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## A computation of invariants of a rational self-map

EKATERINA AMERIK<sup>(1)</sup>

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**ABSTRACT.** — I prove the algebraic stability and compute the dynamical degrees of C. Voisin’s rational self-map of the variety of lines on a cubic fourfold.

**RÉSUMÉ.** — Je démontre la stabilité algébrique et calcule les degrés dynamiques de l’auto-application rationnelle (construite par C. Voisin) de la variété des droites sur une cubique dans  $\mathbb{P}^5$ .

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Let  $V$  be a smooth cubic in  $\mathbb{P}^5$  and let  $X = \mathcal{F}(V)$  be the variety of lines on  $V$ . Thus  $X$  is a smooth four-dimensional subvariety of the grassmannian  $G(1, 5)$ , more precisely, the zero locus of a section of  $S^3U^*$ , where  $U$  is the tautological rank-two bundle over  $G(1, 5)$ . It is immediate from this description that the canonical class of  $X$  is trivial. Let  $\mathcal{F} \subset V \times X$  be the universal family of lines on  $V$ , and let  $p : \mathcal{F} \rightarrow V$ ,  $q : \mathcal{F} \rightarrow X$  be the projections. Beauville and Donagi prove in [BD] that the Abel-Jacobi map  $AJ : q_*p^* : H^4(V, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$  is an isomorphism, at least after tensoring up with  $\mathbb{Q}$ ; since this is also a morphism of Hodge structures, we obtain, by the Noether-Lefschetz theorem for  $V$ , that  $Pic(X) = \mathbb{Z}$  for a sufficiently general  $X$ .

Lines on cubics have been studied in [CG]. It is shown there that the normal bundle of a general line on a smooth 4-dimensional cubic is  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$  ( $l$  is then called “a line of the first kind”), and that some special

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lines (“lines of the second kind”) have normal bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ . The lines of the second kind form a two-parameter subfamily  $S \subset X$ , and all lines on  $V$  are either of the first or of the second kind.

An alternative description of lines on a smooth cubic in  $\mathbb{P}^n$  from [CG] is as follows: let  $F(x_0, \dots, x_n) = 0$  define a smooth cubic  $V$  in  $\mathbb{P}^n$  and consider the Gauss map

$$D_V : \mathbb{P}^n \rightarrow (\mathbb{P}^n)^* : x \mapsto \left( \frac{\partial F}{\partial x_0}(x) : \dots : \frac{\partial F}{\partial x_n}(x) \right)$$

(so  $D_V(x)$  is the tangent hyperplane to  $V$  at  $x$ ). For a line  $l \subset V$  there are only two possibilities: either  $D_V|_l$  maps  $l$  bijectively onto a plane conic, or  $D_V|_l$  is two-to-one onto a line in  $(\mathbb{P}^n)^*$ . In the first case,  $l$  is of the first kind and the intersection of the hyperplanes  $\cap_{x \in l} D_V(x) \subset \mathbb{P}^n$  is a subspace of dimension  $n - 3$ , tangent to  $V$  along  $l$ ; in the other case,  $l$  is of the second kind and the linear subspace  $Q_l = \cap_{x \in l} D_V(x) \subset \mathbb{P}^n$ , which is of course still tangent to  $V$  along  $l$ , is of greater dimension  $n - 2$ .

As remarked in [V],  $X$  always admits a rational self-map. This map is constructed as follows: from the description of the normal bundle, one sees that for a general line  $l \subset V$ , there is a unique plane  $P_l \subset \mathbb{P}^5$  which is tangent to  $V$  along  $l$ . So  $P_l \cap V$  is the union of  $l$  and another line  $l'$ , and one defines  $f : X \rightarrow X$  by sending  $l$  to  $l'$ . This is only a rational map: indeed, if  $l$  is of the second kind, we have a one-parameter family, more precisely, a pencil of planes  $\{P : l \subset P \subset Q_l\}$  tangent to  $V$  along  $l$ . So the surface  $S$  is the indeterminacy locus of the map  $f$ . Blowing  $S$  up, one obtains a resolution of singularities of  $f$ ; it introduces the exceptional divisor  $E$  which is a  $\mathbb{P}^1$ -bundle over  $S$ , each  $\mathbb{P}^1$  corresponding to the pencil of planes tangent to  $V$  along  $l$ . That is, the “image” by  $f$  of every exceptional point is a (possibly singular) rational curve on  $X$ .

