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A p-adic approach to local analytic dynamics: analytic conjugacy of analytic maps tangent to the identity(*)

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Abstract. — In this note, we consider the question of local analytic equivalence of analytic functions which fix the origin and are tangent to the identity. All mappings and equivalences are considered in the non-archimedean context e.g. all norms can be considered p-adic norms. We show that any two mappings f and g which are formally equivalent are also analytically equivalent. We consider the related questions of roots and centralizers for analytic mappings. In this setting, anything which can be done formally can also be done analytically.

Résumé. — Nous considérons la question d'équivalence locale de fonctions analytiques qui fixent l'origine et sont tangentes à l'identité. Toutes les fonctions et équivalences sont dans le contexte nonarchimédien, c'est-à-dire que nous pouvons considérer les normes comme étant des normes p-adiques. Nous démontrons que deux fonctions f et g formellement équivalentes sont aussi équivalentes analytiquement. Nous considérons la question des racines et centraliseurs pour les fonctions analytiques. Dans ce contexte, tout ce qui peut être prouvé formellement peut aussi être prouvé analytiquement.

1. Introduction

The goal of this paper is to consider the local analytic equivalence of mappings f which are tangent to the identity, but whose convergence is with respect to a non-archimedean norm $|\cdot|$ (for example, a p-adic norm).

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Here and in what follows, we will refer to the "local analytic classification" as simply the "analytic classification." We discover the interesting fact that the non-archimedean case yields very simple invariants for this classification, as opposed to the substantially more difficult (both in construction and interpretation) invariants present in the theory for **C**. These results have been known for mappings with other multipliers for some time – we cite some of the relevant results in both the archimedean and non-archimedean settings.

The question of the local conjugacy classes of holomorphic mappings (analytic in $\mathbb{C}$) has a long history. In 1884, the first such results regarding equivalences were proven by Koenigs [13]. In particular, it was shown that given an holomorphic function $f(z) = az + O(z^2)$ defined in a neighborhood U of $\mathbf{C}$, where $|a| \neq 1$, then in a neighborhood $V \subseteq U$ of the origin, there is a conformal mapping $h(z) = z + O(z^2)$ such that $(h \circ f \circ h^{-1})(z) = az$. Thus, any such mapping can be linearized, and moreover, the linearizing biholomorphism h can be taken to be tangent to the identity.

Significantly later, the question of the linearization of holomorphic mappings f whose multiplier has norm 1 but is not a root of unity was settled in famous papers of Siegel [18], Bryuno [4] and Yoccoz [21], [22]. We focus on Siegel's result here: let the mapping f have the form $f(z) = \lambda z + \cdots$, where λ has norm one but is not a root of unity. Define the quantity $\Omega_\lambda(m) = \min_{1 \leqslant k \leqslant m} |\lambda^k - 1|$. Then, the map f is linearizable provided that there are constants $\beta > 1$ and $\gamma > 0$ so that $\Omega_\lambda(m) \geqslant \gamma m^{-\beta}$ for all $m \geqslant 2$. The idea of the proof is to estimate the coefficients in the formal power series conjugating f with its linear part. It is worth noting that Siegel's result does not characterize all such linearizable maps. Although all mappings f whose multipliers are not roots of unity are formally linearizable, there exist such maps which are not holomorphically linearizable.

However, if one instead considers mappings in the non-archimedean category, defined on fields of characteristic 0, similar results can be proven. The proof of Koenig's result can be modified easily in the non-archimedean setting. Moreover, it has been shown by Herman and Yoccoz [9] that the Siegel estimate holds for all p-adic numbers with norm 1 that are not roots of unity. Thus, in both of these settings, the formal classification and the analytic classification coincide.

Of course, one cannot naively expect linearization if $|a| = 1$; as an obvious (and important) example, if $a = 1$, then linearization is impossible for any function $f(z) \neq z$. Thus, an interesting problem is to determine the invariants present in such a classification of mappings tangent to the identity. It is easy to acquire formal invariants for this equivalence (we review this in Section 2.2). Nonetheless, the holomorphic classification remains very delicate; after initial attempts by Fatou [7] in the late 1910's to determine the invariants, the problem remained unsolved until the early 1980's, when Ecalle [6] and Voronin [20] independently developed the moduli space of invariants for such mappings (see also the work of Malgrange [14] and Il'yashenko [10] for a different approach). We note here that such holomorphic classification relied on a topological conjugacy, provided independently by Camacho [5] and Shcherbakov [16].

Strangely enough, the following problem is still very much open: given two mappings f and g, are these two mappings equivalent via an analytic map which is tangent to the identity? While the analytic classifications cited above provide the theoretical invariants present, as Ahern and Rosay note [1], they are too difficult to be computed for even the simplest of mappings. There are partial results present (for example, it is known which *entire* functions are analytically equivalent to $f(z) = \frac{z}{1-z}$, and which are equivalent to $f(z) = z + z^2$), but as a whole, the problem is poorly understood. It is easy to construct formal power series H conjugating f to g, but showing that any such power series converges (or diverges) is generally very difficult.

If we restrict ourselves to the case where the mappings f and g have *rational* coefficients, then we may often take the conjugating power series H to have rational coefficients. In this situation it is natural to study the *p-adic* convergence of H for a given prime p. This analyzes the largest power of p which divides the denominators of the coefficients of H. We may view the rational coefficients of H as sitting inside the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$ rather than the archimedean completion $\mathbb{R}$, and do our work there. This study complements the classical question of holomorphic convergence; it is an instance of what is popularly known as the Lefschetz principle, which roughly says that interesting questions for real and complex numbers should have interesting analogues in the p-adic setting. This principle has found application in harmonic analysis, algebraic number theory, and more recently in dynamical systems (see for example [3], [8], etc).

Indeed, for power series with coefficients in a complete, non-archimedean valued field, it becomes reasonable to test for the convergence of a given conjugating map H. The reason for this is two-fold: first, a series $\sum a_n$ converges with respect to a non-archimedean norm if and only if $a_n \to 0$ as $n \to \infty$. Second (in a sense to be made precise later), the convergence of a power series depends solely on the decay of denominators - growth in the numerator is not detrimental to convergence.

For any field K with norm $|\cdot|$, we denote the ring of absolutely convergent power series centered at 0 with coefficients in K as $\mathcal{O}_0^K$. In this paper, we consider fields K of characteristic 0 which are complete, non-archimedean valued fields, but most of our interest will be in the field of p-adic numbers $\mathbb{Q}_p$, any finite extension of $\mathbb{Q}_p$, and the analytic completion of any infinite algebraic extension of $\mathbb{Q}_p$.

The main result of this paper is the following:

Theorem 1.1. — *Let $f \in \mathcal{O}_0^K$ be an analytic function which is tangent to the identity, $f(x) = x + a_m x^m + \cdots$, with $a_m \neq 0$. Write $\widetilde{K} = K[\sqrt[m-1]{a_m}]$. Then, there are a $\mu \in \widetilde{K}$ and an analytic function $h \in \mathcal{O}_0^{\widetilde{K}}$ so that $(h \circ f \circ h^{-1})(x) = x + x^m + \mu x^{2m-1}$. Moreover, m and μ are analytic invariants for f.*

The algebraic technicality of adjoining a root is convenient, although it is unnecessary if K is algebraically closed. We will usually drop the tilde in practice – this should cause no confusion).

