2 + 4(1-\lambda)^2 - 4\lambda + 4(2(1-\lambda)^2 - 1)^2 - 8\lambda(1-\lambda)(2-\lambda) \\ &= 2(2\lambda^2 - 4\lambda + 1)g(\lambda/2) = 0. \end{aligned}$$

The last factorization was obtained by observing that $w''(1)$ is a polynomial in $\lambda/2$ and that numerical evaluation indicates that this polynomial vanishes at $\lambda/2 = (1 - \cos 2\pi/5)/2$. Division by the polynomial g then produced the desired factorization.

Thus we will have $w > 0$ on $(1, 1.42)$ (i.e., item (2) above) if we can prove that the third derivative of w is positive on $[1, 1.42)$. The third derivative is

$$w'''(s) = 12v'(s)v''(s) + 4v(s)v'''(s) + h(s)$$

with

$$\begin{aligned} h(s) &= 12[2v'(v(s))v'(s) - 1][2v''(v(s))v'(s)^2 + 2v'(v(s))v''(s)] \\ &\quad +4[2v(v(s)) - s] \\ &\qquad \times[2v'''(s)v'(v(s)) + 6v'(s)v''(s)v''(v(s)) + 2v'''(v(s))v'(s)^3]. \end{aligned}$$

The function $v'(s) = 1 - \lambda - \lambda \log s$ is decreasing on $[1, 1.42)$ and with value in $(0, .31)$. Moreover, v'' is negative and v''' positive on $[1, 1.42)$. It easily follows that h is positive on $[1, 1.42)$. To finish the proof that w''' is positive on $[1, 1.42)$, we observe that

$$\begin{aligned} 3v'(s)v''(s) + v(s)v'''(s) &= -\frac{3\lambda}{s}(1 - \lambda - \lambda \log s) + \frac{\lambda}{s^2}(s - \lambda s \log s) \\ &= \frac{\lambda}{s}(-2 + 3\lambda + 2\lambda \log s) > 0 \qquad \square \end{aligned}$$

Remark 3.8. — By itself, Lemma 3.7 is not sufficient to prove Theorem 3.2. Indeed, it is easy to see that a potential minimizer on the 3-point stick must be monotone (in fact, L. Miclo [29] proved the remarkable result that monotonicity holds for any birth and death chain with no holding). However, Lemma 3.7 only treats monotone minimizers with a maximum at the loop-less end of the stick. This is sufficient because of Lemmas 3.5-3.6 which involve lifting to the 5 cycle. A direct proof of Theorem 3.2 (without lifting to the 5 cycle) involves the study of the case where the vector (a, b, c) in Lemma 3.7 satisfies $0 < a \leqslant b \leqslant d$ instead and, in particular, $a \leqslant 1$. This case can be treated by an argument similar to the one given above but involving additional computations.

4. Some other 3-point chains

By collapsing 4, 5 and 6 cycles, we have obtained in Sections 1.2 and 3 the equality $\alpha = \lambda/2$ for the three chains on the 3-point stick described in Figure 8.

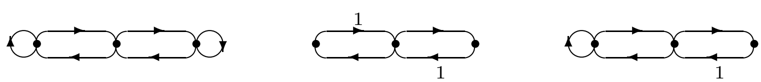


Figure 8. — Three chains on the 3-point stick.
All edges have weight 1/2 except when marked otherwise
In all cases $\alpha = \lambda/2$

In this section, we focus on some specific Markov chains on the 3-point stick and the main results are listed in Table 1.

THEOREM 4.1. — *For $0 \leqslant p < 1$, let K_p be the Markov kernel on the 3-point space $\{1, 2, 3\}$ defined by*

$$K_p = \begin{pmatrix} p & 1-p & 0 \\ .5 & 0 & .5 \\ 0 & 1-p & p \end{pmatrix}$$

with stationary distribution

$$\mu_p = \left(\frac{1}{4-2p}, \frac{2-2p}{4-2p}, \frac{1}{4-2p} \right).$$

Then $\alpha_p = \lambda_p/2 = (1-p)/2$.

THEOREM 4.2. — *For* $0 \leqslant p < 1$*, let* K_p *be the Markov kernel on the* 3*-point space* $\{1,2,3\}$ *defined by*

$$K_p = \begin{pmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 0 & 1-p & p \end{pmatrix}$$

with stationary measure

$$\mu_p = \left(\frac{1-p}{4-3p}, \frac{2-2p}{4-3p}, \frac{1}{4-3p} \right).$$

Then $\lambda_p = \frac{1}{2}\left(3 - p - \sqrt{p^2+1}\right)$ *and the log Sobolev constant* α_p *is strictly decreasing in* p *and satisfies*

$$\alpha_p = \lambda_p/2$$

only when $p = 0$ *or* $p = 1/2$.

THEOREM 4.3. — *For* $0 < p \leqslant 1$*, let* $K_p : \{1,2,3\} \times \{1,2,3\} \mapsto [0,1]$ *be the Markov kernel defined by*

$$K_p = \begin{pmatrix} 0 & 1 & 0 \\ p/2 & 1-p & p/2 \\ 0 & 1 & 0 \end{pmatrix}$$

with stationary distribution

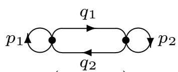
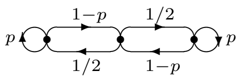
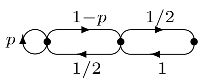
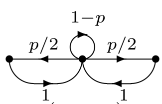
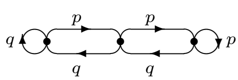
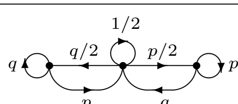
$$\mu_p = \left(\frac{p}{2(1+p)}, \frac{1}{1+p}, \frac{p}{2(1+p)} \right).$$

Then $\lambda_p = 1$ *for all* $p \in (0,1]$ *and we have*

$$\alpha_p = \lambda_p/2$$

if and only if $p \in [3/4, 1]$*. Moreover, the map* $p \mapsto \alpha_p$ *is strictly increasing on* $(0, 3/4)$.

Table 1. — The spectral gap λ and the log-Sobolev constant α of some Markov chains on the 2-point space and the 3-stick

Markov chain	λ	α	$\lambda = 2\alpha$
 $(q_1 q_2 > 0)$	$q_1 + q_2$	$\frac{q_2 - q_1}{\log(q_2/q_1)}$	$q_1 = q_2$
 $(0 \le p < 1)$	$1 - p$	$\frac{1-p}{2}$	$0 \le p < 1$
 $(0 \le p < 1)$	$\frac{3 - p - \sqrt{p^2+1}}{2}$	unknown except for $p = 0$ or $p = 1/2$	$p \in \{0, 1/2\}$
 $(0 < p \le 1)$	1	unknown except for $\frac{3}{4} \le p \le 1$	$p \in [3/4, 1]$
 $(p+q=1,\ 0<p<1)$	$1 - \sqrt{pq}$	$\frac{p-q}{2\log(p/q)}$	$p = 1/2$
 $(p+q=1,\ 0<p<1)$	$\frac{1}{2}$	$\frac{p-q}{2\log(p/q)}$	$p = 1/2$

