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Stability of foliations induced by rational maps

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Stability of foliations induced by rational maps

F. CUKIERMAN⁽¹⁾, J. V. PEREIRA⁽²⁾, I. VAINSENER⁽³⁾

RÉSUMÉ. — Nous montrons que les feuilletages holomorphes induits par les applications rationnelles quasi-homogènes remplissent les composantes irréductibles de l'espace $\mathcal{F}_q(r, d)$ des feuilletages de codimension q et degré d de l'espace projectif \mathbb{P}^r pour tout $1 \leq q \leq r - 2$. Nous étudions la géométrie de telles composantes irréductibles. Nous montrons que ce sont des variétés rationnelles et calculons leur degré dans plusieurs cas.

ABSTRACT. — We show that the singular holomorphic foliations induced by dominant quasi-homogeneous rational maps fill out irreducible components of the space $\mathcal{F}_q(r, d)$ of singular foliations of codimension q and degree d on the complex projective space \mathbb{P}^r , when $1 \leq q \leq r - 2$. We study the geometry of these irreducible components. In particular we prove that they are all rational varieties and we compute their projective degrees in several cases.

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1. Introduction

1.1. The space of codimension one holomorphic foliations on \mathbb{P}^r

Let us consider a differential 1-form in \mathbb{C}^{r+1}

$$\omega = \sum_{i=0}^r a_i dx_i$$

where the a_i are homogeneous polynomials of degree $d + 1$ in variables x_0, \dots, x_r , with complex coefficients. Assume that $\sum_{i=0}^r a_i x_i = 0$, so that ω descends to the complex projective space \mathbb{P}^r and defines a global section of the twisted sheaf of 1-forms $\Omega_{\mathbb{P}^r}^1(d + 2)$.

The space of codimension one foliations of degree d on \mathbb{P}^r is the algebraic subset of $\mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^1(d + 2)))$ consisting of the 1-forms ω that satisfy the Frobenius integrability condition and have zero set of codimension at least two, i.e.,

$$\mathcal{F}(r, d) = \{ \omega \in \mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^1(d + 2))) \mid \omega \wedge d\omega = 0 \text{ and } \text{codim } \text{sing}(\omega) \geq 2 \}.$$

For the study of the irreducible components of $\mathcal{F}(r, d)$ we refer to e. g. [2] and [11].

1.2. Stability of quasi-homogeneous pencils

One of the first results on the subject is due to Gómez-Mont and Lins Neto [7] who proved that there are irreducible components $\mathcal{R}(r, d, d) \subset \mathcal{F}(r, 2d - 2)$, $r \geq 3$, whose generic element is a foliation tangent to a Lefschetz pencil of degree d hypersurfaces. Their proof explores the topology of the underlying real foliation and relies on the stability of the Kupka components of the singular set and on Reeb's Leaf Stability Theorem. Using similar methods they recognized for $r \geq 4$ other irreducible components $\mathcal{R}(r, d_0, d_1) \subset \mathcal{F}(r, d_0 + d_1 - 2)$ with generic member tangent to a quasi-homogeneous pencil $\langle \lambda F^{p_0} - \mu G^{p_1} \rangle$ with p_0 and p_1 relatively prime natural numbers satisfying $p_0 d_0 = p_1 d_1$, $d_i = \deg F_i$. Later Calvo-Andrade [1] extended Gómez-Mont-Lins Neto result about quasi-homogeneous pencils to dimension three. His proof has an extra dynamical ingredient –the stability of leaves carrying non-trivial holonomy.

In fact in both of the above mentioned papers the authors do not restrict to \mathbb{P}^r and prove their results for foliations on an arbitrary projective manifold M with $\dim M \geq 3$ and $\mathbb{H}^1(M, \mathbb{C}) = 0$. Alternative proofs of the above results may be found in [14, 16].

1.3. Infinitesimal stability of quasi-homogeneous pencils

Although full of geometric insights the above mentioned works do not seem to shed any light on the scheme structure or the geometry of $\mathcal{R}(r, d_0, d_1)$. The present article stems from an attempt to understand these problems.

Using infinitesimal techniques, as in [4], we describe the Zariski tangent space of $\mathcal{R}(r, d_0, d_1)$ at a generic point and arrive at a proof that $\mathcal{R}(r, d_0, d_1)$ –with the natural scheme structure given by the Frobenius integrability condition– is *generically reduced*. More precisely if $\mathcal{R}(r, d_0, d_1)$ denotes the closure of the image of the rational map

$$\begin{aligned} \rho : \mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d_0))) \times \mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d_1))) &\dashrightarrow \mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \Omega^1(d_0 + d_1))) \\ (F_0, F_1) &\longmapsto d_0 F_0 d F_1 - d_1 F_1 d F_0. \end{aligned}$$

then our first result reads as follows.

THEOREM 1.1. — *If $r \geq 3$ then $\mathcal{R}(r, d_0, d_1)$ is an irreducible and generically reduced component of $\mathcal{F}(r, d_0 + d_1 - 2)$.*

As explained above the only novelty in Theorem 1.1, besides the method of its proof, is what concerns the scheme structure over a generic point. For a more precise statement see Theorem 2.1 in §2.

The main content of this article is the generalization of Theorem 1.1 to foliations of higher codimension.

1.4. Foliations on \mathbb{P}^r of higher codimension

Let ω be a homogeneous q -form on \mathbb{C}^{r+1} with coefficients of degree $d + 1$ that is annihilated by Euler’s vector field. As before ω can be interpreted as a section of the sheaf of twisted differential q -forms $\Omega_{\mathbb{P}^r}^q(d + q + 1)$.

We recall from [13] (see also [4]) that ω defines a degree d holomorphic foliation of codimension q on \mathbb{P}^r if it satisfies both Plücker’s decomposability condition

$$(i_v \omega) \wedge \omega = 0 \quad \text{for every } v \in \bigwedge^{q-1} \mathbb{C}^{r+1}, \tag{1.1}$$

and the integrability condition

$$(i_v \omega) \wedge d\omega = 0 \quad \text{for every } v \in \bigwedge^{q-1} \mathbb{C}^{r+1}. \tag{1.2}$$

It is therefore natural to set $\mathcal{F}_q(r, d)$, the space of codimension q holomorphic foliations of degree d on \mathbb{P}^r , as

$$\{\omega \in \mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^q(d+q+1))) \mid \omega \text{ satisfies (1.1), (1.2) and } \text{codim sing}(\omega) \geq 2\}.$$

1.5. Infinitesimal stability of quasi-homogeneous rational maps

If one interprets the elements of $\mathcal{R}(r, d_0, d_1)$ as foliations tangent to the fibers of rational maps

$$\begin{aligned} \mathbb{P}^r & \dashrightarrow \mathbb{P}^1 \\ x & \longmapsto (F^{p_0} : G^{p_1}) \end{aligned}$$

then a possible counterpart in the higher codimension case are the foliations tangent to dominant rational maps $\mathbb{P}^r \dashrightarrow \mathbb{P}^q$.

When $q = r - 1$ there is no hope to establish a stability result even for a generic rational map. Indeed, under this constraint both Plücker's condition and the integrability condition are vacuous. Thus $\mathcal{F}_{r-1}(r, d)$ can be identified with an open subset of $\mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^{r-1}(d+r))) = \mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, T\mathbb{P}^r(d-1)))$. It is well known that for $d \geq 2$ a generic element of this space has no algebraic leaves, see for instance [3].

For $1 \leq q \leq r - 2$ fix integers d_0, \dots, d_q and consider homogeneous polynomials F_i of degree d_i for $i = 0, \dots, q$. Assume that the q -form

$$\omega = i_R(dF_0 \wedge \dots \wedge dF_q), \tag{1.3}$$

is non-zero. It is easy to check that ω satisfies both (1.1) and (1.2) since $i_v \omega = \sum a_{ij} i_R(dF_i \wedge dF_j)$, where the a_{ij} are homogeneous polynomials. Moreover, it defines a foliation tangent to the fibers of the map

$$\begin{aligned} \mathbb{P}^r & \dashrightarrow \mathbb{P}^q \\ x & \longmapsto (F_0^{e_0} : \dots : F_q^{e_q}) \end{aligned}$$

with $e_i = \text{lcm}(d_0, \dots, d_q)/d_i$. We set

$$d = \sum d_i - q - 1$$

and denote by

$$\mathcal{R}(r, d_0, \dots, d_q) \subset \mathcal{F}_q(r, d)$$

the closure of the set of foliations that can be written in the form (1.3). It is the closure of the image of the rational map

$$\begin{aligned} \rho : \prod_i \mathbb{P}(\mathbb{H}^0(\mathcal{O}_{\mathbb{P}^r}(d_i))) & \dashrightarrow \mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \Omega^1(d+q+1))) \\ (F_i) & \longmapsto i_R(dF_0 \wedge \dots \wedge dF_q). \end{aligned}$$

Notice that for $q = 1$ we recover the definition of $\mathcal{R}(r, d_0, d_1)$.

THEOREM 1.2. — *If $r \geq 4$ and $1 \leq q \leq r - 2$ then $\mathcal{R}(r, d_0, \dots, d_q)$ is an irreducible and generically reduced component of $\mathcal{F}_q(r, \sum d_i - q - 1)$.*

As far as we know there is no information in the literature concerning the geometry of the irreducible components of $\mathcal{F}_q(r, d)$ so far.

1.6. Geometry of the rational components

In Section 3 we initiate this study through an investigation of the parametrization ρ . Besides computing the dimension of $\mathcal{R}(r, d_0, \dots, d_q)$, we prove the following.

THEOREM 1.3. — *The irreducible components $\mathcal{R}(r, d_0, \dots, d_q)$ are rational varieties.*

By its definition, $\mathcal{R}(r, d_0, \dots, d_q)$ is unirational. The proof of rationality relies on the construction of a variety X that sits as an open set in the total space of a tower of Grassmann bundles, together with a birational morphism $p : X \rightarrow \mathcal{R}(r, d_0, \dots, d_q)$.

In general we do not know how to naturally compactify X to a projective variety where p extends to a morphism. Albeit, in a number of cases we are able to do that and obtain, with the aid of Schubert Calculus, formulas for the degree of the projective subvarieties

$$\mathcal{R}(r, d_0, \dots, d_q) \subset \mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \Omega^q(d + q + 1))).$$

For example the first few values for the degree of $\mathcal{R}(r, 2, 2, 2)$ are listed below.

r	Degree
3	1324220
4	2860923458080
5	243661972980477736263
6	728440733705107831789517245858
7	704613096513585123585398408696231899176183

Several other cases are treated in Section 5.