In other words, the formal and analytic classifications agree in the non-archimedean setting. This is in stark contrast to the analytic classification in $\mathbf{C}$. Theorem 1.1 was obtained in the integral case (i.e., for series whose coefficients have norm less than one) in the thesis of Dominique Vieugué [19], via techniques similar to our own (i.e. via formal power series estimation). However, the general case is not handled there. It is worth noting that, since analytic functions with respect to a non-archimedean norm are continuous, this shows that formal equivalence does indeed imply topological equivalence, which is consistent with the theory in $\mathbf{C}$.

With this result in place, one can then answer questions regarding roots and centralizers of analytic mappings tangent to the identity. Using known reults of Herman and Yoccoz [9], we can then prove the following:

Corollary 1.2. — *Let f be an analytic map tangent to the identity. Then, f admits analytic roots of all orders. The formal and analytic centralizers of f agree.*

The structure of the paper is as follows: Section 2 discusses basic results and notation for non-archimedean analysis and local dynamics. The short Section 3 is used to summarize basic results of analytic flows in the p-adic setting, as proven by Herman and Yoccoz [9]. Most of the proof of Theorem 1.1 is in Section 4. In this section we introduce and apply the useful "sigma function" $\sigma_m(n)$, which estimates rather well the growth of the denominators of the conjugating function. Finally, using the previous work, Section 5 finishes the proof of the Theorem 1.1, Corollary 1.2, and a few other tidbits regarding conjugating maps.

Theorem 1.1 provides a complete analytic classification of mappings which are tangent to the identity and convergent with respect to a non-archimedean norm. As mentioned, this shows that formally equivalent mappings are also topologically equivalent. In a future work, the authors plan to give a complete topological classification of analytic mappings tangent to the identity, and to study what smoothness conditions may be imposed on such a conjugating map. Recently, Jenkins [11] has given a full formal classification of so-called semi-hyperbolic mappings in $\mathbf{C}^n$, and has shown that the formal classification differs wildly from the holomorphic one. There are few algebraic restrictions on the formal classification; the techniques used there would work, for the most part, if the coefficients lay in a field of characteristic 0. It would be of interest to determine whether the formal and analytic classifications agree if one considers non-archimedean norms $|\cdot|$. Finally, we do not consider the case of fields with characteristic p; the methods used here will fail in that setting.

It is possible that the results here can be found along other lines, *à la* the theory of Ecalle and Voronin, providing a more "conceptual" proof. This approach does not seem to be in the literature however. We point out that the proofs here are elementary and self-contained, and our hope is that they will provide insight into the holomorphic case. In particular, we believe that such techniques can be modified in order to give precise examples of convergence or divergence of formal power series conjugating maps f and g which are holomorphic in $\mathbb{C}$ and tangent to the identity (for at least those formal series with rational coefficients).

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2. Preliminaries

This section is devoted to an explanation of the non-archimedean setting in which we work, together with some basic notions of formal dynamics. We also take the opportunity to fix some notation. In the introduction and throughout the paper, we have used the convention that if the norm is archimedean, then we will write sets in the bold style (i.e. $\mathbf{R}, \mathbf{C}$, etc.), whereas if the norm is non-archimedean (or if the field is the rational numbers), we will write in blackboard bold style (i.e. $\mathbb{Q}, \mathbb{Q}_p, \mathbb{C}_p$, etc.). Power series with coefficients in $\mathbf{C}$ will be denoted $f(z)$ as usual, while power series with coefficients in a non-archimedean field K will be denoted $f(x)$.

2.1. Non-archimedean Fields

The bulk of this paper involves not the complex numbers $\mathbb{C}$ but rather a non-archimedean complete (nontrivial) valued field K of characteristic 0. We give a survey of the pertinent facts. For proofs see [17] or [15].

Definition 2.1. — *Let K be a field. A non-archimedean valuation (or norm) on K is a map $|\cdot| : K \to \mathbb{R}$ satisfying the following rules, for all $x, y \in K$:*

1. $|x| \geqslant 0, |x| = 0$ *if and only if* $x = 0$.
2. $|x + y| \leqslant \max\{|x|, |y|\}$.
3. $|xy| = |x||y|$.

The pair $(K, |\cdot|)$ is a non-archimedean valued field.

We will simply write K when the valuation is implicit. Of course the usual absolute value in $\mathbb{C}$ does not satisfy the second condition. The constant valuation, $|x| = 1$ for all $x \neq 0$, is called trivial. We do not consider these.

Let $K = \mathbb{Q}$ and choose a prime $p \in \mathbb{Z}$ and a real number $0 < \alpha < 1$. Consider the map

$$\left|\frac{m}{n}\right|_{p,\alpha} = \alpha^{{}_p(m) - {}_p(n)}, \tag{2.1}$$

where ${}_p(n)$ is the exponent of p in the prime factorization of n. Then $|\cdot|_{p,\alpha}$ is a non-archimedean valuation on $\mathbb{Q}$.

The following is a well-known theorem of Ostrowski:

Proposition 2.2. — *Any nontrivial non-archimedean valuation on $\mathbb{Q}$ is of the form $|\cdot|_{p,\alpha}$ for some p and α as above.*

Given a valuation on a field K, there is a natural topology on K compatible with $|\cdot|$. We define it in the usual way with balls.

Definition 2.3. — *Given a positive number $r \in \mathbb{R}$, and $x \in K$, define $B_r(x) = \{y \in K : |x - y| \leqslant r\}$.*

Then we give K the topology generated by the basis $\{B_r(x) : r \in \mathbb{R}, x \in K\}$. For a given p, the topology of $(\mathbb{Q}, |\cdot|_{p,\alpha})$ does not depend on the choice of α.

Definition 2.4. — $\Delta = B_1(0)$.

Note that Δ is a subring of K by the definition of valuation; we will refer to Δ as the ring of integers of K.

A non-archimedean valued field K is considered complete if it is complete as a metric space. Recall that if $\widetilde{K}$ is a finite-degree field extension of a complete, non-archimedean valued field K, then the norm $|\cdot|$ on K extends uniquely to $\widetilde{K}$, and furthermore, this extension is complete. In particular, if α is any algebraic element over such a field K, then $K[\alpha]$ is a complete, non-archimedean valued field.

For example, $(\mathbb{Q}, |\cdot|_{p,\alpha})$ is not complete, being countable. In fact the completion of any such K will be a complete non-archimedean valued field. The completion of $(\mathbb{Q}, |\cdot|_{p,1/p})$ is called $\mathbb{Q}_p$. Note that $|p| = \frac{1}{p}$ in this case.

From now on we take K to be a non-archimedean complete valued field with characteristic 0. In this case $\mathbb{Q}$ is a subfield, and becomes a valued field by restriction of $|\cdot|$.