In [V], it is shown that the degree of  $f$  is 16. The proof is very short: the space  $h^{2,0}(X)$  is generated by a nowhere-degenerate form  $\sigma$  ([BD]); C.Voisin then uses a Mumford-style argument (saying that a family of rationally equivalent cycles induces the zero map on certain differential forms) to show that  $f^*\sigma = -2\sigma$ . As  $(\sigma\bar{\sigma})^2$  is a volume form, the result follows.

In holomorphic dynamics, one considers the *dynamical degrees* of a rational self-map (the definition below seems to appear first in [RS]). Those are defined as follows: let  $X$  be a Kähler manifold of dimension  $k$ . Fix a Kähler class  $\omega$ . Define

$$\delta_l(f, \omega) = [f^*\omega^l] \cdot [\omega^{k-l}]$$

(the product of cohomology classes on  $X$ ), and the  $l$ th *dynamical degree*

$$\lambda_l(f) = \limsup(\delta_l(f^n, \omega))^{1/n}$$

(this does not depend on  $\omega$ ). It turns out that this is the same as

$$\rho_l(f) = \limsup(r_l(f^n))^{1/n},$$

where  $r_l(f^n)$  is the spectral radius of the action of  $f^n$  on  $H^{l,l}(X)$ . If  $X$  is projective, one can restrict oneself to the subspace of algebraic classes, replacing  $\rho_l$  and  $r_l$  by the correspondent spectral radii  $r_l^{alg}$  and  $\rho_l^{alg}$ . Moreover, Dinh and Sibony [DS] recently proved that the lim sup in the definition of  $\lambda_l(f)$  is actually a limit, and also the invariance of the dynamical degrees by birational conjugation.

The map  $f$  is said to be *cohomologically hyperbolic* if there is a dynamical degree which is strictly greater than all the others. In fact the dynamical degrees satisfy a convexity property which says that if  $\lambda_l(f)$  is such a maximal degree, then  $\lambda_i$  grows together with  $i$  until  $i$  reaches  $l$ , and decreases thereafter. It is conjectured that the cohomological hyperbolicity implies certain important equidistribution properties for  $f$ . The case when the maximal dynamical degree is the topological degree  $\lambda_k$  is studied by Guedj in [G] (building on the work of Briend-Duval [BrDv]); see [DS2] for a proof in a more general situation.

In general it is very difficult to compute the dynamical degrees of a rational self-map, because for rational maps it is not always true that  $(f^n)^* = (f^*)^n$  (where  $f^*$  denotes the transformation induced by  $f$  in the cohomology.). In fact few examples of such computations are known.

If  $(f^n)^* = (f^*)^n$ , the map  $f$  is called *algebraically stable*. The computation of its dynamical degrees is then much easier: one only has to study the linear map  $f^*$ , and it is not necessary to look at the iterations of  $f$ .

The purpose of this note is to show that the example  $f$  considered in the beginning is algebraically stable and to compute its dynamical degrees. The map  $f$  turns out to be cohomologically hyperbolic, and the dominating dynamical degree is in this case  $\lambda_2$ .

I am very much indebted to A. Kuznetsov and C. Voisin for crucially helpful discussions.

## 1. Algebraic stability of the self-map

From now on let  $X = \mathcal{F}(V)$  and  $f$  be as described in the beginning. Let  $S$  be the indeterminacy locus  $I(f)$ , that is, the surface of lines of the second kind.

We assume that  $V$  is *sufficiently general*. The first goal is to prove that  $f$  is algebraically stable.

LEMMA 1. — *For a general  $V$ , the surface  $S$  is smooth.*

*Proof.* — Let  $T$  be the parameter space of cubics in  $\mathbb{P}^5$ . Consider the incidence variety  $\mathcal{I} \subset G(1, 5) \times T$ :  $\mathcal{I} = \{(l, V) : l \text{ is of the 2nd kind on } V\}$ . It is enough to show that  $\mathcal{I}$  is smooth in codimension two. By homogeneity, it is enough to verify that the fiber  $I_l = pr_1^{-1}l$  is smooth in codimension two for some  $l \in G(1, 5)$ . Choose coordinates  $(x_0 : \dots : x_5)$  on  $\mathbb{P}^5$  so that  $l$  is given by  $x_2 = x_3 = x_4 = x_5 = 0$ ; then  $l \subset V$  means that four coefficients (those of  $x_0^3, x_0^2x_1, x_0x_1^2$  and  $x_1^3$ ) of the equation  $F = 0$  of  $V$  vanish, and  $l$  being of the second type means that the 4 quadratic forms  $\partial F / \partial x_i|_l, i = 2, 3, 4, 5$  in two variables  $x_0, x_1$ , span a linear space of dimension 2, rather than 3. This in turn means that a  $4 \times 3$  matrix  $M$ , whose coefficients are (different) coordinates on  $T' \subset T$ , the projective space of cubics containing  $l$ , has rang 2. It follows from general facts about determinantal varieties that  $Sing(I_l)$  is the locus where  $rg(M) \leq 1$ , and it has codimension four in  $I_l$  (in fact,  $Sing(I_l)$  is also contained in the locus of singular cubics, [CG]).  $\square$