2. Infinitesimal stability of quasi-homogeneous pencils

In this first section we present our proof of Theorem 1.1. All the arguments will be reworked later in greater generality. We felt the exposition of this particular case of Theorem 1.2 would improve the clarity of the paper.

For simplicity, let us denote by

$$\mathbf{S}_e = \mathbf{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(e)) \tag{2.1}$$

the vector space of homogeneous polynomials of degree e in $r + 1$ variables, and

$$\mathcal{F} = \mathcal{F}(r, d)$$

so that our rational map ρ is

$$\rho : \mathbb{P}(\mathbf{S}_{d_0}) \times \mathbb{P}(\mathbf{S}_{d_1}) \dashrightarrow \mathcal{F} \subset \mathbb{P}(\mathbf{H}^0(\mathbb{P}^r, \Omega^1(d+2))). \tag{2.2}$$

If p_0 and p_1 denote the unique coprime natural numbers such that $p_0d_0 = p_1d_1$ then

$$\rho(F_0, F_1) = d_0F_0dF_1 - d_1F_1dF_0 = p_1F_0dF_1 - p_0F_1dF_0$$

where the last equality of differential forms is up to multiplicative constant.

We remark that

$$d\left(\frac{F_0^{p_0}}{F_1^{p_1}}\right) = \frac{F_0^{p_0-1}}{F_1^{p_1+1}}(p_1F_0dF_1 - p_0F_1dF_0).$$

Therefore, the closure of the leaves of the singular foliation defined by the integrable 1-form $\rho(F_0, F_1)$ are irreducible components of the members of the pencil of hypersurfaces of degree $p_0d_0 = p_1d_1$ generated by $F_0^{p_0}$ and $F_1^{p_1}$.

2.1. The Zariski tangent space of \mathcal{F}

For a scheme X and a point $x \in X$ we denote by T_xX the Zariski tangent space of X at x . If $\mathbb{P}(V)$ is the projective space associated to a \mathbb{C} -vector space V and denoting $\pi : V - \{0\} \rightarrow \mathbb{P}(V)$ the canonical projection, for each $v \in V$ we have a natural identification

$$T_{\pi(v)}\mathbb{P}(V) = V/(v)$$

where (v) denotes the one-dimensional subspace generated by v . With slight abuse of notations, the Zariski tangent space $T_\omega\mathcal{F}$ of \mathcal{F} at a point ω is represented by the forms $\eta \in \mathbf{H}^0(\mathbb{P}^r, \Omega^1(d+2))/(\omega)$ such that

$$(\omega + \epsilon\eta) \wedge (d\omega + \epsilon d\eta) = 0 \text{ mod } \epsilon^2$$

that is, such that

$$\omega \wedge d\eta + \eta \wedge d\omega = 0 \text{ or, equivalently } d\omega \wedge d\eta = 0,$$

where the equivalence is implied by the following variant of Euler's formula for homogeneous polynomials.

LEMMA 2.1. — *If η is a homogeneous q -form with degree d coefficients then*

$$i_R d\eta + d(i_R \eta) = (q + d)\eta$$

where R is the radial or Euler vector field and i_R denotes the interior product or contraction with R .

Proof. — See [11, Lemme 1.2, pp. 3]. □

Therefore to determine $T_\omega \mathcal{F}$ is equivalent to solve $d\omega \wedge d\eta = 0$. Notice that in the situation under scrutiny $d\omega = (d_0 + d_1)dF_0 \wedge dF_1$. The first step towards the general η satisfying $d\omega \wedge d\eta = 0$ is given by Saito's generalization of DeRham's division Lemma. In Lemma 2.2 we state variants of both DeRham's and Saito's Lemmas fine tuned up for our purposes.

LEMMA 2.2 ([15]). — *Let F_0, \dots, F_q be homogeneous polynomial functions on \mathbb{C}^{r+1} and let $\Theta \in \Omega^{q+1}(\mathbb{C}^{r+1})$ be the $(q + 1)$ -form given by*

$$\Theta = dF_0 \wedge \dots \wedge dF_q.$$

- (a) *Suppose that $q < r$ and $\text{codim sing}(\Theta) \geq 2$. If $\eta \in \Omega^1(\mathbb{C}^{r+1})$ is a homogeneous polynomial 1-form such that $\Theta \wedge \eta = 0$ then there exist homogeneous polynomials a_0, \dots, a_q such that*

$$\eta = \sum_{i=0}^q a_i dF_i.$$

- (b) *Suppose that $q < r - 1$ and $\text{codim sing}(\Theta) \geq 3$. If $\eta \in \Omega^2(\mathbb{C}^{r+1})$ is a homogeneous polynomial 2-form such that $\Theta \wedge \eta = 0$ then there exist homogeneous polynomial 1-forms $\alpha_0, \dots, \alpha_q$ such that*

$$\eta = \sum_{i=0}^q \alpha_i \wedge dF_i.$$

Remark 2.3. — The hypothesis $q < r$ in (a) and $q < r - 1$ in (b) are not really necessary. For instance in item (b) the singular set $\text{sing}(\Theta)$ equals the locus where the $(q + 1) \times (r + 1)$ Jacobian matrix $(\partial F_i / \partial x_j)$ has rank $\leq q$. Hence $\text{sing}(\Theta)$ is empty or has codimension at most $r + 1 - q$. When $q \geq r - 1$ it follows that $\text{codim sing}(\Theta) \geq 3$ implies that Θ has no singularities. We conclude that F_0, \dots, F_q are linearly independent linear forms and the conclusion trivially holds true in this case.

In face of Lemma 2.2 it is natural to define the open subset

$$\mathcal{U} = \{\omega \in \mathcal{R}(r, d_0, d_1) \mid \text{codim sing}(d\omega) \geq 3 \text{ and } \text{codim sing}(\omega) \geq 2\}. \quad (2.3)$$

The next result will imply the infinitesimal stability of quasi-homogeneous pencils corresponding to points of \mathcal{U} . It is a simple particular case of Proposition 3.3. The iteration argument in the proof is generalized in Lemma 3.2. We feel it is worthwhile to write it here for the sake of clarity.

PROPOSITION 2.4. — *Let $(F_0, F_1) \in \mathbb{P}(\mathbf{S}_{d_0}) \times \mathbb{P}(\mathbf{S}_{d_1})$ be such that $\rho(F_0, F_1) = \omega \in \mathcal{U}$. Then the derivative*

$$d\rho(F_0, F_1) : T_{(F_0, F_1)}(\mathbb{P}(\mathbf{S}_{d_0}) \times \mathbb{P}(\mathbf{S}_{d_1})) \rightarrow T_\omega \mathcal{F}$$

is surjective. In other words, ρ is a submersion over \mathcal{U} .

Proof. — It is convenient to write

$$\rho(F_0, F_1) = d_0 F_0 dF_1 - d_1 F_1 dF_0 = i_R(dF_0 \wedge dF_1).$$

Then, the derivative of ρ at the point (F_0, F_1)

$$d\rho(F_0, F_1) : \mathbf{S}_{d_0}/(F_0) \times \mathbf{S}_{d_1}/(F_1) \rightarrow T_\omega \mathcal{F}$$

is calculated as

$$d\rho(F_0, F_1)(F'_0, F'_1) = i_R(dF'_0 \wedge dF_1 + dF_0 \wedge dF'_1).$$

Let $\eta \in H^0(\mathbb{P}^r, \Omega^1(d+2))$ represent an element of $T_\omega \mathcal{F}$, that is, $d\omega \wedge d\eta = 0$. We shall prove that η belongs to the image of $d\rho(F_0, F_1)$, *i.e.*,

$$\eta = i_R(dF'_0 \wedge dF_1 + dF_0 \wedge dF'_1)$$

for some $F'_0 \in \mathbf{S}_{d_0}$ and $F'_1 \in \mathbf{S}_{d_1}$.

Since $d\omega = dF_0 \wedge dF_1$, applying the division Lemma 2.2 to $d\eta$ it follows that there exist homogeneous 1-forms α and β such that

$$d\eta = \alpha \wedge dF_0 + \beta \wedge dF_1.$$

Notice that $d\eta$ is a 2-form with coefficients homogeneous polynomials of degree $d = d_0 + d_1 - 2$. Hence the coefficients of α (resp. β) are homogeneous of degree $d_1 - 1$ (resp. $d_0 - 1$). Applying exterior derivative we find

$$d\alpha \wedge dF_0 + d\beta \wedge dF_1 = 0.$$

Multiplying by dF_1 we get $d\alpha \wedge dF_0 \wedge dF_1 = 0$. From lemma 2.2 applied to $d\alpha$ we deduce

$$d\alpha = \alpha' \wedge dF_0 + \alpha'' \wedge dF_1$$

where α' and α'' are 1-forms with coefficients homogeneous polynomials of respective degrees $d_1 - 2 - (d_0 - 1) = d_1 - d_0 - 1$ and $d_1 - 2 - (d_1 - 1) = -1$. Hence $\alpha'' = 0$. Similarly,

$$d\beta = \beta' \wedge dF_0 + \beta'' \wedge dF_1$$

where β' and β'' are 1-forms with coefficients homogeneous polynomials of respective degrees $d_0 - 2 - (d_0 - 1) = -1$ and $d_0 - 2 - (d_1 - 1) = d_0 - d_1 - 1$. Hence $\beta' = 0$.

Suppose that $d_0 = d_1$. By the considerations above regarding degrees, $\alpha' = \beta'' = 0$. Thus α and β are closed 1-forms. Therefore $\alpha = -dF'_1$ and $\beta = dF'_0$ where F'_i is some homogeneous polynomial of degree d_i . It follows that $d\eta = dF'_0 \wedge dF_1 + dF_0 \wedge dF'_1$ and since $i_R(d\eta) = (d+1)\eta$ we obtain that η is a scalar multiple of $i_R(dF'_0 \wedge dF_1 + dF_0 \wedge dF'_1)$. Therefore the Proposition is proved in the case $d_0 = d_1$.