We record a simple lower estimate on $|n!|$ in this context.

Proposition 2.5. — *If the valuation of K restricts trivially to $\mathbb{Q}$ then $|n!| = 1$. Otherwise, $|n!| = |n!|_{p,\alpha} \geqslant \alpha^n$.*

Proof. — The only thing to prove is the last inequality. It is well-known that ${}_p(n!) = \frac{n-S_n}{p-1}$, where S_n is the sum of the digits of n in base p. Therefore ${}_p(n!) \leqslant n$, and the result follows. □

Since the valuation on K is nontrivial, there is an element $\pi \in K$ with $0 < |\pi| < 1$. Since $\mathbb{R}$ is an archimedean field, for every $\varepsilon > 0$ there is a $k \in \mathbf{N}$ so that if $q = \pi^k$, then $|q| < \varepsilon$.

Finally, we would like to point out that any algebraically closed field of characteristic 0 with the same cardinality as $\mathbb{R}$ is isomorphic as a field to $\mathbb{C}$, by transcendence theory. This applies, for example, to the algebraic closure $\overline{\mathbb{Q}}_p$ and its completion $\mathbb{C}_p$. This means that much of the formal algebraic theory of $\mathbb{C}$ applies to a general non-archimedean field K of characteristic 0.

Of course there is no reason to expect any topological relationship.

2.2. Power Series

We denote the ring of formal power series $K[[x]]$ as usual. If two formal power series $f, g \in K[[x]]$ have zero constant term, then they may be composed to yield another formal power series $f \circ g \in K[[x]]$ with zero constant term.

An interesting feature of non-archimedean analysis is the following: a series $\sum_n a_n$ converges if and only if a_n converge to 0.

Given a power series $f(x) = \sum_n a_n x^n \in K[[x]]$, its radius of convergence about 0 is given by

$$\rho = \left(\limsup_{n \to \infty} \sqrt[n]{|a_n|} \right)^{-1} \tag{2.2}$$

Definition 2.6. — *The power series $f(x) \in K[[x]]$ is called locally analytic at* 0 *if $\rho > 0$. The set of such functions is denoted $\mathcal{O}_0^K$.*

For example if $\gamma \in K$ with $|\gamma| = c$, and $a_n = \gamma^n$, then $\rho = \frac{1}{c}$. On the other hand, if $c > 1$ and $a_n = \gamma^{n^2}$, then $\rho = 0$. Therefore if $K = \mathbb{Q}_p$ the power series

$$f(x) = \sum_n \frac{1}{p^{n^2}} x^n \tag{2.3}$$

is not in $\mathcal{O}_0^K$.

As usual, if the linear term $a_1 \neq 0$, then f will be formally invertible, in the sense that there is a unique power series $g(x) \in K[[x]]$ with $(f \circ g)(x) = (g \circ f)(x) = x$. In particular, power series of the form $f(x) = x + O(x^2)$ are invertible. Moreover, an implicit function theorem implies that if f is locally analytic, then the formal inverse g is itself locally analytic.

Let $f(x) = x + a_m x^m + O(x^{m+1})$ be a power series in $K[|x|]$, where K is any field of characteristic 0. As mentioned above, we write $f \circ g$ to be the composition of f and g, while writing fg to mean the standard multiplicative, pointwise product. Furthermore, given $n \in \mathbf{Z}$, we write $f^{\circ n}$ to be the nth iterate of f, and write f^n to be the nth multiplicative power of f.

Given two such power series f and g, we say that f and g are equivalent (or conjugate) if there is an h satisfying $h \circ f \circ h^{-1} = g$. We are deliberately vague here - as mentioned in the introduction, the degree of smoothness on the map h can have a huge effect on the equivalence classes present. In this paper, we will concern ourselves with two cases: h can be a formal power series, or an analytic one (if h is analytic, then obviously K will have some associated norm).

By considering the conjugating map $x \mapsto (\sqrt[m-1]{a_m})x$, we assume that $a_m = 1$, and this assumption will be present throughout the paper. We show here that any such mapping may be reduced formally to $f_{0,m}(x) = x + x^m + \mu x^{2m-1}$, and moreover, the numbers m and μ provide formal invariants for the mapping f. The proof of this fact is known to many, and is impossible to ascribe to a single source. We include the proof here, however, as the conjugating map constructed will always converge in the non-archimedean setting (as we shall show later).

Proposition 2.7. — *Let $f \in K[|x|]$ have the form*

$$f(x) = x + x^m + \sum_{j=m+1}^{\infty} a_j x^j. \tag{2.4}$$

Then, there exists $\mu \in K$ and a formal power series $H(x) = x + \cdots$ so that $H \circ f \circ H^{-1}(x) = x + x^m + \mu x^{2m-1}$. Moreover, these numbers m and μ are uniquely defined.

Proof. — We consider polynomials of the form $h_n(x) = x + c_n x^n$, where c_n is to be determined for $n \geqslant 2$. We will define inductively $H_2(x) = h_2(x)$, and $H_n(x) = h_n \circ H_{n-1}(x)$ for $n > 2$. Finally, we will define $F_n = H_n \circ F \circ H_n^{-1}$.

We begin with $n = 2$. Let $g = x + x^m + \mu x^{2m-1}$, where μ is to be determined. We will define each c_n so that F_n and g agree through order $m + n - 1$. For $n = 2$, we define

$$c_2 = \frac{a_{m+1}}{m-2}. \tag{2.5}$$

We now suppose that c_l has been chosen for $2 \leqslant l \leqslant n-1$ so that F_l and g agree up to order $l + m - 1$, and will define c_n so that

$$h_n \circ F_{n-1} \circ h_n^{-1}(x) = g(x) x^{n+m}. \tag{2.6}$$

We write $[F_{n-1}]_{n+m-1}$ to be the coefficient of the $(n+m-1)$-degree term of F_{n-1}. With this stipulation, we now define c_n as:

$$c_n = \frac{[F_{n-1}]_{n+m-1}}{m-n}. \tag{2.7}$$

Note that when $n = m$, the formula given above is of no use. The coefficient c_m will have no effect on the $2m-1$ coefficient of F_m. Although one may define c_m to be any element of K, for simplicity of the later presentation, we will define $c_m = 0$. This completes our inductive definition of the polynomials h_l and H_l, $l \geqslant 2$. In addition, we will write $\mu = [F_m]_{2m-1}$.

Finally, the formal map H is defined to be $H = \lim_{n\to\infty} H_n$, taken in the formal sense. Since the nth coefficient of H_l is unchanged for all H_l with $l > n$, we see that each coefficient in the formal series H depends algebraically on a finite number of terms, and thus is well-defined.

In order to see that the numbers m and μ are in fact invariants, suppose that $f(x) = x + x^m + O(x^{m+1})$ and $g(x) = x + x^n + O(x^{n+1})$ are conjugated via a map $h(x) = x + \sum a_l x^l$. From the equation $h \circ f = g \circ h$, we have

$$h(x + x^m + O(x^{m+1})) = h(x) + (h(x))^n + O(x^{n+1}). \tag{2.8}$$

Via trivial computation, this can be reformulated as follows:

$$h(x) + x^m + O(x^{m+1}) = h(x) + x^n + O(x^{n+1}). \tag{2.9}$$

This forces $m = n$. We fix this number m.