Let us now look at the map  $f : X \rightarrow X$ . The first thing to observe is as follows:  $K_X = 0$  means that  $f$  cannot contract a divisor. Indeed, let  $\pi : Y \rightarrow X, g : Y \rightarrow X$  be the resolution of the indeterminacies of  $f$ . Then the ramification divisor (i.e. the vanishing locus of the Jacobian determinant) of  $\pi$  is equal to that of  $g$ , so  $g$  can only contract a divisor which is already  $\pi$ -exceptional. By the same token, any subset contracted by  $g$  must lie in the exceptional divisor of  $\pi$ , in other words, anything contracted by  $f$  is in  $S$ .

LEMMA 2. — *The map  $g$  does not contract a surface (neither to a point nor to a curve), unless possibly some surface already contracted by  $\pi$ .*

*Proof (C. Voisin).* — If it does, this surface  $Z$  projects onto  $S$ . Recall that  $X$  is holomorphic symplectic, with symplectic form  $\sigma = AJ(\eta)$ , where  $\eta$  generates  $H^{3,1}$  of the cubic. Notice that  $S$  is not lagrangian:  $\sigma|_S \neq 0$ . This is because, as we shall see in the next section, the cohomology class

of  $S$  is  $5(H^2 - \Delta)$ , where  $H$  is the hyperplane section class for the Plücker embedding, and  $\Delta$  is the Schubert cycle of lines contained in a hyperplane. We have  $[\sigma] \cdot \Delta = 0$  in the cohomologies, by projection formula (because  $[\eta]$  is a primitive cohomology class). But  $[\sigma] \cdot H^2$  is non-zero, because  $[\sigma\bar{\sigma}] \cdot H^2$  is the Bogomolov square of an ample  $H$ . Thus so is  $\sigma|_S$ .

Now recall ( $[V]$ ) that  $f^*\sigma = -2\sigma$ , that is,  $g^*\sigma = -2\pi^*\sigma$ . This is a contradiction, since  $g^*\sigma|_Z = 0$ , whereas  $\pi^*\sigma|_Z \neq 0$ .  $\square$

*Remark.* — The same argument with  $\sigma$  shows that  $g$  cannot contract the  $\pi$ -exceptional divisor (even not to a surface). A somewhat more elaborate version of it shows that  $g$  does not contract any surface at all; but we won't need this to prove the algebraic stability. See also the remark after Theorem 3.

**THEOREM 3.** — *The self-map  $f$  of  $X = \mathcal{F}(V)$  described above is algebraically stable, that is, we have  $(f^n)^* = (f^*)^n$ , where the upper star denotes the action on the cohomologies, and  $n$  is a positive integer.*

*Proof.* — The transformation induced by  $f$  on the cohomologies is given by the class of the graph  $\Gamma(f) \in H^8(X \times X)$ . In general,  $(hf)^* = h^*f^*$  is implied by  $\Gamma(f) \circ \Gamma(h)$  having only one 4-dimensional component (which is then, of course,  $\Gamma(hf)$ ). Recall that the correspondence  $\Gamma(f) \circ \Gamma(h)$  is

$$\{(x, z) | \exists y : (x, y) \in \Gamma(f), (y, z) \in \Gamma(h)\}.$$

Let  $h = f^n$ . Suppose that  $\Gamma(f) \circ \Gamma(h)$  has an extra 4-dimensional component  $M$ . There are three possibilities for its image under the first projection:

1) It is a divisor  $D$ . Then  $f$  is defined at a general point  $d \in D$ ; but fibers of  $M$  over  $D$  must be at least one-dimensional. That is,  $D$  is contracted by  $f$  (to the indeterminacy locus of  $h$ ). This is impossible.