Now suppose $d_0 \neq d_1$, say $d_0 > d_1$. Then $d_1 - d_0 - 1 < 0$. Hence $d\alpha = 0$ and $d\beta = \beta'' \wedge dF_1$. Repeating the argument of the previous case we obtain a sequence of 1-forms $\beta_i, i \in \mathbb{N}$, such that

$$d\beta_i = \beta_{i+1} \wedge dF_1$$

Comparing degrees it follows that, for $k \gg 0, \beta_k = 0$. Thus $d\beta_{k-1} = 0$ and there exists a homogeneous polynomial b_{k-1} such that $\beta_{k-1} = db_{k-1}$. Then $d\beta_{k-2} = db_{k-1} \wedge dF_1$ and hence $\beta_{k-2} = b_{k-1}dF_1 + db_{k-2}$ for a suitable homogeneous polynomial b_{k-2} . Then $d\beta_{k-3} = \beta_{k-2} \wedge dF_1 = db_{k-2} \wedge dF_1$. Hence there exists b_{k-3} such that $\beta_{k-3} = b_{k-2}dF_1 + db_{k-3}$. Iterating this, we conclude that $\beta = \beta_0 = b_1dF_1 + db_0$ and therefore

$$d\eta = dF'_1 \wedge dF_0 + dF'_0 \wedge dF_1$$

where $dF'_1 = \alpha$ and $dF'_0 = db_0$, as wanted. \square

2.2. Proof of Theorem 1.1

As a matter of fact we prove the following slightly more precise statement.

THEOREM 2.1. — *If $r \geq 3$ then $\mathcal{R}(r, d_0, d_1)$ is an irreducible component of $\mathcal{F}(r, d)$. Moreover, $\mathcal{F}(r, d)$ is smooth and reduced at the points of \mathcal{U} .*

Proof. — Write as before $\rho : P \dashrightarrow \mathcal{F}$, where $P = \mathbb{P}(\mathbf{S}_{d_0}) \times \mathbb{P}(\mathbf{S}_{d_1})$, $\mathcal{F} = \mathcal{F}(r, d)$ and $\mathcal{R} = \mathcal{R}(r, d_0, d_1)$ is the closure of the image of ρ . Put $F = (F_0, F_1) \in P$. Proposition 2.4 implies that for $\omega = \rho(F)$, the derivative

$$d\rho(F) : T_F P \rightarrow T\mathcal{F}_\omega$$

is surjective and also factors through $T_\omega \mathcal{R} \subseteq T_\omega \mathcal{F}$. Then $T_\omega \mathcal{R} = T_\omega \mathcal{F}$. It follows that \mathcal{R} is an irreducible component of \mathcal{F} and \mathcal{F} is reduced at the generic point of \mathcal{R} . \square

3. Stability of quasi-homogeneous rational maps

In this section we exhibit some previously unknown irreducible components $\mathcal{R}(r, d_0, \dots, d_q)$ of $\mathcal{F}_q(r, d)$, generalizing the case $q = 1$ of the previous section.

A point of $\mathcal{R}(r, d_0, \dots, d_q)$ will be a twisted q -form $\omega \in H^0(\mathbb{P}^r, \Omega^q(d + q + 1))$ of type

$$\omega = i_R(dF_0 \wedge \dots \wedge dF_q) = \sum_{0 \leq j \leq q} (-1)^j d_j F_j dF_0 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_q \quad (3.1)$$

where $F_j \in \mathbf{S}_{d_j}$ is a homogeneous polynomial of degree d_j in $r + 1$ variables, and

$$d_0 + \dots + d_q = d + q + 1. \quad (3.2)$$

We call ω a *rational q -form* in \mathbb{P}^r of type (d_0, \dots, d_q) .

More precisely, $\mathcal{R}(r, d_0, \dots, d_q)$ is defined as the closure of the image of the rational map

$$\rho : \mathbb{P}(\mathbf{S}_{d_0}) \times \dots \times \mathbb{P}(\mathbf{S}_{d_q}) \dashrightarrow \mathbb{P}(H^0(\mathbb{P}^r, \Omega^q(d + q + 1))) \quad (3.3)$$

induced by the multilinear map

$$\mu : \mathbf{S}_{d_0} \times \dots \times \mathbf{S}_{d_q} \rightarrow H^0(\mathbb{P}^r, \Omega^q(d + q + 1))$$

such that $\mu(F_0, \dots, F_q) = i_R(dF_0 \wedge \dots \wedge dF_q)$. The base locus of ρ is described in (4.1) below.

As in the previous section, we define the open subset

$$\mathcal{U} = \{\omega \in \mathcal{R}(r, d_0, \dots, d_q) \mid \text{codim sing}(d\omega) \geq 3 \text{ and } \text{codim sing}(\omega) \geq 2\}. \quad (3.4)$$

With notation as above, our main purpose in this section is to prove the following Theorem 3.1, which is a more precise version of Theorem 1.2 of the Introduction.

THEOREM 3.1. — *Suppose $r \geq 3$ and $1 \leq q \leq r - 2$. Then $\mathcal{R}(r, d_0, \dots, d_q)$ is an irreducible component of $\mathcal{F}_q(r, d)$. Moreover, $\mathcal{F}_q(r, d)$ is smooth and reduced at the points of \mathcal{U} .*

The strategy is the same as the one used to prove Theorem 2.1. Let us denote by $\mathcal{F} = \mathcal{F}_q(r, d)$. The scheme \mathcal{F} is defined by the quadratic equations

$$i(v_J)\omega \wedge \omega = 0 \quad \text{and} \quad i(v_J)\omega \wedge d\omega = 0 \tag{3.5}$$

for all $J \subset \{0, \dots, r\}$ of cardinality $q - 1$.

The tangent space $T_\omega \mathcal{F}$ of \mathcal{F} at a point ω is represented by the forms $\omega' \in \mathbb{H}^0(\mathbb{P}^r, \Omega^q(d+q+1))/(\omega)$ such that $\omega_\epsilon = \omega + \epsilon\omega'$ satisfies the conditions (3.5) modulo ϵ^2 , that is

$$i(v_J)\omega_\epsilon \wedge \omega_\epsilon = 0 \quad \text{and} \quad i(v_J)\omega_\epsilon \wedge d\omega_\epsilon = 0$$

modulo ϵ^2 , for all $J \subset \{0, \dots, r\}$ of cardinality $q - 1$. Expanding, one obtains

$$i(v_J)\omega' \wedge \omega + i(v_J)\omega \wedge \omega' = 0 \quad \text{and} \quad i(v_J)\omega' \wedge d\omega + i(v_J)\omega \wedge d\omega' = 0. \tag{3.6}$$

In order to work out ω' from (3.6) we will need a pair of technical results.

3.1. Lemmata

The first technical Lemma is a generalization of Lemma 2.2 that will be a central tool in the rest of this article.

LEMMA 3.1. — *Let F_0, \dots, F_q be homogeneous polynomial functions on \mathbb{C}^{r+1} and let $\Theta \in \Omega^{q+1}(\mathbb{C}^{r+1})$ be the $(q + 1)$ -form given by*

$$\Theta = dF_0 \wedge \dots \wedge dF_q.$$

Suppose that $\text{codim sing}(\Theta) \geq 3$. If $\eta \in \Omega^{q+1}(\mathbb{C}^{r+1})$ is such that $\eta \wedge dF_i \wedge dF_j = 0$ for every $0 \leq i < j \leq q$ then there exist holomorphic 1-forms $\alpha_0, \dots, \alpha_q \in \Omega^1(\mathbb{C}^{r+1})$ such that

$$\eta = \sum_{i=0}^q \alpha_i \wedge dF_0 \wedge \dots \wedge \widehat{dF_i} \wedge \dots \wedge dF_q.$$

Proof. — For the second item let \mathcal{U} be an open covering of $\mathbb{C}^{r+1} \setminus \text{sing}(\Theta)$. Since $\text{codim sing}(\Theta) \geq 3$ we can assume that over each open set $U \in \mathcal{U}$ our set of functions is part of a coordinate system on U . It is then clear that

$$\eta|_U = \sum \alpha_{i,U} \wedge dF_0 \wedge \dots \wedge \widehat{dF_i} \wedge \dots \wedge dF_q$$

for suitable 1-forms $\alpha_{0,U}, \dots, \alpha_{q,U} \in \Omega^1(U)$.

A simple computation shows that over $U \cap V$

$$(\alpha_{i,U} - \alpha_{i,V}) \wedge \Theta = 0.$$

It follows from Saito's Lemma [15] that there exists a unique $(q+1) \times (q+1)$ matrix $A_{U \cap V}$ with entries in $\mathcal{O}(U \cap V)$ such that

$$\begin{bmatrix} \alpha_{0,U} - \alpha_{0,V} \\ \vdots \\ \alpha_{q,U} - \alpha_{q,V} \end{bmatrix} = A_{U \cap V} \cdot \begin{bmatrix} dF_0 \\ \vdots \\ dF_q \end{bmatrix}$$

Of course the collection of matrices $A_{U \cap V}$ with (U, V) ranging in \mathcal{U}^2 defines an element of $H^1(\mathbb{C}^{r+1} \setminus \text{sing}(\Theta), \mathbb{M} \otimes \mathcal{O}) \cong H^1(\mathbb{C}^{r+1} \setminus \text{sing}(\Theta), \mathcal{O}) \otimes \mathbb{M}$, with \mathbb{M} being the vector space of $(q+1) \times (q+1)$ matrices.

The hypothesis $\text{codim} \text{sing}(\Theta) \geq 3$ implies that this cohomology group is trivial, see for instance [8, pg. 133]. Therefore we may write $A_{U \cap V} = A_U - A_V$ where A_U, A_V are matrices of holomorphic functions in U resp. V . We can thus set

$$\begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_q \end{bmatrix} = \begin{bmatrix} \alpha_{0,U} \\ \vdots \\ \alpha_{q,U} \end{bmatrix} - A_U \cdot \begin{bmatrix} dF_0 \\ \vdots \\ dF_q \end{bmatrix} = \begin{bmatrix} \alpha_{0,V} \\ \vdots \\ \alpha_{q,V} \end{bmatrix} - A_V \cdot \begin{bmatrix} dF_0 \\ \vdots \\ dF_q \end{bmatrix}$$

as the sought global 1-forms at least over $\mathbb{C}^{r+1} \setminus \text{sing}(\Theta)$. To conclude one has just to invoke Hartog's extension Theorem to ensure that these 1-forms extend to \mathbb{C}^{r+1} . \square

By expanding in its homogeneous components both sides of the equality

$$\eta = \sum_{i=0}^q \alpha_i \wedge dF_0 \wedge \dots \widehat{dF_i} \dots \wedge dF_q,$$

it can be easily seen that if η is a homogeneous polynomial q -form then the 1-forms $\alpha_0, \dots, \alpha_q$ can be assumed homogeneous polynomial 1-forms.