Proof of Theorem 4.1. — First observe that an easy computation gives $\lambda_p = 1-p$. By Theorem 2.1 it suffices to show that for $\beta < \lambda_p/2$, the system (2.2) has no non-constant positive solution. Suppose on the contrary that (a, b, c) is a non-constant positive solution normalized by

$$a^2 + (2-2p)b^2 + c^2 = 4 - 2p. \tag{4.1}$$

The system (2.2) is equivalent to (using the function u defined at (2.4))

$$\frac{2\beta}{1-p} u(a) = a - b \tag{4.2}$$

$$4\beta u(b) = 2b - a - c \tag{4.3}$$

$$\frac{2\beta}{1-p} u(c) = c - b. \tag{4.4}$$

Assume first that $a \neq c$. By symmetry, we can then assume that $a > c$. Subtract (4.4) from (4.2) to obtain

$$u(a) - u(c) = \frac{1-p}{2\beta}(a-c) > a - c.$$

By (2.7), it follows that $a + c > 2$. This implies

$$a^2 + c^2 > 2 \tag{4.5}$$

and thus, by (4.1),

$$b < 1. \tag{4.6}$$

Now, add (4.2) divided by a to (4.4) divided by c and subtract (4.3) divided by b to obtain

$$\frac{2\beta}{1-p}\log ac - 4\beta \log b = \frac{a}{b} - \frac{b}{a} + \frac{c}{b} - \frac{b}{c}.$$

Rearranging the terms yields

$$\frac{4p\beta}{1-p}\log b = \left(\frac{a}{b} - \frac{b}{a} - \frac{2\beta}{1-p}\log\frac{a}{b}\right) - \left(\frac{b}{c} - \frac{c}{b} - \frac{2\beta}{1-p}\log\frac{b}{c}\right). \tag{4.7}$$

Consider the function $h(t) = t - t^{-1} - k\log t$ on $(0,\infty)$ and note that $h'(t) = t^{-2}(t-1)^2 + t^{-1}(2-k)$ is positive on $(0,\infty)$ if $k < 2$. In the present case, we take $k = 2\beta/(1-p)$ which, by hypothesis, is less than 1. Hence h is increasing. The left-hand side of (4.7) is negative since $b < 1$ by (4.6). Hence $h(a/b) - h(b/c) < 0$ and thus $a/b < b/c$ or, equivalently,

$$ac < b^2 < 1.$$

By (4.1) and (4.5), we have

$$\begin{aligned} 4 - 2p &= a^2 + 2(1-p)b^2 + c^2 > a^2 + 2(1-p)ac + c^2 \\ &= (a+c)^2 - 2pac > 4 - 2pac > 4 - 2p, \end{aligned}$$

a contradiction (in the case $a \neq c$).

Second, consider the case $a = c$. Rewrite (4.7) as

$$g(b) = \frac{4p\beta}{1-p}\log b + 2h(b/a) = 0$$

where h is as above with $k = \frac{2\beta}{1-p}$. Note that g is strictly increasing on $(0,\infty)$. If $a = c = 1$ then (4.1) contradicts the fact that (a,b,c) is not constant. If $a \in (0,1)$, then $g(1) = 2h(1/a) > 2h(1) = 0 = g(b)$ and thus

$b < 1$. This implies that a, b, c are all less than 1 which contradicts (4.1). Finally, if $a \in (1, \infty)$ then $g(1) = 2h(1/a) < 2h(1) = 0 = g(b)$ and thus $b > 1$. This implies that a, b, c are all larger than 1 which, again, contradicts (4.1).

It follows that (2.2) has no non-constant solutions for $\beta < \lambda_p/2$ and thus $\alpha_p = \lambda_p/2 = (1-p)/2$. □

Proof of Theorem 4.2. — We first consider the identity $\alpha_p = \lambda_p/2$. Referring to the family of chains in Theorem 4.2, the facts that $\alpha_p = \lambda_p/2$ when $p = 0$ and $p = 1/2$ are contained respectively in Theorem 4.1 and in Theorem 3.2. To prove $\alpha_p < \lambda_p/2$ when $p \neq 0, 1/2$, we use the criteria contained in Theorem 1.1. A simple computation yields

$$\lambda_p = 1 - \frac{p - 1 + \sqrt{1+p^2}}{2}$$

with eigenfunction

$$\phi = \left(1, \frac{p-1+\sqrt{1+p^2}}{2}, (p-1)(p+\sqrt{1+p^2})\right).$$

Thus, we compute

$$\mu_p(\phi^3) = \frac{p(1-p)(p-1/2)[3 - 3p + 6p^2 - 4p^3 + \sqrt{1+p^2}(-1+6p-4p^2)]}{4-3p}.$$

Notice $p \mapsto 3 - 3p + 6p^2 - 4p^3$ is decreasing and thus no smaller than 2 on $[0,1]$. We have $-1 + 6p - 4p^2 = -(2p - 3/2)^2 + 5/4$. It follows that $\sqrt{1+p^2}(-1+6p-4p^2) \geqslant 0$ on $[(3-\sqrt{5})/4, 1]$ and greater than $-\sqrt{2}$ on $[0, (3-\sqrt{5})/4]$. Combining these observations, we see that

$$3 - 3p + 6p^2 - 4p^3 + \sqrt{1+p^2}(-1+6p-4p^2) > 0 \ \text{ on } (0,1).$$

Hence $\mu_p(\phi^3) \neq 0$ unless $p = 0$ or $p = 1/2$. By Theorem 1.1, we must have $\alpha_p < \lambda_p/2$ for $p \neq 0, 1/2$.

We prove the monotonicity of α_p in two steps. The first step is to show that α_p is non-increasing in $(0,1)$. For $p \in (0,1)$, let $\mathcal{E}_p$ be the Dirichlet form associated to K_p and $\mathcal{L}_p$ be the quantity defined in (1.3) with respect to μ_p. For any non-constant vector $\psi = (a,b,c) \in [0,\infty)^3$, we compute

$$\frac{\mathcal{E}_p(\psi,\psi)}{\mathcal{L}_p(\psi)} = \frac{(a-b)^2 + (b-c)^2}{a^2 \log a^2 + 2b^2 \log 2b^2 + h(p)}$$

where

$$h(p) = \frac{c^2 \log c^2}{1-p} - \frac{4-3p}{1-p} n(p) \log n(p)$$

and $n(p) = \mu_p(\psi^2) = \frac{(1-p)a^2+(2-2p)b^2+c^2}{4-3p}$. As

$$h'(p) = \frac{n(p)}{(1-p)^2} \left(\frac{c^2}{n(p)} \log \frac{c^2}{n(p)} - \frac{c^2}{n(p)} + 1 \right) \geqslant 0, \tag{4.8}$$

it follows that $p \mapsto \alpha_p$ is non-increasing as desired.