2) It is a surface  $Z$ ; fibers of  $M$  over  $Z$  must be two-dimensional. But if  $Z \not\subset I(f)$ , there is only one  $y$  corresponding to generic  $x \in Z$ , and furthermore, since  $Z$  is not contracted by  $f$ ,  $h(y)$  is at most a curve. Indeed, the locus  $I_2(h) = \{y : \dim(h(y)) \geq 2\}$  is of codimension at least three in  $X$ . Thus a fiber of  $M$  over a generic  $x \in Z$  cannot be two-dimensional. If  $Z \subset I(f)$ , one concludes in a similar way: for any  $x \in Z$ , the corresponding  $y$  are parametrized by a rational curve  $D_x$ ; for a general  $x$ , the curve  $D_x$  is not the indeterminacy locus of  $h$ , and, moreover, since no surface is contracted,  $D_x$  intersects  $I(h)$  in a finite number of points, all outside  $I_2(h)$ . Thus again the general fiber of  $\Gamma(f) \circ \Gamma(h)$  over  $T$  is at most one-dimensional.

3) It is a curve  $C$ . By a simple dimension count as in the previous case, we see that then  $C$  must be contracted by  $f$  to a point  $y$  such that  $\dim(h(y)) = 3$ . Recall that  $h = f^n$ . We may assume that  $n$  is the smallest number with the property  $\dim(h(y)) = 3$ . As for any  $x \in X$ ,  $f(x)$  is either a point or a rational curve,  $S$  must be contained in  $f^{n-1}(y)$  as a component. This implies that  $S$  is covered by rational curves. But, as we shall see in the next section,  $K_S$  is ample; so this case is also impossible.  $\square$

*Remark.* — In fact one can show that  $g$  is finite (and this obviously implies the algebraic stability). Indeed, by the argument of Lemma 2, if  $g(C)$  is a point, then  $\sigma|_S$  vanishes on  $\pi(C)$  and so  $\pi(C)$  is a component of a canonical divisor on  $S$ . On the other hand, all lines on  $V$  corresponding to the points of  $\pi(C)$  intersect the line corresponding to the point  $g(C)$ . This means that  $\pi(C)$  is contained in a hyperplane section of  $S$ . But it turns out (see the remark in the end of next section) that  $K_S$  is even ample than the hyperplane section class  $H_S$ . To get a contradiction, it remains to show that the vanishing locus of  $\sigma|_S$  is smooth and irreducible for generic  $V$ . C. Voisin indicates that this is not very difficult to calculate explicitly using the Griffiths' description of the primitive cohomology of hypersurfaces (here cubics) in terms of residues.

## 2. Some cohomology classes on $X$

Recall that  $X \subset G(1, 5)$  is the zero-set of a section of the bundle  $S^3U^*$  (of rang 4). We shall describe the action of  $f^*$  on those cohomology classes which are restrictions of classes on  $G(1, 5)$ .

So let  $H = c_1(U^*)|_X = [\mathcal{O}_{G(1,5)}(1)]|_X$ ;  $\Delta = c_2(U^*)|_X$  (so  $\Delta$  is the class of the subset of points  $x \in X$  such that the corresponding lines  $l_x \subset \mathbb{P}^5$  lie in a given hyperplane).

LEMMA 4. —  $H^4 = 108$ ;  $H^2\Delta = 45$ ;  $\Delta^2 = 27$ .

*Proof.* —  $[X] = c_4(S^3U^*)$  and

$$c_4(S^3U^*) = 9c_2(U^*)(2c_1^2(U^*) + c_2(U^*)).$$

Let  $\sigma_i = c_i(U^*)$ ; those are Schubert classes on  $G(1, 5)$ :  $\sigma_1 = \{\text{lines intersecting a given } \mathbb{P}^3\}$ ,  $\sigma_2 = \{\text{lines lying in a given hyperplane}\}$ .

Notice that  $c_1(U^*)$  is the hyperplane section class in the Plücker imbedding and restricts as such onto a “sub-grassmanian” of lines lying in a linear subspace. So

$$H^4 = 9\sigma_2\sigma_1^4(2\sigma_1^2 + \sigma_2) = 18\deg(G(1, 4)) + 9\deg(G(1, 3)) = 108.$$

The other equalities are proved similarly.  $\square$

For generic  $X$ , the class  $H$  generates  $H_{alg}^{1,1}$  over  $\mathbb{Z}$ , and  $H^3$  generates  $H_{alg}^{3,3}$  over  $\mathbb{Q}$ .

One can also show that  $H^2$  and  $\Delta$  generate  $H_{alg}^{1,1}$  over  $\mathbb{Q}$ ; we do not need this for our computation, but see the remark in the end of the last section.

The next step is to compute the indeterminacy surface  $S$ .