The second technical Lemma in this subsection replaces the iteration argument in the proof of Theorem 2.1

LEMMA 3.2. — *For $j = 0, \dots, q$ let $F_j \in \mathbf{S}_{d_j}$ be a homogeneous polynomial of degree d_j . Suppose $\omega = i_R(dF_0 \wedge \dots \wedge dF_q)$ satisfies $\text{codim} \text{sing}(\omega) \geq 3$. Then, for $\alpha \in H^0(\mathbb{P}^r, \Omega^1(e))$ the following conditions are equivalent:*

(a) $d\alpha = \sum_{0 \leq k \leq q} A_k \wedge dF_k$ for some $A_k \in H^0(\mathbb{P}^r, \Omega^1(e - d_k))$.

(b) $\alpha = dG + \sum_{0 \leq k \leq q} H_k dF_k$ for some $G \in \mathbf{S}_e$ and $H_k \in \mathbf{S}_{e-d_k}$.

Proof. — It is clear that (b) implies (a). Let us prove the converse, by induction on $e \in \mathbb{N}$. If (a) holds, applying exterior derivative we get

$$0 = d^2\alpha = \sum_{0 \leq k \leq q} dA_k \wedge dF_k \implies dA_k \wedge dF_0 \wedge \cdots \wedge dF_q = 0.$$

By the hypothesis on the F_j and Lemma 2.2,

$$dA_k = \sum_{0 \leq h \leq q} A_{kh} \wedge dF_h$$

for some $A_{kh} \in H^0(\mathbb{P}^r, \Omega^1(e - d_k - d_h))$. Since $e - d_k < e$, the inductive hypothesis applies to A_k and yields

$$A_k = dG_k + \sum_{0 \leq h \leq q} H_{kh} dF_h$$

for some $G_k \in \mathbf{S}_{e-d_k}$ and $H_k \in \mathbf{S}_{e-d_k-d_h}$. Replacing in (a) we find

$$d\alpha = \sum_k dG_k \wedge dF_k + \sum_{h,k} H_{kh} dF_h \wedge dF_k.$$

Since $i_R\alpha = 0$, we have $e \cdot \alpha = i_R d\alpha$. Applying i_R we obtain, after a little calculation

$$e \cdot \alpha = dG + \sum_{0 \leq k \leq q} H_k dF_k$$

where

$$G = - \sum_k d_k F_k G_k, \quad H_k = (d_k + e)G_k + \sum_h d_h F_h (H_{kh} - H_{hk})$$

as claimed. \square

3.2. Surjectivity of the derivative and proof of Theorem 1.2

Now we are ready to complete the proof of Theorem 3.1 and hence of Theorem 1.2 of the Introduction. The proof follows from Proposition 3.3 below combined with the same argument used in the proof of Theorem 2.1.

PROPOSITION 3.3. — *Suppose $r \geq 3$ and $1 \leq q < r-1$. If $\underline{F} = (F_0, \dots, F_q) \in \prod_i \mathbb{P}(\mathbf{S}_{d_i})$ is such that $\rho(\underline{F}) = \omega \in \mathcal{U}$ then the derivative*

$$d\rho(\underline{F}) : T_{\underline{F}}(\mathbb{P}(\mathbf{S}_{d_0}) \times \cdots \times \mathbb{P}(\mathbf{S}_{d_q})) \rightarrow T_{\omega}\mathcal{F}$$

is surjective.

Proof. — At a point $\underline{F} = (F_0, \dots, F_q)$ belonging to the domain of ρ the derivative

$$d\rho(\underline{F}) : \mathbf{S}_{d_0}/(F_0) \times \dots \times \mathbf{S}_{d_q}/(F_q) \rightarrow T_\omega \mathcal{F} \quad (3.7)$$

is calculated by multilinearity as

$$d\rho(\underline{F})(F'_0, \dots, F'_q) = \sum_{0 \leq j \leq q} i_R(dF_0 \wedge \dots \wedge dF'_j \wedge \dots \wedge dF_q).$$

Let $\omega = \rho(\underline{F}) \in \mathcal{U}$ and $\omega' \in T_\omega \mathcal{F}$. From (3.6) we have

$$i(v_J)\omega' \wedge d\omega = -i(v_J)\omega \wedge d\omega'.$$

Since $d\omega$ is a constant multiple of $dF_0 \wedge \dots \wedge dF_q$ (see Lemma 2.1), by exterior multiplication with dF_j we obtain

$$dF_j \wedge i(v_J)\omega \wedge d\omega' = 0$$

for all j, J .

Let $Y_j, (0 \leq j \leq q)$, be rational vector fields such that $dF_i(Y_j) = \delta_{ij}$. For $J = \{0, \dots, q\} \setminus \{i, j\}$ we have $i(v_J)\omega = \lambda(F_i dF_j - F_j dF_i)$. Then,

$$0 = dF_j \wedge i(v_J)\omega \wedge d\omega' = \lambda dF_j \wedge F_j dF_i \wedge d\omega',$$

which implies that

$$dF_i \wedge dF_j \wedge d\omega' = 0$$

for all $0 \leq i, j \leq q$.

Lemma 3.1 implies that

$$d\omega' = \sum_{0 \leq j \leq q} \alpha_j \wedge dF_0 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_q \quad (3.8)$$

for some $\alpha_j \in H^0(\mathbb{P}^r, \Omega^1(d_j))$. Applying exterior derivative we find

$$0 = d^2\omega' = \sum_{0 \leq j \leq q} d\alpha_j \wedge dF_0 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_q.$$

Taking wedge product with dF_j we get

$$d\alpha_j \wedge (dF_0 \wedge \dots \wedge dF_q) = 0$$

for all j . Therefore, thanks to Lemma 2.2,

$$d\alpha_j = \sum_{0 \leq k \leq q} A_{jk} \wedge dF_k$$

for suitable $A_{jk} \in H^0(\mathbb{P}^r, \Omega^1(d_j - d_k))$. Lemma 3.2 implies that

$$\alpha_j = dG_j + \sum_{0 \leq k \leq q} H_{jk} dF_k$$

for some $G_j \in \mathbf{S}_{d_j}$ and $H_{jk} \in \mathbf{S}_{d_j - d_k}$ (we use the convention $\mathbf{S}_e = 0$ for $e < 0$). Replacing in (3.8) above we have

$$d\omega' = \sum_{0 \leq j \leq q} dG_j \wedge dF_0 \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_q + c dF_0 \wedge \dots \wedge dF_q \quad (3.9)$$

for some $c \in \mathbb{C}$. Since $i_R \omega' = 0$, Lemma 2.1 yields $(\sum_i d_i) \omega' = i_R d\omega'$. Applying i_R to (3.9) and taking (3.7) into account, we obtain

$$\omega' = d\rho(\underline{F})(F'_0, \dots, F'_q)$$

where $F'_j = \frac{(-1)^j}{(\sum_i d_i)} G_j$. Therefore $d\rho(\underline{F})$ is surjective, as claimed. \square

4. Geometry of the parametrization

In this section we analyze the parametrization

$$\rho : \mathbb{P}(\mathbf{S}_{d_0}) \times \dots \times \mathbb{P}(\mathbf{S}_{d_q}) \dashrightarrow \mathcal{R}_q(r, \bar{d}) \subset \mathbb{P}(H^0(\mathbb{P}^r, \Omega^q(d + q + 1))) ,$$

where $\mathbf{S}_{d_i} = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d_i))$, $d = \sum d_i$ and $\bar{d} = (d_0, \dots, d_q)$.

4.1. Base locus

Let us start by describing the base locus $\mathbf{B}(\rho)$ of ρ .

If $i_R(dF_0 \wedge \dots \wedge dF_q) = 0$, applying exterior differentiation and Lemma 2.1 we obtain that $dF_0 \wedge \dots \wedge dF_q = 0$. This means that the Jacobian matrix of F_0, \dots, F_q has rank $< q + 1$ everywhere, that is, the derivative of the map

$$\underline{F} : \mathbb{C}^{r+1} \rightarrow \mathbb{C}^{q+1}$$

defined by $\underline{F}(x) = (F_0(x), \dots, F_q(x))$ has rank $< q + 1$ at every $x \in \mathbb{C}^{r+1}$. This is equivalent to the fact that F is not dominant, that is, $f(F_0, \dots, F_q) = 0$ for some non-zero polynomial $f \in \mathbb{C}[y_0, \dots, y_q]$ (i.e., the F_j are algebraically dependent). We thus obtain

$$\mathbf{B}(\rho) = \{(F_0, \dots, F_q) \in \prod_i \mathbb{P}(\mathbf{S}_{d_i}) \mid \underline{F} : \mathbb{C}^{r+1} \rightarrow \mathbb{C}^{q+1} \text{ is not dominant}\} \quad (4.1)$$

For $q = 1$ the set theoretical description of ρ is rather simple:

$$\mathbf{B}(\rho) = \{(F_0, F_1) \in \mathbb{P}(\mathbf{S}_{d_0}) \times \mathbb{P}(\mathbf{S}_{d_1}) \mid F_0^{d_1} = F_1^{d_0}\}. \quad (4.2)$$

For general q we have a stratification

$$\mathbf{B}(\rho)_1 \subset \mathbf{B}(\rho)_2 \subset \dots \subset \mathbf{B}(\rho)_q = \mathbf{B}(\rho)$$

where $\mathbf{B}(\rho)_k = \{(F_0, \dots, F_q) \mid \dim \text{image}(F) \leq k\}$. The first stratum $\mathbf{B}(\rho)_1$ is set-theoretically equal to

$$\{(F_0, \dots, F_q) \in \prod_i \mathbb{P}(\mathbf{S}_{d_i}) \mid F_0^{\hat{d}_0} = \dots = F_q^{\hat{d}_q}\}$$

where $\hat{d}_j = \prod_{i \neq j} d_i$. For $k > 1$ the same set theoretical description is considerably more complex and we will carry it out only in very particular cases in §5.