The second step is to show the strict monotonicity. This is equivalent to the existence, for each $q \in (0,1/2) \cup (1/2,1)$, of a positive number ϵ such that $\alpha_p < \alpha_q$ for $q < p < q+\epsilon$. We fix $q \in (0,1/2) \cup (1/2,1)$ and choose, by Theorem 2.1, a positive non-constant vector $\psi = (a,b,c)$ such that

$$\begin{cases} b = a - 2\alpha_q a \log a \\ \frac{a+c}{2} = b - 2\alpha_q b \log b \\ b = c - \frac{2\alpha_q}{1-q} c \log c \\ (1-q)(a^2+2b^2) + c^2 = 4-3q \end{cases}$$

Obviously, the assumption that ψ is non-constant implies $c \neq 1$. Let h be the function defined above. By (4.8), we have $h'(q) = \frac{c^2 \log c^2 - c^2 + 1}{(1-q)^2} > 0$ and, hence, we can choose $\epsilon > 0$ such that

$$\forall q < p < q+\epsilon, \quad \alpha_q = \frac{\mathcal{E}_q(\psi,\psi)}{\mathcal{L}_q(\psi)} > \frac{\mathcal{E}_p(\psi,\psi)}{\mathcal{L}_p(\psi)} \geqslant \alpha_p. \qquad \square$$

Proof of Theorem 4.3. — The proof of whether the identity $\alpha_p = \lambda_p/2$ holds contains two steps. First, we prove that $\alpha_p < \lambda_p/2$ if $p \in (0,3/4)$. Second, we show that $\alpha_p = \lambda_p = 1/2$ when $p \in [3/4,1]$. For the first step, we use the following lemma whose straightforward proof is omitted.

LEMMA 4.4. — *Let μ be a probability measure on a finite set. If $f = 1+\epsilon g$ with $\mu(g) = 0$ and $\|g\|_2 \leqslant 1$ then*

$$\mathcal{L}(f) = \mu\left(|f|^2 \log \frac{|f|^2}{\mu(|f|^2)} \right) = 2\|g\|_2^2 \epsilon^2 + \frac{2}{3}\mu(g^3)\epsilon^3 - \left(\frac{1}{2}\|g\|_2^4 + \frac{1}{6}\|g\|_4^4 \right) \epsilon^4 + O(\epsilon^5)$$

where O is uniform overall such functions g.

The chain in Theorem 4.3 has eigenvalues $1, 0, -p$ with associated eigenvectors $\psi_0 \equiv 1$, $\psi_1 = (1,0,-1)$, $\psi_2 = (1,-p,1)$. The vectors ψ_1, ψ_2 are not normalized and $\|\psi_1\|_2^2 = p/(1+p)$, $\|\psi_2\|_2^2 = p$. Set $f = 1 + \epsilon g$ with $\epsilon g = x\psi_1 + y\psi_2$,

$$\epsilon = \|x\psi_1 + y\psi_2\|_2 = \sqrt{\frac{x^2 p}{1+p} + y^2 p}.$$

Observe that

$$\mathcal{E}_p(f,f) = \frac{x^2 p}{1+p} + y^2 p(1+p).$$

Now, Lemma 4.4 yields

$$\begin{aligned}
\mathcal{L}_p(f) &= 2\left(\frac{x^2 p}{1+p} + y^2 p\right) + \frac{2p}{3(1+p)}(3x^2 y + (1-p^2)y^3) \\
&\quad -\frac{1}{2}\left(\frac{x^2 p}{1+p} + y^2 p\right)^2 - \frac{p}{6(1+p)}(x^4 + 6x^2y^2 + (1+p^3)y^4) \\
&\quad +O\left(\left(\frac{x^2 p}{1+p} + y^2 p\right)^{5/2}\right) \\
&= 2\mathcal{E}_p(f,f) - 2p^2y^2 + \frac{2px^2y}{1+p} - \left(\frac{p^2}{2(1+p)^2} + \frac{p}{6(1+p)}\right)x^4 \\
&\quad +\frac{2p(1-p)y^3}{3} - x^2y^2p - \left(\frac{p^2}{2} + \frac{p(1+p^3)}{6(1+p)}\right)y^4 \\
&\quad +O\left(\left(\frac{x^2 p}{1+p} + y^2 p\right)^{5/2}\right) \\
&= 2\mathcal{E}_p(f,f) - 2\left(py - \frac{x^2}{2(1+p)}\right)^2 + \frac{(3-4p)x^4}{6(1+p)} + \frac{2p(1-p)y^3}{3} \\
&\quad -x^2y^2p - \left(\frac{p^2}{2} + \frac{p(1+p^3)}{6(1+p)}\right)y^4 + O\left(\left(\frac{x^2 p}{1+p} + y^2 p\right)^{5/2}\right)
\end{aligned}$$

Letting $y = \frac{x^2}{2p(1+p)}$ gives

$$\mathcal{L}_p(f) = 2\mathcal{E}_p(f,f) + \frac{(3-4p)x^4}{6(1+p)} + O(x^5).$$

Hence, for each $p \in (0, 3/4)$, we may choose x so small that

$$\mathcal{L}_p(f) > 2\mathcal{E}_p(f,f).$$

This implies $\alpha_p \leqslant \mathcal{E}_p(f,f)/\mathcal{L}_p(f) < 1/2 = \lambda_p/2$.

The monotonicity of α_p is proved as in the case of Theorem 4.2. Namely, we let $\mathcal{E}_p$ be the Dirichlet form associated to K_p and $\mathcal{L}_p$ be the quantity defined in (1.3) with respect to μ_p. For any non-constant and non-negative vector $\psi = (a, b, c)$, we have

$$\frac{\mathcal{E}_p(\psi, \psi)}{\mathcal{L}_p(\psi)} = \frac{(a-b)^2 + (b-c)^2}{a^2 \log a^2 + c^2 \log c^2 + h(p)},$$

where $h : (0, 1) \to (0, \infty)$ is defined by

$$h(p) = \frac{2}{p} b^2 \log b^2 - \frac{2+2p}{p} n(p) \log n(p),$$

and $n(p) = \mu_p(\psi^2) = \frac{p(a^2+c^2)+2b^2}{2+2p}$. We have

$$h'(p) = -\frac{2n(p)}{p^2} \left(\frac{b^2}{n(p)} \log \frac{b^2}{n(p)} - \frac{b^2}{n(p)} + 1 \right) \leqslant 0. \tag{4.9}$$

This implies that, for fixed ψ, $\mathcal{E}_p(\psi, \psi)/\mathcal{L}_p(\psi)$ is non-decreasing in p and, by the definition of α_p, so is α_p.

To prove strict monotonicity on $(0, 3/4)$ we show that for each $q \in (0, 3/4)$, there exists $\epsilon > 0$ such that $\alpha_p < \alpha_q$ when $q - \epsilon < p < q$. To see this, let $\psi = (a, b, c)$ be the non-constant vector solving the Euler-Lagrange equations (2.1), that is,

$$\begin{cases} 2\alpha_q a \log a = a - b \\ 2\alpha_q c \log c = c - b \\ 2\alpha_q b \log b = q(b - (a+c)/2) \\ q(a^2 + c^2) + 2b^2 = 2(1+q). \end{cases}$$

By Lemma 2.6, if $b = 1$, then $a = c = 1$, which contradicts the assumption that ψ is non-constant. This implies that $b \neq 1$ and hence, by (4.9), $h'(q) = -\frac{2}{q^2}(b^2 \log b^2 - b^2 + 1) < 0$. Therefore, we may choose $\epsilon > 0$ such that

$$\forall q - \epsilon < p < q, \quad \alpha_q = \frac{\mathcal{E}_q(\psi, \psi)}{\mathcal{L}_q(\psi)} > \frac{\mathcal{E}_p(\psi, \psi)}{\mathcal{L}_p(\psi)} \geqslant \alpha_p. \qquad \square$$

The second step in the proof of Theorem 4.3 consists in proving Lemma 4.5 and 4.6 below.

