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Eigenvalue distribution for non-self-adjoint operators with small multiplicative random perturbations

Eigenvalue distribution for non-self-adjoint operators with small multiplicative random perturbations

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Résumé. — Dans ce travail nous continuons l'étude de l'asymptotique de Weyl de la distribution des valeurs propres d'opérateurs (pseudo-)différentiels avec des perturbations aléatoires petites, en traitant le cas des perturbations multiplicatives en dimension quelconque. Nous avons été amenés à faire des améliorations essentielles des aspects probabilistes.

Abstract. — In this work we continue the study of the Weyl asymptotics of the distribution of eigenvalues of non-self-adjoint (pseudo)differential operators with small random perturbations, by treating the case of multiplicative perturbations in arbitrary dimension. We were led to quite essential improvements of many of the probabilistic aspects.

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1. Introduction

In [6] Mildred Hager considered a class of randomly perturbed semi-classical unbounded (pseudo-)differential operators of the form

$$P_\delta = P(x, hD_x; h) + \delta Q_\omega, \ 0 < h \ll 1, \tag{1.1}$$

on $L^2(\mathbf{R})$, where $P(x, hD_x; h)$ is a non-self-adjoint pseudodifferential operator of some suitable class (including differential operators) with leading symbol $p(x, \xi)$ and where $Q_\omega u(x) = q_\omega(x)u(x)$ is a random multiplicative perturbation and $\delta > 0$ is a small parameter.

Let $\Gamma \Subset \mathbf{C}$ have smooth boundary and assume that $p^{-1}(z)$ is finite for every $z \in \Gamma$ and also that $\{p, \overline{p}\}(\rho) \neq 0$ for every $\rho \in p^{-1}(\Gamma)$. Then under some additional assumptions Hager showed that for $\delta = e^{-\epsilon/h}$, the number $\#(\sigma(P_\delta) \cap \Gamma)$ of eigenvalues of P_δ in Γ satisfies

$$\left| \#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{2\pi h} \mathrm{vol}\,(p^{-1}(\Gamma)) \right| \leqslant \frac{C\sqrt{\epsilon}}{h}, \tag{1.2}$$

with a probability very close to 1 in the limit of small h.

Recently, W. Bordeaux-Montrieux [1] established almost sure Weyl asymptotics for the large eigenvalues of elliptic operators and systems on S^1 under assumptions quite similar to those of Hager. The one-dimensional nature of the problems is essential in the proofs in [6, 1].

In [7], Hager and the author found a new approach and extended the results of [6] to the case of operators on $\mathbf{R}^n$ and replaced the assumption about the non-vanishing of $\{p, \overline{p}\}$ by a weaker condition, allowing Γ to contain boundary points of $\overline{p(\mathbf{R}^{2n})}$. In dimension $\geqslant 2$, it turned out to be simpler to consider general random perturbations of the form

$$\delta Q_\omega u = \delta \sum \sum \alpha_{j,k}(\omega)(u|f_k)e_j, \tag{1.3}$$

where $\{e_j\}$, $\{f_k\}$ are orthonormal families of eigenfunctions of certain elliptic h-pseudodifferential operators of Hilbert Schmidt class and $\alpha_{j,k}(\omega)$ are independent complex Gaussian random variables. With some exageration, the results of [7] show that most non-self-adjoint pseudodifferential operators obey Weyl-asymptotics, but since the perturbations are no more multiplicative, we did not have the same conclusion for the differential operators.

The purpose of the present paper is to treat the case of multiplicative perturbations in any dimension. Several elements of [7] carry over to the multiplicative case, while the study of a certain effective Hamiltonian, here a finite random matrix, turned out to be more difficult. Because of that we were led to abandon the fairly explicit calculations with Gaussian random variables and instead resort to arguments from complex analysis. A basic difficulty was then to find at least one perturbation within the class of permissible ones, for which we have a lower bound on the determinant of the associated effective Hamiltonian. This is achieved via an iterative ("renormalization") procedure, with estimates on the singular values at each step. An advantage with the new approach is that we can treat more general random perturbations.

We next state the main result of this work. For simplicity we shall work on $\mathbf{R}^n$, where some results from [7] are already available. In principle the extension of our results to the case of compact manifolds should only present moderate technical difficulties.

Let us first specify the assumptions about the unperturbed operator.

Let $m \geqslant 1$ be an order function on $\mathbf{R}^{2n}$ in the sense that

$$m(\rho) \leqslant C_0 \langle \rho - \mu \rangle^{N_0} m(\mu), \ \rho, \mu \in \mathbf{R}^{2n}$$

for some fixed positive constants C_0, N_0, where we use the standard notation $\langle \rho \rangle = (1 + |\rho|^2)^{1/2}$.

Let

$$p \in S(m) := \{a \in C^\infty(\mathbf{R}^{2n}); |\partial_\rho^\alpha a(\rho)| \leqslant C_\alpha m(\rho), \forall \rho \in \mathbf{R}^{2n}, \alpha \in \mathbf{N}^{2n}\}.$$

We assume that $p - z$ is elliptic (in the sense that $(p - z)^{-1} \in S(m^{-1})$) for at least one value $z \in \mathbf{C}$. Put $\Sigma = \overline{p(\mathbf{R}^{2n})} = p(\mathbf{R}^{2n}) \cup \Sigma_\infty$, where Σ_∞ is the set of accumulation values of $p(\rho)$ near $\rho = \infty$. Let $P(\rho) = P(\rho; h)$, $0 < h \leqslant h_0$ belong to $S(m)$ in the sense that $|\partial_\rho^\alpha P(\rho; h)| \leqslant C_\alpha m(\rho)$ as above, with contants that are independent of h. Assume that there exist

$p_1, p_2, ... \in S(m)$ such that

$$P \sim p + h p_1 + ... \text{ in } S(m), \ h \to 0.$$

By $P = P(x, hD_x; h)$ we also denote the Weyl quantization of $P(x, h\xi; h)$ (see for instance [2]). Let $\Omega \Subset \mathbf{C}$ be open simply connected with $\overline{\Omega} \cap \Sigma_\infty = \emptyset$, $\Omega \not\subset \Sigma$. Then for $h > 0$ small enough, the spectrum $\sigma(P)$ of P is discrete in Ω and constituted of eigenvalues of finite algebraic multiplicity. We will also need the symmetry assumption,

$$P(x, -\xi; h) = P(x, \xi; h). \tag{1.4}$$

Let $V_z(t) := \mathrm{vol}\,(\{\rho \in \mathbf{R}^{2n}; |p(\rho) - z|^2 \leqslant t\})$. For $\kappa \in]0, 1]$, $z \in \Omega$, we consider the property that

$$V_z(t) = \mathcal{O}(t^\kappa), \ 0 \leqslant t \ll 1. \tag{1.5}$$

Let K be a compact neighborhood of $\pi_x p^{-1}(\Omega)$, where π_x denotes the natural projection from the cotangent bundle to the base space. The random potential will be of the form

$$q_\omega(x) = \chi_0(x) \sum_{0 < \mu_k \leqslant L} \alpha_k(\omega) \epsilon_k(x), \ |\alpha|_{\mathbf{C}^D} \leqslant R, \tag{1.6}$$

where ϵ_k is the orthonormal basis of eigenfunctions of $h^2 \widetilde{R}$, where $\widetilde{R}$ is an h-independent positive elliptic 2nd order operator with smooth coefficients on a compact manifold of dimension n, containing an open set diffeomorphic to an open neighborhood of supp χ_0. Here $\chi_0 \in C_0^\infty(\mathbf{R}^n)$ is equal to 1 near K. μ_k^2 denote the corresponding eigenvalues, so that $h^2 \widetilde{R} \epsilon_k = \mu_k^2 \epsilon_k$. We choose $L = L(h)$ and $R = R(h)$ in the intervals

$$h^{\frac{\kappa - 3n}{s - \frac{n}{2} - \epsilon}} \ll L \leqslant C h^{-M}, \qquad M \geqslant \frac{3n - \kappa}{s - \frac{n}{2} - \epsilon}, \tag{1.7}$$

$$\frac{1}{C} h^{-(\frac{n}{2} + \epsilon)M + \kappa - \frac{3n}{2}} \leqslant R \leqslant C h^{-\widetilde{M}}, \qquad \widetilde{M} \geqslant \frac{3n}{2} - \kappa + (\frac{n}{2} + \epsilon)M,$$

for some $\epsilon \in]0, s - \frac{n}{2}[$, $s > \frac{n}{2}$, so by Weyl's law for the large eigenvalues of elliptic self-adjoint operators, the dimension D is of the order of magnitude $\mathcal{O}((L/h)^n)$. We introduce the small parameter[1] $\delta = \tau_0 h^{N_1 + n}$, $\tau_0 = \tau_0(h) \in]0, \sqrt{h}]$, where

$$N_1 := \widetilde{M} + sM + \frac{n}{2}. \tag{1.8}$$

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[1] In the proof of the main result, we get $\delta = \tau_0 h^{N_1 + n}/C$ for some large constant C, but a dilation in τ_0 can easily be absorbed in the constants later on.

The randomly perturbed operator is

$$P_\delta = P + \delta h^{N_1} q_\omega =: P + \delta Q_\omega. \tag{1.9}$$

We have chosen the exponent N_1 so that $\|h^{N_1}q\|_{L^\infty} \leqslant \mathcal{O}(1)h^{-n/2}\|h^{N_1}q\|_{H^s} \leqslant \mathcal{O}(1)$, when q is an admissible potential as in (1.6), (1.7) and H^s is the semiclassical Sobolev space in Section 2. The lower bounds on L, R are dictated by the construction of a special admissible potential in Sections 6, 7.

The random variables $\alpha_j(\omega)$ will have a joint probability distribution

$$P(d\alpha) = C(h)e^{\Phi(\alpha;h)}L(d\alpha), \tag{1.10}$$

where for some $N_4 > 0$,

$$|\nabla_\alpha \Phi|_{\mathbf{C}^D} = \mathcal{O}(h^{-N_4}), \tag{1.11}$$

and $L(d\alpha)$ is the Lebesgue measure. ($C(h)$ is the normalizing constant, assuring that the probability of $B_{\mathbf{C}^D}(0, R)$ is equal to 1.)

We also need the parameter

$$\epsilon_0(h) = (h^\kappa + h^n \ln \frac{1}{h})(\ln \frac{1}{\tau_0} + (\ln \frac{1}{h})^2) \tag{1.12}$$

and assume that $\tau_0 = \tau_0(h)$ is not too small, so that $\epsilon_0(h)$ is small. The main result of this work is:

THEOREM 1.1. — *Under the assumptions above, let $\Gamma \Subset \Omega$ have smooth boundary, let $\kappa \in]0, 1]$ be the parameter in (1.6), (1.7), (1.12) and assume that (1.5) holds uniformly for z in a neighborhood of $\partial\Gamma$. Then there exists a constant $C > 0$ such that for $C^{-1} \geqslant r > 0$, $\widetilde{\epsilon} \geqslant C\epsilon_0(h)$ we have with probability*

$$\geqslant 1 - \frac{C\epsilon_0(h)}{rh^{n+\max(n(M+1),N_4+\widetilde{M})}}e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} \tag{1.13}$$

that:

$$|\#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma))| \leqslant \tag{1.14}$$

$$\frac{C}{h^n}\left(\frac{\widetilde{\epsilon}}{r} + C\left(r + \ln(\frac{1}{r})\mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0, r)))\right)\right).$$

Here $\#(\sigma(P_\delta) \cap \Gamma)$ denotes the number of eigenvalues of P_δ in Γ, counted with their algebraic multiplicity.

Actually, we shall prove the theorem for the slightly more general operators, obtained by replacing P by P_0 in (7.6).

The second volume in (1.14) is $\mathcal{O}(r^{2\kappa-1})$ which is of interest when $\kappa > 1/2$. In that case

$$\ln \frac{1}{r} \operatorname{vol} \left(p^{-1}(\partial\Gamma + D(0,r)) \right) = \mathcal{O}(r^\beta), \tag{1.15}$$

for any $\beta \in]0, 2\kappa-1[$. Even if $\kappa < 1/2$ we can reasonably assume that (1.15) holds for some $\beta > 0$. (For instance if p is real-valued and Γ does not contain any critical values of p, then (1.5) holds uniformly for z in a neighborhood of $\partial\Gamma$ with $\kappa = 1/2$, but if we choose Γ so that its boundary can only intersect the real axis transversally, then $\operatorname{vol} \left(p^{-1}(\partial\Gamma + D(0,r)) \right) = \mathcal{O}(r)$.) Assuming (1.15) for some $\beta > 0$ we choose $r = \widetilde{\epsilon}^{\frac{1}{\beta+1}}$ and the right hand side of (1.14) is $\leqslant C h^{-n} \widetilde{\epsilon}^{\beta/(1+\beta)}$, which gives Weyl asymptotics, if $\widetilde{\epsilon}$ is small.

If we assume that

$$\exp(-h^{-\kappa_0}) \leqslant \tau_0 \leqslant \sqrt{h}, \text{ for some } \kappa_0 \in]0, \kappa[, \tag{1.16}$$

then

$$\epsilon_0 = \mathcal{O}(h^{\kappa-\kappa_0} \ln \frac{1}{h}) \tag{1.17}$$

is small. Now take $\widetilde{\epsilon} = h^{\widetilde{\kappa}}$, for some $\widetilde{\kappa} \in]0, \kappa-\kappa_0[$. Then, we get the following corollary:

COROLLARY 1.2. — *We make the general assumptions of Theorem 1.1. Assume (1.15) for some $\beta > 0$ and recall that this is automatically the case when $\kappa > 1/2$ and $0 < \beta < 2\kappa - 1$. Choose δ as prior to (1.9) with τ_0 as in (1.16). Let $0 < \widetilde{\kappa} < \kappa - \kappa_0$. Then, with probability*

$$\geqslant 1 - \frac{C h^{\kappa-\kappa_0} \ln \frac{1}{h}}{h^{\frac{\widetilde{\kappa}}{1+\beta} + n + \max(n(M+1), N_4+\widetilde{M})}} e^{-h^{\widetilde{\kappa}-(\kappa-\kappa_0)}/(C \ln \frac{1}{h})}, \tag{1.18}$$

we have

$$\left| \#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{(2\pi h)^n} \operatorname{vol} \left(p^{-1}(\Gamma) \right) \right| \leqslant \frac{C}{h^n} h^{\frac{\widetilde{\kappa}\beta}{1+\beta}}. \tag{1.19}$$

As in [7] we also have a result valid simultaneously for a family $\mathcal{C}$ of domains $\Gamma \subset \Omega$ satisfying the assumptions of Theorem 1.1 uniformly in the natural sense: With a probability

$$\geqslant 1 - \frac{\mathcal{O}(1)\epsilon_0(h)}{r^2 h^{n+\max(n(M+1), N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}}, \tag{1.20}$$

the estimate (1.14) holds simultaneously for all $\Gamma \in \mathcal{C}$. The corresponding variant of Corollary 1.2 holds also; just replace $\frac{\widetilde{\kappa}}{1+\beta}$ in the exponent of the denominator in (1.18) by $\frac{2\widetilde{\kappa}}{1+\beta}$.

Remark 1.3. — When $\widetilde{R}$ has real coefficients, we may assume that the eigenfunctions ϵ_j are real. Then (cf Remark 8.3) we may restrict α in (1.6) to be in $\mathbf{R}^D$ so that q_ω is real, still with $|\alpha| \leqslant R$, and change $C(h)$ in (1.10) so that P becomes a probability measure on $B_{\mathbf{R}^D}(0, R)$. Then Theorem 1.1 remains valid. This might be of interest in resonance counting problems, where self-adjointness of the operator should be preserved in the interior region where no complex scaling is performed.

Remark 1.4. — We believe that the main result of this paper can also be proved in the case when $\mathbf{R}^n$ is replaced by a compact manifold. Taking this for granted, we see that the assumption (1.4) cannot be completely eliminated. Indeed, let $P = hD_x + g(x)$ on $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$ where g is smooth and complex valued. Then (cf Hager [5]) the spectrum of P is contained in the line $\Im z = \int_0^{2\pi} \Im g(x) dx/(2\pi)$. This line will vary only very little under small multiplicative perturbations of P so Theorem 1.1 cannot hold in this case.

When $z \in \Sigma \setminus \Sigma_\infty$ and $(\Re z, \Im z)$ is not a critical value of the map $(x, \xi) \to (\Re p, \Im p)$, then (1.5) holds with $\kappa = 1$. Since the critical values form a set of Lebesgue measure zero by Sard's theorem, this is what we expect for most z. However such points are necessarily interior points of Σ (by the implicit function theorem) and it is particularly important to study the distribution of eigenvalues near the boundary. When $z \in \partial\Sigma \setminus \Sigma_\infty$, and $\{p, \{p, \overline{p}\}\} \neq 0$ at every point of $p^{-1}(z)$, then we saw in [7], Example 12.1, that (1.5) holds with $\kappa = \frac{3}{4}$.

Example 1.5. — Let $1 \leqslant m_0(x)$ be an order function on $\mathbf{R}^n$, let $V \in S(m_0)$ be a smooth potential which is elliptic in the sense that $|V(x)| \geqslant m_0(x)/C$ and assume that $-\pi + \epsilon_0 \leqslant \arg(V(x)) \leqslant \pi - \epsilon_0$ for some fixed $\epsilon_0 > 0$. Then it is easy to see that $p(x, \xi) := \xi^2 + V(x)$ is an elliptic element of $S(m)$, where $m(x, \xi)$ is the order function $m_0(x) + \xi^2$. Let $\Sigma_\infty(V)$ be the set of accumulation points of $V(x)$ at infinity and define $\Sigma(V) = \overline{V(\mathbf{R}^n)} = V(\mathbf{R}^n) \cup \Sigma_\infty(V)$. Then with Σ and Σ_∞ defined for p as above, we get $\Sigma = \Sigma(V) + [0, +\infty[$, $\Sigma_\infty = \Sigma_\infty(V) + [0, +\infty[$. Using the fact that $\partial_{\xi_1}^2 \Re p \geqslant 1/C$, we further see that if $\widetilde{K} \subset \mathbf{C}$ is compact and disjoint from Σ_∞, then (1.5) holds uniformly for $z \in \widetilde{K}$ with $\kappa = 1/4$. The non-self-adjoint Schrödinger operator $P := -h^2\Delta + V(x)$ has $P(x, \xi) = p(x, \xi)$ as its symbol and (1.4) is fulfilled. This means that Theorem 1.1 is applicable, but to have

an interesting conclusion, we have to look for domains Γ for which (1.15) holds for some $\beta > 0$.