Since the proof above shows that f is formally conjugate to $x + x^m + \mu x^{2m-1}$ for some choice of μ, we fix this value of μ. We can now assume that $f(x) = x + x^m + \mu x^{2m-1}$. Let $g(x) = x + x^m + \nu x^{2m-1}$, and suppose there is an $h(x) = x + c_\ell x^\ell + \ldots$, with $c_\ell \neq 0$, satisfying $h \circ f = g \circ h$. Then, we have

$$\begin{aligned} &h(x) + x^m + \mu x^{2m-1} + \ell c_\ell x^{\ell+m-1} + O(x^{\ell+m}) \\ &= h(x) + x^m + m c_\ell x^{\ell+m-1} + \nu x^{2m-1} + O(x^{\ell+m}). \end{aligned}$$

From this we see that $\ell = m$, and then $\mu = \nu$. The numbers m and μ are therefore uniquely determined, and provide invariants for the formal classification. $\square$

We make the following remarks:

1. Since the coefficient c_m has no effect on the process outlined above, it can be considered a “free term”. We have defined $c_m = 0$, but it will be shown that the series above converges locally for any choice of c_m.

2. One consequence of the formal classification is that any mapping f of the form (2.4) can be taken to the form $\tilde{f}(x) = x + x^m + \mu x^{2m-1} + O(x^{2m})$ by a polynomial change of variable of degree $m-1$, and moreover, the proof shows that this change of variable is unique, if chosen so that it is tangent to the identity. Therefore, in much of what follows, we will assume that $f(x) = x + x^m + \mu x^{2m-1} + \cdots$, and therefore that $H(x) = x + A_{m+1}x^{m+1} + \cdots$.

3. While it is not important to our theory to find a precise formula for the formal invariant μ, it is worth mentioning that in the case $m = 2$, we have that $\mu = \frac{a_3}{a_2^2}$.

2.3. Miscellaneous Notation

We will often need to study the process of raising a power series to a given exponent (multiplicatively). Consider for example the power series

$$f(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Then, for any natural number ℓ, $f(x)^\ell$ will be a sum of terms of the form

$$a_{i_1} a_{i_2} \cdots a_{i_\ell} x^{i_1+i_2+\cdots+i_\ell},$$

where $i_1, i_2, \ldots, i_\ell$ is a (finite) sequence of positive integers, not necessarily distinct.

Definition 2.8. — *Given a finite sequence $\underline{i} = (i_1, \ldots, i_\ell)$, write $|\underline{i}| = i_1 + \cdots + i_\ell$. Also write $\ell(\underline{i}) = \ell$, the "length" of $\underline{i}$.*

We also adopt the following notation.

Definition 2.9. — *Given a power series f, we write $[f]_n$ for the coefficient of the nth-degree term of f.*

Thus, any power series f may be written as $f(x) = \sum_n [f]_n x^n$.

The following lemma will be useful later. Its proof is immediate.

Lemma 2.10. — *Let $\eta(x) = \alpha_1 x + \alpha_d x^d + \alpha_{d+1} x^{d+1} + \cdots \in K[[x]]$, and $j, T \in \mathbb{N}$. Then if $j \neq T$ and $T < j + d - 1$, then $[\eta(x)^j]_T = 0$.*

3. Vector Fields and Flows in K

This short paragraph is devoted to the formal and analytic theory of flows and vector fields in K. We merely cite theorems in this section; a reference for these results is Herman and Yoccoz [9].

Proposition 3.1. — *Let $T_V^t(x) = T_V(t,x)$ be the formal flow of a (possibly formal) vector field V of the form*

$$V(x) = \sum_{n=1}^{\infty} v_n x^n, \tag{3.1}$$

with coefficients $v_n \in \Delta$. Then

1. *This flow has the form*

$$T_V(t,x) = \sum_{n=1}^{\infty} a_n(t) x^n, \tag{3.2}$$

   where $a_1(t) = \exp(tv_1)$, and $a_n(t)$ is a formal power series in t satisfying $a_n(0) = 0$ for all $n \geqslant 2$

2. *If V is a locally analytic vector field near 0, then for any value of t, the time-t map $T^t(x)$ is locally analytic in x near 0.*

In part (ii) of Proposition 3.1, we will often say that the flow is locally analytic in x near 0. However, from the example of $V(x) = x^2$, it is clear that the neighborhood on which $T^t(x)$ is defined can shrink for different values of t.

Our interest is in the case when $v_1 = 0$; in this setting, for any value of t, the flow T_V^t will be a map tangent to the identity.

Example: Consider the vector field $V(z) = \frac{x^m}{1-(\mu-1)x^{m-1}} \frac{\partial}{\partial x}$. This vector field is locally analytic near $0 \in K$. Moreover, the time-one map T_V^1 of this vector field takes the form

$$T_V^1(x) = x + x^m + \mu x^{2m-1} + \cdots, \tag{3.3}$$

and is necessarily locally analytic.

Moreover, if a map f has the form $f = T_V^{t_0}(x)$ for some value $t_0 \neq 0$, the centralizer $Z(f)$ of f is completely determined. In particular, we shall make use of the following lemma in Section 4.

Lemma 3.2. — *Let $V(x)$ be a vector field which generates a formal flow $T_V^t(x)$. Fix $t = t_0$, and write $f(x) = T_V^{t_0}(x)$. Then, if $g(x) = x + \cdots$ is any formal map satisfying $g \circ f = f \circ g$, then there is a t_1 so that $g(x) = T_V^{t_1}(x)$; i.e. g is in the flow of $V(x)$.*

Note that in our setting, this implies that any formal map g centralizing the time-t map of an analytic vector field V must itself be analytic.

4. The Sigma Function

This section is devoted to the bulk of the proof of Theorem 1.1.

Let $f \in \mathcal{O}_0^K$ be a power series of the form (2.4). By the second remark following Proposition 2.7, there are $m \in \mathbf{N}$, $\mu \in K$, and a polynomial change of variable which is tangent to the identity conjugating any such series to $f(x) = x + x^m + \mu x^{2m-1} + O(x^{2m})$. Thus, when treating analytic equivalence, we can assume that f takes the form

$$f(x) = x + x^m + \mu x^{2m-1} + \sum_{n=2m}^{\infty} a_n x^n, \tag{4.1}$$

so that f is formally equivalent to $f_0(x) = x + x^m + \mu x^{2m-1}$. Then, via Proposition 2.7 there is a formal series $H(x) = x + A_{m+1}x^{m+1} + O(x^{m+2})$ conjugating f with f_0, where c_k is given by Equation (2.7) for all k (and note that the series is unique, since we have chosen $c_2 = c_3 = \ldots = c_m = 0$). We show that this series converges in some neighborhood of $0 \in K$.