Lemma 4.5. — *Let K_p, μ_p be as in* Theorem 4.3 *and assume that $p \in [3/4, 1]$. Then, for any $\beta \leqslant 1/2$, the corresponding equation* (2.2) *has no non-constant positive solutions.*

Proof. — We let $\psi = (a, b, c)$ be a potential positive non-constant solution of (2.2) normalized by $\|\psi\|_2 = 1$. This means that (a, b, c) solves

$$\begin{cases} 2\beta a \log a = a - b \\ 2\beta c \log c = c - b \\ 2\beta b \log b = p(b - (a+c)/2) \\ p(a^2 + c^2) + 2b^2 = 2(1+p). \end{cases} \tag{4.10}$$

By symmetry, we can assume that $a \geqslant c$. Our first step is to show that we cannot have $a = c$. Indeed, assume that $a = c$. Then

$$2\beta \log a = 1 - b/a, \quad 2\beta \log b = p(1 - a/b).$$

Subtracting the second equation from the first yields $w(b/a, \beta, p) = 0$ with

$$w(t, \beta, p) = 2\beta \log t - t + pt^{-1} + 1 - p.$$

As $w(1, \beta, p) = 0$ and

$$\frac{\partial w}{\partial t}(t, \beta, p) = 2\beta t^{-1} - 1 - pt^{-2} = -(\beta t^{-1} - 1)^2 + (\beta^2 - p)t^{-2}$$

is negative if $\beta \in (0, 1/2]$ and $p \in (1/4, 1]$, we conclude that we must have $a = b$. This contradicts the hypothesis that $\psi = (a, b, c)$ is not constant.

From now on we thus assume that $a > c$ and set $s = c/a \in (0, 1)$. From the first and second equations in (4.10), we deduce

$$\frac{b}{c} = 2\beta \frac{\log s}{s - 1}.$$

From the second and third equations in (4.10), we obtain

$$\frac{b}{c} - 2\beta \log \frac{b}{c} + (p - 1) - \frac{p}{2}\left(1 + \frac{1}{s}\right)\frac{c}{b} = 0.$$

Using these two equations to eliminate b/c we find that $s = c/a$ is solution of

$$V(s, \beta, p) = 0$$

where

$$V(t, \beta, p) = -2\beta \log(2\beta) + (p - 1) + 2\beta\left(\frac{\log t}{t - 1} - \log \frac{\log t}{t - 1}\right) + \frac{p(t^{-1} - t)}{4\beta \log t} \tag{4.11}$$

for $t \in (0, 1)$ and, by continuity,

$$V(1, \beta, p) = 2\beta(1 - \log 2\beta) - \frac{p}{2\beta} + p - 1.$$

Lemma 4.6 below shows that for $p \in [3/4, 1]$ and $\beta \in (0, 1/2]$ the equation

$$V(s, \beta, p) = 0$$

has no solutions in $(0, 1)$. This finishes the proof of Lemma 4.5. $\square$

LEMMA 4.6. — *The function $V : (0, \infty)^3 \to \mathbb{R}$ defined above has the following properties.*

(i) $V(t, \beta, p) = V(1/t, \beta, p)$.

(ii) For fixed $t, p \in (0, \infty)^2$, the function $\beta \mapsto V(t, \beta, p)$ is increasing on the interval $(0, 1/2]$.

(iii) For $(t, \beta, p) \in (0, \infty) \times (0, 1/2] \times [3/4, 1]$, $V(t, \beta, p) \leqslant 0$ with equality only if $t = 1$ and $\beta = 1/2$.

(iv) For $(t, p) \in (0, \infty) \times (0, 3/4)$ and $\beta \in \left(0, \sqrt{p/3}\right]$, $V(t, \beta, p) < 0$.

Proof. — Part (i) is obvious by a direct computation. Using (i), it suffices to prove (ii) for $t \in (0, 1]$. The case $t = 1$ is clear. For $t \in (0, 1)$ we have

$$\frac{\partial V}{\partial \beta}(t, \beta, p) = -2\log(2\beta) + 2\left(\frac{\log t}{t-1} - \log\frac{\log t}{t-1} - 1\right) + \frac{p(t - t^{-1})}{4\beta^2 \log t}.$$

For $(t, \beta, p) \in (0, 1) \times (0, 1/2] \times (0, \infty)$, the first and last term are clearly nonnegative. The middle term is positive because the function $t \mapsto (t - 1)^{-1}\log t$ is decreasing on $(0, 1)$ with value in $(1, \infty)$ and the function $t \mapsto t - \log t$ is increasing on $(1, \infty)$.

Part (iii) and (iv) are more difficult. A simple computation shows that for $(\beta, p) \in (0, 1/2] \times (0, \infty)$ we have

$$V(1, \beta, p) \leqslant 0$$

with equality only if $\beta = 1/2$. Observe that for any $t \in (0, \infty)$, the function $p \mapsto V(t, 1/2, p)$ is decreasing. Thus, using (i) and (ii), it suffices to prove (iii) and (iv) simultaneously with $p \in (0, 3/4]$, $\beta = \sqrt{p/3}$ and $t \in (0, 1]$. That is, it remains to show that, for fixed $p \in (0, 3/4]$,

$$\begin{aligned} V(t) = V(t, \sqrt{p/3}, p) \;&=\; -\sqrt{p/3}\log(4p/3) + p - 1 \\ &\quad + \sqrt{\frac{4p}{3}}\left(\frac{\log t}{t-1} - \log\frac{\log t}{t-1} + \frac{3(t^{-1} - t)}{8\log t}\right) \end{aligned}$$

satisfies $V(t) < 0$ on $(0,1)$. As $V(1) \leqslant 0$ with equality only if $p = 3/4$, it suffices to show that $V'(t) > 0$ on $(0,1)$. We compute

$$\begin{aligned}\sqrt{\frac{3}{4p}}V'(t) &= \left(1-\frac{t-1}{\log t}\right)\left(\frac{t-1-t\log t}{t(t-1)^2}\right) \\ &\qquad +\frac{3}{8}\left(\frac{t^2-1-(t^2+1)\log t}{(t\log t)^2}\right) \\ &= \frac{(1-t)^3}{(t\log t)^2}(W_1(t)+W_2(t))\end{aligned}$$

with

$$\begin{aligned}W_1(t) &= (1-t)^{-3}\left(\frac{\log t}{t-1}-1\right)\left(1-t\frac{\log t}{t-1}\right)(t\log t) \\ W_2(t) &= \frac{3}{8}(1-t)^{-3}\left(t^2-1-(t^2+1)\log t\right).\end{aligned}$$

To finish the proof of (iii) and (iv), we will show that all the coefficients of the Taylor series of $W_1 + W_2$ at $t = 1$ are nonnegative. Namely, observe that $W_1(t) + W_2(t) = \sum_0^\infty c_k(1-t)^k$ where the series converges for each $t \in (0,1)$ and $c_k = d_k + e_k + f_k$ with

$$\begin{aligned}d_k &= \sum_{\substack{cn,m\geqslant 0\\ n+m=k}} \frac{-1}{(n+2)(m+1)(m+2)} \\ e_k &= \sum_{\substack{cn,m,\ell\geqslant 0\\ n+m+\ell=k-1}} \frac{1}{(n+2)(m+1)(m+2)(\ell+1)(\ell+2)} \\ f_k &= \frac{3[(k+1)(k+2)+2]}{8(k+1)(k+2)(k+3)}.\end{aligned}$$

Note that for $k \geqslant 0$,

$$d_{k+1}-d_k = \sum_{m=1}^{k}\frac{1}{(k-m+3)(k-m+2)(m+1)(m+2)} \geqslant 0.$$

This implies

$$e_k = \sum_{\ell=0}^{k-1}\frac{-d_{k-\ell-1}}{(\ell+1)(\ell+2)} \geqslant -\frac{k}{k+1}d_k.$$

A simple computation shows that $((k+1)f_k)_{k\geqslant 0}$ is an increasing sequence and, hence,

$$f_k + \frac{d_k}{k+1} = \frac{(k+1)f_k+d_k}{k+1} \geqslant \frac{1}{k+1}(f_0+d_0) = 0.$$

Therefore, $c_k \geqslant 0$ for $k \geqslant 0$ as desired. □

The following corollary is another useful application of Lemma 2.6 and 4.6 in bounding α_p.