The conditions on the random perturbations are clearly not the most general ones attainable with the methods of this paper and further generalizations may come naturally when looking at new problems. It should be possible to consider infinite sums in (1.6) and drop the upper bound on the size of α, provided that we add assumptions on the probability in (1.10), (1.11). Here, we just give an example where the upper bound $|\alpha|_{\mathbf{C}^D} \leqslant R$ can be removed: Consider

$$q_\omega(x) = \chi_0(x) \sum_1^D \alpha_k(\omega) \epsilon_k(x), \qquad (1.21)$$

as in (1.6). We now assume that $\alpha_k(\omega)$ are independent Gaussian $\mathcal{N}(0, \sigma_k^2)$-laws, i.e. with probability distribution

$$\frac{1}{\pi \sigma_k^2} e^{-\frac{|\alpha_k|^2}{\sigma_k^2}} L(d\alpha_k). \qquad (1.22)$$

Then $P(d\alpha)$ is of the form (1.10) (now normalized on $\mathbf{C}^D$ rather than on the ball $B_{\mathbf{C}^D}(0, R)$) with

$$\Phi(\alpha; h) = -\sum_{k=1}^D \frac{|\alpha_k|^2}{\sigma_k^2}.$$

On $B_{\mathbf{C}^D}(0, R)$, we have

$$\|\nabla \Phi\| = \mathcal{O}(1) \frac{R}{\min \sigma_k^2},$$

so (1.11) holds for some N_4, provided that R is bounded by some negative power of h as in (1.6) and

$$\min \sigma_k \text{ is bounded from below by some power of } h. \qquad (1.23)$$

As we saw in [7] and further improved and simplified by Bordeaux Montrieux [1], the probability that $|\alpha|_{\mathbf{C}^D} \geqslant R$ is

$$\leqslant \exp\left(\frac{C_0}{2 \min \sigma_j^2} \sum \sigma_j^2 - \frac{R^2}{2 \min \sigma_j^2} \right),$$

so

$$P(|\alpha|_{\mathbf{C}^D} \geqslant R) \leqslant e^{-h^{-\hat{\kappa}}}, \qquad (1.24)$$

for h small enough, where $\widehat{\kappa}$ is any given fixed positive number, provided that $\max \sigma_j$ is bounded from above by some power of h and we choose $R \asymp h^{-\widetilde{M}}$ for $\widetilde{M}$ large enough. Hence Theorem 1.1 is applicable.

The remainder of this paper is devoted to the proof of Theorem 1.1. Much of the proof follows the strategy of [7] but there are also some essential differences, since we had to abandon the fairly explicit random matrix considerations there. As in [7] we identify the eigenvalues with the zeros of a holomorphic function, here $F_\delta(z; h) = \det(P_{\delta,z})$, where $P_{\delta,z} = (\widetilde{P}_\delta - z)^{-1}(P_\delta - z) = 1 + (\widetilde{P}_\delta - z)^{-1}(P - \widetilde{P})$, $\widetilde{P}_\delta = P_\delta + \widetilde{P} - P$ and $\widetilde{P}$ is a new pseudodifferential operator, whose symbol coincides with the one of P outside a compact set and such that $\widetilde{P} - z$ is elliptic for all $z \in \Omega$. In *Sections 2, 3* we prepare this approach by showing that δQ_ω is bounded and has small norm: $H^\sigma \to H^\sigma$ for $-s \leqslant \sigma \leqslant s$, where H^σ is the standard Sobolev space equipped with a natural semi-classical h-dependent norm). We also need to understand some localization and boundedness properties of the resolvent and the spectral projections corresponding to small eigenvalues of the self-adjoint operators $S_{\delta,z} = P_{\delta,z}^* P_{\delta,z}$ and $S_\delta = (P_\delta - z)^*(P_\delta - z)$.

In *Section 4*, we apply results from [7] to estimate the number of small eigenvalues of $S_{\delta,z}$ and S_δ. Using this, we set up an auxiliary invertible "Grushin" matrix

$$\mathcal{P}_\delta = \begin{pmatrix} P_{\delta,z} & R_- \\ R_+ & 0 \end{pmatrix} : L^2(\mathbf{R}^n) \times \mathbf{C}^N \to L^2(\mathbf{R}^n) \times \mathbf{C}^N,$$

where $N = \mathcal{O}(\alpha^\kappa h^{-n})$ is the number of eigenvalues of $S_{\delta,z}$ that are $\leqslant \alpha$ where $\alpha = Ch$ for some large constant C, and we establish (4.43) saying roughly that

$$\ln|\det \mathcal{P}_\delta| \approx \frac{1}{(2\pi h)^n} \iint \ln|p_z(x, \xi)|dxd\xi, \quad p_z = \frac{p - z}{\widetilde{p} - z},$$

where $\widetilde{p}$ denotes the leading symbol of $\widetilde{P}$. If E_{-+}^δ denotes the lower right entry in the block matrix of $\mathcal{P}_\delta^{-1}$ then $\det P_\delta = \det \mathcal{P}_\delta + \det E_{-+}^\delta$ as we showed in [7] using some calculation from [13]. Using that the size N of E_{-+}^δ is $\ll h^{-n}$, we get a nice upper bound on $\ln|\det E_{-+}^\delta|$ and it follows that for z in a neighborhood of $\partial\Gamma$,

$$\ln|F_\delta| \leqslant \frac{1}{(2\pi h)^n} \iint \ln|p_z(x, \xi)|dxd\xi + \text{"small"}. \tag{1.25}$$

See (7.48) for a more precise statement.

The crucial step (as in [6, 7]) is to get a corresponding lower bound with probability close to 1 for each z, and this amounts to getting a corresponding

lower bound for $\ln|\det E^{\delta}_{-+}|$. In [7] we did so by showing that E^{δ}_{-+} (there) was quite close to a random matrix with independent Gaussian entries. In the case of multiplicative perturbations, such an explicit approach seems out of reach even if we assume the α_j to be independent Gaussian random variables. Instead we choose a different approach based on complex analysis and Jensen's formula in the α-variables. The main step in this new approach is then to construct one admissible potential as in (1.6), (1.7) (ie to find one special value of $\alpha \in B_{\mathbf{C}^D}(0, R)$), for which $|\det E^{\delta}_{-+}|$ is not too small. When trying to do so, one is led to consider the singular values of E^{δ}_{-+} or equivalently (as we shall see) the small singular values of $P_{\delta} - z$.

In *Section 5* this is carried out for a model matrix that would correspond to a leading term in the perturbative expansion of E^{δ}_{-+}, however with q_{ω} replaced by a a sum of N delta functions. Then in *Section 6* we approximate such δ-functions with admissible potentials and get corresponding estimates for a true leading term in the expansion of E^{δ}_{-+}. Due to the approximation we only get good lower bounds for the first roughly $N/2$ singular values.

In *Section 7* we make an iterative procedure. Let $0 < \theta < 1/4$ be fixed. and consider the first θN values of E_{-+} appearing in the inverse of the Grushin matrix for the unperturbed problem. (For simplicity we here treat θN and similar numbers as if they were integers.) If they are all conveniently large, we add no further perturbation in this step, or more precisely we choose the zero potential as the admissible perturbation. If not, we consider the perturbation P_{δ} given by the special admissible potential q constructed in the preceding section. Then with appropriate choices of the parameters, we get the desired lower bound on the first θN singular values of the matrix E^{δ}_{-+}, corresponding to this perturbation. In both cases we get a perturbed operator P_{δ} (which may or may not be equal to P) and we next consider the natural Grushin problem for P_{δ} now with N replaced by $(1 - \theta)N$. For the new E_{-+} of size $(1 - \theta)N$ we again consider the first $\theta(1 - \theta)N$ singular values. If they are all larger than a new bound, obtained from the preceding one by multiplication by a suitable power of h, then the next perturbation is zero, if not, use again the result of the preceding section to find a convenient perturbation and so on. In the end we get the desired admissible perturbation as a geometrically convergent sum of perturbations, and for this perturbation we get

$$\ln|F_{\delta}| \geqslant (\frac{1}{2\pi h})^n \iint \ln|p_z(x, \xi)|dxd\xi - \text{``small''}. \qquad (1.26)$$

In *Section 8,* the spectral parameter is still fixed, and we perform a complex analysis argument in the α-variables to show that if we have (1.26) for one value of α then it holds with probability close to 1. In *Section 9* it then

only remains to let z become variable and to apply a result of [7] (extending one of [6]) about counting zeros of holomorphic functions with exponential growth. Very roughly, this result says that if $u(z) = u(z, \widetilde{h})$ is holomorphic in a fixed neighborhood of $\overline{\Gamma}$ such that $|u(z; \widetilde{h})| \leqslant e^{\phi(z)/\widetilde{h}}$ for all z in a neighborhood of $\partial\Gamma$ and satisfying the lower bound $|u(z_j; \widetilde{h})| \geqslant e^{(\phi(z_j) - \text{small})/\widetilde{h}}$ at finitely many points z_j, nicely spread along the boundary of Γ, then the number of zeros of u in Γ is approximately equal to $(2\pi\widetilde{h})^{-1} \iint_\Gamma \Delta\phi(z) L(dz)$. Here, as in [6, 7] we take $\widetilde{h} = (2\pi h)^n$, $\phi(z)$ equal to the integral in (1.25), (1.26) and use the fact that $\frac{1}{2\pi}$ times the Laplacian of this function can be identified with the push forward under p of the symplectic volume element.

In *Section 10,* we review some h-pseudodifferential and functional calculus.

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2. Semiclassical Sobolev spaces and multiplication

We let $H^s(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n)$, $s \in \mathbf{R}$, denote the semiclassical Sobolev space of order s equipped with the norm $\|\langle hD\rangle^s u\|$ where the norms are the ones in L^2, ℓ^2 or the corresponding operator norms if nothing else is indicated. Here $\langle hD\rangle = (1 + (hD)^2)^{1/2}$. Let $\widehat{u}(\xi) = \int e^{-ix\cdot\xi} u(x) dx$ denote the Fourier transform of the tempered distribution u on $\mathbf{R}^n$.

PROPOSITION 2.1. — *Let $s > n/2$. Then there exists a constant $C = C(s)$ such that for all $u, v \in H^s(\mathbf{R}^n)$, we have $u \in L^\infty(\mathbf{R}^n)$, $uv \in H^s(\mathbf{R}^n)$ and*

$$\|u\|_{L^\infty} \leqslant Ch^{-n/2}\|u\|_{H^s}, \tag{2.1}$$

$$\|uv\|_{H^s} \leqslant Ch^{-n/2}\|u\|_{H^s}\|v\|_{H^s}. \tag{2.2}$$

Proof. — The fact that $u \in L^\infty$ and the estimate (2.1) follow from Fourier's inversion formula and the Cauchy-Schwartz inequality:

$$|u(x)| \leqslant \frac{1}{(2\pi)^n} \int \langle h\xi\rangle^{-s} (\langle h\xi\rangle^s |\widehat{u}(\xi)|) d\xi \leqslant \frac{1}{(2\pi)^{n/2}} \|\langle h\cdot\rangle^{-s}\| \|u\|_{H^s}.$$

It then suffices to use that $\|\langle h\cdot\rangle^{-s}\| = C(s)h^{-n/2}$.

In order to prove (2.2) we pass to the Fourier transform side, and we see that it suffices to show that

$$\int \langle h\xi\rangle^s w(\xi) (\langle h\cdot\rangle^{-s}\widetilde{u} * \langle h\cdot\rangle^{-s}\widetilde{v})(\xi) d\xi \leqslant C(s)h^{-\frac{n}{2}} \|\widetilde{u}\| \|\widetilde{v}\| \|w\|, \tag{2.3}$$

for all non-negative $\widetilde{u}, \widetilde{v}, w \in L^2$, where $*$ denotes convolution. Here the left hand side can be written

$$\iint_{\eta+\zeta=\xi} \frac{\langle h\xi\rangle^s}{\langle h\eta\rangle^s \langle h\zeta\rangle^s} w(\xi)\widetilde{u}(\eta)\widetilde{v}(\zeta) d\xi d\zeta \leqslant \mathrm{I} + \mathrm{II},$$

where I, II denote the corresponding integrals over the sets $\{|\eta| \geqslant |\xi|/2\}$ and $\{|\zeta| \geqslant |\xi|/2\}$ respectively. Here

$$
\begin{aligned}
\mathrm{I} &\leqslant C(s)\int\left(\int w(\xi)\widetilde{u}(\xi-\zeta)d\xi\right)\frac{\widetilde{v}(\zeta)}{\langle h\zeta\rangle^s}d\zeta \\
&\leqslant C(s)\|w\|\|\widetilde{u}\|\|\frac{\widetilde{v}}{\langle h\cdot\rangle^s}\|_{L^1}.
\end{aligned}
$$

As in the proof of (2.1) we see that $\|\frac{\widetilde{v}}{\langle h\cdot\rangle^s}\|_{L^1} \leqslant C(s)h^{-\frac{n}{2}}\|\widetilde{v}\|$, so I is bounded by a constant times $h^{-\frac{n}{2}}\|\widetilde{u}\|\|\widetilde{v}\|\|w\|$. The same estimate holds for II and (2.3) follows. $\quad\square$

Let $\widetilde{\Omega}$ be a compact n-dimensional manifold. We cover $\widetilde{\Omega}$ by finitely many coordinate neighborhoods $M_1, ..., M_p$ and for each M_j, we let $x_1, ..., x_n$ denote the corresponding local coordinates on M_j. Let $0 \leqslant \chi_j \in C_0^\infty(M_j)$ have the property that $\sum_1^p \chi_j > 0$ on $\widetilde{\Omega}$. Define $H^s(\widetilde{\Omega})$ to be the space of all $u \in \mathcal{D}'(\widetilde{\Omega})$ such that

$$\|u\|_{H^s}^2 := \sum_1^p \|\chi_j\langle hD\rangle^s \chi_j u\|^2 < \infty. \tag{2.4}$$

It is standard to show that this definition does not depend on the choice of the coordinate neighborhoods or on χ_j. With different choices of these quantities we get norms in (2.4) which are uniformly equivalent when $h \to 0$. In fact, this follows from the h-pseudodifferential calculus on manifolds with symbols in the Hörmander space $S_{1,0}^m$. (This calculus has been used in several papers like [9, 13, 15] and for completeness we discuss it in the appendix, Section 10.)

An equivalent definition of $H^s(\widetilde{\Omega})$ is the following: Let

$$h^2\widetilde{R} = \sum (hD_{x_j})^* r_{j,k}(x) hD_{x_k} \tag{2.5}$$

be a non-negative elliptic operator with smooth coefficients on $\widetilde{\Omega}$, where the star indicates that we take the adjoint with respect to some fixed positive smooth density on $\widetilde{\Omega}$. Then $h^2\widetilde{R}$ is essentially self-adjoint with domain

$H^2(\widetilde{\Omega})$, so $(1 + h^2\widetilde{R})^{s/2} : L^2 \to L^2$ is a well-defined closed densely defined operator for $s \in \mathbf{R}$, which is bounded precisely when $s \leqslant 0$. Standard methods allow to show that $(1 + h^2\widetilde{R})^{s/2}$ is an h-pseudodifferential operator with symbol in $S_{1,0}^s$ and semiclassical principal symbol given by $(1 + r(x,\xi))^{s/2}$, where $r(x,\xi) = \sum_{j,k} r_{j,k}(x)\xi_j\xi_k$ is the semiclassical principal symbol of $h^2\widetilde{R}$. See Section 10. The h-pseudodifferential calculus gives for every $s \in \mathbf{R}$:

PROPOSITION 2.2. — $H^s(\widetilde{\Omega})$ *is the space of all* $u \in \mathcal{D}'(\widetilde{\Omega})$ *such that* $(1 + h^2\widetilde{R})^{s/2}u \in L^2$ *and the norm* $\|u\|_{H^s}$ *is equivalent to* $\|(1 + h^2\widetilde{R})^{s/2}u\|$, *uniformly when* $h \to 0$.

Remark 2.3. — From the first definition we see that Proposition 2.1 remains valid if we replace $\mathbf{R}^n$ by a compact n-dimensional manifold $\widetilde{\Omega}$.

3. H^s-perturbations and eigenfunctions

Let $m \geqslant 1$ be an order function on $\mathbf{R}^{2n}$ in the sense that

$$m(\rho) \leqslant C_0\langle\rho - \mu\rangle^{N_0}m(\mu), \ \rho, \mu \in \mathbf{R}^{2n}$$

for some fixed positive constants C_0, N_0, and let

$$p \in S(m) := \{a \in C^\infty(\mathbf{R}^{2n}); |\partial_\rho^\alpha a(\rho)| \leqslant C_\alpha m(\rho), \forall\rho \in \mathbf{R}^{2n}, \alpha \in \mathbf{N}^{2n}\}.$$

We assume that $p - z$ is elliptic (in the sense that $(p - z)^{-1} \in S(m^{-1})$) for at least one value $z \in \mathbf{C}$. Put $\Sigma = \overline{p(\mathbf{R}^{2n})} = p(\mathbf{R}^{2n}) \cup \Sigma_\infty$, where Σ_∞ is the set of accumulation values of p near $\rho = \infty$. Let $p_1, p_2, ... \in S(m)$,

$$P \sim p + hp_1 + ... \text{ in } S(m), \ h \to 0.$$

Let $\Omega \Subset \mathbf{C}$ be open simply connected with $\overline{\Omega} \cap \Sigma_\infty = \emptyset$, $\Omega \not\subset \Sigma$. Then as in [6, 7], we can construct $\widetilde{p} \in S(m)$, such that

$$\widetilde{p} = p \text{ away from a compact set.} \tag{3.1}$$

$$\widetilde{p} - z \text{ is elliptic in } S(m), \text{ uniformly for } z \in \overline{\Omega}. \tag{3.2}$$

The construction also shows that $\widetilde{p}$ can be chosen so that $\widetilde{p} = p$ away from any given neighborhood of $p^{-1}(\overline{\Omega})$.

Let

$$\widetilde{P} = P + \widetilde{p} - p \sim \widetilde{p} + hp_1 + ... \in S(m)$$

By P, $\widetilde{P}$ we also denote the corresponding h-Weyl quantizations i.e. the Weyl quantizations of $P(x, h\xi; h)$ and $\widetilde{P}(x, h\xi; h)$ respectively. (Sometimes

it will also be convenient to indicate the quantization so that if a is a symbol, then $\mathrm{Op}\,(a)$ denotes the corresponding h-pseudodifferential operator.) Then we know that $(\widetilde{P} - z)^{-1}$ is a well-defined uniformly bounded operator when h is small, uniformly for $z \in \overline{\Omega}$, and that P has discrete spectrum in Ω which is contained in any given neighborhood of $\overline{\Omega} \cap \Sigma$ when h is small enough.

We also recall that the eigenvalues in Ω, counted with their algebraic multiplicity, coincide with the zeros of the function $z \mapsto \det(\widetilde{P} - z)^{-1}(P - z) = \det(1 - (\widetilde{P} - z)^{-1}(\widetilde{P} - P))$, counted with their multiplicity. In fact, if $z_0 \in \Omega$, then its multiplicity $m(z_0)$ as a zero of the determinant is

$$= \mathrm{tr}\, \frac{1}{2\pi i} \int_\gamma (1 + K(z))^{-1} \dot{K}(z) dz = \mathrm{tr}\, \frac{1}{2\pi i} \int_\gamma (z - P)^{-1}(z - \widetilde{P})\dot{K}(z)dz,$$

where γ is a small circle centered at z_0, $K(z) = (z - \widetilde{P})^{-1}(\widetilde{P} - P)$, $\dot{K}(z) = (z - \widetilde{P})^{-1} - (z - \widetilde{P})^{-2}(z - P)$ and the dots indicate derivatives with respect to z, so

$$m(z_0) = \mathrm{tr}\, \frac{1}{2\pi i} \int_\gamma (z - P)^{-1}dz - \mathrm{tr}\, \frac{1}{2\pi i} \int_\gamma (z - P)^{-1}(z - \widetilde{P})^{-1}(z - P)dz.$$

Here the first term to the right is the rank of the spectral projection of P at the eigenvalue z_0 ie the multiplicity of z_0 as an eigenvalue of P, and from Lemma 2.2 of [12], we see that the second term is equal to

$$-\mathrm{tr}\, \frac{1}{2\pi i} \int_\gamma (z - \widetilde{P})^{-1}dz = 0.$$

Now, consider the perturbed operator

$$P_\delta = P + \delta Q, \tag{3.3}$$

where $0 \leqslant \delta \ll 1$ will depend on h and Q is the operator of multiplication with $q \in H^s(\mathbf{R}^n)$, satisfying

$$\|q\|_{H^s} \leqslant h^{\frac{n}{2}}. \tag{3.4}$$

Here $s > n/2$ is fixed and we systematically use the semiclassical Sobolev spaces in Section 2.