Let  $A$  denote the underlying vector space for  $\mathbb{P}^5 \supset V$ , and let  $L \subset A$  be the underlying vector space for a line  $l \subset V$ . Write the Gauss map associated to  $F$  defining  $V$  as  $D_F : A \rightarrow S^2 A^*$ . The condition  $l \subset V$  implies that this map descends to  $\phi : A/L \rightarrow S^2 L^*$ :

$$\begin{array}{ccc} A & \rightarrow & S^2 A^* \\ \downarrow & & \downarrow \\ A/L & \rightarrow & S^2 L^* \end{array}$$

For a general  $L$ , the map  $\phi$  is surjective with one-dimensional kernel; and it has two-dimensional kernel exactly when  $L$  is the underlying vector space of a line of the second kind. Now  $L$  is the fiber over a point  $x_l \in X$  of the restriction  $U_X$  of  $U$  to  $X$ ; so, globalizing and dualizing, we get the following resolution for  $S$ :

$$0 \rightarrow S^2 U_X \rightarrow Q_X^* \rightarrow M \rightarrow M \otimes \mathcal{O}_S \rightarrow 0,$$

where  $Q_X^*$  is the restriction to  $X$  of the universal quotient bundle over  $G(1, 5)$ , and  $M$  is a line bundle. In fact

$$[M] = \det(Q_X^*) - \det(S^2 U_X) = -H + 3H = 2H.$$

So one has a resolution for the ideal sheaf  $\mathcal{I}_S$ :

$$0 \rightarrow S^2 U_X(-2H) \rightarrow Q_X^*(-2H) \rightarrow \mathcal{I}_S \rightarrow 0.$$

The cohomology class  $[S]$  can be computed by the Thom-Porteous formula: a partial case of this formula identifies the cohomology class of the degeneracy locus  $D_{e-1}(\phi)$ , where a vector bundle morphism  $\phi : E \rightarrow F$  ( $rg(E) = e \leq rg(F) = f$ ) is not of maximal rang, as  $c_{f-e+1}(F - E)$ , where we put formally

$$c(F - E) = 1 + c_1(F - E) + c_2(F - E) + \dots = c(F)/c(E).$$

So

$$\begin{aligned} [S] &= c_2(Q_X^* - S^2U_X) = c_2(Q_X^*) - c_2(S^2U_X) + 2Hc_1(S^2U_X) = \\ &= H^2 - \Delta - 4\Delta - 2H^2 + 6H^2 = 5(H^2 - \Delta). \end{aligned}$$

One can also make a direct calculation using the equality

$$[S] = -c_2(\mathcal{O}_S) = c_2(\mathcal{I}_S)$$

(see e.g. [F], chapter 15.3). Later, we shall need its extension

$$i_*c_1(N_{S,X}) = i_*K_S = c_3(\mathcal{I}_S)$$

where  $i : S \rightarrow X$  is the usual embedding. From the resolution of the ideal sheaf, one computes

$$\begin{aligned} c_3(\mathcal{I}_S) &= c_3(Q_X^*(-2H)) - c_3(S^2U_X(-2H)) - c_2(\mathcal{I}_S)c_1(S^2U_X(-2H)) \\ &= 20H^3 - 27H\Delta. \end{aligned}$$

Thus we have obtained the following

PROPOSITION 5. — *The Chern classes of  $\mathcal{I}_S$  are:  $c_1(\mathcal{I}_S) = 0$ ;  $c_2(\mathcal{I}_S) = 5(H^2 - \Delta)$ ;  $c_3(\mathcal{I}_S) = 20H^3 - 27H\Delta = \frac{35}{4}H^3$ .*

The last equality is obtained by taking the intersection with  $H$  and using lemma 4.

*Remark.* — One can also remark that  $S$  being the degeneration locus of a map  $\phi : S^2U \rightarrow Q^*$ , its normal bundle is isomorphic to  $(Ker(\phi|_S))^* \otimes (Coker(\phi|_S))$ , so its determinant is  $c_1(Coker(\phi|_S)) - 2c_1(Ker(\phi|_S)) = c_1(Q^*|_S) - c_1(S^2U|_S) - c_1(Ker(\phi|_S)) = 2H_S - c_1(Ker(\phi|_S))$ , in particular it is at least as positive than  $2H_S$ . It is easy to see that the cohomology class of the intersection  $HS$  is  $\frac{35}{12}H^3$ ; this suggests that the canonical class of  $S$  is equal to  $3H_S$ , but I have not checked this (one probably can write an analogue of the Thom-Porteous formula, but I could not find a reference, and in our particular case the direct calculation is easy).