COROLLARY 4.7. — *For $p \in (0,1]$, let K_p be the Markov kernel in Theorem 4.3 with the associated logarithmic Sobolev constant α_p. Then α_p satisfies*

$$\left(\sqrt{\frac{p}{3}} \vee c_p\right) \leqslant \alpha_p \leqslant \left(\frac{p-1}{\log p} \wedge \frac{1}{2}\right),$$

where $s \wedge t = \min\{s,t\}$, $s \vee t = \max\{s,t\}$ and, for $0 < p \leqslant 1$, c_p is the unique zero of the identity $2x = pe^{(1-p)/(2x)}$ with $x > 0$.

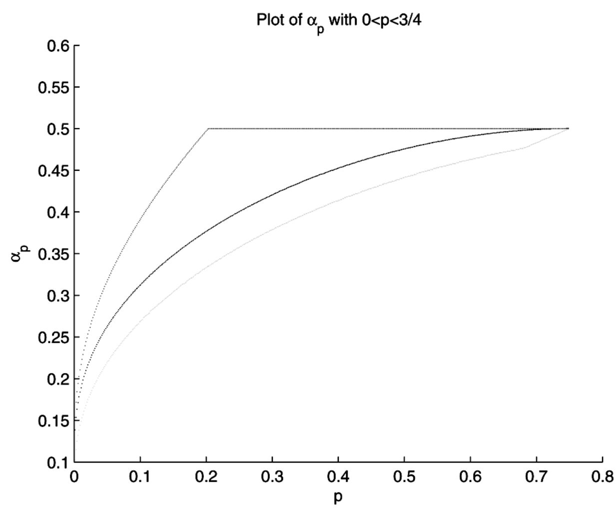


Figure 9. — These curves display in order from above the upper bound, α_p and the lower bound in Corollary 4.7. The curve for α_p shows a numerical approximation

Proof of Corollary 4.7. — By Theorem 4.3, both inequalities are obviously true for $p \in [3/4, 1]$. In the case $p \in (0, 3/4)$, the upper bound of α_p is immediately from Theorem 1.1, Theorem 1.4 and Corollary 2.4, where Theorem 1.4 uses the projection map $p(1) = p(3) = 0$ and $p(2) = 1$.

For the lower bound of α_p, since $\alpha_p < \lambda_p/2 = 1/2$ for $p \in (0, 3/4)$, there exists a positive non-constant vector $\psi = (a, b, c)$ solving the Euler-Lagrange equations in (2.1), that is,

$$\begin{cases} b = a - 2\alpha_p a \log a = c - 2\alpha_p c \log c \\ (a+c)/2 = b - (2\alpha_p/p) b \log b \\ p(a^2+c^2) + 2b^2 = 2(1+p). \end{cases} \tag{4.12}$$

On one hand, Lemma 4.6 (iv) implies that $\alpha_p > \sqrt{p/3}$, otherwise $a = b = c$. On the other hand, in the notation of Lemma 2.6 with $\beta = 2\alpha_p$, it is clear that $a, c \in \{t_1(b), t_2(b)\}$. Observe that $t_1(b) \neq 1$. Otherwise, the third identity in (4.12) implies $a = b = c = 1$. Hence, $t_1(b) < 1$. As a consequence of Lemma 2.6, if $a \neq c$, then $a + c = t_1(b) + t_2(b) \geqslant 2e^{1/(2\alpha_p)-1}$. If $a = c$, we must have $a = t_2(b)$, since the assumption $a = t_1(b) < 1$ and the fact $2\alpha_p \leqslant 1$ give $b < 1$, which contradicts the third identity of (4.12). In this case, we have

$$a + c = 2t_2(b) > t_1(b) + t_2(b) \geqslant 2e^{1/(2\alpha_p)-1}.$$

Note that, for $\beta > 0$, the map $t \mapsto t - \beta t \log t$ with domain $(0, \infty)$ has its maximum $\beta e^{1/\beta - 1}$. This gives

$$b - (2\alpha_p/p) b \log b \leqslant (2\alpha_p/p) e^{p/(2\alpha_p)-1}.$$

Using the second identity of (4.12) and the above two inequalities, we get $2\alpha_p \geqslant pe^{(1-p)/2\alpha_p}$. Hence, $c_p \leqslant \alpha_p$ since the map $t \mapsto t - pe^{(1-p)/t}$ is increasing for $t > 0$. This proves the lower bound. □

Next, we study one of the most natural chain on a 3-point stick where transitions are to the left with probability $q = 1 - p$ and to the right with probability p. For $p \neq q$, we show that $\alpha_p < \lambda_p/2$ and compute α_p. The case $p \neq 1/2$ and the asymmetric two-point space are the only cases in this paper where $\alpha_p < \lambda_p/2$ and we are able to compute α_p. Indeed, the proof of the following theorem is rather miraculous as it uses a crude comparison technique to achieve an exact computation!

Theorem 4.8. — *Fix $0 < p < 1$ and $q = 1 - p$. Let $K_p : \{1,2,3\} \times \{1,2,3\} \mapsto [0,1]$ be the Markov kernel defined by*

$$K_p = \begin{pmatrix} q & p & 0 \\ q & 0 & p \\ 0 & q & p \end{pmatrix}$$

with stationary distribution

$$\mu_p = \left(c_p, c_p(p/q), c_p(p/q)^2\right), \quad c_p = \left(1 + (p/q) + (p/q)^2\right)^{-1}.$$

Then $\lambda_p = 1 - \sqrt{pq}$ *and*

$$\alpha_p = \frac{p-q}{2(\log p - \log q)}$$

with minimizer

$$\psi = (p/q, 1, q/p).$$

Proof. — We compare this chain with another 3-point chain,

$$\widetilde{K}_p = \begin{pmatrix} q & p & 0 \\ q/2 & 1/2 & p/2 \\ 0 & q & p \end{pmatrix} \tag{4.13}$$

whose stationary distribution is

$$\widetilde{\mu}_p = \left(\widetilde{c}_p, 2\widetilde{c}_p(p/q), \widetilde{c}_p(p/q)^2\right), \quad \widetilde{c}_p = \left(1 + 2(p/q) + (p/q)^2\right)^{-1}.$$

The Dirichlet form associated with $(\widetilde{K}_p, \widetilde{\mu}_p)$ is

$$\widetilde{\mathcal{E}}_p(u,u) = \widetilde{c}_p p\left((u_1-u_2)^2 + (p/q)(u_2-u_3)^2\right).$$