Beware that the scheme structure of $\mathbf{B}(\rho)$ is often non-reduced, see §5.6.

At any rate, we register the following easy consequence of Lemma 2.1.

PROPOSITION 4.1. — *Let*

$$\begin{aligned} \tilde{\rho} : \prod_i \mathbb{P}(\mathbf{S}_{d_i}) & \dashrightarrow \mathbb{P}\left(\mathbf{S}_{d-1} \otimes \bigwedge^{q+1} \mathbf{S}_1^*\right) \\ (F_0, \dots, F_q) & \longmapsto dF_0 \wedge \dots \wedge F_q. \end{aligned}$$

Then the base loci of $\tilde{\rho}$ and ρ are one and the same as schemes.

Proof. — Let $V \subset \mathbf{S}_e \otimes \bigwedge^q \mathbf{S}_1^*$ be the subspace of closed q -forms with coefficients of degree e . Put $W = i_R(V) \subset \mathbf{S}_{e+1} \otimes \bigwedge^{q-1} \mathbf{S}_1^*$. Then $i_R : V \rightarrow W$ is a linear isomorphism. We still denote by $i_R : \mathbb{P}(V) \rightarrow \mathbb{P}(W)$ the projectivization. Since the image of $\tilde{\rho}$ lies in $\mathbb{P}(V)$ and $\rho = i_R \circ \tilde{\rho}$, the assertion follows. \square

4.2. Weighted homogeneous polynomials

Fix $\bar{d} = (d_0, \dots, d_q) \in \mathbb{N}^{q+1}$ and $e \in \mathbb{N}$. A polynomial f in $\mathbb{C}[y_0, \dots, y_q]$ is said to be weighted homogeneous of type \bar{d} and degree e if

$$f(\lambda^{d_0} y_0, \dots, \lambda^{d_q} y_q) = \lambda^e f(y_0, \dots, y_q)$$

for any $\lambda \in \mathbb{C}$. Equivalently, f is a linear combination of monomials

$$\prod_{0 \leq j \leq q} y_j^{\alpha_j} \quad \text{such that} \quad \bar{d} \cdot \alpha := \sum_{0 \leq j \leq q} d_j \alpha_j = e.$$

This is tantamount to declaring each variable y_i to be of degree d_i .

We denote by

$$\mathbf{S}_{q, \bar{d}, e}$$

the \mathbb{C} -vector space of all such polynomials and write its dimension as $N(q, \bar{d}, e)$. Notice that $N(q, \bar{d}, e) = \dim \mathbf{S}_{q, \bar{d}, e}$ can be expressed by the Hilbert series

$$H(t) = \sum_e N(q, \bar{d}, e) t^e = \frac{1}{\prod_{i=1}^q (1 - t^{d_i})}.$$

Throughout we will assume that the vector of natural numbers $\bar{d} \in \mathbb{N}^{q+1}$ is non-decreasingly ordered, *i.e.*, $d_0 \leq d_1 \leq \dots \leq d_q$.

Define $\bar{e} = \bar{e}(\bar{d}) = (e_1, \dots, e_k)$ such that $e_i < e_{i+1}$ and $\cup_{0 \leq i \leq q} \{d_i\} = \cup_{1 \leq i \leq k} \{e_i\}$. If n_i stands for the number of times the natural number e_i appears in \bar{d} then the pair (\bar{e}, \bar{n}) , where $\bar{n} = (n_1, \dots, n_k)$, determines \bar{d} .

Set $q_j = -1 + \sum_{1 \leq i \leq j} n_i$, and for $l = 1, \dots, k$

$$\bar{d}_l = (\underbrace{e_1, \dots, e_1}_{n_1 \text{ times}}, \underbrace{e_2, \dots, e_2}_{n_2 \text{ times}}, \dots, \underbrace{e_l, \dots, e_l}_{n_l \text{ times}}).$$

Clearly, for each $f \in \mathbf{S}_{q, \bar{d}, e_j}$, no variable y_i with weight $d_i > e_j$ occurs in f ; thus

$$\mathbf{S}_{q, \bar{d}, e_j} \cong \mathbf{S}_{q_j, \bar{d}_j, e_j}.$$

Denote by $\mathbb{E}^{q+1} = \text{End}(\mathbb{C}^{q+1})$ the set of all polynomial maps $f : \mathbb{C}^{q+1} \rightarrow \mathbb{C}^{q+1}$. It is a ring under sum and composition of maps. If $f = (f_0, \dots, f_q) \in \mathbb{E}^{q+1}$, we say that f is of type \bar{d} if f_i is weighted homogeneous of type \bar{d} and degree d_i , for all $i = 0, \dots, q$.

LEMMA 4.2. — *Maps of type \bar{d} form a subring of \mathbb{E}^{q+1} . More precisely, if $f, g \in \mathbb{E}^{q+1}$ are of type \bar{d} then $f \circ g$ is of type \bar{d} . Moreover, the set*

$$\text{GL}(q, \bar{d}) = \{f \in \mathbb{E}^{q+1} | f \text{ is of type } \bar{d} \text{ and } df(0) \text{ is invertible}\}$$

is a group.

Proof. — Clearly $G = \text{GL}(q, \bar{d})$ is closed under compositions. It remains to show that every element is invertible in G . Let us denote the block of variables of weight e_i by

$$\underline{y}_1 = \underbrace{y_0, \dots, y_{q_1}}_{(\text{weight } e_1)}, \quad \underline{y}_2 = \underbrace{y_{q_1+1}, \dots, y_{q_2}}_{(\text{weight } e_2)}, \quad \dots, \quad \underline{y}_k = \underbrace{y_{q_{k-1}}, \dots, y_{q_k}}_{(\text{weight } e_k)}.$$

The main point is that each $f \in G$ has the following triangular shape,

$$(\underline{f}_1(\underline{y}_1), \underline{f}_2(\underline{y}_1, \underline{y}_2), \dots, \underline{f}_k(\underline{y}_1, \dots, \underline{y}_k)).$$

Here

$$\underline{f}_i(\underline{y}_1, \dots, \underline{y}_i) = (f_{i1}(\underline{y}_1, \dots, \underline{y}_2), \dots, f_{2n_i}(\underline{y}_1, \dots, \underline{y}_i)),$$

with

$$f_{ij}(\underline{y}_1, \dots, \underline{y}_i) = g_{ij}(\underline{y}_1, \dots, \underline{y}_{i-1}) + h_{ij}(\underline{y}_i) \in \mathbf{S}_{q_i, \bar{d}_i, e_i}$$

where $h_{ij}(\underline{y}_i)$ is in fact linear in the block of variables \underline{y}_i of weight e_i . Indeed, since $e_{i+1} > e_i$, no \underline{y}_{i+1} occurs in \underline{f}_i . Thus f can be written as

$$(\underline{h}_1(\underline{y}_1), \underline{h}_2(\underline{y}_2) + \underline{g}_2(\underline{y}_1), \dots, \underline{h}_k(\underline{y}_k) + \underline{g}_k(\underline{y}_1, \dots, \underline{y}_{k-1})).$$

Now we see that $df(0)$ is made up of blocks of the linear maps $\underline{h}_i = d\underline{h}_i : \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_i}$. Hence invertibility of the former is equivalent to $d\underline{h}_i \in \text{GL}_{n_i} \forall i$. Thus, given $(z_1, \dots, z_q) = (f(y))$, one can solve successively

$$\begin{cases} \underline{y}_1 = \underline{h}_1^{-1}(\underline{z}_1), \text{ then} \\ \underline{y}_2 = \underline{h}_2^{-1}(\underline{z}_2 - \underline{g}_2(\underline{y}_1)), \\ \vdots \\ \underline{y}_k = \underline{h}_k^{-1}(\underline{z}_k - \underline{g}_k(\underline{y}_1, \dots, \underline{y}_{k-1})). \end{cases}$$

□

The group $\text{GL}(q, \bar{d})$ naturally acts on the domain of μ (cf. 3.3):

$$\begin{aligned} \text{GL}(q, \bar{d}) \times \prod_{0 \leq j \leq q} \mathbf{S}_{d_j} &\longrightarrow \prod_{0 \leq j \leq q} \mathbf{S}_{d_j} \\ (f, (F_0, \dots, F_q)) &\longmapsto (f_0(\underline{F}), \dots, f_q(\underline{F})). \end{aligned}$$

In other words, considering \underline{F} as a polynomial map $\underline{F} : \mathbb{C}^{r+1} \rightarrow \mathbb{C}^{q+1}$, the action is just composition with a polynomial map $f : \mathbb{C}^{q+1} \rightarrow \mathbb{C}^{q+1}$ which belongs to $\text{GL}(q, \bar{d})$.

4.3. The fibers of ρ

The key tool for the description of the fiber of ρ and the proof of Theorem 1.3 is the following Proposition.

PROPOSITION 4.3. — *Let $\underline{F} = (F_0, \dots, F_q), \underline{G} = (G_0, \dots, G_q) \in \mathbf{S}_{d_0} \times \dots \times \mathbf{S}_{d_q}$. Suppose that both $dF_0 \wedge \dots \wedge dF_q$ and $dG_0 \wedge \dots \wedge dG_q$ are non-zero $(q+1)$ -forms. If $\text{codim sing}(dF_0 \wedge \dots \wedge dF_q) \geq 2$ then the following conditions are equivalent:*

- (a) $i_R(dF_0 \wedge \dots \wedge dF_q) = i_R(dG_0 \wedge \dots \wedge dG_q)$ up to a constant multiple.
- (b) $dF_0 \wedge \dots \wedge dF_q = dG_0 \wedge \dots \wedge dG_q$ up to a constant multiple.
- (c) $dG_j = \sum_{0 \leq k \leq q} A_{jk} dF_k$ for some $A_{jk} \in \mathbf{S}_{d_j - d_k}$, for all j .
- (d) $G_j = f_j(F_0, \dots, F_q)$ for some $f_j \in \mathbb{C}[y_0, \dots, y_q]$, for all j .
- (e) $G_j = f_j(F_0, \dots, F_q)$, for all j for a unique $f_j \in \mathbf{S}_{q, \bar{d}, d_j}$. Moreover, (f_0, \dots, f_q) belongs to $\mathrm{GL}(q, \bar{d})$.