First, we make the following observation: since the radius of convergence is positive, the sequence $\{1/\sqrt[n]{|a_n|}\}$ is bounded below by some $\varepsilon > 0$. Pick $q \in K$ with $0 < |q| \leqslant \varepsilon$ so that $b_n = a_n q^n \in \Delta$ for all $n \geqslant 2m-1$. (Here $a_{2m-1} = \mu$.) Thus, we are reduced to the study of series f of the form

$$f(x) = x + x^m + \sum_{n=2m-1}^{\infty} \frac{b_n}{q^n} x^n, \tag{4.2}$$

where $b_n \in \Delta$. The idea here will be to estimate the decay of the denominators in the coefficients of h_n and H_n.

We first begin with some algebraic results which shall be of use to us in our estimation of the coefficients of the formal conjugating maps H. We need to study how the power series H in the proof of Proposition 2.7 combines the coefficients c_j of the polynomials h_j. The following lemma, which is purely algebraic, determines which products may occur in a given degree.

Let $\underline{c} = c_2, c_3, \ldots$ be a sequence of indeterminates. We assign a degree (or a weight) to each c_j, which we define to be j. Write $\mathcal{A} = \mathbb{Z}[c_2, \ldots]$ for the $\mathbb{Z}$-module generated by the products $\prod_j c_j$. Suppose that $\underline{i} = (i_1, \ldots, i_\ell)$ is a finite sequence of natural numbers (not necessarily distinct). We define $\ell(\underline{i}) = \ell$ and $|\underline{i}| = i_1 + \ldots + i_\ell$ as in Definition 2.8. Write $c_{\underline{i}}$ for the monomial $c_{i_1} \cdots c_{i_\ell} \in \mathcal{A}$; its degree is $|\underline{i}|$.

Then a typical element of $\mathcal{A}$ may be written as $a(\underline{c}) = \sum_{\underline{i}} \alpha_{\underline{i}} c_{\underline{i}}$, with $\alpha_{\underline{i}} \in \mathbb{Z}$.

Lemma 4.1. — *The Sigma Function For $j \geqslant 2$ let $h_j(x) = x + c_j x^j \in \mathcal{A}[x]$, and $H_j = h_j \circ h_{j-1} \circ \cdots \circ h_2 \in \mathcal{A}[x]$. Write $H_j(x) = x + \sum_n A_n^j(\underline{c})x^n$, with $A_n^j(\underline{c}) \in \mathcal{A}$. Suppose for a given $n \geqslant 2$, $A_n^j(\underline{c}) = \sum_{\underline{i}} \alpha_{\underline{i}}^j c_{\underline{i}}$, for integers $\alpha_{\underline{i}}^j$. Then for any $\underline{i}$ such that $\alpha_{\underline{i}}^j \neq 0$, one has $n = |\underline{i}| - \ell(\underline{i}) + 1$.*

Proof. — We induct on j. The statement is clear if $j = 2$. Given $\underline{i}$, let $n(\underline{i}) = |\underline{i}| - \ell(\underline{i}) + 1$. Then $H_{j+1}(x) = H_j(x) + c_{j+1}(H_j(x))^{j+1}$. The terms of the first part, $H_j(x)$, satisfy the proposition by induction. By our inductive hypothesis, the second part is a sum of monomials of the form

$$c_{j+1}(\alpha_{\underline{i}_1}^j c_{\underline{i}_1}) \cdots (\alpha_{\underline{i}_L}^j c_{\underline{i}_L}) x^{\sum_{k=1}^L n(\underline{i}_k)} \cdot x^{(j+1)-L}.$$

Therefore the exponent of x in this monomial is given by

$$\begin{aligned}\left(\sum_{k=1}^L n(\underline{i}_k)\right) + (j+1) - L &= \left(\sum_{k=1}^L |\underline{i}_k| - \ell(\underline{i}_k) + 1\right) + (j+1) - L \\ &= \left(\sum_{k=1}^L |\underline{i}_k| - \ell(\underline{i}_k)\right) + (j+1).\end{aligned}$$

On the other hand, write $\underline{i}'$ for the new sequence formed by concatenating $j+1, \underline{i}_1, \cdots$, and $\underline{i}_L$. We have

$$n(\underline{i}') = \left((j+1) + \sum_{k=1}^L |\underline{i}_k|\right) - \left(1 + \sum_{k=1}^L \ell(\underline{i}_k)\right) + 1,$$

which is equal to the previous expression. □

We now introduce a function which governs the growth of the coefficients of H_n. Fix $m \geqslant 2$. Let us define, for $n \in \mathbf{N}$ with $n \geqslant m+1$,

$$\sigma_m(n) = (n-1) + m\left[\frac{n-2}{m-1}\right], \tag{4.3}$$

where $[x]$ denotes the greatest integer less than or equal to x.

Lemma 4.2. — *Let the function σ_m be given by Equation (4.3), and let $n \geqslant m+1$. Then the following properties hold.*

1. *$\sigma_m(n)$ is a strictly increasing, integer-valued function of n, $\sigma_m(n + (m-1)) = \sigma_m(n) + (2m-1)$, and*

$$\left(\frac{2m-1}{m-1}\right)n - \left(m + 2 + \frac{1}{m-1}\right) \leqslant \sigma_m(n) \leqslant \left(\frac{2m-1}{m-1}\right)n - \left(\frac{3m-1}{m-1}\right).$$

2. *If $a, b \in \mathbf{N}$, and $b-a \geqslant m-1$, then $\sigma_m(b) - \sigma_m(a) \geqslant (b-a)+m$.*
3. *Let $\underline{i} = (i_1, \cdots, i_l)$ be an ℓ-tuple of positive integers and let $n = |\underline{i}| - \ell + 1$. Then,*

$$\sum_{j=1}^{\ell} \sigma_m(i_j) \leqslant \sigma_m(n).$$

Proof. — The first two statements are elementary; we prove the last statement. The problem reduces to proving that $\sum_{j=1}^{\ell} \left[\frac{i_j-2}{m-1}\right] \leqslant \left[\frac{n-2}{m-1}\right]$. Since $\ell \geqslant 1$, we have $\sum_j (i_j - 2) \leqslant n-2$. The property then follows from more general fact that for any ℓ integers $a_1, \ldots, a_\ell$, and positive integer N, one has

$$\sum_j \left[\frac{a_j}{N}\right] \leqslant \left[\frac{\sum_j a_j}{N}\right].$$

□

Now, let f be of the form (4.2) with formal invariants m and $\mu = \dfrac{b_{2m-1}}{q^{2m-1}}$. Associated to m, we have the function σ_m; we drop the m for convenience.