The Dirichlet form associated with (K_p, μ_p) is

$$\mathcal{E}_p(u,u) = c_p p\left((u_1-u_2)^2 + (p/q)(u_2-u_3)^2\right).$$

Hence

$$\widetilde{\mathcal{E}}_p = (\widetilde{c}_p/c_p)\mathcal{E}_p \ \text{ and } \ (\widetilde{c}_p/c_p)\mu_p \leqslant \widetilde{\mu}_p.$$

By a classical comparison technique (see, e.g., [13, Lemma 3.4]), it follows that

$$\alpha_p \geqslant \widetilde{\alpha}_p. \tag{4.14}$$

Next, on $\{0,1\}^2$, we consider the product chain (with weights $(1/2,1/2)$) of two copies of 2-point asymmetric chains in Theorem 2.2. In details, transitions in this product chain have probability 0 except

$$K((0,0),(0,0)) = q, K((1,1),(1,1)) = p, K((0,0),(0,1)) = K((0,0),(1,0)) = p/2,$$
$$K((1,0),(1,1)) = K((0,1),(1,1)) = p/2, K((1,1),(0,1)) = K((1,1),(1,0)) = q/2,$$

and

$$K((0,1),(0,0)) = K((1,0),(0,0)) = q/2,$$

$$K((0,1),(0,1)) = K((1,0),(1,0)) = 1/2.$$

By Theorem 1.3 and Theorem 2.2, its logarithmic Sobolev constant is

$$\frac{p-q}{2\log(p/q)}.$$

This chain projects to the 3-point space $\{1,2,3\}$ using the mapping

$$p : \{0,1\}^2 \to \{1,2,3\}, \quad (x,y) \mapsto 1 + |x| + |y|$$

and the projected chain is the chain $\widetilde{K}_p$ considered above. Hence, by Theorem 1.4,

$$\widetilde{\alpha}_p \geqslant \frac{p-q}{2(\log p - \log q)}.$$

Therefore, by (4.14),

$$\alpha_p \geqslant \frac{p-q}{2(\log p - \log q)}.$$

To show that this is in fact an equality, it suffices to find a good test function. Let $\psi = (p/q, 1, q/p)$. Then

$$\frac{\mathcal{E}_p(\psi,\psi)}{\mathcal{L}_p(\psi)} = \frac{p-q}{2(\log p - \log q)}.$$

Thus,

$$\frac{p-q}{2(\log p - \log q)} \leqslant \alpha_p \leqslant \frac{p-q}{2(\log p - \log q)},$$

proving Theorem 4.8. □

Remark 4.9. — Note that the proof of Theorem 4.8 also determines the logarithmic Sobolev constant of the Markov kernel $\widetilde{K}_p$ defined in (4.13), which is $\widetilde{\alpha}_p = \frac{p-q}{2\log(p/q)}$.

5. Some 4-point chains

Theorem 1.4 is a useful technique to study the logarithmic Sobolev constant. Markov chains in Figure 8 and Theorem 4.8 are typically examples. This section concentrates on some 4-point chains. Most of the results use Theorem 1.4 and the computation done for 3-stick chains in the previous section. Table 2 lists all results of 4-point chains discussed in this paper.

THEOREM 5.1. — *Let* $p, q \in (0, 1]$ *and* $K_{p,q} : \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \mapsto [0, 1]$ *be a Markov kernel defined by*

$$K_{p,q} = \begin{pmatrix} 1-p & p/2 & 0 & p/2 \\ q/2 & 1-q & q/2 & 0 \\ 0 & p/2 & 1-p & p/2 \\ q/2 & 0 & q/2 & 1-q \end{pmatrix}$$

with stationary distribution

$$\mu_{p,q} = \left(\frac{q}{2(p+q)}, \frac{p}{2(p+q)}, \frac{q}{2(p+q)}, \frac{p}{2(p+q)} \right).$$

Then $2\alpha_{p,q} = \lambda_{p,q} = \min\{p, q\}$.

Remark 5.2. — Theorem 4.1, 4.3 and 5.1 illustrate the fact that α and λ behave differently under collapse. Observe that if $K_{p,1}$ is the Markov kernel defined in Theorem 5.1, then collapsing states 2 and 4 gives K_{1-p} from Theorem 4.1 whereas collapsing states 1 and 3 gives K_p from Theorem 4.3. As a consequence of Theorem 4.1 and 4.3, the identity $\alpha = \lambda/2$ is preserved for all $0 < p \leqslant 1$ in the former collapse but not for all p in the latter case. The main reason is that the eigenvector of $K_{p,1}$ associated to the spectral gap is $f = (f(1), f(2), f(3), f(4)) = (1, 0, -1, 0)$. Hence, the collapse of 2 and 4 preserves the spectral gap and, consequently by Theorem 1.4, the identity $\alpha = \lambda/2$. However, the collapse of 1 and 3 enlarges the spectral gap from p to 1. In this case, the only contribution of Theorem 1.4 is to provide a lower bound on the logarithmic Sobolev constant for the collapsed chain.

THEOREM 5.3. — *For* $p \in [0, 1)$, *let* K_p *be a Markov chain on the set* $\{1, 2, 3, 4\}$ *defined by*

$$K_p = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1-p}{2} & p & \frac{1-p}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

with stationary distribution

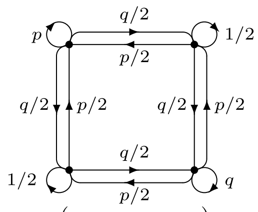
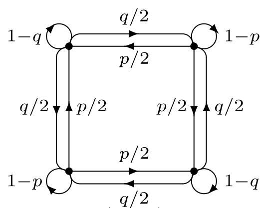
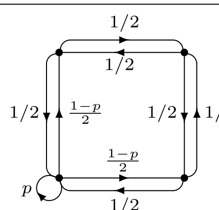
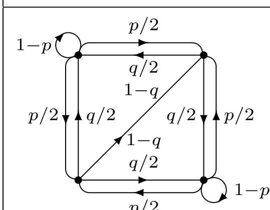
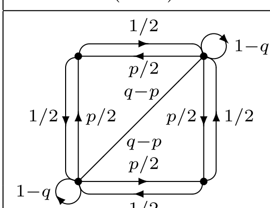
$$\mu_p = \left(\frac{1-p}{4-3p}, \frac{1-p}{4-3p}, \frac{1}{4-3p}, \frac{1-p}{4-3p} \right).$$

Then the spectral gap is equal to

$$\lambda_p = 1 - \frac{p - 1 + \sqrt{p^2 + 1}}{2},$$

and the logarithmic Sobolev constant satisfies $\alpha_p = \lambda_p/2$ *if and only if* $p \in \{0, 1/2\}$.