Proof. — (a) \Leftrightarrow (b): Use the identity $d(i_R(dF_0 \wedge \dots \wedge dF_q)) = (q + d)(dF_0 \wedge \dots \wedge dF_q)$ from Lemma 2.1.

(b) \Rightarrow (c): Multiplying by dG_j we obtain $dG_j \wedge dF_0 \wedge \dots \wedge dF_q = 0$. Since \underline{F} is generic, it follows by the division lemma that the dG_j are linear combinations of the dF_k . The coefficients may be chosen as homogeneous polynomials, necessarily of the stated degree.

(c) \Rightarrow (b): Using the hypothesis and calculating wedges we have

$$dG_0 \wedge \dots \wedge dG_q = \det(A) dF_0 \wedge \dots \wedge dF_q.$$

Now $\det(A)$ is a non-zero homogeneous polynomial, and its degree is zero, so it is a constant, thereby proving the claim.

(d) \Rightarrow (e): Let $f_j = \sum_{\alpha} c_{\alpha} y^{\alpha}$, where $\alpha \in \mathbb{N}^{q+1}$ and $c_{\alpha} \in \mathbb{C}$, so that $G_j = \sum_{\alpha} c_{\alpha} F^{\alpha}$. Write $f_j = g_j + h_j$ where g_j is the sum over the exponents α such that $\bar{d} \cdot \alpha = d_j$. We have $h_j(\underline{F}) = 0$ by the homogeneity of G_j and of the F_k . Therefore we may take $f_j = g_j$, the weighted homogeneous polynomial that we needed. Uniqueness is clear since the F_k are algebraically independent. Finally, setting $f = (f_0, \dots, f_q)$, since

$$dG_0 \wedge \dots \wedge dG_q = \det(df) dF_0 \wedge \dots \wedge dF_q$$

it follows that $\det(df) = \det(df(0))$ is a nonzero constant.

(e) \Rightarrow (d): obvious.

(d) \Rightarrow (c): If $G_j = \sum_{\alpha} c_{\alpha} F^{\alpha}$, taking exterior derivative we immediately get dG_j as a linear combination of the dF_k .

(c) \Rightarrow (d): It suffices to use Lemma 4.4 below. □

LEMMA 4.4. — Let $\underline{F} = (F_0, \dots, F_q) \in \mathbf{S}_{d_0} \times \dots \times \mathbf{S}_{d_q}$ be generic. Let G be a homogeneous polynomial of degree e such that $dG = \sum_{0 \leq k \leq q} A_k dF_k$

for some $A_k \in \mathbf{S}_{e-d_k}$. Then $G = f(F_0, \dots, F_q)$ for a unique polynomial $f \in \mathbf{S}_{q, \bar{d}, e}$.

Proof. — We proceed by induction on e . The assertion is clear for $e = 0$. Taking exterior derivative we have $d^2G = \sum_k dA_k \wedge dF_k = 0$. Thus $dA_k \wedge dF_0 \wedge \dots \wedge dF_q = 0$ for all k . Since \underline{F} is generic, we get $dA_k = \sum_h B_{kh} dF_h$ for some $B_{kh} \in \mathbf{S}_{e-d_k-d_h}$. By the inductive hypothesis, $A_k = f_k(F_0, \dots, F_q)$ for some polynomial f_k . On the other hand, applying i_R to $dG = \sum_k A_k dF_k$ we obtain $eG = \sum_k A_k d_k F_k$. Replacing here A_k by $f_k(F_0, \dots, F_q)$ we obtain the claim. Uniqueness and weighted homogeneity were argued before. \square

PROPOSITION 4.5. — For general $\underline{F} = (F_0, \dots, F_q) \in \prod_{0 \leq j \leq q} \mathbf{S}_{d_j}$ we have a bijective map

$$\begin{aligned} \mathrm{GL}(q, \bar{d}) &\longrightarrow \mu^{-1}\mu(\underline{F}) \\ (f_0, \dots, f_q) &\longmapsto (f_0(\underline{F}), \dots, f_q(\underline{F})) \end{aligned}$$

with μ the multilinear map inducing ρ as in (3.3).

Proof. — The assertion follows from the equivalence (a) \iff (e) in 4.3. \square

COROLLARY 4.6. — We have the formula for the fiber dimension,

$$\dim \rho^{-1}\rho(\underline{F}) = \sum_{0 \leq j \leq q} (N(q, \bar{d}, d_j) - 1).$$

4.4. A natural factorization and proof of Theorem 1.3

We will now proceed to describe a tower of open subsets of Grassmann bundles birational to $\mathcal{R}(r, \bar{d})$. We preserve the notation of Subsection 4.2.

Start with $Y_0 = G(n_1, \mathbf{S}_{e_1})$, the grassmannian of n_1 -planes in \mathbf{S}_{e_1} . Let $X_1 \subset Y_1$ be the open subset defined as

$$X_1 = \{F_1 \wedge \dots \wedge F_{n_1} \in G(n_1, \mathbf{S}_{e_1}) \mid \mathrm{codim} \mathrm{sing}(dF_0 \wedge \dots \wedge dF_{n_1}) \geq 2\}.$$

Now let $\mathcal{A}_2 \rightarrow X_1$ be the vector subbundle of the trivial bundle $\mathbf{S}_{e_2} \times X_1$ with fiber over $\underline{F}_1 = F_1 \wedge \dots \wedge F_{n_1} \in X_1$ given by

$$\mathcal{A}_2(\underline{F}_1) = \{G \in \mathbf{S}_{e_2} \mid dF_1 \wedge \dots \wedge dF_{n_1} \wedge dG = 0\}.$$

Recalling Lemma 2.2(a), and the above considerations on weighted homogeneity, we have in fact

$$\mathcal{A}_2(\underline{F}_1) = \{G \in \mathbf{S}_{e_2} \mid G = f(\underline{F}_1), f \in \mathbf{S}_{q_1, \bar{d}_1, e_2}\} \cong \mathbf{S}_{q_1, \bar{d}_1, e_2}.$$

Let $Y_2 = G(n_2, \mathbf{S}_{e_2}/\mathcal{A}_2)$ be the Grassmann bundle over X_1 . Notice that, for an element $\underline{G}_2 = [G_1] \wedge \cdots \wedge [G_{n_2}] \in G(n_2, \mathbf{S}_{e_2}/\mathbf{S}_{q, \bar{d}, e_2}(p))$ over a point $\underline{F}_1 = F_1 \wedge \cdots \wedge F_{n_1} \in X_1$, the $(n_1 + n_2)$ -form

$$\eta(\underline{G}_2) = dF_1 \wedge \cdots \wedge dF_{n_1} \wedge dG_1 \wedge \cdots \wedge dG_{n_2}$$

is well-defined up to a non zero multiplicative constant. Therefore we can set $X_2 \subset Y_2$ as the open subset defined by

$$X_2 = \{\underline{G}_2 \in Y_2 \mid \text{codim sing } \eta(\underline{G}_2) \geq 2\}$$

Continuing, we have a vector subbundle \mathcal{A}_3 of $\mathbf{S}_{e_3} \times X_2$ with fiber

$$\mathcal{A}_3(\underline{F}_1, \underline{G}_2) = \{H \in \mathbf{S}_{e_3} \mid dF_1 \wedge \cdots \wedge dF_{n_1} \wedge dG_1 \wedge \cdots \wedge dG_{n_2} \wedge dH = 0\}.$$

As before, this is isomorphic to $\mathbf{S}_{q_2, \bar{d}_2, e_3}$. Proceeding this way, we arrive at an open subset $X = X_k \subset Y_k$ where $Y_k \rightarrow X_{k-1}$ is the Grassmann bundle $G(n_k, \mathbf{S}_{e_k}/\mathcal{A}_{k-1})$. Clearly X is a rational variety just like all Grassmann bundles over rational varieties. Using Proposition 4.3, we arrive at a birational map from X to $\mathcal{R}(r, \bar{d})$. It follows that $\mathcal{R}(r, \bar{d})$ is rational and this concludes the proof of Theorem 1.3 \square

5. Degree calculations

5.1. Input from intersection theory

Let $q : X \rightarrow \mathbb{P}^N$ be a proper, generically finite map. Then the degree of the image $Y = q(X) \subseteq \mathbb{P}^N$ is given by

$$\int_Y \mathbf{h}^r = \frac{1}{\delta} \int_X q^* \mathbf{h}$$

where \mathbf{h} =hyperplane class, $\delta = \deg q$ is the number of points in $q^{-1}(q(x))$ for general $x \in X$.

Suppose next that $\pi : \mathcal{E} \rightarrow Z$ is a holomorphic vector bundle over a smooth projective variety Z . The total Chern class of E can be written as $c(E) = 1 + c_1(E) + \cdots + c_k(E)$, with $c_i(E) \in A^i(Z)$, the Chow group of codimension i cycles. Chern classes can be thought of as operators on homology or on the total Chow group $A(Z) = \bigoplus A_j(Z)$ by taking cap products: $A_j(Z) \ni z \mapsto c_i(E) \cap z \in A_{j-i}(Z)$. The total Segre class is given by the inverse operator, $s(E) = (1 - \eta)^{-1} = 1 + \eta + \eta^2 + \cdots$, with $\eta = 1 - c(E)$. Thus, $s(E) = 1 - c_1(E) + c_1(E)^2 - c_2(E) + \cdots$. Segre classes give the Gysin

map for a projective bundle, $p : \mathbb{P}(\mathcal{E}) \rightarrow Z$. If \mathbf{h} denotes the relative hyperplane class, we have $p_*(\mathbf{h}^{e+i} \cap p^*(z)) = s_i(E) \cap z$, where $e = \text{rank } \mathcal{E} - 1$. Suppose $\mathcal{E} \rightarrow \mathbb{C}^{N+1} \times Z$ is a vector subbundle. Put $X = \mathbb{P}(\mathcal{E})$. Let $q : X \subset \mathbb{P}^N \times Z \rightarrow \mathbb{P}^N$ be induced by projection. The relative hyperplane class is equal to the pullback of the hyperplane class of \mathbb{P}^N , $q^*\mathbf{h}$, written simply \mathbf{h} . If q is generically injective then we have $\text{deg } q(X) = \int_Z s_n(\mathcal{E})$, with $n = \dim Z$. Indeed, we have $\dim q(X) = \dim X = n + e$. Hence $\text{deg } q(X) = \int_X \mathbf{h}^{e+n} = \int_Z p_*(\mathbf{h}^{e+n}) = \int_Z s_n(\mathcal{E})$.