Proposition 4.3. — *Fix a natural number $m \geqslant 2$, and let $c_j \in K$ for $j = m+1, \cdots$ satisfy $(j-m)!q^{\sigma(j)}c_j \in \Delta$. Define $h_j = x + c_j x^j \in K[x]$, and write $H_j(x) = h_j \circ \cdots \circ h_{m+1}(x) = x + \sum_n A_n^j x^n$. Then for all n, $(n-m)!q^{\sigma(n)}A_n^j \in \Delta$.*

Proof. — As in Lemma 4.1, we know that $H_j(x) = x + \sum_n A_n^j(\underline{c})x^n$, whose nth term is $A_n^j(\underline{c}) = \sum_{\underline{i}} \alpha_{\underline{i}}^j c_{\underline{i}}$. The coefficients $\alpha_{\underline{i}}^j$ will be nonzero only when $n = |\underline{i}| - \ell(\underline{i}) + 1$. So for the nth term we need only consider products of the form $c_{i_1} \cdots c_{i_\ell}$, with

$$n = (i_1 + \cdots + i_\ell) - \ell + 1.$$

By hypothesis we have $(i_1 - m)! \cdots (i_\ell - m)! q^{\sigma(i_1)+\cdots+\sigma(i_\ell)} c_{i_1} \cdots c_{i_\ell} \in \Delta$. First, we deal with the factorials. We know that the multinomial coefficient

$$i_1 + i_2 + \cdots + i_\ell - \ell m i_1 - m, i_2 - m, \ldots, i_\ell - m$$

$$= \frac{(|\underline{i}| - \ell m)!}{(i_1 - m)!(i_2 - m)! \cdots (i_\ell - m)!}$$

is an integer. It is therefore enough to prove that $n - m \geqslant |\underline{i}| - \ell m$. By the equation for n this reduces to showing that

$$|\underline{i}| - \ell + 1 \geqslant |\underline{i}| - \ell(m-1),$$

which is true since $\ell \geqslant 1$ and $m \geqslant 2$. Finally, $|q^{\sigma(n)}| \leqslant |q^{\sigma(i_1)+\cdots+\sigma(i_\ell)}|$ by Part (iii) of Lemma 4.2. $\square$

Similarly, for the coefficients c_n, we have the following:

PROPOSITION 4.4. — *Let f be an analytic mapping of the form (4.2), where $b_n \in \Delta$. Let h_n, H_n, and c_n be defined as in Proposition 2.7. Then, $(n-m)!q^{\sigma(n)}c_n \in \Delta$ for all $n \geqslant m+1$.*

Proof. — Much of the proof of this proposition is based on the following simple fact: if $|ac| \leqslant 1$ and $|b| \leqslant |a|$, then $|bc| \leqslant 1$. In what follows, all of our computations are motivated by replacing a particular a with a b of smaller norm.

We induct on n. For $n = m+1$, we have $c_{m+1} = -\dfrac{b_{2m}}{q^{2m}}$ as in the proof of Proposition 2.7. Since $\sigma(m+1) = 2m$, we note that c_{m+1} satisfies the estimate, and take $c_2 = c_3 = \cdots = c_m = 0$. Thus, we assume that c_n satisfies the estimate $\text{(n-m)!q}^{\sigma(n)}c_n \in \Delta$, and we show that $\text{(n-m+1)!q}^{\sigma(n+1)}c_{n+1} \in \Delta$.

From Proposition 4.3, we can write H_n in the form $\mathrm{H}_n(x) = x + \sum_{k\geqslant m+1} A_k^n x^k$, where $q^{\sigma(k)}(k-m)!A_k^n \in \Delta$ for all $k \geqslant m+1$. Writing $H_{n+1} = H_n + c_{n+1}H_n^{n+1}$, the formal classification theorem shows that, up to order $O(x^{n+m+1})$, we must have $H_{n+1} \circ f = f_0 \circ H_{n+1}$. Therefore, up to this order we must have

$$\begin{aligned} H_n \circ f + c_{n+1}(H_n \circ f)^{n+1} &= H_n + c_{n+1}H_n^{n+1} + (H_n + c_{n+1}H_n^{n+1})^m \\ &\quad + \frac{b_{2m-1}}{q^{2m-1}}(H_n + c_{n+1}H_n^{n+1})^{2m-1}. \end{aligned}$$

We consider the $(n+m)$-degree coefficient of each side. Once we expand the powers, we can see how this simplifies the expression.

We have $(\mathrm{H}_n + c_{n+1}H_n^{n+1})^m = H_n^m + mc_{n+1}H_n^{n+m} + \sum_{j\geqslant n+m+1} \alpha_j H_n^j$, where $\alpha_j \in K$. Now $H_n^{n+m} = x^{n+m} + O(x^{n+m+1})$, and for $j \geqslant n+m+1$, it is clear that $[H_n^j]_{n+m} = 0$.

Therefore $[(\mathrm{H}_n + c_{n+1}H_n^{n+1})^m]_{n+m} = mc_{n+1} + [H_n^m]_{n+m}$.

In the same way, we have $[(\mathrm{H}_n + c_{n+1}H_n^{n+1})^{2m-1}]_{n+m} = [H_n^{2m-1}]_{n+m}$.

Thirdly, we consider the expression $[H_n^{n+1}]_{m+n}$. We may apply Lemma 2.10, with $T = n+m$, $j = n+1$, and $d = m+1$, so that $T < j+d-1$ for all terms. Therefore $[c_{n+1}H_n^{n+1}]_{m+n} = 0$.

Finally, by writing $f(x) = x + x^m + \cdots$, we see that $[(H_n \circ f)^{n+1}]_{n+m} = n+1$, since $H_n \circ f = x + x^m + O(x^{m+1})$.

Therefore, the induction reduces to the study of the equation

$$(n-m+1)c_{n+1} = [H_n - H_n \circ f]_{n+m} + [H_n^m]_{n+m} + \frac{b_{2m-1}}{q^{2m-1}}\left[H_n^{2m-1}\right]_{n+m}. \quad (4.4)$$

We show that the sum of the terms on the right-hand side of (4.4) lies in $\dfrac{q^{-\sigma(n+1)}}{(n-m)!}\Delta$, breaking the argument into three claims.

Claim 4.5. —

$$(n-m)!q^{\sigma(n+1)}\frac{b_{2m-1}}{q^{2m-1}}[H_n^{2m-1}]_{n+m} \in \Delta.$$

The coefficient $[H_n^{2m-1}]_{n+m}$ comes from a sum of terms of the form

$$x^{k_0}\prod_{t=1}^{\ell}(A_{s_t}x^{s_t})^{i_t}, \quad (4.5)$$

where we have the sums

$$k_0 + i_1 + \cdots + i_\ell = 2m-1 \quad (4.6)$$

and

$$k_0 + i_1 s_1 + \cdots + i_\ell s_\ell = n+m. \quad (4.7)$$

Let us call the coefficient of this $(n+m)$-degree term B_{n+m}. This coefficient satisfies the estimate

$$\prod_{t=1}^{\ell}((s_t-m)!)^{i_t}\, q^{i_t\sigma(s_t)}B_{n+m} \in \Delta. \quad (4.8)$$

We consider the multinomial coefficient

$$i_1(s_1-m) + \cdots + i_\ell(s_\ell - m) s_1 - m, \ldots, s_1 - m, \cdots, s_\ell - m, \cdots, s_\ell - m$$

where $s_k - m$ appears i_k times, for $k = 1, 2, \ldots, \ell$. Since this is an integer, we may replace the product of factorials appearing in (4.8) with $(n-m)!$, provided that

$$i_1(s_1-m) + \cdots + i_\ell(s_\ell - m) \leqslant n-m. \quad (4.9)$$

From (4.7), this is true exactly when

$$2m \leqslant k_0 + (i_1 + \cdots + i_\ell)m. \tag{4.10}$$

This inequality certainly holds if $i_1 + \cdots + i_l \geqslant 2$. This is only possibly false when $\ell = 1$ and $i_1 = 1$. But then by (4.6), we have $k_0 = 2m - 2$. Then (4.10) follows since $m \geqslant 2$.