Put

$$\widetilde{P}_\delta = \widetilde{P} + \delta Q. \tag{3.5}$$

If

$$\delta \ll 1, \quad h \ll 1, \tag{3.6}$$

we know from Section 2 that $\|\delta Q\|_{L^2 \to L^2} = \delta \|q\|_{L^\infty} \ll 1$, and hence $(\widetilde{P}_\delta - z)^{-1}$ is a well-defined bounded operator when h is small enough. The spectrum of P_δ in Ω is discrete and coincides with the zeros of

$$\det((\widetilde{P}_\delta - z)^{-1}(P_\delta - z)) = \det(1 - (\widetilde{P}_\delta - z)^{-1}(\widetilde{P} - P)).$$

Notice here that $(\widetilde{P}_\delta - z)^{-1}(\widetilde{P} - P)$ is a trace class operator and that again the multiplicities of the eigenvalues of P_δ and of the zeros of the determinant agree. It is also clear that $\sigma(P_\delta) \cap \overline{\Omega}$ is contained in any given neighborhood of $\Sigma \cap \Omega$, when h and δ are sufficiently small.

From Section 2 we know that $Q = \mathcal{O}(1) : H^\sigma \to H^\sigma$ for $\sigma = s$, by duality we get the same fact when $\sigma = -s$ and finally by interpolation (or more directly by (2.1) applied to q) we get it also for $\sigma = 0$. Writing

$$\widetilde{P}_\delta - z = (\widetilde{P} - z)(1 + (\widetilde{P} - z)^{-1}\delta Q) = (1 + \delta Q(\widetilde{P} - z)^{-1})(\widetilde{P} - z), \quad (3.7)$$

and observing that $(\widetilde{P} - z)^{-1} \in \mathrm{Op}(S(\frac{1}{m}))$ is uniformly bounded: $H^s \to H^s$, $H^{-s} \to H^{-s}$, when $z \in \overline{\Omega}$, we see that

$$(\widetilde{P}_\delta - z)^{-1} = \mathcal{O}(1) : H^s \to H^s, \ H^{-s} \to H^{-s}, \ H^0 \to H^0, \quad (3.8)$$

uniformly when $z \in \overline{\Omega}$ and (3.6) holds, and similarly for $(1 + (\widetilde{P} - z)^{-1}\delta Q)^{-1}$, $(1 + \delta Q(\widetilde{P} - z)^{-1})^{-1}$.

Put

$$P_{\delta,z} := (\widetilde{P}_\delta - z)^{-1}(P_\delta - z) = 1 - (\widetilde{P}_\delta - z)^{-1}(\widetilde{P} - P) =: 1 - K_{\delta,z}, \quad (3.9)$$

$$S_{\delta,z} := P_{\delta,z}^* P_{\delta,z} = 1 - (K_{\delta,z} + K_{\delta,z}^* - K_{\delta,z}^* K_{\delta,z}) =: 1 - L_{\delta,z}. \quad (3.10)$$

Notice that

$$K_{\delta,z}, L_{\delta,z} = \mathcal{O}(1) : H^{-s} \to H^s, \quad (3.11)$$

when (3.6) holds. For $0 \leqslant \alpha \leqslant 1/2$, let $\pi_\alpha = 1_{[0,\alpha]}(S_{\delta,z})$ be the spectral projection corresponding to the spectrum of $S_{\delta,z}$ in the interval $[0,\alpha]$.

We shall study H^s regularization and localization of π_α and of the analogous spectral projections for $(P_\delta - z)^*(P_\delta - z)$. The reader who is not too much interested in the technicalities may proceed directly to Proposition 3.2 at the end of this section.

Apply π_α to (3.10):

$$\pi_\alpha(1 - S_{\delta,z}\pi_\alpha) = L_{\delta,z}\pi_\alpha.$$

Here $\|S_{\delta,z}\pi_\alpha\| \leqslant 1/2$, so $1 - S_{\delta,z}\pi_\alpha$ is invertible with inverse of norm $\leqslant 2$. It follows that

$$\pi_\alpha = L_{\delta,z}\pi_\alpha(1 - S_{\delta,z}\pi_\alpha)^{-1}, \tag{3.12}$$

so under the assumption (3.6), we see that

$$\pi_\alpha = \mathcal{O}(1) : L^2 \to H^s, \tag{3.13}$$

and since $\pi_\alpha = \pi_\alpha\pi_\alpha^*$ we even get $\pi_\alpha = \mathcal{O}(1) : H^{-s} \to H^s$.

Since $L_{\delta,z}$ is compact, we know that the range $\mathcal{R}(\pi_\alpha)$ of π_α is of finite dimension, N. Let $e_1, ..., e_N$ be an orthonormal basis in this space. An equivalent way of stating (3.13) is then

$$\|\sum_1^N \lambda_j e_j\|_{H^s} \leqslant \mathcal{O}(1)\|\lambda\|_{\ell^2}, \ \forall \lambda = (\lambda_1, .., \lambda_N) \in \mathbf{C}^N \simeq \ell^2(\{1, 2, .., N\}). \tag{3.14}$$

If $\chi \in C_b^\infty(\mathbf{R}^n) = \{f \in C^\infty(\mathbf{R}^n); \partial^\alpha f$ is bounded for every $\alpha \in \mathbf{N}^n\}$, we have

$$[\widetilde{P}_\delta, \chi] = [\widetilde{P}, \chi] \in h\mathrm{Op}\,(S(m)).$$

Combining this with (3.7) and the fact mentioned right after (3.8), we see that

$$(\widetilde{P}_\delta - z)^{-1}[\widetilde{P}_\delta, \chi], \ [\widetilde{P}_\delta, \chi](\widetilde{P}_\delta - z)^{-1} = \mathcal{O}(h) : H^\sigma \to H^\sigma, \ \sigma = \pm s, 0. \tag{3.15}$$

From this, it is standard to deduce that

$$\chi_1(\widetilde{P}_\delta - z)^{-1}\chi_0 = \mathcal{O}(h^\infty) : \ H^\sigma \to H^\sigma, \ \sigma = \pm s, 0, \tag{3.16}$$

if $\chi_1, \chi_0 \in C_b^\infty(\mathbf{R}^n)$ and dist $(\mathrm{supp}\,\chi_0, \mathrm{supp}\,\chi_1) > 0$. In fact, for any $M \in \mathbf{N}^*$, choose $\psi_1, ..., \psi_M \in C_b^\infty(\mathbf{R}^n)$, such that $\mathrm{supp}\,\psi_M \cap \mathrm{supp}\,\chi_1 = \emptyset$, $\psi_{j+1} = 1$ on $\mathrm{supp}\,\psi_j$, $\psi_1 = 1$ on $\mathrm{supp}\,\chi_0$, and use the telescopic formula,

$$\chi_1(\widetilde{P}_\delta - z)^{-1}\chi_0 = \pm\chi_1(\widetilde{P}_\delta - z)^{-1}[\widetilde{P}_\delta, \psi_M](\widetilde{P}_\delta - z)^{-1}...[\widetilde{P}_\delta, \psi_1](\widetilde{P}_\delta - z)^{-1}\chi_0. \tag{3.17}$$

Let

$$K = \pi_x(\mathrm{supp}\,(\widetilde{p} - p)) \tag{3.18}$$

be the x-space projection of supp $(\widetilde{p} - p)$, so that K is compact. Combining (3.9), (3.16), we see that

$$\chi K_{\delta,z}, \ K_{\delta,z}\chi = \mathcal{O}(h^\infty) : \ H^\sigma \to H^\sigma, \ \sigma = \pm s, 0, \tag{3.19}$$

when $\chi \in C_b^\infty(\mathbf{R}^n)$ satisfies $\operatorname{supp}\chi \cap K = \emptyset$. From (3.10) we get the same conclusion for $L_{\delta,z}$ and then we get from (3.12) that

$$\chi\pi_\alpha = \mathcal{O}(h^\infty): \ L^2 \to H^s, \qquad (3.20)$$

if $\chi \in C_b^\infty(\mathbf{R}^n)$, and $\operatorname{supp}\chi \cap K = \emptyset$. Using that $\pi_\alpha = \pi_\alpha^2$ and that $\pi_\alpha = \mathcal{O}(1): H^{-s} \to H^s$, this can be sharpened to the statement that

$$\chi\pi_\alpha, \ \pi_\alpha\chi = \mathcal{O}(h^\infty): \ H^{-s} \to H^s.$$

We also need to establish the corresponding results for $P_\delta - z$. Let

$$S_\delta = (P_\delta - z)^*(P_\delta - z), \quad \widetilde{S}_\delta = (\widetilde{P}_\delta - z)^*(\widetilde{P}_\delta - z), \qquad (3.21)$$

viewed as self-adjoint Friedrichs extensions from $(\widetilde{P}_\delta - z)^{-1}(H(m))$ with quadratic form domain $H(m)$. Then

$$S_\delta = \widetilde{S}_\delta + R,$$

where

$$R = (P - \widetilde{P})^*(\widetilde{P}_\delta - z) + (\widetilde{P}_\delta - z)^*(P - \widetilde{P}) + (P - \widetilde{P})^*(P - \widetilde{P}), \quad (3.22)$$

and we see that

$$R = \mathcal{O}(1): H^{-s} \to H^s. \qquad (3.23)$$

It follows that

$$\begin{aligned}(w - S_\delta)^{-1} &= (w - \widetilde{S}_\delta)^{-1} + (w - S_\delta)^{-1}R(w - \widetilde{S}_\delta)^{-1} \qquad (3.24)\\ &= (w - \widetilde{S}_\delta)^{-1} - (w - \widetilde{S}_\delta)^{-1}R(w - S_\delta)^{-1}.\end{aligned}$$

If $\widetilde{m}$ is an order function on $\mathbf{R}^{2n}$, we define $H(\widetilde{m})$ for $h > 0$ small enough, to be the space $\widetilde{M}^{-1}L^2(\mathbf{R}^n)$, where $\widetilde{M} \in \operatorname{Op}(S(\widetilde{m}))$ is an elliptic operator, so that $\widetilde{M}^{-1} \in \operatorname{Op}(S(\frac{1}{m}))$.

Remark 3.1. — For future reference we notice that S_δ coincides with $\widehat{S}_\delta := (P_\delta - z)^*(P_\delta - z)$ with domain $\mathcal{D}(\widehat{S}_\delta) = \{u \in H(m); (P_\delta - z)u \in H(m)\}$. In fact, $\widehat{S}_\delta$ is a closed operator, with domain contained in the quadratic form domain $H(m)$ of S_δ, so it suffices to check that $\widehat{S}_\delta$ is self-adjoint. Clearly this operator is symmetric so it suffices to check that $\widehat{S}_\delta^* \subset \widehat{S}_\delta$. To shorten notations, assume that $z = 0$. If $u \in \mathcal{D}(\widehat{S}_\delta^*)$, $\widehat{S}_\delta^* u = v$, then $(\widehat{S}_\delta\phi | u) = (\phi | v)$ for all $\phi \in \mathcal{D}(\widehat{S}_\delta)$, so $(P_\delta\phi | P_\delta u) = (\phi | v) = \mathcal{O}(\|\phi\|_{H(m)})$, so $(\widetilde{P}_\delta\phi | P_\delta u) = \mathcal{O}(\|\phi\|_{H(m)})$, implying that $P_\delta u \in L^2$, since $\widetilde{P}_\delta: H(m) \to L^2$ is bijective and $\mathcal{D}(\widehat{S}_\delta)$ is dense in $H(m)$. Using $(P_\delta\phi | P_\delta u) = (\phi | v)$ again, we get $P_\delta^* P_\delta u = v$ in the sense of distributions and since P is elliptic near infinity, we deduce that $u, P_\delta u \in H(m)$, so $u \in \mathcal{D}(\widehat{S}_\delta)$.

Let $f \in C_0^\infty(\text{neigh}\,(0, \mathbf{R}))$ and let $\widetilde{f} \in C_0^\infty(\text{neigh}\,(0, \mathbf{C}))$ be an almost holomorphic extension. Since $\widetilde{S}_\delta$ has no spectrum in a fixed neighborhood of 0, we get (using the Cauchy-Riemann formula

$$f(S_\delta) = -\frac{1}{\pi} \int \overline{\partial}\widetilde{f}(w)(w - S_\delta)^{-1} L(dw))$$

for f supported in that neighborhood,

$$
\begin{aligned}
f(S_\delta) &= -\int \overline{\partial}\widetilde{f}(w)(w - S_\delta)^{-1} R(w - \widetilde{S}_\delta)^{-1} \frac{L(dw)}{\pi} \qquad (3.25)\\
&= \int \overline{\partial}\widetilde{f}(w)(w - \widetilde{S}_\delta)^{-1} R(w - S_\delta)^{-1} \frac{L(dw)}{\pi}
\end{aligned}
$$

Here, $(w - \widetilde{S}_\delta)^{-1} = \mathcal{O}(1) : H^\sigma \to H^\sigma$, $\sigma = \pm s, 0$, so we conclude that

$$f(S_\delta) = \mathcal{O}(1) : H^{-s} \to L^2 \text{ and } L^2 \to H^s.$$

Then $f^2(S_\delta) = \mathcal{O}(1) : H^{-s} \to H^s$. Let $\pi_\alpha = 1_{[0,\alpha]}(S_\delta)$. It follows that for $0 \leqslant \alpha \ll 1$:

$$\pi_\alpha = \mathcal{O}(1) : H^{-s} \to H^s, \qquad (3.26)$$

so (3.14) remains valid. Using the same telescopic formula as above, we shall next show that

$$\chi\pi_\alpha, \pi_\alpha\chi = \mathcal{O}(h^\infty) : H^{-s} \to H^s, \qquad (3.27)$$

if $\chi \in C_b^\infty(\mathbf{R}^n)$ has the property that $\text{supp}\,(\chi) \cap K = \emptyset$.

For $w \in \text{neigh}\,(0)$, we can write $\widetilde{S}_0 - w = \Lambda_1 \Lambda_2$, where $\Lambda_j \in \text{Op}\,(S(m))$ are elliptic. On the other hand (for $\delta \ll 1$), we have

$$\widetilde{S}_\delta - w = \widetilde{S}_0 - w + (\widetilde{P} - z)^* \delta q + \delta\overline{q}(\widetilde{P} - z) + \delta^2|q|^2.$$

We get

$$\widetilde{S}_\delta - w = \Lambda_1(1 + \underbrace{\Lambda_1^{-1}((\widetilde{P} - z)^* \delta q + \delta\overline{q}(\widetilde{P} - z) + \delta^2|q|^2)\Lambda_2^{-1}}_{=\mathcal{O}(\delta):\, H^\sigma \to H^\sigma})\Lambda_2,$$

so

$$(\widetilde{S}_\delta - w)^{-1} = \Lambda_2^{-1} A \Lambda_1^{-1},$$

where $A = \mathcal{O}(1) : H^\sigma \to H^\sigma$ and consequently

$$(\widetilde{S}_\delta - w)^{-1} = \mathcal{O}(1) : H(\frac{\langle\xi\rangle^\sigma}{m}) \to H(m\langle\xi\rangle^\sigma). \qquad (3.28)$$

Next, consider $(w - S_\delta)^{-1}$ in (3.22)–(3.24). Using (3.28), we see that

$$(w - S_\delta)^{-1} = \mathcal{O}(\frac{1}{|\Im w|}) : \; H(\frac{\langle \xi \rangle^\sigma}{m}) \to L^2 + H(m \langle \zeta \rangle^\sigma).$$

Reinjecting this information into the last expression in (3.24), we see that

$$(w - S_\delta)^{-1} = \mathcal{O}(\frac{1}{|\Im w|}) : \; H(\frac{\langle \xi \rangle^\sigma}{m}) \to H(m \langle \xi \rangle^\sigma). \tag{3.29}$$

If $\psi \in C_b^\infty(\mathbf{R}^n)$ we next see that

$$
\begin{aligned}
[\widetilde{S}_\delta, \psi] &= [\widetilde{P}^*, \psi](\widetilde{P} - z + \delta q) + (\widetilde{P}^* - \overline{z} + \delta \overline{q})[\widetilde{P}, \psi] \tag{3.30} \\
&= \mathcal{O}(h) : \; H(m \langle \xi \rangle^\sigma) \to H(\frac{1}{m} \langle \xi \rangle^\sigma),
\end{aligned}
$$

and similarly with $\widetilde{S}_\delta$ replaced by S_δ. We conclude that

$$(w - \widetilde{S}_\delta)^{-1}[\widetilde{S}_\delta, \psi] = \mathcal{O}(h) : H(m \langle \xi \rangle^\sigma) \to H(m \langle \xi \rangle^\sigma), \tag{3.31}$$

$$[\widetilde{S}_\delta, \psi](w - \widetilde{S}_\delta)^{-1} = \mathcal{O}(h) : H(\frac{\langle \xi \rangle^\sigma}{m}) \to H(\frac{\langle \xi \rangle^\sigma}{m}), \tag{3.32}$$

and we have the analogous estimates with $\widetilde{S}_\delta$ replaced by S_δ and $\mathcal{O}(h)$ replaced by $\mathcal{O}(h/|\Im w|)$.

Now, let χ be as in (3.27) and choose $\chi_0 \in C_0^\infty(\mathbf{R}^n)$ such that $\chi_0 = 1$ near K, $\operatorname{supp} \chi \cap \operatorname{supp}(\chi_0) = \emptyset$. Choose $\psi_1, ..., \psi_M$ as in the telescopic formula (3.17) with χ_1 there equal to χ. Then we get

$$\chi(w - \widetilde{S}_\delta)^{-1}\chi_0 = \tag{3.33}$$
$$\pm \chi(w - \widetilde{S}_\delta)^{-1}[\widetilde{S}_\delta, \psi_M](w - \widetilde{S}_\delta)^{-1}[\widetilde{S}_\delta, \psi_{M-1}]...(w - \widetilde{S}_\delta)^{-1}[\widetilde{S}_\delta, \psi_1](w - \widetilde{S}_\delta)^{-1}\chi_0$$
$$= \mathcal{O}(h^M) : \; H(\frac{\langle \xi \rangle^\sigma}{m}) \to H(m \langle \xi \rangle^\sigma).$$

Write $R = \chi_0 R + (1 - \chi_0)R$. Here $(1 - \chi_0)(P - \widetilde{P})^* = \mathcal{O}(h^\infty) : H(m_1) \to H(m_2)$ for all order functions, m_1, m_2, so (cf (3.22))

$$(1 - \chi_0)(P - \widetilde{P})^*(\widetilde{P}_\delta - z), \; (1 - \chi_0)(P - \widetilde{P})^*(P - \widetilde{P}) = \mathcal{O}(h^\infty) : \; H^{-s} \to H(m_2).$$

Moreover,

$$
\begin{aligned}
&(1 - \chi_0)(\widetilde{P}_\delta - z)^*(P - \widetilde{P}) \\
=\; &(1 - \chi_0)(\widetilde{P} - z)^*(P - \widetilde{P}) + \delta \overline{q}(1 - \chi_0)(P - \widetilde{P}) \\
=\; &\mathcal{O}(h^\infty) : \; H(m_1) \to H^s,
\end{aligned}
$$

and we conclude that

$$(1 - \chi_0)R = \mathcal{O}(h^\infty): \ H^{-s} \to H^s. \tag{3.34}$$

Combining this with (3.33), we get

$$\chi(w - \widetilde{S}_\delta)^{-1}R = \mathcal{O}(h^\infty): H^{-s} \to H(m\langle\xi\rangle^s). \tag{3.35}$$

Using this and (3.29) in the second expression for $f(S_\delta)$ in (3.25), we see that

$$\chi f(S_\delta) = \mathcal{O}(h^\infty): \ H(\frac{1}{m\langle\xi\rangle^s}) \to H(m\langle\xi\rangle^s). \tag{3.36}$$

Choosing $f = 1$ on $[0, \alpha]$, we see that

$$\chi\pi_\alpha = \chi f(S_\delta)\pi_\alpha = \mathcal{O}(h^\infty): \ H^{-s} \to H(m\langle\xi\rangle^s),$$

which implies the estimate on $\chi\pi_\alpha$ in (3.27), now with $\pi_\alpha = 1_{[0,\alpha]}(S_\delta)$. Passing to the adjoints we get the estimate on $\pi_\alpha\chi$ and this completes the verification of (3.27).