### 3. Computation of the inverse images

Let  $\pi : \tilde{X} \rightarrow X$  be the blow-up of  $S$ ; this resolves the singularities of  $f$ , that is,  $g = f \circ \pi : \tilde{X} \rightarrow X$  is holomorphic. Let  $E$  denote the exceptional

divisor of  $\pi$ . We have  $g^*H = a\pi^*H - bE$ ; the number  $a$  is, by algebraic stability, the first dynamical degree we are looking for. In fact the numbers  $a$  and  $b$  can be computed by a simple geometric argument; we begin by sketching this computation.

PROPOSITION 6. —  $a = 7$  and  $b = 3$ .

*Proof.* — Consider the two-dimensional linear sections  $V \cup \mathbb{P}^3$  of  $V$ ; they form a family of dimension  $\dim(G(3, 5)) = 8$ . A standard dimension count shows that in this family, there is a one-parameter subfamily of cones over a plane cubic (indeed, one computes that out of  $\infty^{19}$  cubics in  $\mathbb{P}^3$ ,  $\infty^{12}$  are such cones, so being a cone imposes only 7 conditions on a cubic surface in  $\mathbb{P}^3$ ). Furthermore, some (a finite number) of those plane cubics are degenerate, so on  $V$  we have a finite number of cones over a plane cubic with a double point (as it was already mentioned in the first section). Thus on  $X$ , we have a one-parameter family of plane cubic curves  $C_t$ .

Another way to see this is to remark that the lines passing through a point of  $V$  sweep out a surface which is a cone over the intersection of a cubic and a quadric in  $\mathbb{P}^3$  ([CG]). The quadric sometimes degenerates in the union of two planes, so the intersection becomes a union of two plane cubics. Moreover, the two plane cubics have three points in common (e.g. from the arithmetic genus count).

The resulting curves  $C_t$  on  $X$  are clearly  $f$ -invariant, and when smooth, map 4:1 to themselves (indeed, through a given point  $p$  of an elliptic curve  $C \in \mathbb{P}^2$ , there are 4 tangents to  $C$ , because the projection from  $p$  is a 2:1 map from  $C$  to  $\mathbb{P}^1$  and so has 4 ramification points). A smooth  $C_t$  has 3 points in common with  $S$ . Let  $C_0$  be singular; then it has at least one point  $x$  in common with  $S$ , and is the image by  $g$  of the exceptional  $E_x \cong \mathbb{P}^1 \subset \tilde{X}$  over this point;  $g$  is 1:1 from  $E_x$  to  $C_0$ .

It follows that  $E_x g^*H = 3$ , so  $b = 3$  (recall that  $E$  restricts to itself as  $\mathcal{O}_E(-1)$ , hence  $E_x E = -1$ ). Also,  $C_t H = C_t E = 3$  and  $C_t g^*H = 12$ , from where  $a = 7$ .  $\square$

However, I could not find such a simple way to compute the higher dynamical degrees. The following approach is suggested by A. Kuznetsov.

On  $\tilde{X}$ , one has a vector bundle  $P$  of rank three: its fiber over  $x_l$  is the underlying vector space of the plane tangent to  $V$  along  $l$  when  $l$  is of the first kind, and we take the obvious extension to the exceptional divisor. One has  $\pi^*U_X \subset P$ .

First of all, let us describe the quotient: let  $E$  denote the exceptional divisor of  $\pi$ . Then one can see that we obtain  $P$  as the following extension:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \pi^*U_X & \rightarrow & P & \rightarrow & M^*(E) & \rightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \pi^*U_X & \rightarrow & A \otimes \mathcal{O} & \rightarrow & \pi^*Q_X & \rightarrow & 0, \end{array}$$

where the map  $M^*(E) \rightarrow \pi^*Q_X$  is the dual of the pull-back of the map  $Q_X^* \rightarrow M \otimes \mathcal{I}_S$  from the resolution of  $S$  in the last section.

(Alternatively, one can compute the cokernel of the natural inclusion  $\pi^*U_X \rightarrow P$  using the exact sequences from the proof of the next proposition and the knowledge of  $c_1(g^*U_X^*) = 7\pi^*H - 3E$ ).