In order to estimate the power of q appearing in (4.8), we will show that

$$\sum_{t=1}^{\ell} i_t \sigma(s_t) \leqslant \sigma(n+1) - (2m-1). \tag{4.11}$$

(The slightly smaller decay is necessary, since ultimately, we multiply $[H_n^{2m-1}]_{n+m}$ by $q^{-(2m-1)}$.)

Denote by $\underline{s}$ the $(i_1 + \cdots + i_\ell)$-tuple of integers consists of s_1 in the first i_1 components, s_2 in the next i_2 components, etc. Then $|\underline{s}| = i_1 s_1 + \cdots + i_\ell s_\ell$ and $\ell(\underline{s}) = i_1 + \cdots + i_\ell$.

From (4.6) and (4.7), we see —$\underline{\text{s}}$—=n+m-k$_0$ $= n+m-((2m-1)-\ell(\underline{s}))$. Therefore, we have n+1=(—$\underline{\text{s}}$—-$\ell(\underline{s}) + 1) + (m-1)$. Therefore, from part (i) of Lemma 4.2, $\sigma(n+1) = \sigma(|\underline{s}| - \ell(\underline{s}) + 1) + (2m-1)$. Finally, Part (iii) of Lemma 4.2 yields the inequality (4.11).

This finishes Claim 4.5.

Claim 4.6. —

$$(n-m)! q^{\sigma(n+1)} \left([H_n^m]_{n+m} - mA_{n+1}\right) \in \Delta.$$

Again, write the $(n+m)$-degree term of $[H_n^m]_{n+m}$ as a sum of terms of the form (4.5). Equation (4.7) remains the same, k$_0$ $+ i_1 s_1 + \cdots + i_\ell s_\ell = n+m$, but (4.6) becomes

$$k_0 + i_1 + \cdots + i_\ell = m. \tag{4.12}$$

We consider again the estimate (4.8), given by

$$\prod_{t=1}^{\ell} \left((s_t - m)!\right)^{i_t} q^{i_t \sigma(s_t)} B_{n+m} \in \Delta.$$

In order to replace the sigma functions appearing there, here we must show that

$$\sum_{t=1}^{\ell} i_t \sigma(s_t) \leqslant \sigma(n+1).$$

Using the notation $\underline{s}$ to be the $(i_1 + \ldots + i_\ell)$-tuple with s_1 in the first i_1 components, etc., and defining $|\underline{s}| = i_1 s_1 + \ldots + i_\ell s_\ell$ and $\ell(s) = i_1 + \ldots + i_\ell$, we obtain by subtracting Equation (4.12) from (4.7) (and adding 1 to both sides)

$$n + 1 = |\underline{s}| - \ell(\underline{s}) + 1.$$

Therefore, we may apply part (iii) of Lemma 4.3 to replace the sigmas appearing in (4.8) with $\sigma(n+1)$.

Dealing with the factorials is trickier. From (4.10), if $\ell \geqslant 2$, we may replace the product of factorials appearing in (4.8) with $(n-m)!$. If this is not the case, then again $i_1 = 1$, and so $k_0 = m - 1$. This forces $s_1 = n + 1$, and we are left with the single term mA_{n+1}. We will consider this term later - for now, we simply set it aside.

This finishes the proof of Claim 4.6.

Claim 4.7. —

$$(n-m)! q^{\sigma(n+1)} \left([H_n - H_n \circ f]_{n+m} + (n+1)A_{n+1} \right) \in \Delta$$

The computation of $[H_n - H_n \circ f]_{n+m}$ reduces to the study of the $(n+m)$-degree coefficient of

$$H_n(x) - H_n \circ f(x) = \sum_{j=m+1}^{n+m} A_j (x^j - f(x)^j).$$

Let us write $g_j(x) = x^j - f(x)^j$. If $j \geqslant n+2$, then $g_j(x) = O(x^{n+m+1})$, so those terms may be discarded. Therefore, we may assume that $j \leqslant n+1$. If $j = n+1$, the the only term appearing is

$$[A_{n+1}(x^{n+1} - f(x)^{n+1})]_{n+m} = -(n+1)A_{n+1}.$$

This exception is why we subtract this term from $[H_n - H_n \circ f]_{n+m}$ in the claim. We now consider the case of $j \leqslant n$.

We will consider each coefficient $[A_j(f(x))^j]_{n+m}$ individually. We first expand the jth power of f. A typical term will be a sum of terms of the form

$$x^{e_1} (x^m)^{e_m} \left(\frac{b_{2m-1} x^{2m-1}}{q^{2m-1}} \right)^{e_{2m-1}} \cdots \left(\frac{b_\ell x^\ell}{q^\ell} \right)^{e_\ell}, \qquad (4.13)$$

with

$$e_1 + e_m + \sum_{s=2m-1}^{\ell} e_s = j. \tag{4.14}$$

This term will have degree $n+m$ when

$$e_1 + me_m + \sum_{s=2m-1}^{\ell} se_s = n+m. \tag{4.15}$$

Of course,

$$q^{(2m-1)e_{2m-1}+\ldots+\ell e_\ell}\left[x^{e_1}(x^m)^{e_m}\left(\frac{b_{2m-1}x^{2m-1}}{q^{2m-1}}\right)^{e_{2m-1}}\cdots\left(\frac{b_\ell x^\ell}{q^\ell}\right)^{e_\ell}\right]_{n+m} \in \Delta.$$

We will prove that

$$\sigma(n+1) - \sigma(j) \geqslant \sum_{s=2m-1}^{\ell} se_s. \tag{4.16}$$

(We need the extra decay, since we will multiply this term by A_j). By Lemma 4.2(i), we have

$$\sigma(n+1) - \sigma(j) \geqslant \frac{2m-1}{m-1}(n-j+1) + (1-m) + \frac{1}{m-1}. \tag{4.17}$$

Subtracting Equation (4.14) from Equation (4.15) gives

$$n-j+1 = 1-m+(m-1)e_m + \sum_{s=2m-1}^{\ell}(s-1)e_s.$$

Combining this with the above estimate, we see that the inequality (4.16) will be true when

$$(2m-1)e_m + \sum_{s=2m-1}^{\ell}\left((s-1)\left(\frac{2m-1}{m-1}\right) - s\right)e_s \geqslant (3m-2) - \frac{1}{m-1}. \tag{4.18}$$

In fact, all the coefficients on the left hand side are greater or equal to $2m-1$, since we may rewrite each coefficient as

$$(s-1)\left(2+\frac{1}{m-1}\right) - s = s-2+\frac{s-1}{m-1} \geqslant s-2+\frac{2m-2}{m-1} = s.$$

We may therefore conclude the following: If the inequality (4.16) does *not* hold, then only one of the exponents $e_m, \ldots, e_\ell$ may be nonzero, and in fact must be equal to 1.