PROPOSITION 3.2. — *Let $P, p, \widetilde{P}, \widetilde{p}$ be as in the beginning of this section. Let P_δ, $\widetilde{P}_\delta$ be given by (3.3), (3.4), (3.5) (where $s > n/2$ is fixed) and make the assumption (3.6). Define $P_{\delta,z}, S_{\delta,z}$ as in (3.9), (3.10), and S_δ as in (3.21) and realize S_δ as the Friedrichs extension. Let π_α denote either $1_{[0,\alpha]}(S_{\delta,z})$ for $0 \leqslant \alpha \leqslant 1/2$, or $1_{[0,\alpha]}(S_\delta)$ for $0 \leqslant \alpha \ll 1$. In both cases, we have $\pi_\alpha = \mathcal{O}(1): H^{-s} \to H^s$ uniformly with respect to α, h, implying (3.14). Moreover, if $\chi \in C_b^\infty(\mathbf{R}^n)$ is independent of h and $\mathrm{supp}\,\chi \cap \pi_x(\mathrm{supp}\,(\widetilde{p} - p)) = \emptyset$ (cf (3.18)), then $\chi\pi_\alpha, \pi_\alpha\chi$ are $= \mathcal{O}(h^\infty): H^{-s} \to H^s$. In the second case we also have $\chi\pi_\alpha = \mathcal{O}(h^\infty): H^{-s} \to H(m\langle\xi\rangle^s)$.*

4. Grushin problems

Let $P : \mathcal{H} \to \mathcal{H}$ be a bounded operator, where $\mathcal{H}$ is a complex separable Hilbert space. Following the standard definitions (see [3]) we define the singular values of P to be the decreasing sequence $s_1(P) \geqslant s_2(P) \geqslant \dots$ of eigenvalues of the self-adjoint operator $(P^*P)^{1/2}$ as long as these eigenvalues lie above the supremum of the essential spectrum. If there are only finitely many such eigenvalues, $s_1(P), \dots, s_k(P)$ then we define $s_{k+1}(P) = s_{k+2}(P) = \dots$ to be the supremum of the essential spectrum of $(P^*P)^{1/2}$. When $\dim \mathcal{H} = M < \infty$ our sequence is finite (by definition); $s_1 \geqslant s_2 \geqslant \dots \geqslant s_M$, otherwise it is infinite. Using that if $P^*Pu = s_j^2 u$, then $PP^*(Pu) = s_j^2 Pu$ and similarly with P and P^* permuted, we see that $s_j(P^*) = s_j(P)$.

Strictly speaking, $P^*P : \mathcal{N}(P)^\perp \to \mathcal{N}(P)^\perp$ and $PP^* : \mathcal{N}(P^*)^\perp \to \mathcal{N}(P^*)^\perp$ are unitarily equivalent via the map $P(P^*P)^{-1/2} : \mathcal{N}(P)^\perp \to \mathcal{N}(P^*)^\perp$ and its inverse $P^*(PP^*)^{-1/2} : \mathcal{N}(P^*)^\perp \to \mathcal{N}(P)^\perp$. (To check this, notice that the relation $P(P^*P) = (PP^*)P$ on $\mathcal{N}(P)^\perp$ implies $P(P^*P)^\alpha = (PP^*)^\alpha P$ on the same space for every $\alpha \in \mathbf{R}$.)

In the case when P is a Fredholm operator of index 0, it will be convenient to introduce the increasing sequence $0 \leqslant t_1(P) \leqslant t_2(P) \leqslant \dots$ consisting first of all eigenvalues of $(P^*P)^{1/2}$ below the infimum of the essential spectrum and then, if there are only finitely many such eigenvalues, we repeat indefinitely that infimum. (The length of the resulting sequence is the dimension of $\mathcal{H}$.) When $\dim \mathcal{H} = M < \infty$, we have $t_j(P) = s_{M+1-j}(P)$. Again, we have $t_j(P^*) = t_j(P)$ (as reviewed in [7]). Moreover, in the case when P has a bounded inverse, we see that

$$s_j(P^{-1}) = \frac{1}{t_j(P)}. \tag{4.1}$$

Let P be a Fredholm operator of index 0. Let $1 \leqslant N < \infty$ and let $R_+ : \mathcal{H} \to \mathbf{C}^N$, $R_- : \mathbf{C}^N \to \mathcal{H}$ be bounded operators. Assume that

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H} \times \mathbf{C}^N \to \mathcal{H} \times \mathbf{C}^N \tag{4.2}$$

is bijective with a bounded inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \tag{4.3}$$

Recall (for instance from [14]) that P has a bounded inverse precisely when E_{-+} has, and when this happens we have the relations,

$$P^{-1} = E - E_+ E_{-+}^{-1} E_-, \quad E_{-+}^{-1} = -R_+ P^{-1} R_-. \tag{4.4}$$

Recall ([3]) that if A, B are bounded operators, then we have the general estimates,

$$s_{n+k-1}(A + B) \leqslant s_n(A) + s_k(B), \tag{4.5}$$

$$s_{n+k-1}(AB) \leqslant s_n(A)s_k(B), \tag{4.6}$$

in particular for $k = 1$, we get

$$s_n(AB) \leqslant \|A\|s_n(B), \ s_n(AB) \leqslant s_n(A)\|B\|, \ s_n(A + B) \leqslant s_n(A) + \|B\|.$$

Applying this to the second part of (4.4), we get

$$s_k(E_{-+}^{-1}) \leqslant \|R_-\|\|R_+\|s_k(P^{-1}), \ 1 \leqslant k \leqslant N$$

implying

$$t_k(P) \leqslant \|R_-\| \|R_+\| t_k(E_{-+}), \ 1 \leqslant k \leqslant N. \tag{4.7}$$

By a perturbation argument, we see that this holds also in the case when P, E_{-+} are non-invertible.

Similarly from the first part of (4.4), we get

$$s_k(P^{-1}) \leqslant \|E\| + \|E_+\| \|E_-\| s_k(E_{-+}^{-1}),$$

leading to

$$t_k(P) \geqslant \frac{t_k(E_{-+})}{\|E\| t_k(E_{-+}) + \|E_+\| \|E_-\|}. \tag{4.8}$$

Again this can be extended to the non-necessarily invertible case by means of small perturbations.

Next, we recall from [7] a natural construction of an associated Grushin problem to a given operator. Let $P_0 : \mathcal{H} \to \mathcal{H}$ be a Fredholm operator of index 0 as above. Assume that the first N singular values $t_1(P_0) \leqslant t_2(P_0) \leqslant \ldots \leqslant t_N(P_0)$ correspond to discrete eigenvalues of $P_0^* P_0$ and assume that $t_{N+1}(P_0)$ is strictly positive. In the following we sometimes write t_j instead of $t_j(P_0)$ for short.

Recall that t_j^2 are the first eigenvalues both for $P_0^* P_0$ and $P_0 P_0^*$. Let $e_1, \ldots, e_N$ and $f_1, \ldots, f_N$ be corresponding orthonormal systems of eigenvectors of $P_0^* P_0$ and $P_0 P_0^*$ respectively. They can be chosen so that

$$P_0 e_j = t_j f_j, \ P_0^* f_j = t_j e_j. \tag{4.9}$$

Define $R_+ : L^2 \to \mathbf{C}^N$ and $R_- : \mathbf{C}^N \to L^2$ by

$$R_+ u(j) = (u|e_j), \ R_- u_- = \sum_1^N u_-(j) f_j. \tag{4.10}$$

As in [7], the Grushin problem

$$\begin{cases} P_0 u + R_- u_- = v, \\ R_+ u = v_+, \end{cases} \tag{4.11}$$

has a unique solution $(u, u_-) \in L^2 \times \mathbf{C}^N$ for every $(v, v_+) \in L^2 \times \mathbf{C}^N$, given by

$$\begin{cases} u = E^0 v + E_+^0 v_+, \\ u_- = E_-^0 v + E_{-+}^0 v_+, \end{cases} \tag{4.12}$$

where

$$E^0_+ v_+ = \sum_1^N v_+(j)e_j, \quad E^0_- v(j) = (v|f_j), \tag{4.13}$$

$$E^0_{-+} = -\mathrm{diag}\,(t_j), \quad \|E^0\| \leqslant \tfrac{1}{t_{N+1}}.$$

E^0 can be viewed as the inverse of P_0 as an operator from the orthogonal space $(e_1, e_2, ..., e_N)^\perp$ to $(f_1, f_2, ..., f_N)^\perp$.

We notice that in this case, the norms of R_+ and R_- are equal to 1, so (4.7) tells us that $t_k(P_0) \leqslant t_k(E^0_{-+})$ for $1 \leqslant k \leqslant N$, but of course the expression for E^0_{-+} in (4.13) implies equality.

Let $Q \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ and put $P_\delta = P_0 - \delta Q$ (where we sometimes put a minus sign in front of the perturbation for notational convenience). We are particularly interested in the case when $Q = Q_\omega u = q_\omega u$ is the operator of multiplication with a random function q_ω. Here $\delta > 0$ is a small parameter. Choose $R_\pm$ as in (4.10). Then if $\delta < t_{N+1}$ and $\|Q\| \leqslant 1$, the perturbed Grushin problem

$$\begin{cases} P_\delta u + R_- u_- = v, \\ R_+ u = v_+, \end{cases} \tag{4.14}$$

is well posed and has the solution

$$\begin{cases} u = E^\delta v + E^\delta_+ v_+, \\ u_- = E^\delta_- + E^\delta_{-+} v_+, \end{cases} \tag{4.15}$$

where

$$\mathcal{E}^\delta = \begin{pmatrix} E^\delta & E^\delta_+ \\ E^\delta_- & E^\delta_{-+} \end{pmatrix} \tag{4.16}$$

is obtained from $\mathcal{E}^0$ by

$$\mathcal{E}^\delta = \mathcal{E}^0 \left(1 - \delta \begin{pmatrix} QE^0 & QE^0_+ \\ 0 & 0 \end{pmatrix} \right)^{-1}. \tag{4.17}$$

Using the Neumann series, we get

$$E^\delta_{-+} = E^0_{-+} + \delta E^0_- Q E^0_+ + \delta^2 E^0_- Q E^0 Q E^0_+ + \delta^3 E^0_- Q (E^0 Q)^2 E^0_+ + ... \tag{4.18}$$

We also get

$$E^\delta = E^0 + \sum_1^\infty \delta^k E^0 (QE^0)^k \tag{4.19}$$

$$E^\delta_+ = E^0_+ + \sum_1^\infty \delta^k (E^0 Q)^k E^0_+ \tag{4.20}$$

$$E_-^\delta = E_-^0 + \sum_1^\infty \delta^k E_-^0 (QE^0)^k. \tag{4.21}$$

The leading perturbation in E_{-+}^δ is δM, where $M = E_-^0 Q E_+^0 : \mathbf{C}^N \to \mathbf{C}^N$ has the matrix

$$M(\omega)_{j,k} = (Qe_k|f_j), \tag{4.22}$$

which in the multiplicative case reduces to

$$M(\omega)_{j,k} = \int q(x)e_k(x)\overline{f_j(x)}dx. \tag{4.23}$$

Put $\tau_0 = t_{N+1}(P_0)$ and recall the assumption

$$\|Q\| \leqslant 1. \tag{4.24}$$

Then, if $\delta \leqslant \tau_0/2$, the new Grushin problem is well posed with an inverse $\mathcal{E}^\delta$ given in (4.16)–(4.21). We get

$$\|E^\delta\| \leqslant \frac{1}{1 - \frac{\delta}{\tau_0}}\|E^0\| \leqslant \frac{2}{\tau_0}, \quad \|E_\pm^\delta\| \leqslant \frac{1}{1 - \frac{\delta}{\tau_0}} \leqslant 2, \tag{4.25}$$

$$\|E_{-+}^\delta - (E_{-+}^0 + \delta E_-^0 Q E_+^0)\| \leqslant \frac{\delta^2}{\tau_0}\frac{1}{1 - \frac{\delta}{\tau_0}} \leqslant 2\frac{\delta^2}{\tau_0}. \tag{4.26}$$

Using this in (4.7), (4.8) together with the fact that $t_k(E_{-+}^\delta) \leqslant 2\tau_0$, we get

$$\frac{t_k(E_{-+}^\delta)}{8} \leqslant t_k(P_\delta) \leqslant t_k(E_{-+}^\delta). \tag{4.27}$$

Remark 4.1. — Under suitable assumptions, the preceding discussion can be extended to the case of unbounded operators. The purpose of this remark is to make one such extension that will be needed later. Let P, m, P_{δ_0} be as in Section 3, satisfying (3.4), (3.6) with δ there equal to δ_0. We fix $z \in \Omega$ with Ω, Σ, Σ_∞ as in that section. For notational convenience, we may assume that $z = 0$. Then we know that $P_{\delta_0} : H(m) \to L^2(\mathbf{R}^n)$ is Fredholm of index 0, and the same holds for the formal adjoint $P_{\delta_0}^*$.

Let

$$S_{\delta_0} = P_{\delta_0}^* P_{\delta_0}, \quad T_{\delta_0} = P_{\delta_0} P_{\delta_0}^* \tag{4.28}$$

be the unbounded operators equipped with their natural domains,

$$\mathcal{D}(S_{\delta_0}) = \{u \in L^2; P_{\delta_0}u \in L^2, P_{\delta_0}^*(P_{\delta_0}u) \in L^2\} = \{u \in H(m); P_{\delta_0}u \in H(m)\}, \tag{4.29}$$

and similarly for T_{δ_0}. From Remark 3.1 we know that S_{δ_0} is self-adjoint and we clearly have the same fact for T_{δ_0}.

It is now easy to check that $S_{\delta_0} \geqslant 0$, $T_{\delta_0} \geqslant 0$ have discrete spectra in a fixed neighborhood of 0, using that $S_{\delta_0} - \widetilde{S}_{\delta_0}$ and $T_{\delta_0} - \widetilde{T}_{\delta_0}$ are compact, where $\widetilde{S}_{\delta_0}$ and $\widetilde{T}_{\delta_0}$ are defined as in (4.28) with P_{δ_0} replaced by $\widetilde{P}_{\delta_0}$. Moreover,

$$\mathcal{N}(S_{\delta_0}) = \{u \in H(m); \ P_{\delta_0} u = 0\}, \ \mathcal{N}(T_{\delta_0}) = \{u \in H(m); \ P_{\delta_0}^* u = 0\},$$

and since $P_{\delta_0}, P_{\delta_0}^*$ are Fredholm of index 0, we deduce that

$$\dim \mathcal{N}(S_{\delta_0}) = \dim \mathcal{N}(T_{\delta_0}). \tag{4.30}$$

Further, if $S_{\delta_0} u = \lambda u$, $\|u\| = 1$, $0 < \lambda \ll 1$, then we can apply P_{δ_0} and write

$$P_{\delta_0} P_{\delta_0}^* (P_{\delta_0} u) = \lambda(P_{\delta_0} u). \tag{4.31}$$

Here $P_{\delta_0} u \in H(m)$ (the quadratic form domain of T_{δ_0}) and since the right hand side is (a fortiori) in L^2, we see that $P_{\delta_0} u \in \mathcal{D}(T_{\delta_0})$ and that $T_{\delta_0}(P_{\delta_0} u) = \lambda(P_{\delta_0} u)$. Similarly, if $T_{\delta_0} v = \lambda v$, $\|v\| = 1$, $0 < \lambda \ll 1$, we see that $P_{\delta_0}^* v \in \mathcal{D}(S_{\delta_0})$ and that $S_{\delta_0}(P_{\delta_0}^* v) = \lambda(P_{\delta_0}^* v)$.

It is then clear that if $0 < \alpha \ll 1$, then $S_{\delta_0}, T_{\delta_0}$ have the same eigenvalues in $[0, \alpha]$, and if these eigenvalues are denoted by $0 \leqslant t_1^2 \leqslant t_2^2 \leqslant ... \leqslant t_N^2 \leqslant \alpha$ with $t_j \geqslant 0$, then we can find orthonormal families of eigenfunctions, $e_1, e_2, ..., e_N \in \mathcal{D}(S_{\delta_0})$, $f_1, f_2, ..., f_N \in \mathcal{D}(T_{\delta_0})$, such that

$$P_{\delta_0} e_j = t_j f_j, \quad P_{\delta_0}^* f_j = t_j e_j, \tag{4.32}$$

in analogy with (4.9)

From this point on, the discussion from (4.9) to (4.27) goes through with only minor changes, with P_0 replaced by P_{δ_0} and P_δ replaced by a new perturbation $P_{\delta_0} + \delta Q_{\text{new}}$. End of the remark.

We next collect some facts from [7]. The first result follows from Section 2 in that paper.

PROPOSITION 4.2. — *Let $P : \mathcal{H} \to \mathcal{H}$ be bounded and assume that $P - 1$ is of trace class, so that P is Fredholm of index 0. Let $R_+, R_-, \mathcal{P}, \mathcal{E} = \mathcal{P}^{-1}$ be as in (4.2), (4.3). Then $\mathcal{P}$ is also a trace class perturbation of the identity operator and*

$$\det P = \det \mathcal{P} \det E_{-+}. \tag{4.33}$$

Now consider the operator $P_z = P_{0,z}$ in (3.9) for $z \in \Omega$, and recall that P_z is a trace class perturbation of the identity. Put $s(x,\xi) = s_z(x,\xi) = |p_z(x,\xi)|^2$. Following Section 4 in [7], we introduce $V(t) = V_z(t)$ by

$$V(t) = \iint_{s(x,\xi) \leqslant t} dx d\xi, \ 0 \leqslant t \leqslant \frac{1}{2}. \tag{4.34}$$

For a given $z \in \Omega$, we assume that there exists $\kappa \in]0,1]$, such that

$$V(t) = \mathcal{O}(1) t^\kappa, \ 0 \leqslant t \leqslant \frac{1}{2} \tag{4.35}$$

(Later on we shall also assume that this condition holds uniformly when z varies in some subset of Ω, and then all estimates below will hold uniformly for z in that subset.) Proposition 4.5 in [7] and a subsequent remark there give

PROPOSITION 4.3. — *Assume (4.35). For $0 < h \ll \alpha \ll 1$, the number $N(\alpha)$ of eigenvalues of $P_z^* P_z$ in $[0,\alpha]$ satisfies*

$$N(\alpha) = \mathcal{O}(\alpha^\kappa h^{-n}). \tag{4.36}$$

Moreover,

$$\ln \det P_z^* P_z \leqslant \frac{1}{(2\pi h)^n} \left(\iint \ln(s) dx d\xi + \mathcal{O}(\alpha^\kappa \ln \frac{1}{\alpha}) \right). \tag{4.37}$$

We next consider $P_{\delta,z} = (\widetilde{P}_\delta - z)^{-1}(P_\delta - z) = 1 - K_{\delta,z}$ with P_δ, $\widetilde{P}_\delta$ as in Section 3 and under the assumptions (3.4), (3.6). Put

$$S_{\delta,z} = P_{\delta,z}^* P_{\delta,z} = 1 - K_{\delta,z} - K_{\delta,z}^* + K_{\delta,z}^* K_{\delta,z},$$

where $K_{\delta,z}$ is given by (3.9), so that

$$\|K_{\delta,z}\| \leqslant \mathcal{O}(1), \ \|K_{\delta,z}\|_{\mathrm{tr}} \leqslant \|(\widetilde{P}_\delta - z)^{-1}\| \|\widetilde{P} - P\|_{\mathrm{tr}} \leqslant \mathcal{O}(h^{-n}).$$

Here $\|\cdot\|_{\mathrm{tr}}$ denotes the trace class norm, and we refer for instance to [2] for the standard estimate on the trace class norm of an h-pseudodifferential operator with compactly supported symbol, that we used for the last estimate.