Table 2. — The spectral gap λ and the log-Sobolev constant α of some Markov chains on the 4 point space

Markov chain	λ	α	$\lambda = 2\alpha$
 $(p+q=1, 0<p<1)$	$\frac{1}{2}$	$\frac{p-q}{2\log(p/q)}$	$p = q = \frac{1}{2}$
 $(pq>0)$	$\min\{p,q\}$	$\frac{\min\{p,q\}}{2}$	$pq > 0$
 $(0 \le p < 1)$	$\frac{3-p-\sqrt{p^2+1}}{2}$	unknown except for $p = 0$ or $p = 1/2$	$p \in \{0, 1/2\}$
 $(pq>0)$	p	$p\alpha_{q/p}$	$p/q \le 4/3$
 $(0<p\le 1,\ p\le q\le 1)$	$\min\{2q-p, 1\}$	$\min\left\{\alpha_p, \frac{\lambda}{2}\right\}$	$D_1 \cup D_2$
	$D_1 = \{(p,q) : 0 \le p < 3/4,\ p \le q \le \alpha_p + p/2\}$ $D_2 = \{(p,q) : 3/4 \le p,\ p \le q \le 1\}$		
α_p is the log-Sobolev constant of K_p defined in Theorem 14			

THEOREM 5.4. — *For $p, q \in (0, 1]$, consider the Markov kernel $K_{p,q}$ defined by*

$$K_{p,q} = \begin{pmatrix} 1-p & p/2 & 0 & p/2 \\ q/2 & 0 & q/2 & 1-q \\ 0 & p/2 & 1-p & p/2 \\ q/2 & 1-q & q/2 & 0 \end{pmatrix}$$

with stationary distribution

$$\mu_{p,q} = \left(\frac{q}{2(p+q)}, \frac{p}{2(p+q)}, \frac{q}{2(p+q)}, \frac{p}{2(p+q)} \right).$$

Then $\lambda_{p,q} = p$ and $\alpha_{p,q} = \lambda_{p,q}/2$ if and only if $p/q \leqslant 4/3$. In particular, if $p/q > 4/3$, then $\alpha_{p,q} = p\alpha_{q/p}$, where α_r is the log-Sobolev constant of the Markov kernel K_r in Theorem 4.3*, that is,*

$$K_r = \begin{pmatrix} 0 & 1 & 0 \\ r/2 & 1-r & r/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

THEOREM 5.5. — *For $0 < p \leqslant q \leqslant 1$, let $K_{p,q}$ be a Markov kernel defined by*

$$K_{p,q} = \begin{pmatrix} 1-q & p/2 & q-p & p/2 \\ 1/2 & 0 & 1/2 & 0 \\ q-p & p/2 & 1-q & p/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

with stationary distribution $\mu_{p,q} = \frac{1}{2(1+p)}(1, p, 1, p)$. Then $\lambda_{p,q} = \min\{2q - p, 1\}$ and $\alpha_{p,q} = \min\{\alpha_p, \lambda_{p,q}/2\}$, where α_p is the log-Sobolev constant of K_p in Theorem 4.3*.*

Furthermore, $\alpha_{p,q} = \lambda_{p,q}/2$ if and only if $(p, q) \in D_1 \cup D_2$, where

$$D_1 = \{(p, q) : 0 \leqslant p < 3/4,\ p \leqslant q \leqslant \alpha_p + p/2\},$$

and

$$D_2 = \{(p, q) : 3/4 \leqslant p \leqslant 1,\ p \leqslant q \leqslant 1\}.$$

Proof of Theorem 5.1. — Due to the symmetry of $K_{p,q}$, we may assume that $p \leqslant q$. It is easy to see that $K_{p,q}$ is reversible and has eigenvalues $\{1, 1-p, 1-q, 1-p-q\}$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} p \\ -q \\ p \\ -q \end{pmatrix}. \tag{5.1}$$

This implies that $\lambda_{p,q} = p$.

For the identity $\alpha_{p,q} = \lambda_{p,q}/2$, we will prove, by contradiction, the stronger result that when $\alpha_{p,q} \leqslant \lambda_{p,q}/2$, there is no positive non-constant solution for the Euler-Lagrange equation

$$(I - K_{p,q})\psi = 2\alpha_{p,q}\psi \log(\psi/\|\psi\|_2). \tag{5.2}$$

Assume the converse, that is, $\psi = (a, b, c, d)$ is a positive non-constant solution for this system. Equivalently,

$$\begin{cases} b + d = 2v(2\alpha_{p,q}/p, a) = 2v(2\alpha_{p,q}/p, c) \\ a + c = 2v(2\alpha_{p,q}/q, b) = 2v(2\alpha_{p,q}/q, d) \\ q(a^2 + c^2) + p(b^2 + d^2) = 2(p + q) \end{cases}$$

where $v(\beta, t) = t - \beta t \log t$. We claim that $a = c$ and $b = d$. To show this, observe first that the normalizing equation and the convexity of the map $t \mapsto t^2$ imply that $a + c < 2$ or $b + d < 2$. When $a + c < 2$, the first two equations imply that

$$\frac{b + d}{2} = \frac{v(2\alpha_{p,q}/p, a) + v(2\alpha_{p,q}/p, c)}{2} \leqslant v\left(\frac{2\alpha_{p,q}}{p}, \frac{a + c}{2}\right) < 1,$$

where the first inequality applies the concavity of the function $v(2\alpha_{p,q}/p, \cdot)$ and the second inequality uses the fact $2\alpha_{p,q}/p \leqslant 1$. Similarly, we have $a + c < 2$ when $b + d < 2$. Whatever, it is always the case $a + c < 2$ and $b + d < 2$ and, as a consequence of Lemma 2.6, we must have $a = c$ and $b = d$.

The above observation implies that $K_{p,q}$ and the Markov chain in Corollary 2.4 with $q_1 = p$ and $q_2 = q$ have the same logarithmic Sobolev constant, that is,

$$\alpha_{p,q} = \begin{cases} \frac{p-q}{\log p - \log q} & \textit{if } p < q \\ p & \textit{if } p = q \end{cases}$$

This, however, contradicts Theorem 1.1, which says $\alpha_{p,q} \leqslant \lambda_{p,q}/2$, since

$$\frac{s - t}{\log s - \log t} > s, \quad \forall\, 0 < s < t.$$

Therefore, no non-constant vector attains $\alpha_{p,q}$ and, by Theorem 2.1, we must have $\alpha_{p,q} = \lambda_{p,q}/2$. □

Proof of Theorem 5.3. — Note that K_p is reversible and has eigenvalues $\{1, 0, \frac{1}{2}(p-1 \pm \sqrt{p^2+1})\}$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} p^{-1}(1-p \pm \sqrt{p^2+1}) \\ 1 \\ p^{-1}(p-1)(1+p \pm \sqrt{p^2+1}) \\ 1 \end{pmatrix} \quad \forall p \neq 0.$$

Hence, the spectral gap is equal to $1 - \frac{1}{2}(p-1+\sqrt{p^2+1})$ and the associated eigenvector is

$$\psi = \begin{pmatrix} p^{-1}(1-p+\sqrt{p^2+1}) \\ 1 \\ p^{-1}(p-1)(1+p+\sqrt{p^2+1}) \\ 1 \end{pmatrix}.$$

To see whether $\alpha_p = \lambda_p/2$, observe first that the case $p = 0$ is the simple random walk on the 4-cycle and the desired identity is known (see the introduction and [10]). For $p = 1/2$, we let K be the Markov kernel of the simple random walk on the 5-cycle with spectral gap λ and logarithmic Sobolev constant α. As a consequence of Theorem 3.1, one has

$$\lambda_{1/2} = \lambda = 2\alpha.$$

It is a simple exercise to show that $\mathcal{E}_{K_{1/2}}(f,f) = \mathcal{E}_K(g_f, g_f)$ and $\mathcal{L}_{\mu_{1/2}}(f) = \mathcal{L}_U(g_f)$ for all $f = (a,b,c,d)$ and $g_f = (a,b,c,c,d)$, where U is the uniform probability measure on the 5-cycle and $\mathcal{L}_{\mu_{1/2}}, \mathcal{L}_U$ are quantities defined in (1.3) with respect to $\mu_{1/2}$ and U. This implies $\alpha \leqslant \alpha_{1/2}$, and hence $\alpha_{1/2} = \lambda_{1/2}/2$.