PROPOSITION 7. — *The bundle  $g^*U_X^*$  fits into an exact sequence*

$$0 \rightarrow (M^*)^{\otimes 2}(2E) \rightarrow P^* \rightarrow g^*U_X^* \rightarrow 0.$$

*Proof.* — Consider the projective bundle  $p : \mathbb{P}_{\tilde{X}}(P) \rightarrow \tilde{X}$ . It is equipped with a natural map  $\psi : \mathbb{P}_{\tilde{X}}(P) \rightarrow \mathbb{P}(A)$ , and the inverse image of  $\mathcal{O}_{\mathbb{P}(A)}(1)$  is the tautological line bundle  $\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1)$ , meaning that its direct image under the projection to  $\tilde{X}$  is  $P^*$ . Let  $h = c_1(\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1))$ . We have a natural inclusion  $\mathbb{P}_{\tilde{X}}(\pi^*U_X) \subset \mathbb{P}_{\tilde{X}}(P)$ . Denote by  $d$  the class of  $\mathbb{P}_{\tilde{X}}(\pi^*U_X)$  considered as a divisor on  $\mathbb{P}_{\tilde{X}}(P)$ . The subvariety of  $\mathbb{P}_{\tilde{X}}(P)$  swept out by the lines  $f(l)$  (where  $l$  is a fiber of  $\mathbb{P}_{\tilde{X}}(\pi^*U_X)$ ), is a divisor; let us denote it by  $\Sigma$ . We clearly have  $2d + [\Sigma] = 3h$ . Furthermore,  $\Sigma$  itself is a projective bundle over  $\tilde{X}$ , namely, it is the projectivization of  $g^*U_X$ , in the sense that the vector bundle  $g^*U_X^*$  is the direct image of  $\mathcal{O}_{\Sigma}(1) = \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1)|_{\Sigma}$  under the projection to  $\tilde{X}$ . On  $\mathbb{P}_{\tilde{X}}(P)$ , we have the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - [\Sigma]) \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1) \rightarrow \mathcal{O}_{\Sigma}(1) \rightarrow 0.$$

Also,  $h - [\Sigma] = 2(d - h)$ . So our proposition follows by taking the direct image of the above exact sequence, once we show that

$$\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - d) = p^*M(-E).$$

As for this last statement, it is clear that  $\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - d) = p^*F$  for some line bundle  $F$  on  $\tilde{X}$ , because it is trivial on the fibers. We deduce that  $F = M(-E)$  from the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - d) \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1) \rightarrow \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1)|_{\mathbb{P}_{\tilde{X}}(\pi^*U_X)} \rightarrow 0 :$$

indeed, it pushes down to

$$0 \rightarrow F \rightarrow P^* \rightarrow \pi^*U_X^* \rightarrow 0.$$

We are ready to prove the main result:

**THEOREM 8.** — *We have  $f^*H = 7H$ ,  $f^*H^2 = 4H^2 + 45\Delta$ ,  $f^*\Delta = 31\Delta$  and  $f^*H^3 = 28H^3$ . The dynamical degrees of  $f$  are therefore 7, 31, 28 and 16. In particular,  $f$  is cohomologically hyperbolic.*

*Proof.* — We have already seen (proposition 6) that  $f^*H = \pi_*g^*H = 7H$ . Notice that  $\pi_*(\pi^*HE) = 0$  and  $\pi_*E^2 = -S$  (this is because  $\mathcal{O}(E)$  restricts to  $E$  as  $\mathcal{O}_E(-1)$ , where we view  $E$  as a projective bundle over  $S$ ). This gives

$$f^*H^2 = 49H^2 - 9S = 49H^2 - 45(H^2 - \Delta) = 4H^2 + 45\Delta.$$

Further,  $g^*\Delta = c_2(g^*U_X)$ , and we can find the latter from the exact sequences describing  $g^*U_X$  and  $P$ . From

$$0 \rightarrow \pi^*U_X \rightarrow P \rightarrow M^*(E) \rightarrow 0,$$

we get  $c_2(P) = \pi^*\Delta - \pi^*H(-2\pi^*H + E) = \pi^*\Delta - \pi^*HE + 2\pi^*H^2$ , and from

$$0 \rightarrow g^*U_X \rightarrow P \rightarrow M^{\otimes 2}(-2E) \rightarrow 0,$$

$c_2(g^*U_X) = c_2(P) + (4\pi^*H - 2E)(7\pi^*H - 3E) = \pi^*\Delta + 30\pi^*H^2 + 6E^2 - 27\pi^*HE$ , that is,

$$f^*\Delta = \Delta + 30H^2 - 30(H^2 - \Delta) = 31\Delta.$$

Finally,

$$g^*H^3 = (7\pi^*H - 3E)^3 = 343\pi^*H^3 - 441\pi^*H^2E + 189\pi^*HE^2 - 27E^3.$$