Write $\dot{K}_{\delta,z} = \frac{\partial}{\partial \delta} K_{\delta,z}$. Then

$$\dot{K}_{\delta,z} = -(z - \widetilde{P}_\delta)^{-1} Q (z - \widetilde{P}_\delta)^{-1} (\widetilde{P} - P),$$

so

$$\|\dot{K}_{\delta,z}\| \leqslant \mathcal{O}(\|Q\|), \quad \|\dot{K}_{\delta,z}\|_{\mathrm{tr}} \leqslant \mathcal{O}(\|Q\|h^{-n}).$$

It follows that

$$\|\dot{S}_{\delta,z}\| \leqslant \mathcal{O}(\|Q\|), \quad \|\dot{S}_{\delta,z}\|_{\mathrm{tr}} \leqslant \mathcal{O}(\|Q\|h^{-n}).$$

Let $N = N(\alpha, \delta)$ denote the number of singular values of $P_{\delta,z}$ in the interval $[0, \sqrt{\alpha}[$ for $h \ll \alpha \ll 1$. Strengthen the assumption (3.6) to

$$\delta \leqslant \mathcal{O}(h). \tag{4.38}$$

Then $\|S_{\delta,z} - S_{0,z}\| \leqslant \mathcal{O}(h)$ and from (4.36) we get

$$N(\alpha, \delta) = \mathcal{O}(\alpha^{\kappa} h^{-n}). \tag{4.39}$$

Define

$$\mathcal{P}_\delta = \begin{pmatrix} P_{\delta,z} & R_{-,\delta} \\ R_{+,\delta} & 0 \end{pmatrix}$$

as in (4.9)–(4.11), so that $\mathcal{P} = \mathcal{P}_0$. As in (5.10) in [7] we have

$$|\det \mathcal{P}_\delta|^2 = \alpha^{-N} \det 1_\alpha(S_{\delta,z}), \quad 2\ln|\det \mathcal{P}_\delta| = \ln \det 1_\alpha(S_{\delta,z}) + N \ln \frac{1}{\alpha}, \tag{4.40}$$

where $1_\alpha(t) = \max(\alpha, t)$, $t \geqslant 0$. (The different power of α is due to the normalizing factor $\sqrt{\alpha}$, used in the definition of $R_\pm$ in [7].)

For $0 < \epsilon \ll 1$, let $C^\infty(\overline{\mathbf{R}}_+) \ni 1_{\alpha,\epsilon} \geqslant 1_\alpha$ be equal to t outside a small neighborhood of $t = 0$ and converge to 1_α uniformly when $\epsilon \to 0$. For any fixed $\epsilon > 0$, we put $f(t) = 1_{\alpha,\epsilon}(t)$ for $t \geqslant 0$ and extend f to $\mathbf{R}$ in such a way that $f(t) = t + g(t)$, $g \in C_0^\infty(\mathbf{R})$. Let $\tilde{f}(t) = t + \tilde{g}(t)$ be an almost holomorphic extension of f with $\tilde{g} \in C_0^\infty(\mathbf{C})$. Then we have the Cauchy-Riemann formula (see for instance [2] and further references given there):

$$f(S_{\delta,z}) = S_\delta - \frac{1}{\pi} \int (w - S_{\delta,z})^{-1} \overline{\partial} \tilde{g}(w) L(dw).$$

From this we see that

$$\frac{\partial}{\partial \delta} f(S_{\delta,z}) = \dot{S}_\delta - \frac{1}{\pi} \int (w - S_{\delta,z})^{-1} \dot{S}_{\delta,z} (w - S_{\delta,z})^{-1} \overline{\partial} \tilde{g}(w) L(dw).$$

Now,

$$\frac{\partial}{\partial \delta} \ln \det f(S_{\delta,z}) = \mathrm{tr}\, f(S_{\delta,z})^{-1} \frac{\partial}{\partial \delta} f(S_{\delta,z}) =$$

$$\mathrm{tr}\,(f(S_\delta)^{-1} \dot{S}_\delta) - \frac{1}{\pi} \int \mathrm{tr}\,(f(S_{\delta,z})^{-1}(w - S_{\delta,z})^{-1} \dot{S}_{\delta,z}(w - S_{\delta,z})^{-1}) \overline{\partial} \tilde{g}(w) L(dw).$$

Here $f(S_{\delta,z})^{-1}$ and $(w - S_{\delta,z})^{-1}$ commute, and using also the cyclicity of the trace, we see that the last term is equal to

$$\operatorname{tr}(f(S_{\delta,z})^{-1}\frac{(-1)}{\pi}\int (w - S_{\delta,z})^{-2}\overline{\partial}_w\widetilde{g}(w)L(dw)\dot{S}_{\delta,z})$$

$$= \operatorname{tr}(f(S_{\delta,z})^{-1}\frac{(-1)}{\pi}\int (w - S_{\delta,z})^{-1}\overline{\partial}_w\partial_w\widetilde{g}(w)L(dw)\dot{S}_{\delta,z})$$

$$= \operatorname{tr}(f(S_{\delta,z})^{-1}g'(S_{\delta,z})\dot{S}_{\delta,z}),$$

leading to the general identity

$$\frac{\partial}{\partial\delta}\ln\det f(S_{\delta,z}) = \operatorname{tr}(f(S_{\delta,z})^{-1}f'(S_{\delta,z})\dot{S}_{\delta,z}).$$

Now we can choose $f = 1_{\alpha,\epsilon}$ such that $|f'(t)| \leqslant 1$ for $t \geqslant 0$. Then we get the estimate

$$\frac{\partial}{\partial\delta}\ln\det(1_{\alpha,\epsilon}(S_{\delta,z})) = \operatorname{tr}(1_{\alpha,\epsilon}(S_{\delta,z})^{-1}1'_{\alpha,\epsilon}(S_{\delta,z})\dot{S}_{\delta,z})$$

$$= \mathcal{O}(\frac{\|\dot{S}_{\delta,z}\|_{\operatorname{tr}}}{\alpha})$$

$$= \mathcal{O}(1)\frac{\|Q\|}{\alpha h^n}.$$

Since $\ln\det 1_\alpha(S_{\delta,z}) = \lim_{\epsilon\to 0}\ln\det 1_{\alpha,\epsilon}(S_{\delta,z})$, we can integrate the above estimate, pass to the limit and obtain

$$\ln\det 1_\alpha(S_{\delta,z}) = \ln\det 1_\alpha(S_{0,z}) + \mathcal{O}(\frac{\delta\|Q\|}{\alpha h^n}).$$

Using (4.40), (4.39), we get

$$\ln|\det\mathcal{P}_\delta|^2 = \ln|\det\mathcal{P}|^2 + \mathcal{O}(\frac{\delta\|Q\|}{\alpha h^n} + \alpha^\kappa h^{-n}\ln\frac{1}{\alpha}). \qquad (4.41)$$

The estimate (5.13) in [7] is valid in our case:

$$\ln|\det\mathcal{P}| = \frac{1}{(2\pi h)^n}(\iint\ln|p_z|dxd\xi + \mathcal{O}(\alpha^\kappa\ln\frac{1}{\alpha})), \qquad (4.42)$$

and using this in (4.41),we get

$$\ln|\det\mathcal{P}_\delta| = \frac{1}{(2\pi h)^n}(\iint\ln|p_z|dxd\xi + \mathcal{O}(\alpha^\kappa\ln\frac{1}{\alpha} + \frac{\delta}{\alpha}\|Q\|)). \qquad (4.43)$$

5. Singular values and determinants of certain matrices associated to δ potentials

We start with a general observation.

PROPOSITION 5.1. — *If $e_1(x), ..., e_N(x)$ are linearly independent continuous functions on an open domain $\Omega \subset \mathbf{R}^n$, then we can find N different points $a_1, ..., a_N \in \Omega$ so that $\overrightarrow{e}(a_1), ..., \overrightarrow{e}(a_N)$ are linearly independent in $\mathbf{C}^N$, where*

$$\overrightarrow{e}(x) = \begin{pmatrix} e_1(x) \\ e_2(x) \\ .. \\ .. \\ e_N(x) \end{pmatrix}.$$

Proof. — Let $E \subset \mathbf{C}^N$ be the linear subspace spanned by all the $\overrightarrow{e}(x)$, $x \in \Omega$. We claim that $E = \mathbf{C}^N$. Indeed, if that were not the case, there would exist $0 \neq (\lambda_1, ..., \lambda_N) \in \mathbf{C}^N$ such that

$$0 = \langle \lambda, \overrightarrow{e}(x) \rangle := \sum_1^N \lambda_j e_j(x), \quad \forall x \in \Omega.$$

But this means that $e_1, ..., e_N$ are *linearly dependent* functions in contradiction with the assumption, hence $E = \mathbf{C}^N$ and then we can find $a_1, .., a_N \in \Omega$ such that $\overrightarrow{e}(a_1), ..., \overrightarrow{e}(a_N)$ form a basis in $\mathbf{C}^N$ and consequently so that they are linearly independent. $\quad\square$

PROPOSITION 5.2. — *Let $e_1, ..., e_N$ be as in Proposition 5.1 and let $f_1, ..., f_N$ be a second family with the same properties. Assume that we can find $a_1, ..., a_N \in \Omega$ such that both $\{\overrightarrow{e}(a_1), ..., \overrightarrow{e}(a_N)\}$ and $\{\overrightarrow{f}(a_1), ..., \overrightarrow{f}(a_N)\}$ are linearly independent. (We notice that this holds in the special case when $f_j = \overline{e}_j$.) Define $M = \mathbf{C}^N \to \mathbf{C}^N$ by*

$$Mu = \sum_1^N (u | \overrightarrow{f}(a_\nu)) \overrightarrow{e}(a_\nu), \ u \in \mathbf{C}^N, \tag{5.1}$$

where $(\cdot | \cdot \cdot)$ denotes the usual scalar product on $\mathbf{C}^N$. Then M is bijective.

Proof. — Let $u \in \mathbf{C}^N$ belong to the kernel of M. Since $\overrightarrow{e}(a_1), ..., \overrightarrow{e}(a_N)$ form a basis in $\mathbf{C}^N$, we have $(u | \overrightarrow{f}(a_\nu)) = 0$ for all ν. Since $\overrightarrow{f}(a_1), ..., \overrightarrow{f}(a_N)$ form a basis in $\mathbf{C}^N$, it then follows that $u = 0$. $\quad\square$

COROLLARY 5.3. — *Under the assumptions of Proposition 5.2, there exists $q \in C_0^\infty(\Omega; \mathbf{R})$ such that $M_q : \mathbf{C}^N \to \mathbf{C}^N$ is bijective, where*

$$M_q u = \int q(x)(u| \overrightarrow{f}(x)) \overrightarrow{e}(x) dx. \tag{5.2}$$

Proof. — It suffices to let $q(x)$ be very close to $\sum_1^N \delta(x - a_j)$ in the weak measure sense. $\qquad\square$

We observe that M has the matrix

$$M_{j,k} = \sum_{\nu=1}^N e_j(a_\nu) \overline{f}_k(a_\nu) \tag{5.3}$$

and that M_q has the matrix

$$M_{q,j,k} = \int q(x) e_j(x) \overline{f}_k(x) dx.$$

We now look for quantitative versions of the preceding results.

LEMMA 5.4. — *Let $e_1, ..., e_N$ be as in Proposition 5.1 and also square integrable. Let $L \subset \mathbf{C}^N$ be a linear subspace of dimension $M - 1$, for some $1 \leqslant M \leqslant N$. Then there exists $x \in \Omega$ such that*

$$\operatorname{dist}(\overrightarrow{e}(x), L)^2 \geqslant \frac{1}{\operatorname{vol}(\Omega)} \operatorname{tr}((1 - \pi_L)\mathcal{E}_\Omega), \tag{5.4}$$

where $\mathcal{E}_\Omega = ((e_j|e_k)_{L^2(\Omega)})_{1 \leqslant j,k \leqslant N}$ and π_L is the orthogonal projection from $\mathbf{C}^N$ onto L.

Proof. — Let $\nu_1, ..., \nu_N$ be an orthonormal basis in $\mathbf{C}^N$ such that L is spanned by $\nu_1, ..., \nu_{M-1}$ (and equal to 0 when $M = 1$). Let $(\cdot|\cdot)$ denote the usual scalar product on $\mathbf{C}^N$ and let $(\cdot|\cdot)_\Omega$ be the scalar product on $L^2(\Omega)$. Write

$$\nu_\ell = \begin{pmatrix} \nu_{1,\ell} \\ .. \\ .. \\ \nu_{N,\ell} \end{pmatrix}.$$

We have

$$\operatorname{dist}(\overrightarrow{e}(x), L)^2 = \sum_{\ell=M}^N |(\overrightarrow{e}(x)|\nu_\ell)|^2$$

$$= \sum_{\ell=M}^{N} |\sum_{j} e_j(x)\overline{\nu}_{j,\ell}|^2$$

$$= \sum_{\ell=M}^{N} \sum_{j,k} \overline{\nu}_{j,\ell} e_j(x)\overline{e}_k(x)\nu_{k,\ell}.$$

It follows that

$$\int_{\Omega} \text{dist}\,(\overrightarrow{e}(x), L)^2 dx = \sum_{\ell=M}^{N} (\mathcal{E}_{\Omega}\nu_{\ell}|\nu_{\ell}) = \text{tr}\,((1 - \pi_L)\mathcal{E}_{\Omega}).$$

It then suffices to estimate the integral from above by

$$\text{vol}\,(\Omega) \sup_{x\in\Omega} \text{dist}\,(\overrightarrow{e}(x), L)^2.$$

If $\text{dist}\,(\overrightarrow{e}(x), L)^2$ is constant, then any $x \in \Omega$ will satisfy (5.4), if not,

$$\text{tr}\,((1 - \pi_L)\mathcal{E}_{\Omega}) < \text{vol}\,(\Omega) \sup_{\Omega} \text{dist}\,(\overrightarrow{e}(x), L)^2$$

and we can find an $x \in \Omega$ satisfying (5.4).   □

If we make the assumption that

$$e_1, ..., e_N \text{ is an orthonormal family in } L^2(\Omega), \qquad (5.5)$$

then $\mathcal{E}_{\Omega} = 1$ and (5.4) simplifies to

$$\max_{x\in\Omega} \text{dist}\,(\overrightarrow{e}(x), L)^2 \geqslant \frac{N - M + 1}{\text{vol}\,(\Omega)}. \qquad (5.6)$$

In the general case, let $0 \leqslant \varepsilon_1 \leqslant \varepsilon_2 \leqslant ... \leqslant \varepsilon_N$ denote the eigenvalues of $\mathcal{E}_{\Omega}$. Then we have

$$\inf_{\dim L = M-1} \text{tr}\,((1 - \pi_L)\mathcal{E}_{\Omega}) = \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_{N-M+1} =: E_M. \qquad (5.7)$$

Indeed, the min-max principle shows that

$$\varepsilon_k = \inf_{\dim L'=k} \sup_{\substack{\nu\in L' \\ \|\nu\|=1}} (\mathcal{E}_{\Omega}\nu|\nu),$$

so for a general subspace L of dimension $M - 1$, the eigenvalues of $(1 - \pi_L)\mathcal{E}_{\Omega}(1 - \pi_L)$ are $\varepsilon_1' \leqslant ... \leqslant \varepsilon_{N-M+1}'$, with $\varepsilon_j' \geqslant \varepsilon_j$.

Now, we can use the lemma to choose successively $a_1, ..., a_N \in \Omega$ such that

$$\|\overrightarrow{e}(a_1)\|^2 \;\geqslant\; \frac{E_1}{\text{vol}\,(\Omega)},$$

$$\text{dist}\,(\overrightarrow{e}(a_2), \mathbf{C}\overrightarrow{e}(a_1))^2 \;\geqslant\; \frac{E_2}{\text{vol}\,(\Omega)},$$

$$...$$

$$\text{dist}\,(\overrightarrow{e}(a_M), \mathbf{C}\overrightarrow{e}(a_1) \oplus ... \oplus \mathbf{C}\overrightarrow{e}(a_{M-1}))^2 \;\geqslant\; \frac{E_M}{\text{vol}\,(\Omega)},$$

$$...$$

Let $\nu_1, \nu_2, ..., \nu_N$ be the Gram-Schmidt orthonormalization of the basis $\overrightarrow{e}(a_1), \overrightarrow{e}(a_2), ..., \overrightarrow{e}(a_N)$, so that

$$\overrightarrow{e}(a_M) \equiv c_M \nu_M \,\text{mod}\,(\nu_1, ..., \nu_{M-1}), \text{ where } |c_M| \geqslant \left(\frac{E_M}{\text{vol}\,(\Omega)}\right)^{\frac{1}{2}}. \quad (5.8)$$

Consider the $N \times N$ matrix $E = (\overrightarrow{e}(a_1)\,\overrightarrow{e}(a_2)\,...\,\overrightarrow{e}(a_N))$ where $\overrightarrow{e}(a_j)$ are viewed as columns. Expressing these vectors in the basis $\nu_1, ..., \nu_N$ will not change the absolute value of the determinant and E now becomes an upper triangular matrix with diagonal entries $c_1, ..., c_N$. Hence

$$|\det E| = |c_1 \cdot ... \cdot c_N|, \quad (5.9)$$

and (5.8) implies that

$$|\det E| \geqslant \frac{(E_1 E_2 ... E_N)^{1/2}}{(\text{vol}\,(\Omega))^{N/2}}. \quad (5.10)$$

We now return to M in (5.1), (5.3) and observe that

$$M = E \circ F^*, \quad (5.11)$$

where

$$F = (\overrightarrow{f}(a_1)...\overrightarrow{f}(a_N)). \quad (5.12)$$

Now, we assume

$$f_j = \overline{e}_j, \; \forall j. \quad (5.13)$$

Then $F^* = {}^t E$, so

$$M = E \circ {}^t E. \quad (5.14)$$

We get from (5.10), (5.14), that

$$|\det M| \geqslant \frac{E_1 E_2 ... E_N}{\text{vol}\,(\Omega)^N}. \quad (5.15)$$

Under the assumption (5.5), this simplifies to

$$|\det M| \geqslant \frac{N!}{\text{vol}\,(\Omega)^N}. \tag{5.16}$$

It will also be useful to estimate the singular values $s_1(M) \geqslant s_2(M) \geqslant \ldots \geqslant s_N(M)$ of the matrix M (by definition the decreasing sequence of eigenvalues of the matrix $(M^*M)^{\frac{1}{2}}$). Clearly,

$$s_1^N \geqslant s_1^{k-1}s_k^{N-k+1} \geqslant \prod_1^N s_j = |\det M|, \quad 1 \leqslant k \leqslant N, \tag{5.17}$$

and we recall that

$$s_1 = \|M\|. \tag{5.18}$$

Combining (5.15) and (5.17), we get

PROPOSITION 5.5. — *Under the above assumptions,*

$$s_1 \geqslant \frac{(E_1 \ldots E_N)^{\frac{1}{N}}}{\text{vol}\,(\Omega)}, \tag{5.19}$$

$$s_k \geqslant s_1 \left(\prod_1^N \left(\frac{E_j}{s_1 \text{vol}\,(\Omega)} \right) \right)^{\frac{1}{N-k+1}}. \tag{5.20}$$

6. Singular values of matrices associated to suitable admissible potentials

In this section, we let $P, \widetilde{P}, p, \widetilde{p}$ be as in the introduction. (The assumption (1.5) will not be used here.) We also choose $\chi_0(x)$, ϵ_k, μ_k, $D = D(h)$, $L = L(h)$ as in and around (1.6), (1.7).