5.2. Linear projections of grassmannians

We keep the notation as in the previous section. Here we proceed to find the degree of the projective variety

$$\mathcal{R}(r, \bar{d}) \subset \mathbb{P}(\mathbb{H}^0(\mathbb{P}^r, \Omega^q(d + q + 1)))$$

in some cases. Recalling the proof of Proposition 4.1, we see that all degree calculations can be lifted from $\mathbb{P}(W)$ to $\mathbb{P}(V)$.

When $q_1 = q$, i.e. all the degrees d_i are equal to e_1 , the variety X constructed in §4.4 is an open subset of the grassmannian $G(q, \mathbf{S}_{e_1})$. It follows that the morphism $\bar{\rho} : X \rightarrow \mathcal{R}(r, \bar{d})$ gives rise to a rational map

$$\tilde{\rho} : G(q + 1, \mathbf{S}_{e_1}) \dashrightarrow \tilde{\mathcal{R}}(r, \bar{d}) \subset \mathbb{P}(\mathbf{S}_{d-1} \otimes \wedge^{q+1} \mathbf{S}_1^*)$$

Notice that $\bar{\rho}$ is the composition of Plücker's embedding with a central projection

$$\begin{aligned} \mathbb{P}(\wedge^{q+1} S_{e_1}) &\dashrightarrow \mathbb{P}(\mathbf{S}_{d-1} \otimes \wedge^{q+1} \mathbf{S}_1^*) \\ F_0 \wedge \cdots \wedge F_q &\mapsto dF_0 \wedge \cdots \wedge dF_q. \end{aligned}$$

It is a simple exercise to show that $G(q + 1, \mathbf{S}_{e_1})$ is disjoint from the center of this projection if, and only if, $q = 1$ or $d_0 = \cdots = d_q = 1$. In both cases the degree of these components is equal to the degree of the corresponding grassmannians under Plücker's embedding. More precisely, setting $N = (q + 1)(r - q) = \dim G(q + 1, r + 1)$, we have

$\text{deg}(\mathcal{R}(q, 1, \dots, 1)) = \text{deg } G(q + 1, \mathbf{S}_1) = \frac{1!2! \cdots q!N!}{(r - q)!(r - q + 1)! \cdots r!}$
$\text{deg}(\mathcal{R}(1, d, d)) = \text{deg } G(2, \mathbf{S}_{d_1}) = \frac{1}{N_d - 1} \binom{2N_d - 2}{N_d},$
where $N_d = \binom{r + d}{r - 1}.$

(5.1)

5.3. Correction due to base locus

The scheme-theoretic structure of the base locus of a rational map $\phi : Y \dashrightarrow \mathbb{P}(\mathbb{C}^N)$ is defined as follows (cf. [10, 7.17.3, p. 168]). We are given a line bundle (=invertible sheaf) \mathcal{L} over Y together with a homomorphism $\mathcal{O}_Y^N \rightarrow \mathcal{L}$, surjective over the open dense subset $U \subseteq Y$ where ϕ is a morphism. The image, \mathcal{J} , of the induced homomorphism

$$\begin{array}{ccc} \mathcal{O}_Y^N \otimes \mathcal{L}^V & \longrightarrow & \mathcal{O}_Y \\ & \searrow & \uparrow \\ & & \mathcal{J} \end{array}$$

is the sheaf of ideals defining the base locus. If D denotes an effective Cartier divisor such that $\mathcal{J} = \mathcal{O}_Y(-D) \cdot \mathcal{J}'$ for some ideal sheaf \mathcal{J}' , then the set of zeros, $V(\mathcal{J}')$ is contained in $V(\mathcal{J})$. Clearly ϕ extends to the complement $U' = Y \setminus V(\mathcal{J}') \supseteq U$ in such a way that the pullback of the hyperplane bundle is

$$\phi_{|U}^* \mathcal{O}_{\mathbb{P}(\mathbb{C}^N)}(1) = \mathcal{L} \otimes \mathcal{O}(-D).$$

5.4. Case (2,2,2)

The situation is still manageable. It turns out that the scheme of indeterminacy of the rational map

$$\begin{array}{ccc} \tilde{\rho} : X = G(3, \mathbf{S}_2) & \dashrightarrow & \tilde{\mathcal{R}}(r, \bar{d}) \subset \mathbb{P}(\mathbf{S}_3 \otimes \wedge^3 \mathbf{S}_1^*) \\ F_0 \wedge F_1 \wedge F_2 & \longmapsto & dF_0 \wedge dF_1 \wedge dF_2 \end{array}$$

is equal to the image of the Veronese-like embedding

$$\begin{array}{ccc} Y = G(2, \mathbf{S}_1) & \xrightarrow{\vee} & X = G(3, \mathbf{S}_2) \\ \langle L_0, L_1 \rangle & \longmapsto & \langle L_0^2, L_0 L_1, L_1^2 \rangle. \end{array}$$

Thus a single blowup $\pi : \tilde{X} \rightarrow X$ along Y resolves the indeterminacy *i.e.*, the induced map $\tilde{\rho} : \tilde{X} \rightarrow \tilde{\mathcal{R}}(r, \bar{d})$ is a morphism. The reader may consult the Arxiv version [5] for details. This yields the formula for the pullback of the hyperplane class

$$\tilde{\rho}^* \mathbf{h} = \pi^* \mathbf{q}_1 - \mathbf{e},$$

with $\mathbf{e} = [E]$, class of the exceptional divisor $E = \pi^{-1}Z$ and \mathbf{q}_1 is the hyperplane class of the Plücker embedding. See also [6, §4.4, p. 82].

Since $\tilde{\rho}$ is generically injective, the degree of the image can be calculated as

$$\deg \mathcal{R}(r, 2, 2, 2) = \int_{\tilde{X}} \tilde{\rho}^* \mathbf{h}^{\dim X}.$$

Setting $N = \dim X = \dim G(3, \mathbf{S}_2) = 3\left(\binom{r+2}{2} - 3\right)$, the degree is given by

$$\int_{\tilde{X}} \tilde{\rho}^* \mathbf{h}^N = \int_X \pi_* \sum_0^N \binom{N}{i} \pi^* \mathbf{q}_1^i \cdot (-\mathbf{e})^{N-i}.$$

Using projection formula, we are reduced to the calculation of

- the Plücker's degree of $G(3, \mathbf{S}_2)$ for the term with $i = N$,

and

- the contribution of $\pi_*(\mathbf{e})^j = (-1)^{j-1} v_* s_{j-\delta} \mathcal{N}$,

where \mathcal{N} stands for the normal bundle of the embedding v and

$$\delta = \text{rank } \mathcal{N} = \dim G(3, \mathbf{S}_2) - \dim G(2, \mathbf{S}_1).$$

The minus signs come from the formula

$$\iota^* \mathcal{O}_{\tilde{X}}(E) = \mathcal{O}_{\mathcal{N}}(-1).$$

The Segre classes of the normal bundle are obtained from the usual exact sequence

$$TY \succrightarrow TX|_Y \twoheadrightarrow \mathcal{N}.$$

Details can be seen in [5]. We find,

r	deg
3	1324220
4	2860923458080
5	243661972980477736263
6	728440733705107831789517245858
7	704613096513585123585398408696231899176183

$$d_0 = d_1 = d_2 = 2$$

5.5. Bundles of projective spaces

When $k = 2$ and $n_2 = 1$, the variety X constructed in §4.4 is an open subset of a projective bundle over an open subset of a grassmannian. In

general we do not know a manageable compactification. Even when we can compactify X as above, the scheme structure of the base locus of $\bar{\rho}$ can be non reduced and is far from being understood in general.

Nevertheless in the following three cases we are able to handle the degree:

- $q = 1$ and d_0 divides d_1 .
- arbitrary q but $k = 2$ and $d_1 = 1$, *i.e.*, $\bar{d} = (1, \dots, 1, e)$.
- $q = 1$, $d_0 = 2$ and $d_1 = 3$.

5.5.2. $q = 1$ and d_0 divides d_1

This is in fact the only case for which we got a closed formula. Now the natural parameter space, X , is a projective bundle

$$X \longrightarrow \mathbb{P}(\mathbf{S}_{d_0})$$

described in the sequel.

Write the tautological line subbundle over the projective space $\mathbb{P}(\mathbf{S}_{d_0})$,

$$\mathcal{O}_{\mathbf{S}_{d_0}}(-1) \twoheadrightarrow \mathbf{S}_{d_0}.$$

Set $\kappa = d_1/d_0$. Taking symmetric power, we have the exact sequence

$$\mathcal{O}_{\mathbf{S}_{d_0}}(-\kappa) \twoheadrightarrow \mathbf{S}_{d_1} \twoheadrightarrow \bar{\mathbf{S}}_{d_1},$$

which defines the vector bundle $\bar{\mathbf{S}}_{d_1}$. The fiber of $\bar{\mathbf{S}}_{d_1}$ over each $F_0 \in \mathbb{P}(\mathbf{S}_{d_0})$ is the quotient vector space $\mathbf{S}_{d_1}/\langle F_0^\kappa \rangle$. Thus we have

$$\begin{aligned} \tilde{\rho}: X = \mathbb{P}(\bar{\mathbf{S}}_{d_1}) &\longrightarrow \tilde{\mathcal{R}}(r, d_0, d_1) \subseteq \mathbb{P}\left(\mathbf{S}_{d_1+d_0-2} \otimes \wedge^2 \mathbf{S}_1^*\right). \\ (F_0, \bar{F}_1) &\longmapsto dF_0 \wedge dF_1. \end{aligned}$$

The pullback of the hyperplane class via the map $\tilde{\rho}$ is obtained as

$$H = \mathbf{h} + \mathbf{h}'$$

where $\mathbf{h} = c_1 \mathcal{O}_{\mathbf{S}_{d_0}}(1)$, which comes from the base $\mathbb{P}(\mathbf{S}_{d_0})$, and $\mathbf{h}' = c_1 \mathcal{O}_{\bar{\mathbf{S}}_{d_1}}(1)$, the relative hyperplane class. With the notation as in (18), we have

$$\text{rank } \bar{\mathbf{S}}_{d_1} - 1 = N_{d_1} - 2$$

for the fiber dimension of $\mathbb{P}(\overline{\mathbf{S}}_{d_1}) \rightarrow \mathbb{P}(\mathbf{S}_{d_0})$. The sought for degree is

$$\begin{aligned} \deg \mathcal{R}(r, d_0, d_1) &= \int_{\mathbb{P}(\overline{\mathbf{S}}_{d_1})} H^{N_{d_1} + N_{d_0} - 1} = \sum_i (N_{d_1} + N_{d_0} - 1i) \mathbf{h}^i s_{N_{d_0} - i}(\overline{\mathbf{S}}_{d_1}) \\ &= \binom{N_{d_1} + N_{d_0} - 1}{N_{d_0}} - \frac{d_1}{d_0} \binom{N_{d_1} + N_{d_0} - 1}{N_{d_0} - 1}. \end{aligned}$$

The last equality follows from the calculation of the Segre class $s(\overline{\mathbf{S}}_{d_1}) = 1 - \kappa \mathbf{h}$, so $s_i(\overline{\mathbf{S}}_{d_1})$ is zero in degrees $i \geq 2$, cf. [6, p. 47].