We have  $\pi_*(\pi^*H^2E) = 0$  and

$$\pi_*(\pi^*HE^2) = -5H(H^2 - \Delta) = -\frac{35}{12}H^3.$$

Let us compute  $\pi_*E^3$ : this is  $\pi_*\xi^2$ , where  $\xi$  is the tautological class on  $E$  viewed as a projective bundle  $\mathbb{P}_S(N)$  over  $S$ . Let  $r$  be the projection of  $E$  to  $S$ , i.e. the restriction of  $\pi$  to  $E$ . We have  $\xi^2 + c_1(r^*N)\xi + c_2(r^*N) = 0$ , which yields  $\pi_*E^3 = -i_*c_1(N)$ ,  $i$  being the embedding of  $S$  into  $X$ . But the latter is just  $c_3(i_*\mathcal{O}_S)$  ([F]). We have computed this class in the last section:

$$\pi_*E^3 = 27H\Delta - 20H^3 = -\frac{35}{4}H^3.$$

Putting all this together, we get

$$f^*H^3 = 343H^3 - \frac{35}{4}(63 - 27)H^3 = 28H^3.$$

This finishes the proof of Theorem 8.  $\square$

*Remark 9.* — Notice that the eigenvectors of  $f^*$  on the invariant subspace generated by  $H^2$  and  $\Delta$  are orthogonal with respect to the intersection form. There are reasons for this; moreover, at least for  $X$  generic the map  $f^*$  on the whole cohomology group  $H^4(X)$  is self-adjoint with respect to the intersection form. One can see it as follows:  $X$  is a deformation of the punctual Hilbert scheme  $Hilb^2(S)$  of a  $K3$  surface [BD]; this implies a decomposition of  $H^4(X)$  into an orthogonal direct sum of Hodge substructures

$$H^4(X) = \langle H^2 \rangle \oplus (H \cdot H^2(X)^0) \oplus S^2(H^2(X)^0),$$

where  $H^2(X)^0$  denotes the orthogonal to  $H$  in  $H^2(X)$ . For  $X$  generic, the second summand is an irreducible Hodge structure and the third one is a sum of an irreducible one and a one-dimensional subspace: this follows from the fact that  $H^2(X)^0 = H^4(V)^{prim}$  (where  $V$  is the corresponding cubic fourfold and “prim” is primitive cohomologies) and a theorem of Deligne which says that the closure of the monodromy group (of a general Lefschetz pencil) acting on  $H^4(V)^{prim}$  is the full orthogonal group. Therefore the decomposition rewrites as

$$H^4(X) = \langle H^2, \Delta \rangle \oplus H \cdot H^2(X)^0 \oplus V,$$

where the last two summands are simple, and so  $f^*$  must be a homothety on each of them.

It also follows from this decomposition that on  $X$  generic, the space of algebraic cycles of codimension two is only two-dimensional, and that all lagrangian surfaces must have the cohomology class proportional to  $\Delta$ . In particular, since  $f^*$  and  $f_*$  must preserve the property of being lagrangian, this explains why  $\Delta$  is an eigenvector of  $f^*$  and  $f_*$ .

## Bibliography

- [BD] BEAUVILLE (A.), DONAGI (R.). — La variété des droites d’une hypersurface cubique de dimension 4, C. R. Acad. Sci. Paris Sér. I Math., 301, no. 14, p. 703-706 (1985).
- [BrDv] BRIEND (J.-Y.), DUVAL (J.). — Deux caractérisations de la mesure d’équilibre d’un endomorphisme de  $\mathbb{P}^k(\mathbb{C})$ , Publ. Math. Inst. Hautes études Sci. No. 93, p. 145-159 (2001).
- [CG] CLEMENS (H.), GRIFFITHS (Ph.). — The intermediate jacobian of the cubic threefold, Ann. Math. 95, p. 281-356 (1972).
- [DS] DINH (T. C.), SIBONY (N.). — Une borne supérieure pour l’entropie topologique d’une application rationnelle, Annals of Maths. 161, p. 1637-1644 (2005).
- [DS2] DINH (T. C.), SIBONY (N.). — Distribution des valeurs de transformations méromorphes et applications, Comment. Math. Helvetici 81, p. 221-258 (2006).

A computation of invariants of a rational self-map

- [F] FULTON (W.). — Intersection theory, Springer-Verlag, Berlin 1984.
- [G] GUEDJ (V.), Ergodic properties of rational mappings with large topological degree, *Ann. of Math. (2)* 161, no. 3, p. 1589-1607 (2005).
- [RS] RUSSAKOVSKII (A.), SCHIFFMAN (B.), Value distribution for sequences of rational mappings and complex dynamics, *Indiana Univ. Math. J.* 46 (1997), no. 3, 897-932.
- [V] VOISIN (C.). — Intrinsic pseudovolume forms and  $K$ -correspondences. The Fano Conference, p. 761-792, Univ. Torino, Turin, (2004).