Suppose this is the case. Recall that $f(x) = x + x^m + O(x^{2m-1})$. If $e_m = 1$ and all higher exponents are 0, then the term (4.13) is simply $x^{j-1}(x^m)$, and we must show that $q^{\sigma(n+1)-\sigma(j)} \in \Delta$. But in this case $n+1 = j$, so this is clear.

On the other hand, suppose some $e_s = 1$ for $s \geqslant 2m-1$.

The term (4.13) now has the form

$$x^{j-1} b_{n+m-j+1} \left(\frac{x}{q}\right)^{n+m-j+1},$$

with $n+m-j+1 \geqslant 2m-1$. Thus we only need to check that if $m-1 \leqslant n+1-j$, then $\sigma(n+1) - \sigma(j) \geqslant (n+1-j) + m$. This follows immediately from statement (ii) of Lemma 4.2. Putting it all together, we see that

$$(n-m)! q^{\sigma(n+1)} [A_j (f(x))^j]_{n+m} \in \Delta$$

for all values $m+1 \leqslant j \leqslant n$.

This finishes Claim 4.7.

We now complete the proof of Proposition 4.4. We rewrite the right-hand side of Equation (4.4) as

$$([H_n - H_n \circ f]_{n+m} + (n+1)A_{n+1}) + ([H_n^m]_{n+m} - mA_{n+1}) + \frac{b_{2m-1}}{q^{2m-1}} [H_n^{2m-1}]_{n+m}$$
$$-(n-m+1)A_{n+1}.$$

By Proposition 4.3, we have that

$$(n-m+1)! q^{\sigma(n+1)} (-A_{n+1}) = (n-m)! q^{\sigma(n+1)} (-(n-m+1)A_{n+1}) \in \Delta.$$

Therefore, putting this together with Claims 4.5, 4.6 and 4.7, we see that

$$(n-m+1)! q^{\sigma(n+1)} c_{n+1} \in \Delta. \tag{4.19}$$

This completes the induction, and the proof of Proposition 4.4. $\square$

5. Theorem and Corollaries

In this last section, we wrap up the proof of Theorem 1.1. We record several corollaries, including an observation regarding formal conjugating maps.

Proof of Theorem 1.1. — We write $H_n(x) = (h_n \circ h_{n-1} \circ \cdots \circ h_{m+1})(x)$, where $h_k(x) = x + c_k x^k$. From Propositions 4.3 and 4.4, we note that the conjugating map $H = \lim_{n\to\infty} H_n$ will have coefficients A_n satisfying (n-m)!$\mathrm{q}^{\sigma_m(n)} A_n \in \Delta$, where q is chosen as in the beginning of Section 4. By Proposition 2.5, there is a real number $0 < \alpha \leqslant 1$ so that $|(n-m)!|^{-1} < \alpha^{-n}$ for some real number $0 < \alpha \leqslant 1$. Thus by choosing q to satisfy also $0 < |q| < \alpha$, we obtain that $|(n-m)!|^{-1} < |q|^{-n}$. From Lemma 4.2, we have that $\sigma_m(n) \leqslant 3n$ for all $m \geqslant 2$, $n \geqslant m+1$. Thus, $A_n x^n$ will tend to 0 if $|x|$ is sufficiently small, and hence our series converges.

Let f take the form (4.1) and let $\varepsilon(f) = \min \left\{ \dfrac{1}{\sqrt[n]{|a_n|}} \right\}$. From the proof we have the following:

Corollary 5.1. — *Suppose the norm on K restricts to the usual p-adic norm $|\cdot|_{p,\frac{1}{p}}$ on $\mathbb{Q}$. Choose $e \in \mathbb{N}$ so that $p^{-e} \leqslant \varepsilon(f)$. Then $H_n(x)$ converges for $|x| < p^{-(\lambda e+1)}$, where $\lambda = \frac{2m-1}{m-1}$.*

With a full analytic classification in place, we now settle the questions of centralizers and root extraction for a typical analytic map f of the form (4.1).

Let us begin with centralizers of f, both formal and analytic. We write $\mathrm{Z}_F(f) = \{g \in K[[x]] : g(0) = 0, g'(0) = 1, g \circ f = f \circ g\}$, and $\mathrm{Z}_A(f) = \{g \in \mathcal{O}_0^K : g(0) = 0, g'(0) = 1, g \circ f = f \circ g\}$.

The centralizers of two conjugate elements are themselves conjugate subgroups. Therefore we only need to compute the centralizer for one representative from each conjugacy class.

Let f be of the form (4.1) with formal invariants m and μ.

In fact f is equivalent to the time-one map of the flow of a vector field. This is useful because the centralizers of these maps are completely understood.

As in Section 3, consider the vector field $V(z) = \frac{x^m}{1-(\mu-1)x^{m-1}} \frac{\partial}{\partial x}$. The time-one map T_V^1 of this vector field takes the form

$$T_V^1(x) = x + x^m + \mu x^{2m-1} + \cdots.$$

Thus, f and T_V^1 are analytically conjugate. Since $Z_F(T_V^1) = Z_A(T_V^1) = \{T_V^t : t \in K\}$, we may now compute the centralizers $Z_F(f)$ and $Z_A(f)$. Let h be an analytic map tangent to the identity satisfying $h \circ f \circ h^{-1} = T_V^1$. We have the following:

Corollary 5.2. — $Z_A(f) = \{h^{-1} \circ T_V^t \circ h\}_{t \in K}$.

Note also that $Z_A(f) = Z_F(f)$.

Root extraction is now a simple consequence.

Corollary 5.3. — *Let f be of the form (4.1), and let $n \geqslant 1$ be a natural number. Then, there is a unique $g \in \mathcal{O}_0^K$ tangent to the identity satisfying $g^{\circ n} = f$.*

Proof. — An easy induction constructs a unique *formal* map g tangent to the identity satisfying $g^{\circ n} = f$. Any such root necessarily belongs to the centralizer of f, and since the formal centralizer agrees with the analytic one, we conclude that g is in fact analytic. Thus, f admits analytic nth-root extraction of all orders. □

Corollaries 5.2 and 5.3 together yield Corollary 1.2.

Finally, if f and g are analytically equivalent, then there is an analytic map h satisfying $h \circ f \circ h^{-1} = g$. Let $\tilde{h}$ be any other formal map satisfying $\tilde{h} \circ f \circ \tilde{h}^{-1} = g$. Then, since $\tilde{h}^{-1} \circ h = k$ is a formal map centralizing f, it must be analytic as well. Thus, we have proven

Corollary 5.4. — *Let h be a formal map which conjugates two formally equivalent maps f and g, both of which are analytic in some neighborhood of the origin. Then, h is in fact analytic in some neighborhood of $0 \in K$.*

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