If $r = 3$, $d_1 = 2$, $d_0 = 1$, one finds $\binom{3+8}{3} - 2 \binom{11}{2} = 55$. By contrast, the degree of the Segre variety $\check{\mathbb{P}}^3 \times \mathbb{P}^9 \subset \mathbb{P}^{39}$ of which the image of ρ is a rational projection, is equal to $\binom{12}{3}$.

5.5.2. $k = 2$ and $d_0 = 1$

We are now looking at foliations defined by

$$\omega = i_R(dF_0 \wedge \cdots \wedge dF_q)$$

where $\deg F_0 = \cdots = \deg F_{q-1} = 1$; $\deg F_q = d \geq 2$. A natural parameter space is the projective bundle over the grassmannian

$$G = G(q, \mathbf{S}_1)$$

defined as follows. Write the tautological sequence

$$R_q \twoheadrightarrow \mathbf{S}_1 \longrightarrow Q.$$

The fiber of R_q over $\underline{F} \in G$ is the space $\langle F_0, \dots, F_{q-1} \rangle$ spanned by linear forms. Now the last polynomial F_q is taken as a class in the projective space $\mathbb{P}(\mathbf{S}_d / \langle F_0^d, F_0 \cdot F_1^{d-1}, \dots, F_{q-1}^d \rangle)$. The natural homomorphism $\text{Sym}_d R_q \rightarrow \mathbf{S}_d$ is injective; it corresponds to an instance of the vector bundle \mathcal{A}_2 described in 4.4. Form the projective bundle

$$\pi : X = \mathbb{P}(\mathbf{S}_d / \text{Sym}_d R_q) \longrightarrow G.$$

Note that the rational map

$$\begin{array}{ccc} X & \xrightarrow{\bar{\rho}} & \mathbb{P}(\mathbf{S}_{d-1} \otimes \bigwedge^q \mathbf{S}_1^*) \\ (\langle F_0, \dots, F_{q-1} \rangle, \overline{F}_q) & \longmapsto & dF_0 \wedge \dots \wedge dF_{q-1} \wedge dF_q \end{array}$$

is in fact regular everywhere. Indeed, regularity is an open condition; the map is invariant under the natural action of GL_{r+1} and is regular at the

representative $(\langle x_0, \dots, x_{q-1} \rangle, \overline{x_q^{d-1} x_0})$ of the unique closed orbit. Thus the sought for degree can be computed by Schubert calculus in the following manner. Set

$$g = q(r + 1 - q) = \dim G$$

$$N = \binom{r + d}{r} - \binom{q - 1 + d}{q - 1} - 1, \tag{5.3}$$

so that presently the dimension of the component is $\delta = N + g$. The pullback of the hyperplane class from $\mathbb{P}(\mathbf{S}_{d-1} \otimes \bigwedge^q \mathbf{S}_1^*)$ is equal to $\mathbf{h} + \mathbf{q}_1$, where \mathbf{h} stands for the relative hyperplane class of the projective bundle $X \rightarrow G$ and $\mathbf{q}_1 = c_1 Q$. By general principles, the degree is given by

$$\int_X (\mathbf{h} + \mathbf{q}_1)^\delta = \sum_0^g \binom{\delta}{i} \int_G \pi_* (\mathbf{h}^{\delta-i}) \mathbf{q}_1^i = \sum_0^g \binom{\delta}{i} \int_G s_{g-i} \cdot \mathbf{q}_1^i.$$

Here $s_i = c_i(\text{Sym}_d R_q)$. For $q = 2, r = 3$ we find

$$d^2(d - 1)(d + 3)(d^2 + 2)(d^2 + 4d + 6)(d + 2)^2(d + 1)^2 / (2^6 \cdot 3^5),$$

a polynomial of degree 12 in d . For $q = 2; r = 4, 5, 6, 7, 8$ we find polynomial formulas of respective degrees 24, 40, 60, 84, 112. This suggests a polynomial degree like $2r(r - 1)$. Now for $q = 3, r = 4, 5, 6, 7, 8$ we get polynomial formulas of degrees $3r(r - 2)$ with respect to d . Further experiments (cf. [17]) suggest polynomial formulas of degrees $qr(r - q + 1)$. Here is a sample for small values of r, q, d .

$(r, q) = (5, 2)$				
d	2	3	4	5
deg	2390850	10457430102	9654013512864	3099059696318355

$(r, q) = (6, 2)$				
d	2	3	4	5
deg	1139133688	91451421683006	1118409272891730904	3524857658574891999976

$(r, q) = (6, 3)$				
2	3	4	5	
8983484048	9350781792221835	1060759743612735149417	22044166363067583367287424	

5.6. $(2, 2m + 1)$

Assume $q = 1, d_0 = 2$ and $d_1 = 2m + 1$. Set for short $\mathbf{X} = \mathbb{P}(\mathbf{S}_2) \times \mathbb{P}(\mathbf{S}_{2m+1})$. Put as before $N_d = \binom{r + d}{d} - 1$. We have

$$\dim \mathbf{X} = N_2 + N_{2m+1}.$$

The indeterminacy locus of

$$\begin{aligned} \tilde{\rho} : \mathbf{X} & \dashrightarrow \mathbb{P} \left(\mathbf{S}_{2m+1} \otimes \wedge^2 \mathbf{S}_1^* \right) \\ (F, G) & \longmapsto dF \wedge dG. \end{aligned}$$

is, set-theoretically, the “bi-Veronese”,

$$\mathbf{V} = \{(L^2, L^{2m+1}) \mid L \in \mathbb{P}(\mathbf{S}_1)\}.$$

There is compelling computer algebra evidence indicating that the indeterminacy locus of the rational map $\tilde{\rho}$ is a thickening of \mathbf{B} . Blowing up the reduced structure, the new indeterminacy locus, \mathbf{B}' , of the induced rational map $\mathbf{X}' \dashrightarrow \mathbb{P}(\mathbf{S}_{2m+1} \otimes \wedge^2 \mathbf{S}_1^*)$ is reduced only for $m = 1$. Nevertheless, it still is a rather manageable complete intersection. In fact, we find local equations of \mathbf{B}' of the form e^m, f_1, \dots, f_u , with e denoting the equation of the exceptional divisor, and the f_i 's define a projective subbundle of the exceptional divisor. We find that the reduced structure $\mathbf{B}'_{\text{red}} = \mathbb{P}(\mathcal{N}_{\mathbb{P}(S_1)/\mathbf{B}})$ is the projectivization of the normal bundle of $\mathbb{P}(S_1)$ in \mathbf{B} , as indicated in the diagram

$$\begin{array}{ccccc} T\mathbb{P}(S_1) & \xlongequal{\quad} & T\mathbb{P}(S_1) & & \\ \downarrow & & \downarrow & & \\ T\mathbf{B} & \xrightarrow{\quad} & TX & \xrightarrow{\quad} & \mathcal{N}_{\mathbf{B}} \text{ (rank} = \dim \mathbb{P}(S_q)) \\ \downarrow & & \downarrow & & \parallel \\ \mathcal{N}_{\mathbb{P}(S_1)/\mathbf{B}} & \xrightarrow{\quad} & \mathcal{N}_{\mathbb{P}(S_1)/X} & \xrightarrow{\quad} & \mathcal{N}_{\mathbf{B}}, \end{array}$$

all restricted to $\mathbb{P}(S_1)$. Using this, we find the following table for the first few degrees in dimensions 3 and 4. Details can be read in [5].

$\deg \mathcal{R}(r, d_0 = 2, d_1 = 2m + 1)$	
d_1	$\deg(\mathbb{P}^3)$
5	27500627268
7	19062120397608
9	3910289698588916
11	341013122932980120
d_1	$\deg(\mathbb{P}^4)$
5	5858652068789831804
7	2734930355086609774678630
9	118796991387599661786404269060
11	955667356931740162987705236374200

Interpolating the first few values of odd d_1 , we find for \mathbb{P}^3 the polynomial $(t-1)(t^{26}+55t^{25}+1450t^{24}+24616t^{23}+305020t^{22}+2961172t^{21}+23561656t^{20}+158392960t^{19}+918866662t^{18}+4670514826t^{17}+21033417148t^{16}+84615935632t^{15}+305921226844t^{14}+998318576836t^{13}+2949392111320t^{12}+7903552056256t^{11}+19229223618721t^{10}+41774679574903t^9+72390849730794t^8+15945324910344t^7-541088235621216t^6-2539188961011216t^5-315410776482528t^4+14933666207688192t^3+85822791395378688t^2-247712474710388736t+162893498195312640)/3656994324480$.

It fits all values of $\deg \mathcal{R}(3, 2, t)$, $t = 2m + 1$, up to $m = 35$, presently the physical limit of our computer's memory. It should be noted that $\deg \mathcal{R}(3, 2, 2t) = \binom{N_{2t} + N_2 - 1}{N_2} - \frac{2t}{2} \binom{N_{2t} + N_2 - 1}{N_2 - 1}$ is a polynomial in t of the same degree 27 as in (19) above.

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