DEFINITION 6.1. — *An admissible potential is a potential of the form*

$$q(x) = \chi_0(x) \sum_{0 < \mu_k \leqslant L} \alpha_k \epsilon_k(x), \ \alpha \in \mathbf{C}^D. \tag{6.1}$$

Here we shall take another step in the construction of an admissible potential q for which the singular values of $P + \delta h^{N_1} q$ (cf (1.9)) satisfy nice lower bounds. More precisely, we shall approximate δ-potentials in H^{-s} with admissible ones and then apply the results of the preceding two sections. Let us start with the approximation. As in the introduction we let $s > n/2$, $0 < \epsilon < s - n/2$.

PROPOSITION 6.2. — *Let $a \in \{x \in \mathbf{R}^n; \chi_0(x) = 1\}$. Then $\exists \alpha \in \mathbf{C}^D$, $r \in H^{-s}$ such that*

$$\delta_a(x) = \chi_0(x) \sum_{\mu_k \leqslant L} \alpha_k \epsilon_k + \chi_0(x) r(x), \qquad (6.2)$$

where

$$\|\chi_0 r\|_{H^{-s}} \leqslant C_{s,\epsilon} L^{-(s-\frac{n}{2}-\epsilon)} h^{-\frac{n}{2}}, \qquad (6.3)$$

$$\left(\sum |\alpha_k|^2\right)^{\frac{1}{2}} \leqslant \langle L \rangle^{\frac{n}{2}+\epsilon} \left(\sum_{\mu_k \leqslant L} \langle \mu_k \rangle^{-2(\frac{n}{2}+\epsilon)} |\alpha_k|^2\right)^{\frac{1}{2}} \leqslant C L^{\frac{n}{2}+\epsilon} h^{-\frac{n}{2}}. \qquad (6.4)$$

Proof. — Observe first that if $\delta_a = \delta(x-a)$ for some fixed $a \in \mathbf{R}^n$, and $s > \frac{n}{2}$ is fixed as in the introduction,

$$\|\delta_a\|_{H^{-s}} = \mathcal{O}(1)\|\langle h\xi\rangle^{-s}\|_{L^2} = \mathcal{O}_s(1) h^{-\frac{n}{2}}. \qquad (6.5)$$

In general, if $u \in H^{-s_1}(\widetilde{\Omega})$, $s_1 > \frac{n}{2}$, then Proposition 2.2 (where s is arbitrary) shows that

$$u = \sum_1^\infty \alpha_k \epsilon_k, \quad \sum \langle \mu_k \rangle^{-2s_1} |\alpha_k|^2 \asymp \|u\|_{H^{-s_1}}^2.$$

Thus, if $s > s_1$:

$$u = \sum_{\mu_k \leqslant L} \alpha_k \epsilon_k + r, \qquad (6.6)$$

where

$$\|r\|_{H^{-s}}^2 = \sum_{\mu_k > L} \langle \mu_k \rangle^{-2s} |\alpha_k|^2 \leqslant C L^{-2(s-s_1)} \|u\|_{H^{-s_1}}^2, \qquad (6.7)$$

$$\left(\sum_{\mu_k \leqslant L} |\alpha_k|^2\right)^{\frac{1}{2}} \leqslant \langle L \rangle^s \left(\sum_{\mu_k \leqslant L} \langle \mu_k \rangle^{-2s} |\alpha_k|^2\right)^{\frac{1}{2}} \leqslant C L^s \|u\|_{H^{-s}}. \qquad (6.8)$$

In particular, when $u = \delta_a$, $a \in K$, we can multiply (6.6) with χ_0 and we get the proposition with $s_1 = \epsilon + n/2$ ☐

Let P_δ be as in (3.3) and assume (3.4), (3.6). Let $\mathcal{R}(\pi_\alpha) = \mathbf{C}e_1 \oplus \ldots \oplus \mathbf{C}e_N$ be as in one of the two cases of Proposition 3.2. By the minimax principle and standard spectral asymptotics (see [2]), we know that $N = \mathcal{O}(h^{-n})$ and if we want to use the assumption (1.5) we even have $N = \mathcal{O}((\max(\alpha, h))^\kappa h^{-n})$ by Proposition 4.3. For the moment we shall only use that N is bounded by a negative power of h. Recall that we have (3.14), where $s > \frac{n}{2}$ is the fixed number appearing in (3.4).

Let V be a fixed neighborhood of the set K in (3.18), which, as we have seen, can be assumed to be contained in any fixed given neighborhood of $\overline{\pi_x p^{-1}(\Omega)}$, where Ω is the set in the introduction. Let $a = (a_1, ..., a_N) \in V^N$ and put

$$q_a(x) = \sum_1^N \delta(x - a_j), \qquad (6.9)$$

$$M_{q_a;j,k} = \int q_a(x) e_k(x) e_j(x) dx, \ 1 \leqslant j, k \leqslant N. \qquad (6.10)$$

Then using (3.14), (2.2) and the fact that $\|q_a\|_{H^{-s}} = \mathcal{O}(1) N h^{-n/2}$, we get for all $\lambda, \mu \in \mathbf{C}^n$,

$$
\begin{aligned}
\langle M_{q_a} \lambda, \mu \rangle &= \int q_a(x) (\sum \lambda_k e_k)(\sum \mu_j e_j) dx \\
&= \mathcal{O}(1) N h^{-n} \|\lambda\| \|\mu\|
\end{aligned}
$$

and hence

$$s_1(M_{q_a}) = \|M_{q_a}\|_{\mathcal{L}(\mathbf{C}^N, \mathbf{C}^N)} = \mathcal{O}(1) N h^{-n}. \qquad (6.11)$$

We now choose a so that (5.19), (5.20) hold, where we recall that s_k is the k:th singular value of M_{q_a} and E_j is defined in (5.7), where $0 \leqslant \varepsilon_1 \leqslant \varepsilon_2 \leqslant ... \leqslant \varepsilon_N$ are the eigenvalues of the Gramian $\mathcal{E}_V = ((e_j | e_k)_{L^2(V)})_{1 \leqslant j, k \leqslant N}$.

From Proposition 3.2 we see that $e_{j|_V}$ almost form an orthonormal system in $L^2(V)$: $\mathcal{E}_V = 1 + \mathcal{O}(h^\infty)$. Hence,

$$E_j = N - j + 1 + \mathcal{O}(h^\infty). \qquad (6.12)$$

Then (5.19) gives the lower bound

$$s_1 \geqslant \frac{(1 + \mathcal{O}(h^\infty))(N!)^{\frac{1}{N}}}{\mathrm{vol}\,(V)} = (1 + \mathcal{O}(\frac{\ln N}{N})) \frac{N}{e \,\mathrm{vol}\,(V)}, \qquad (6.13)$$

where the last identity follows from Stirling's formula.

Rewriting (5.20) as

$$s_k \geqslant s_1^{-\frac{k-1}{N-k+1}} \left(\prod_1^N \frac{E_j}{\mathrm{vol}\,(V)} \right)^{\frac{1}{N-k+1}},$$

and using (6.11), we get

$$s_k \geqslant \frac{(1 + \mathcal{O}(h^\infty))}{C^{\frac{k-1}{N-k+1}} (\mathrm{vol}\,(V))^{\frac{N}{N-k+1}}} \left(\frac{h^n}{N} \right)^{\frac{k-1}{N-k+1}} (N!)^{\frac{1}{N-k+1}}. \qquad (6.14)$$

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Summing up, we get

PROPOSITION 6.3. — *Let V be a fixed neighborhood of the set K in (3.18) (which can be assumed to be contained in any fixed given neighborhood of $\pi_x^{-1}(\overline{\Omega})$). We can find $a_1, ..., a_N \in V$ such that if $q_a = \sum_1^N \delta(x - a_j)$ and $M_{q_a;j,k} = \int q_a(x)e_k(x)e_j(x)dx$, then the singular values $s_1 \geqslant s_2 \geqslant ... \geqslant s_N$ of M_{q_a}, satisfy (6.11), (6.13) and (6.14).*

We shall next approximate q_a with an admissible potential. Apply Proposition 6.2 to each δ-function in q_a, to see that

$$q_a = q + r, \quad q = \chi_0(x) \sum_{\mu_k \leqslant L} \alpha_k \epsilon_k, \tag{6.15}$$

where

$$\|q\|_{H^{-s}} \leqslant C h^{-\frac{n}{2}} N, \tag{6.16}$$

$$\|r\|_{H^{-s}} \leqslant C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)} h^{-\frac{n}{2}} N, \tag{6.17}$$

$$\left(\sum |\alpha_k|^2\right)^{\frac{1}{2}} \leqslant C L^{\frac{n}{2}+\epsilon} h^{-\frac{n}{2}} N. \tag{6.18}$$

Below, we shall have $N = \mathcal{O}(h^{\kappa-n})$ so if we choose L as in (1.7), we get

$$|\alpha|_{\mathbf{C}^D} \leqslant C h^{-(\frac{n}{2}+\epsilon)M+\kappa-\frac{3n}{2}}$$

and q becomes an admissible potential in the sense of (1.6), (1.7).

In order to estimate M_r, we write

$$\langle M_r \beta, \gamma \rangle = \int r(x)(\sum \beta_k e_k)(\sum \gamma_j e_j)dx,$$

so that

$$|\langle M_r \beta, \gamma \rangle| \leqslant C\|r\|_{H^{-s}} h^{-\frac{n}{2}} \|\sum \beta_k e_k\|_{H^s} \|\sum \gamma_j e_j\|_{H^s}.$$

Applying (3.14) to the last two factors, we get with a new constant $C > 0$:

$$|\langle M_r \beta, \gamma \rangle| \leqslant C\|r\|_{H^{-s}} h^{-\frac{n}{2}} \|\beta\|\|\gamma\|,$$

so

$$\|M_r\| \leqslant C h^{-\frac{n}{2}} \|r\|_{H^{-s}}. \tag{6.19}$$

Using (6.17), we get for every $\epsilon > 0$

$$\|M_r\| \leqslant C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)} h^{-n} N. \tag{6.20}$$

For the admissible potential q in (6.15), we thus obtain from (6.14), (6.20):

$$s_k(M_q) \geqslant \frac{(1 + \mathcal{O}(h^\infty))}{C^{\frac{k-1}{N-k+1}}(\operatorname{vol}(V))^{\frac{N}{N-k+1}}} \left(\frac{h^n}{N}\right)^{\frac{k-1}{N-k+1}} (N!)^{\frac{1}{N-k+1}} - C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)} h^{-n} N.$$

(6.21)

Similarly, from (6.11), (6.20) we get for $L \geqslant 1$:

$$\|M_q\| \leqslant CNh^{-n}.$$

(6.22)

Using Proposition 2.2, we get for all $s_1 > n/2$,

$$
\begin{aligned}
\|q\|_{H^s} &\leqslant \mathcal{O}(1)(\sum_{\mu_k \leqslant L} \langle \mu_k \rangle^{2s} |\alpha_k|^2)^{\frac{1}{2}} \\
&\leqslant \mathcal{O}(1)(\sum_{\mu_k \leqslant L} \langle \mu_k \rangle^{-2s_1} |\alpha_k|^2)^{\frac{1}{2}} L^{s+s_1} \\
&\leqslant \mathcal{O}(1)h^{-\frac{n}{2}} N L^{s+s_1},
\end{aligned}
$$

where we used (6.16) or rather its proof in the last step. Thus for every $\epsilon > 0$,

$$\|q\|_{H^s} \leqslant \mathcal{O}(1)NL^{s+\frac{n}{2}+\epsilon} h^{-\frac{n}{2}}, \ \forall \epsilon > 0.$$

(6.23)

Summing up, we have obtained

PROPOSITION 6.4. — *Fix $s > n/2$ and P_δ as in (3.3), (3.4), (3.6) and let $\pi_\alpha, e_1, ..., e_N$ be as in one of the two cases in Proposition 6.2. Let $V \Subset \mathbf{R}^n$ be a fixed open neighborhood of K in (3.18) and let $\chi_0 \in C_0^\infty(V)$ be equal to 1 near K. Choose the h-dependent parameter L with $1 \ll L \leqslant \mathcal{O}(h^{-N_0})$ for some fixed $N_0 > 0$. Then we can find an admissible potential q as in (6.15) (different from the one in (3.3), (3.4)) such that the matrix M_q, defined by*

$$M_{q;j,k} = \int qe_k e_j dx,$$

satisfies (6.21), (6.22). Moreover the H^s-norm of q satisfies (6.23).

Notice also that if we choose $\widetilde{R}$ with real coefficients, then we can choose q real-valued.

7. Lower bounds on the small singular values for suitable perturbations

As before, we let

$$P \sim p + hp_1 + ... \in S(m), \ p, p_j \in S(m), \tag{7.1}$$

where $m \geqslant 1$ is an order function on $\mathbf{R}^{2n}$. We assume that $p - z$ is elliptic for at least one value of $z \in \mathbf{C}$ and define Σ, Σ_∞ as in the introduction. Let $\Omega \Subset \mathbf{C}$ be open, simply connected with $\Omega \not\subset \Sigma$, $\overline{\Omega} \cap \Sigma_\infty = \emptyset$.

In this section, we fix a $z \in \Omega$. We will use Proposition 6.4 iteratively to construct a special admissible perturbation P_δ for which we have nice lower bounds on the small singular values of $P_\delta - z$, that will lead to similar bounds for the ones of $P_{\delta,z}$ and to a lower bound on $|\det P_{\delta,z}|$.

We will need the symmetry assumption (1.4). Recall that P also denotes the h-Weyl quantization of the symbol P. On the operator level, (1.4) is equivalent to the property

$$P^* = \Gamma \circ P \circ \Gamma, \tag{7.2}$$

where $\Gamma u = \overline{u}$ denotes the antilinear operator of complex conjugation. Notice that the equivalent conditions (1.4), (7.2) remain unchanged if we add a multiplication operator to P.

As in the introduction, we introduce

$$V_z(t) = \mathrm{vol}\left(\{\rho \in \mathbf{R}^{2n}; \ |p(\rho) - z|^2 \leqslant t\}\right), \tag{7.3}$$

and assume (for our fixed value of z) that

$$V_z(t) = \mathcal{O}(t^\kappa), \ 0 \leqslant t \ll 1, \tag{7.4}$$

for some $\kappa \in]0,1]$. It is easy to see that this assumption is equivalent to (4.35). Moreover, from Proposition 4.3 (or directly from [7]) it is easy to get,

PROPOSITION 7.1. — *Assume (7.4) (or equivalently (4.35)) and recall Remark 4.1. For $0 < h \ll \alpha \ll 1$, the number $N(\alpha)$ of eigenvalues of $(P-z)^*(P-z)$ in $[0, \alpha]$ satisfies*

$$N(\alpha) = \mathcal{O}(\alpha^\kappa h^{-n}). \tag{7.5}$$

Proof. — If $e \in \mathcal{D}(P)$ is normalized in L^2 and $\|(P - z)e\| \leqslant \alpha^{\frac{1}{2}}$ then $\|(\widetilde{P} - z)^{-1}(P - z)e\| \leqslant (C\alpha)^{\frac{1}{2}}$ for some constant $C > 0$. By the mini-max

principle, it follows that the number of eigenvalues of $(P-z)^*(P-z)$ in $[0, \alpha]$ is smaller than or equal to the number of eigenvalues of $P_z^* P_z$ in $[0, C\alpha]$, (where $P_z = (\widetilde{P} - z)^{-1}(P - z)$) and it suffices to apply Proposition 4.3. $\square$

Let $\epsilon > 0$, $s > \frac{n}{2} + \epsilon$ be fixed as in the introduction and consider

$$P_0 = P + \delta_0 q_0, \text{ with } 0 \leqslant \delta_0 \ll h, \; \|q_0\|_{H^s} \leqslant h^{\frac{n}{2}}. \tag{7.6}$$

From the mini-max principle, we see that Proposition 7.1 still applies after replacing P by P_0.

Choose $\tau_0 \in]0, (Ch)^{\frac{1}{2}}]$ and let $N = \mathcal{O}(h^{\kappa - n})$ be the number of singular values of $P_0 - z$; $0 \leqslant t_1(P_0 - z) \leqslant ... \leqslant t_N(P_0 - z) < \tau_0$ in the interval $[0, \tau_0[$. As in the introduction we put

$$N_1 = \widetilde{M} + sM + \frac{n}{2}, \tag{7.7}$$

where $M, \widetilde{M}$ are the parameters in (1.7). Fix $\theta \in]0, \frac{1}{4}[$ and recall that N is determined by the property $t_N(P_0 - z) < \tau_0 \leqslant t_{N+1}(P_0 - z)$. Fix $\epsilon_0 > 0$.

PROPOSITION 7.2. — a) *If q is an admissible potential as in (1.6), (1.7), we have*

$$\|q\|_\infty \leqslant Ch^{-n/2}\|q\|_{H^s} \leqslant \widetilde{C}h^{-N_1}. \tag{7.8}$$

b) *If N is sufficiently large, there exists an admissible potential q as in (1.6), (1.7), such that if*

$$P_\delta = P_0 + \frac{\delta h^{N_1}}{\widetilde{C}} q =: P_0 + \delta Q, \; \delta = \frac{\tau_0}{C} h^{N_1 + n}$$

(so that $\|Q\| \leqslant 1$) then

$$t_\nu(P_\delta - z) \geqslant t_\nu(P_0 - z) - \frac{\tau_0 h^{N_1 + n}}{C} \geqslant (1 - \frac{h^{N_1 + n}}{C})t_\nu(P_0 - z), \; \nu > N, \tag{7.9}$$

$$t_\nu(P_\delta - z) \geqslant \tau_0 h^{N_2}, \; [N - \theta N] + 1 \leqslant \nu \leqslant N. \tag{7.10}$$

Here, we put

$$N_2 = 2(N_1 + n) + \epsilon_0, \tag{7.11}$$

and we let $[a] = \max(\mathbf{Z} \cap]-\infty, a])$ denote the integer part of the real number a. When $N = \mathcal{O}(1)$, we have the same result provided that we replace (7.10) by

$$t_N(P_\delta) \geqslant \tau_0 h^{N_2}. \tag{7.12}$$

Proof. — The part a) follows from Section 2, the definition of admissible potentials in the introduction and from the definition of N_1 in (7.7). (See also (6.23).) We shall therefore concentrate on the proof of b).