It remains to consider the case $p \notin \{0, 1/2\}$. Note that, for $p \in [0,1)$, the chain given in Theorem 4.2 is collapsed from K_p through the projection map $p : \{1,2,3,4\} \mapsto \{1,2,3\}$ defined by

$$p(1) = 1, \quad p(2) = p(4) = 2, \quad p(3) = 3.$$

Since $\psi(2) = \psi(4)$, both Markov kernels have the same spectral gap and, by Theorem 1.4 and 4.2, we have $\alpha_p < \lambda_p/2$ if $p \notin \{0, 1/2\}$. $\square$

Proof of Theorem 5.4. — It is an easy exercise to show that $K_{p,q}$ is reversible and has eigenvalues $\{1, 1-p, q-1, 1-p-q\}$ with eigenvectors in (5.1), and hence $\lambda_{p,q} = p$.

For $p/q \leqslant 4/3$, we prove $\alpha_{p,q} = \lambda_{p,q}/2$ by contradiction. Assume the inverse $\alpha_{p,q} < \lambda_{p,q}/2$ and let, by Theorem 2.1, $\psi = (a,b,c,d)$ be a positive non-constant solution of the following equations.

$$\begin{cases} b+d = 2v(2\alpha_{p,q}/p, a) = 2v(2\alpha_{p,q}/p, c) \\ b-(1-q)d - q(\frac{a+c}{2}) = 2\alpha_{p,q} b\log b \\ d-(1-q)b - q(\frac{a+c}{2}) = 2\alpha_{p,q} d\log d \\ q(a^2+c^2)+p(b^2+d^2) = 2(p+q) \end{cases} \tag{5.3}$$

where $v(\beta,t) = t-\beta t\log t$. Note that the last equation implies that $a+c<2$ or $b+d<2$ and, by the first two identities, we have

$$\frac{b+d}{2} = \frac{v(2\alpha_{p,q}/p,a)+v(2\alpha_{p,q}/p,c)}{2} \leqslant v\left(\frac{2\alpha_{p,q}}{p}, \frac{a+c}{2}\right) < 1$$

if $a+c<2$. Combining both cases, one always has $b+d<2$. Note also that the second and third identities in (5.3) implies that

$$v\left(\frac{2\alpha_{p,q}}{2-q}, b\right) = v\left(\frac{2\alpha_{p,q}}{2-q}, d\right) > 0.$$

Then, by Lemma 2.6, the fact $2\alpha_{p,q}/(2-q) \leqslant p/(2-q) \leqslant 1$ gives $b=d$.

Let $K_r, \widetilde{K}_r$ be the Markov kernels in Theorem 4.3 and 4.1 with associated Dirichlet forms $\mathcal{E}_r, \widetilde{\mathcal{E}}_r$ and let $\mathcal{L}_r, \widetilde{\mathcal{L}}_r$ be the quantities defined in (1.3) with respect to the stationary distributions of K_r and $\widetilde{K}_r$. Then the conclusion of the previous paragraph implies that

$$\forall p\leqslant q, \quad \alpha_{p,q} = q\times\inf\left\{\frac{\widetilde{\mathcal{E}}_{1-p/q}(f,f)}{\widetilde{\mathcal{L}}_{1-p/q}(f)} : \widetilde{\mathcal{L}}_{1-p/q}(f)>0\right\} = \frac{p}{2}$$

and

$$\forall q<p\leqslant 4q/3, \quad \alpha_{p,q} = p\times\inf\left\{\frac{\mathcal{E}_{q/p}(f,f)}{\mathcal{L}_{q/p}(f)} : \mathcal{L}_{q/p}(f)>0\right\} = \frac{p}{2}.$$

This contradicts the assumption $\alpha_{p,q} < \lambda_{p,q}/2 = p/2$ and, hence, we must have $\alpha_{p,q} = \lambda_{p,q}/2$.

For the case $p/q > 4/3$, Theorem 4.3 implies that

$$\alpha_{p,q} \leqslant p\times\inf\left\{\frac{\mathcal{E}_{q/p}(f,f)}{\mathcal{L}_{q/p}(f)} : \mathcal{L}_{q/p}(f)>0\right\} = p\alpha_{q/p} < \frac{p}{2} = \frac{\lambda_{p,q}}{2}.$$

Then, by Theorem 2.1, the Euler-Lagrange equations in (5.3) must has a positive non-constant solution, say $\psi = (a,b,c,d)$, and by the discussion after (5.3), we have $b=d$. This means that the first inequality above is an equality. □

Proof of Theorem 5.5. — Obviously, $K_{p,q}$ is reversible and has eigenvalues $\{1, 1+p-2q, 0, -p\}$ with associated eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} p \\ -1 \\ p \\ -1 \end{pmatrix}.$$

Thus, $\lambda_{p,q} = \min\{2q-p, 1\}$. Let α_p be the log-Sobolev constant associated to K_p as defined in Theorem 4.3. On one hand, if $\alpha_{p,q} = \lambda_{p,q}/2$, then

$$\alpha_{p,q} \leqslant \inf\left\{ \frac{\mathcal{E}(f,f)}{\mathcal{L}(f)} : f = (a,b,a,d) \right\} = \alpha_p.$$

This implies that $\alpha_{p,q} = \min\{\lambda_{p,q}/2, \alpha_p\}$. On the other hand, as a consequence of Theorem 2.1, if $\psi = (a,b,c,d)$ is a minimizer for $\alpha_{p,q}$, then ψ is positive and non-constant and its entries must satisfy

$$\begin{cases} a+c = 2v(2\alpha_{p,q}, b) = 2v(2\alpha_{p,q}, d) \\ qa - (q-p)c - p(\frac{b+d}{2}) = 2\alpha_{p,q} a \log a \\ qc - (q-p)a - p(\frac{b+d}{2}) = 2\alpha_{p,q} c \log c \\ a^2 + c^2 + p(b^2+d^2) = 2+2p \end{cases}$$

where $v(\beta, t) = t - \beta t \log t$. Note that the last equation implies that $a+c<2$ or $b+d<2$. By the concavity of $v(2\alpha_{p,q}, \cdot)$, if $b+d<2$, then

$$\frac{a+c}{2} = \frac{v(2\alpha_{p,q}, b) + v(2\alpha_{p,q}, d)}{2} \leqslant v\left(2\alpha_{p,q}, \frac{b+d}{2}\right) < 1,$$

where the last inequality also uses the fact $2\alpha_{p,q} \leqslant \lambda_{p,q} \leqslant 1$. Hence, we always have $a+c<2$. Observe that the second and the third equalities above imply

$$v\left(\frac{2\alpha_{p,q}}{2q-p}, a\right) = v\left(\frac{2\alpha_{p,q}}{2q-p}, c\right) > 0.$$

Then Lemma 2.6 and the fact $2\alpha_{p,q}/(2q-p) \leqslant 2\alpha_{p,q}/\lambda_{p,q} \leqslant 1$ give $a=c$. In this case, the log-Sobolev constant $\alpha_{p,q}$ is just equal to α_p. This finishes the proof of the first part.

The second part is easily proved by considering the two cases $p \in (0, 3/4)$ and $p \in [3/4, 1]$. We omit the details. □

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