Let $e_1, ..., e_N$ be an orthonormal family of eigenfunctions corresponding to $t_\nu(P_0 - z)$, so that

$$(P_0 - z)^*(P_0 - z)e_j = (t_j(P_0 - z))^2 e_j. \tag{7.13}$$

Using the symmetry assumption (1.4) $\Leftrightarrow$ (7.2), we see that a corresponding family of eigenfunctions of $(P - z)(P - z)^*$ is given by

$$\widetilde{f}_j = \Gamma e_j. \tag{7.14}$$

If the non-vanishing t_j are not all distinct it is not immediately clear that we can arrange so that $\widetilde{f}_j = f_j$ in (4.32), but we know that $\widetilde{f}_1, ..., \widetilde{f}_N$ and $f_1, ..., f_N$ are orthonormal families that span the same space F_N. Let E_N be the span of $e_1, ..., e_N$. We then know that

$$(P_0 - z) : E_N \to F_N \text{ and } (P_0 - z)^* : F_N \to E_N \tag{7.15}$$

have the same singular values $0 \leqslant t_1 \leqslant t_2 \leqslant ... \leqslant t_N$.

Define $R_+ : L^2 \to \mathbf{C}^N$, $R_- : \mathbf{C}^N \to L^2$ by

$$R_+ u(j) = (u|e_j), \quad R_- u_- = \sum_1^N u_-(j)\widetilde{f}_j. \tag{7.16}$$

Then

$$\mathcal{P} = \begin{pmatrix} P_0 - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}(P_0) \times \mathbf{C}^N \to L^2 \times \mathbf{C}^N \tag{7.17}$$

has a bounded inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}.$$

Since we do not necessarily have (4.32) we cannot say that $E_{-+} = \mathrm{diag}\,(t_j)$ but we know that the singular values of E_{-+} are given by $t_j(E_{-+}) = t_j(P_0 - z)$, $1 \leqslant j \leqslant N$, or equivalently by $s_j(E_{-+}) = t_{N+1-j}(P_0 - z)$, for $1 \leqslant j \leqslant N$.

We will apply Section 4, and recall that N is assumed to be sufficiently large and that θ has been fixed in $]0, 1/4[$. (The case of bounded N will be treated later.) Let N_2 be given in (7.11). Since z is fixed it will also be notationally convenient to assume that $z = 0$.

Case 1. $s_j(E_{-+}) \geqslant \tau_0 h^{N_2}$, for $1 \leqslant j \leqslant N - [(1 - \theta)N]$. Then we get the proposition with $q = 0$, $P_\delta = P_0$.

Case 2.

$$s_j(E_{-+}) < \tau_0 h^{N_2} \text{ for some } j \text{ such that } 1 \leqslant j \leqslant N - [(1-\theta)N]. \quad (7.18)$$

Recall that for the special admissible potential q in (6.15), we have (6.21). For $k \leqslant N/2$, we have $N - k + 1 > N/2$, so

$$\frac{k-1}{N-k+1} \leqslant 1,$$

and (6.21) gives

$$s_k(M_q) \geqslant \frac{1 + \mathcal{O}(h^\infty)}{C} \frac{h^n}{N} (N!)^{\frac{1}{N}} - C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)} \frac{N}{h^n}.$$

By Stirling's formula, we have $(N!)^{\frac{1}{N}} \geqslant N/\text{Const}$, so for $1 \leqslant k \leqslant N/2$, we obtain with a new constant $C > 0$:

$$s_k(M_q) \geqslant \frac{h^n}{C} - C_\epsilon L^{-(s-\frac{n}{2}-\epsilon)} \frac{N}{h^n}.$$

Here, we recall from Proposition 7.1 (which also applies to P_0) that $N = \mathcal{O}(h^{\kappa-n})$ and choose L so that

$$L^{-(s-\frac{n}{2}-\epsilon)} h^{\kappa-2n} \ll h^n,$$

i.e. so that (in agreement with (1.7))

$$L \gg h^{\frac{\kappa-3n}{s-\frac{n}{2}-\epsilon}}. \quad (7.19)$$

We then get

$$s_k(M_q) \geqslant \frac{h^n}{C}, \ 1 \leqslant k \leqslant \frac{N}{2}, \quad (7.20)$$

for a new constant $C > 0$.

From (6.22) and the fact that $N = \mathcal{O}(h^{\kappa-n})$ we get

$$s_1(M_q) \leqslant \|M_q\| \leqslant CNh^{-n} \leqslant \widetilde{C}h^{\kappa-2n}. \quad (7.21)$$

In addition to the lower bound (7.19) we assume as in (1.7) (in all cases) that

$$L \leqslant Ch^{-M}, \text{ for some } M \geqslant \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}. \quad (7.22)$$

As we saw after (6.18), q is indeed an admissible potential as in (1.6), (1.7), so that by (7.8)

$$\|q\|_\infty \leqslant Ch^{-\frac{n}{2}} \|q\|_{H^s} \leqslant \widetilde{C}h^{-N_1}. \quad (7.23)$$

Put
$$P_\delta = P_0 + \frac{\delta h^{N_1}}{\widetilde{C}} q = P_0 + \delta Q, \ Q = \frac{h^{N_1}}{\widetilde{C}} q, \ \|Q\| \leqslant 1. \qquad (7.24)$$

Then, if $\delta \leqslant \tau_0/2$, we can replace P_0 by P_δ in (7.17) and we still have a well-posed problem with inverse as in (4.16)–(4.21), satisfying (4.25)–(4.27) with $Q_\omega = Q$ as above. Here $E_-^0 Q E_+^0 = h^{N_1} M_q/\widetilde{C}$ so according to (7.20), we have with a new constant C

$$s_k(\delta E_-^0 Q E_+^0) \geqslant \frac{\delta h^{N_1+n}}{C}, \ 1 \leqslant k \leqslant \frac{N}{2}. \qquad (7.25)$$

Playing with the general estimate (4.5), we get

$$s_\nu(A + B) \geqslant s_{\nu+k-1}(A) - s_k(B)$$

and for a sum of three operators

$$s_\nu(A + B + C) \geqslant s_{\nu+k+\ell-2}(A) - s_k(B) - s_\ell(C).$$

We apply this to E_{-+}^δ in (4.26) and get

$$s_\nu(E_{-+}^\delta) \geqslant s_{\nu+k-1}(\delta E_-^0 Q E_+^0) - s_k(E_{-+}^0) - 2\frac{\delta^2}{\tau_0}. \qquad (7.26)$$

Here we use (7.18) with $j = k = N - [(1-\theta)N]$ as well as (7.25), to get for $\nu \leqslant N - [(1-\theta)N]$

$$s_\nu(E_{-+}^\delta) \geqslant \frac{\delta h^{N_1+n}}{C} - \tau_0 h^{N_2} - 2\frac{\delta^2}{\tau_0}. \qquad (7.27)$$

Recall that $\theta < \frac{1}{4}$.

Choose
$$\delta = \frac{1}{C} \tau_0 h^{N_1+n}, \qquad (7.28)$$

where (the new constant) $C > 0$ is sufficiently large.

Then, with a new constant $C > 0$, we get (for $h > 0$ small enough)

$$s_\nu(E_{-+}^\delta) \geqslant \frac{\delta}{C} h^{N_1+n}, \ 1 \leqslant \nu \leqslant N - [(1-\theta)N], \qquad (7.29)$$

implying
$$s_\nu(E_{-+}^\delta) \geqslant 8\tau_0 h^{N_2}, \ 1 \leqslant \nu \leqslant N - [(1-\theta)N]. \qquad (7.30)$$

For the corresponding operator P_δ, we have for $\nu > N$:

$$t_\nu(P_\delta) \geqslant t_\nu(P_0) - \delta = t_\nu(P_0) - \frac{\tau_0 h^{N_1+n}}{C}.$$

Since $t_\nu(P) \geqslant \tau_0$ in this case, we get (7.9).

From (7.30) and (4.27), we get (7.10).

When $N = \mathcal{O}(1)$, we still get (7.27) with $\nu = 1$ and this leads to (7.12).

$\square$

The construction can now be iterated. Assume that $N \gg 1$ and replace (P_0, N, τ_0) by $(P_\delta, [(1-\theta)N], \tau_0 h^{N_2}) =: (P^{(1)}, N^{(1)}, \tau_0^{(1)})$ and keep on, using the same values for the exponents N_1, N_2. Then we get a sequence $(P^{(k)}, N^{(k)}, \tau_0^{(k)})$, $k = 0, 1, ..., k(N)$, where the last value $k(N)$ is determined by the fact that $N^{(k(N))}$ is of the order of magnitude of a large constant. Moreover,

$$t_\nu(P^{(k)}) \geqslant \tau_0^{(k)}, \ N^{(k)} < \nu \leqslant N^{(k-1)}, \tag{7.31}$$

$$t_\nu(P^{(k+1)}) \geqslant t_\nu(P^{(k)}) - \frac{\tau_0^{(k)} h^{N_1+\nu}}{C}, \ \nu > N^{(k)}, \tag{7.32}$$

$$\tau_0^{(k+1)} = \tau_0^{(k)} h^{N_2}, \tag{7.33}$$

$$N^{(k+1)} = [(1-\theta)N^{(k)}], \tag{7.34}$$

$$P^{(0)} = P, \ N^{(0)} = N, \ \tau_0^{(0)} = \tau_0.$$

Here,

$$P^{(k+1)} = P^{(k)} + \delta^{(k+1)} Q^{(k+1)} = P^{(k)} + \frac{\delta^{(k+1)} h^{N_1}}{\widetilde{C}} q^{(k+1)},$$

$$\|Q^{(k+1)}\| \leqslant 1, \ \delta^{(k+1)} = \tfrac{1}{C} \tau_0^{(k)} h^{N_1+n}.$$

Notice that $N^{(k)}$ decays exponentially fast with k:

$$N^{(k)} \leqslant (1-\theta)^k N, \tag{7.35}$$

so we get the condition on k that $(1-\theta)^k N \geqslant C \ll 1$ which gives

$$k \leqslant \frac{\ln \frac{N}{C}}{\ln \frac{1}{1-\theta}}. \tag{7.36}$$

We also have

$$\tau_0^{(k)} = \tau_0 \left(h^{N_2}\right)^k. \tag{7.37}$$

For $\nu > N$, we iterate (7.32), to get

$$
\begin{aligned}
t_\nu(P^{(k)}) &\geqslant t_\nu(P) - \tau_0 \frac{h^{N_1+n}}{C}\left(1 + h^{N_2} + h^{2N_2} + ...\right) \qquad (7.38)\\
&\geqslant t_\nu(P) - \tau_0 \mathcal{O}(\frac{h^{N_1+n}}{C}).
\end{aligned}
$$

For $1 \ll \nu \leqslant N$, let $\ell = \ell(N)$ be the unique value for which $N^{(\ell)} < \nu \leqslant N^{(\ell-1)}$, so that

$$
t_\nu(P^{(\ell)}) \geqslant \tau_0^{(\ell)}, \qquad (7.39)
$$

by (7.31). If $k > \ell$, we get

$$
t_\nu(P^{(k)}) \geqslant t_\nu(P^{(\ell)}) - \tau_0^{(\ell)}\mathcal{O}(\frac{h^{N_1+n}}{C}). \qquad (7.40)
$$

The iteration above works until we reach a value $k = k_0 = \mathcal{O}(\frac{\ln \frac{N}{C}}{\ln \frac{1}{1-\theta}})$ for which $N^{(k_0)} = \mathcal{O}(1)$. After that, we continue the iteration further by decreasing $N^{(k)}$ by one unit at each step.

Summing up the discussion so far, we have obtained

PROPOSITION 7.3. — *Let (P, z) satisfy the assumptions as in the beginning of this section and choose P_0 as in (7.6). Let $s > \frac{n}{2}$, $0 < \epsilon < s - \frac{n}{2}$, $M \geqslant \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}$, $N_1 = \widetilde{M} + sM + \frac{n}{2}$, $N_2 = 2(N_1 + n) + \epsilon_0$, where $\epsilon_0 > 0$. Let L be an h-dependent parameter satisfying*

$$
h^{\frac{\kappa-3n}{s-\frac{n}{2}-\epsilon}} \ll L \leqslant Ch^{-M}. \qquad (7.41)
$$

Let $0 < \tau_0 \leqslant \sqrt{h}$ and let $N^{(0)} = \mathcal{O}(h^{\kappa-n})$ be the number of singular values of $P_0 - z$ in $[0, \tau_0[$. Let $0 < \theta < \frac{1}{4}$ and let $N(\theta) \gg 1$ be sufficiently large. Define $N^{(k)}$, $1 \leqslant k \leqslant k_1$ iteratively in the following way. As long as $N^{(k)} \geqslant N(\theta)$, we put $N^{(k+1)} = [(1-\theta)N^{(k)}]$. Let $k_0 \geqslant 0$ be the last k value we get in this way. For $k > k_0$ put $N^{(k+1)} = N^{(k)} - 1$, until we reach the value k_1 for which $N^{(k_1)} = 1$.

Put $\tau_0^{(k)} = \tau_0 h^{kN_2}$, $1 \leqslant k \leqslant k_1 + 1$. Then there exists an admissible potential $q = q_h(x)$ as in (1.6), (1.7), satisfying (6.18), (6.23), so that,

$$
\|q\|_{H^s} \leqslant \mathcal{O}(1)h^{-N_1+\frac{n}{2}}, \quad \|q\|_{L^\infty} \leqslant \mathcal{O}(1)h^{-N_1},
$$

such that if $P_\delta = P_0 + \frac{1}{C}\tau_0 h^{2N_1+n}q = P_0 + \delta Q$, $\delta = \frac{1}{C}h^{N_1+n}\tau_0$, $Q = h^{N_1}q$, we have the following estimates on the singular values of $P_\delta - z$:

- If $\nu > N^{(0)}$, we have $t_\nu(P_\delta - z) \geqslant (1 - \frac{h^{N_1+n}}{C})t_\nu(P_0 - z)$.

- If $N^{(k)} < \nu \leqslant N^{(k-1)}$, $1 \leqslant k \leqslant k_1$, then $t_\nu(P_\delta - z) \geqslant (1 - \mathcal{O}(h^{N_1+n}))\tau_0^{(k)}$.

- Finally, for $\nu = N^{(k_1)} = 1$, we have $t_1(P_\delta - z) \geqslant (1 - \mathcal{O}(h^{N_1+n}))\tau_0^{(k_1+1)}$.

We shall now obtain the corresponding estimates for the singular values of $P_{\delta,z} = (\widetilde{P}_\delta - z)^{-1}(P_\delta - z)$. Let $e_1, ..., e_N$ be an orthonormal family corresponding to the singular values $t_j(P_\delta)$ in $[0, \sqrt{h}[$, put $\widetilde{f}_j = \overline{e}_j$ and let

$$(P_\delta - z)u + R_- u_- = v, \ R_+ u = v_+$$

be the corresponding Grushin problem so that the solution operators fulfil

$$\|E\| \leqslant \frac{1}{\sqrt{h}}, \ \|E_\pm\| \leqslant 1, \quad t_j(E_{-+}) = t_j(P_\delta) \leqslant \sqrt{h}, \ 1 \leqslant j \leqslant N. \quad (7.42)$$

Still with $z = 0$ we put $\widetilde{R}_- = \widetilde{P}_\delta^{-1} R_-$. Then the problem

$$P_{\delta,z} u + \widetilde{R}_- u_- = v, \ R_+ u = v_+,$$

is wellposed with the solution

$$u = \widetilde{E}v + \widetilde{E}_+ v_+, \ u_- = \widetilde{E}_- v + \widetilde{E}_{-+} v_+,$$

where

$$\widetilde{E} = E\widetilde{P}_\delta \quad , \quad \widetilde{E}_+ = E_+$$
$$\widetilde{E}_- = E_- \widetilde{P}_\delta \quad , \quad \widetilde{E}_{-+} = E_{-+}.$$

Adapting the estimate (4.8) to our situation, we get

$$t_k(P_{\delta,z}) \geqslant \frac{t_k(P_\delta)}{\|E\widetilde{P}_\delta\| t_k(P_\delta) + \|E_+\| \|E_- \widetilde{P}_\delta\|}, \ 1 \leqslant k \leqslant N, \quad (7.43)$$

where we also recall that $t_k(P_\delta) \leqslant \sqrt{h}$.

Write

$$E\widetilde{P}_\delta \ = \ EP_\delta + E(\widetilde{P} - P)$$
$$E_- \widetilde{P}_\delta \ = \ E_- P_\delta + E_-(\widetilde{P} - P)$$

and use that

$$EP_\delta = 1 - E_+ R_+ = \mathcal{O}(1) \text{ in } \mathcal{L}(L^2, L^2)$$
$$E_- P_\delta = -E_{-+} R_+ = \mathcal{O}(\sqrt{h}) \text{ in } \mathcal{L}(L^2, \ell^2)$$

together with (7.42) and the fact that $\|\widetilde{P} - P\| = \mathcal{O}(1)$. It follows that

$$\|E\widetilde{P}_\delta\| = \mathcal{O}(\frac{1}{\sqrt{h}}), \quad \|E_-\widetilde{P}_\delta\| = \mathcal{O}(1).$$

Using this in (7.43), we get

$$t_k(P_{\delta,z}) \geqslant \frac{t_k(P_\delta)}{C\frac{t_k(P_\delta)}{\sqrt{h}} + C} \geqslant \frac{t_k(P_\delta)}{2C}, \tag{7.44}$$

where used that $t_k(P_\delta) \leqslant \sqrt{h}$ when $1 \leqslant k \leqslant N^{(0)}$. Now the choice of N_2 gives us some margin and we can even get rid of the effect of $2C$ and get for $\tau_0 \in]0, \sqrt{h}]$:

PROPOSITION 7.4. — *Proposition 7.3 remains valid if we replace $P_\delta - z$ there with $P_{\delta,z}$.*

Consider the operator $P_{\delta,z}$ in Proposition 7.4, let $\tau_0 \in]0, \sqrt{h}]$ and choose a corresponding associated Grushin problem

$$\mathcal{P}_\delta = \begin{pmatrix} P_{\delta,z} & R_{-,\delta} \\ R_{+,\delta} & 0 \end{pmatrix}$$

as in (4.9)–(4.11) so that (4.33) holds and moreover for the corresponding inverse $\begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}$

$$t_\nu(E_{-+}) = t_\nu(P_{\delta,z}), \quad 1 \leqslant \nu \leqslant N^{(0)}.$$

We have

$$|\det E_{-+}| = \prod_1^{N^{(0)}} t_\nu(E_{-+}), \tag{7.45}$$

and we shall estimate this quantity from below. In the terms of Proposition 7.3 we have for $1 \leqslant k \leqslant k_0$:

$$N^{(k-1)} - N^{(k)} = N^{(k-1)} - [(1-\theta)N^{(k-1)}] \leqslant \theta N^{(k-1)} + 1 \leqslant 1 + \theta(1-\theta)^{k-1}N^{(0)},$$

so according to Proposition 7.4 we know that

$$\prod_{1+N^{(k)}}^{N^{(k-1)}} t_\nu(E_{-+}) \geqslant ((1 - \mathcal{O}(h^{N_1+n}))\tau_0 h^{kN_2})^{1+\theta(1-\theta)^k N^{(0)}}.$$

For the bounded number of k with $k_0 < k \leqslant k_1$, we have $N^{(k-1)} - N^{(k)} = 1$ and $t_{N^{(k-1)}}(E_{-+}) \geqslant (1 - \mathcal{O}(h^{N_1 + n}))\tau_0 h^{kN_2}$. Hence from (7.45):

$$
\begin{aligned}
\ln|\det E_{-+}| \quad \geqslant \quad & -\sum_{k=1}^{k_0}(\mathcal{O}(h^{N_1+n}) + \ln\frac{1}{\tau_0} + kN_2\ln\frac{1}{h})(1 + \theta(1-\theta)^{k-1}N^{(0)}) \\
& -\sum_{k_0+1}^{k_1+1}(\mathcal{O}(h^{N_1+n}) + \ln\frac{1}{\tau_0} + kN_2\ln\frac{1}{h}).
\end{aligned}
$$

Recall also that $k_1 = \mathcal{O}(\ln N^{(0)}) = \mathcal{O}(1)\ln\frac{1}{h}$, and that $N^{(0)} = \mathcal{O}(h^{\kappa-n})$ (by (7.5) with $\alpha = \mathcal{O}(h)$, valid for P_0). We get

$$
\begin{aligned}
\ln|\det E_{-+}| \quad \geqslant \quad & -C(\ln\frac{1}{\tau_0} + (\ln\frac{1}{h})^2)\sum_{k=0}^{k_1}(1 + \theta(1-\theta)^k N^{(0)}) \quad (7.46) \\
\geqslant \quad & -\widetilde{C}(\ln\frac{1}{\tau_0} + (\ln\frac{1}{h})^2)(h^{\kappa-n} + \ln\frac{1}{h}).
\end{aligned}
$$

Combining this estimate with (4.33) and (4.43) for $\alpha = h$, we get when $\tau_0 = \sqrt{h}$:

PROPOSITION 7.5. — *For the special admissible perturbation P_δ in the propositions 7.3, 7.4, we have*

$$
\ln|\det P_{\delta,z}| \geqslant \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.47)
$$
$$
\frac{1}{(2\pi h)^n}\left(\iint \ln|p_z|dxd\xi - \mathcal{O}\left(h^{N_1+n-\frac{1}{2}} + (h^\kappa + h^n\ln\frac{1}{h})(\ln\frac{1}{\tau_0} + (\ln\frac{1}{h})^2)\right)\right).
$$

We also have the upper bound

$$
|\det E_{-+}| \leqslant \|E_{-+}\|^{N^{(0)}} \leqslant \exp(CN^{(0)}),
$$

which together with (4.33), (4.43) gives

$$
\ln|\det P_{\delta,z}| \leqslant \frac{1}{(2\pi h)^n}\left(\iint \ln|p_z|dxd\xi + \mathcal{O}\left(h^{N_1+n-\frac{1}{2}} + h^\kappa\ln\frac{1}{h}\right)\right).
$$
$$
(7.48)
$$

Notice that this bound is more general, it only depends on the fact that the perturbation of P is of the form δQ with $\delta = \tau_0 h^{N_1+n}/C$ and with $\|Q\| = \mathcal{O}(1)$.

When $\tau_0 \leqslant \sqrt{h}$ we keep the same Grushin problem as before and notice that the singular values of E_{-+} that are $\leqslant \tau_0$, obey the estimates in Proposition 7.3. Their contribution to $\ln|\det E_{-+}|$ can still be estimated from

below as in (7.46). The contribution from the singular values of E_{-+} that are $> \tau_0$ to $\ln|\det E_{-+}|$ can be estimated from below by $-\mathcal{O}(h^{\kappa-n}\ln(1/\tau_0))$ and hence (7.46) remains valid in this case. *We conclude that Proposition 7.5 remains valid when $0 < \tau_0 \leqslant \sqrt{h}$. The same holds for the upper bound (7.48).*

8. Estimating the probability that $\det E_{-+}^{\delta}$ is small

In this section we keep the assumptions on (P, z) of the beginning of Section 7 and choose P_0 as in (7.6). We consider general P_δ of the form

$$P_\delta = P_0 + \delta Q, \quad \delta Q = \delta h^{N_1} q(x), \quad \delta = \frac{1}{C} h^{N_1+n} \tau_0, \tag{8.1}$$

where q is an admissible potential as in (1.6), (1.7). Notice that $D := \#\{k; \ \mu_k \leqslant L\}$ satisfies:

$$D \leqslant \mathcal{O}(L^n h^{-n}) \leqslant \mathcal{O}(h^{-N_3}), \quad N_3 := n(M+1). \tag{8.2}$$

With R as in (1.6), we allow α to vary in the ball

$$|\alpha|_{\mathbf{C}^D} \leqslant 2R = \mathcal{O}(h^{-\widetilde{M}}). \tag{8.3}$$

(Our probability measure will be supported in $B_{\mathbf{C}^D}(0, R)$ but we will need to work in a larger ball.)

We consider the holomorphic function

$$F(\alpha) = (\det P_{\delta,z}) \exp\left(-\frac{1}{(2\pi h)^n} \iint \ln|p_z| dx d\xi\right). \tag{8.4}$$

Then by (7.48), we have

$$\ln|F(\alpha)| \leqslant \epsilon_0(h) h^{-n}, \quad |\alpha| < 2R, \tag{8.5}$$

and for one particular value $\alpha = \alpha^0$ with $|\alpha^0| \leqslant \frac{1}{2}R$, corresponding to the special potential in Proposition 7.3:

$$\ln|F(\alpha^0)| \geqslant -\epsilon_0(h) h^{-n}, \tag{8.6}$$

where we put

$$\epsilon_0(h) = C\left(h^{N_1+n-\frac{1}{2}} + (h^{\kappa} + h^n \ln\frac{1}{h})(\ln\frac{1}{\tau_0} + (\ln\frac{1}{h})^2)\right). \tag{8.7}$$

Here $N_1 \geqslant 1/2$ by (1.8) so we can drop the first term in (8.7).

Let $\alpha^1 \in \mathbf{C}^D$ with $|\alpha^1| = R$ and consider the holomorphic function of one complex variable

$$f(w) = F(\alpha^0 + w\alpha^1). \tag{8.8}$$

We will mainly consider this function for w in the disc determined by the condition $|\alpha^0 + w\alpha^1| < R$:

$$D_{\alpha^0,\alpha^1} : \left| w + \left(\frac{\alpha^0}{R} | \frac{\alpha^1}{R} \right) \right|^2 < 1 - \left| \frac{\alpha^0}{R} \right|^2 + \left| \left(\frac{\alpha^0}{R} | \frac{\alpha^1}{R} \right) \right|^2 =: r_0^2, \tag{8.9}$$

whose radius is between $\frac{\sqrt{3}}{2}$ and 1.

From (8.5), (8.6) we get

$$\ln |f(0)| \geqslant -\epsilon_0(h)h^{-n}, \ \ln |f(w)| \leqslant \epsilon_0(h)h^{-n}. \tag{8.10}$$

By (8.5), we may assume that the last estimate holds in a larger disc, say $D(-(\frac{\alpha^0}{R}|\frac{\alpha^1}{R}), 2r_0)$. Let $w_1, ..., w_M$ be the zeros of f in $D(-(\frac{\alpha^0}{R}|\frac{\alpha^1}{R}), 3r_0/2)$. Then it is standard to get the factorization

$$f(w) = e^{g(w)} \prod_1^M (w - w_j), \ w \in D(-(\frac{\alpha^0}{R}|\frac{\alpha^1}{R}), 4r_0/3), \tag{8.11}$$

together with the bounds

$$|\Re g(w)| \leqslant \mathcal{O}(\epsilon_0(h)h^{-n}), \ M = \mathcal{O}(\epsilon_0(h)h^{-n}). \tag{8.12}$$

See for instance Section 5 in [11] where further references are also given.

For $0 < \epsilon \ll 1$, put

$$\Omega(\epsilon) = \{r \in [0, r_0[; \exists w \in D_{\alpha^0,\alpha^1} \text{ such that } |w| = r \text{ and } |f(w)| < \epsilon\}. \tag{8.13}$$

If $r \in \Omega(\epsilon)$ and w is a corresponding point in D_{α^0,α^1}, we have with $r_j = |w_j|$,

$$\prod_1^M |r - r_j| \leqslant \prod_1^M |w - w_j| \leqslant \epsilon \exp(\mathcal{O}(\epsilon_0(h)h^{-n})). \tag{8.14}$$

Then at least one of the factors $|r - r_j|$ is bounded by $(\epsilon e^{\mathcal{O}(\epsilon_0(h)h^{-n})})^{1/M}$. In particular, the Lebesgue measure $\lambda(\Omega(\epsilon))$ of $\Omega(\epsilon)$ is bounded by $2M(\epsilon e^{\mathcal{O}(\epsilon_0(h)h^{-n})})^{1/M}$. Noticing that the last bound increases with M when the last member of (8.14) is $\leqslant 1$, we get

PROPOSITION 8.1. — *Let $\alpha^1 \in \mathbf{C}^D$ with $|\alpha^1| = R$ and assume that $\epsilon > 0$ is small enough so that the last member of (8.14) is $\leqslant 1$. Then*

$$\lambda(\{r \in [0, r_0];\ |\alpha^0 + r\alpha^1| < R,\ |F(\alpha^0 + r\alpha^1)| < \epsilon\}) \leqslant \qquad (8.15)$$
$$\frac{\epsilon_0(h)}{h^n} \exp(\mathcal{O}(1) + \frac{h^n}{\mathcal{O}(1)\epsilon_0(h)} \ln \epsilon).$$

Here and in the following, the symbol $\mathcal{O}(1)$ in a denominator indicates a bounded positive quantity.

Typically, we can choose $\epsilon = \exp -\frac{\epsilon_0(h)}{h^{n+\alpha}}$ for some small $\alpha > 0$ and then the upper bound in (8.15) becomes

$$\frac{\epsilon_0(h)}{h^n} \exp(\mathcal{O}(1) - \frac{1}{\mathcal{O}(1)h^\alpha}).$$

Now we equip $B_{\mathbf{C}^D}(0, R)$ with a probability measure of the form

$$P(d\alpha) = C(h)e^{\Phi(\alpha)}L(d\alpha), \qquad (8.16)$$

where $L(d\alpha)$ is the Lebesgue measure, Φ is a C^1 function which depends on h and satisfies

$$|\nabla \Phi| = \mathcal{O}(h^{-N_4}), \qquad (8.17)$$

and $C(h)$ is the appropriate normalization constant.

Writing $\alpha = \alpha^0 + Rr\alpha^1$, $0 \leqslant r < r_0(\alpha^1)$, $\alpha^1 \in S^{2D-1}$, $\frac{\sqrt{3}}{2} \leqslant r_0 \leqslant 1$, we get

$$P(d\alpha) = \widetilde{C}(h)e^{\phi(r)}r^{2D-1}dr S(d\alpha^1), \qquad (8.18)$$

where $\phi(r) = \phi_{\alpha^0, \alpha^1}(r) = \Phi(\alpha^0 + rR\alpha^1)$ so that $\phi'(r) = \mathcal{O}(h^{-N_5})$, $N_5 = N_4 + \widetilde{M}$. Here $S(d\alpha^1)$ denotes the Lebesgue measure on S^{2D-1}.

For a fixed α^1, we consider the normalized measure

$$\mu(dr) = \widehat{C}(h)e^{\phi(r)}r^{2D-1}dr \qquad (8.19)$$

on $[0, r_0(\alpha^1)]$ and we want to show an estimate similar to (8.15) for μ instead of λ. Write $e^{\phi(r)}r^{2D-1} = \exp(\phi(r) + (2D-1)\ln r)$ and consider the derivative of the exponent,

$$\phi'(r) + \frac{2D - 1}{r}.$$

This derivative is $\geqslant 0$ for $r \leqslant 2\widetilde{r}_0$, where $\widetilde{r}_0 = C^{-1}\min(1, Dh^{N_5})$ for some large contant C, and we may assume that $2\widetilde{r}_0 \leqslant r_0$. Introduce the measure $\widetilde{\mu} \geqslant \mu$ by

$$\widetilde{\mu}(dr) = \widehat{C}(h)e^{\phi(r_{\max})}r_{\max}^{2D-1}dr,\ r_{\max} := \max(r, \widetilde{r}_0). \qquad (8.20)$$

Since $\widetilde{\mu}([0, \widetilde{r}_0]) \leqslant \mu([\widetilde{r}_0, 2\widetilde{r}_0])$, we get

$$\widetilde{\mu}([0, r(\alpha^1)]) \leqslant \mathcal{O}(1). \tag{8.21}$$

We can write

$$\widetilde{\mu}(dr) = \widehat{C}(h) e^{\psi(r)} dr, \tag{8.22}$$

where

$$\psi'(r) = \mathcal{O}(\max(D,^{-N_5})) = \mathcal{O}(h^{-N_6}), \tag{8.23}$$
$$N_6 = N_3 + N_5.$$

Cf (8.2).

We now decompose $[0, r_0(\alpha^1)]$ into $\asymp h^{-N_6}$ intervals of length $\asymp h^{N_6}$. If I is such an interval, we see that

$$\frac{\lambda(dr)}{C\lambda(I)} \leqslant \frac{\widetilde{\mu}(dr)}{\widetilde{\mu}(I)} \leqslant C \frac{\lambda(dr)}{\lambda(I)} \text{ on } I. \tag{8.24}$$

From (8.15), (8.24) we get when the right hand side of (8.14) is $\leqslant 1$,

$$\widetilde{\mu}(\{r \in I; |F(\alpha^0 + rR\alpha^1)| < \epsilon\})/\widetilde{\mu}(I) \leqslant \frac{\mathcal{O}(1)}{\lambda(I)} \frac{\epsilon_0(h)}{h^n} \exp(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)} \ln \epsilon)$$
$$= \mathcal{O}(1) h^{-N_6} \frac{\epsilon_0(h)}{h^n} \exp(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)} \ln \epsilon).$$

Multiplying with $\widetilde{\mu}(I)$ and summing the estimates over I we get

$$\widetilde{\mu}(\{r \in [0, r(\alpha^1)]; |F(\alpha^0 + rR\alpha^1)| < \epsilon\}) \leqslant \mathcal{O}(1) h^{-N_6} \frac{\epsilon_0(h)}{h^n} \exp(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)} \ln \epsilon). \tag{8.25}$$

Since $\mu \leqslant \widetilde{\mu}$, we get the same estimate with $\widetilde{\mu}$ replaced by μ. Then from (8.18) we get

PROPOSITION 8.2. — *Let $\epsilon > 0$ be small enough for the right hand side of (8.14) to be $\leqslant 1$. Then*

$$P(|F(\alpha)| < \epsilon) \leqslant \mathcal{O}(1) h^{-N_6} \frac{\epsilon_0(h)}{h^n} \exp(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)} \ln \epsilon). \tag{8.26}$$

Remark 8.3. — In the case when $\widetilde{R}$ has real coefficients, we may assume that the eigenfunctions ϵ_j are real, and from the observation after Proposition 6.4 we see that we can choose α_0 above to be real. The discussion above can then be restricted to the case of real α^1 and hence to real α. We can then introduce the probability measure P as in (8.16) on the real ball $B_{\mathbf{R}^D}(0, R)$. The subsequent discussion goes through without any changes, and we still have the conclusion of Proposition 8.2.

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9. End of the proof of the main result

We now work under the assumptions of Theorem 1.1. For z in a fixed neighborhood of Γ, we rephrase (8.5) as

$$|\det P_{\delta,z}| \leqslant \exp \frac{1}{h^n}\left(\frac{1}{(2\pi)^n}\iint \ln |p_z|dxd\xi + \epsilon_0(h)\right), \qquad (9.1)$$

where $\epsilon_0(h)$ is given in (8.7). Moreover, Proposition 8.2 shows that with probability

$$\geqslant 1 - \mathcal{O}(1)h^{-N_6-n}\epsilon_0(h)e^{-\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)}\ln\frac{1}{\epsilon}}, \qquad (9.2)$$

we have

$$|\det P_{\delta,z}| \geqslant \epsilon \exp\left(\frac{1}{h^n}\left(\frac{1}{(2\pi)^n}\right)\iint \ln |p_z|dxd\xi\right), \qquad (9.3)$$

provided that $\epsilon > 0$ is small enough so that

$$\text{The right hand side of (8.14) is } \leqslant 1, \forall \alpha^1 \in S^{2D-1}. \qquad (9.4)$$

From (8.7) and the subsequent remark we can take

$$\epsilon_0(h) = C(h^\kappa + h^n \ln\frac{1}{h})(\ln\frac{1}{\tau_0} + (\ln\frac{1}{h})^2). \qquad (9.5)$$

Write $\epsilon = e^{-\widetilde{\epsilon}/h^n}$, $\widetilde{\epsilon} = h^n \ln\frac{1}{\epsilon}$. Then (9.4) holds if

$$\widetilde{\epsilon} \geqslant C\epsilon_0(h), \qquad (9.6)$$

for some large constant C. (9.2), (9.3) can be rephrased by saying that with probability

$$\geqslant 1 - \mathcal{O}(1)h^{-N_6-n}\epsilon_0(h)e^{-\frac{1}{C}\frac{\widetilde{\epsilon}}{\epsilon_0(h)}}, \qquad (9.7)$$

we have

$$|\det P_{\delta,z}| \geqslant \exp\frac{1}{h^n}\left(\frac{1}{(2\pi)^n}\iint \ln |p_z|dxd\xi - \widetilde{\epsilon}\right). \qquad (9.8)$$

This is of interest for $\widetilde{\epsilon}$ in the range

$$\epsilon_0(h) \ll \widetilde{\epsilon} \ll 1. \qquad (9.9)$$

Now, let $\Gamma \Subset \Omega$ be connected with smooth boundary. Recall that $0 < \kappa \leqslant 1$ and that

$$\text{(4.35) holds uniformly for all } z \text{ in some neighborhood of } \partial\Gamma. \qquad (9.10)$$

Then the function

$$\phi(z) = \frac{1}{(2\pi)^n} \iint \ln |p_z| dx d\xi \tag{9.11}$$

is continuous and subharmonic in a neighborhood of $\partial\Gamma$. Moreover it satisfies the assumption (11.37) of [7] uniformly for z in some neighborhood of $\partial\Gamma$ with ρ_0 there equal to 2κ. We shall apply Proposition 11.5 in [7] to the holomorphic function

$$u(z) = \det P_{\delta,z},$$

with "ϵ" there replaced by $C\widetilde{\epsilon}$ for $C > 0$ sufficiently large and "h" there replaced by h^n. Choose $0 < r \ll 1$ and $z_1, ..., z_N \in \partial\Gamma + D(0, \frac{r}{2})$ as in that proposition, so that

$$\partial\Gamma + D(0, r) \subset \cup_1^N D(z_j, 2r), \ N \asymp \frac{1}{r}.$$

Then, according to (9.7), (9.8) we know that with probability

$$\geqslant 1 - \frac{\mathcal{O}(1)\epsilon_0(h)}{rh^{N_6+n}} e^{-\frac{\widetilde{\epsilon}}{\mathcal{O}(1)\epsilon_0(h)}} \tag{9.12}$$

we have

$$h^n \ln |u(z_j)| \geqslant \phi(z_j) - \widetilde{\epsilon}, \ j = 1, ..., N. \tag{9.13}$$

In a full neighborhood of $\partial\Gamma$ we also have

$$h^n \ln |u(z)| \leqslant \phi(z) + C\widetilde{\epsilon}. \tag{9.14}$$

By Proposition 11.5 in [7] we conclude that with probability bounded from below as in (9.12) we have for every $\widehat{M} > 0$:

$$|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{h^n 2\pi} \int_\Gamma \Delta\phi L(dz)| \leqslant \tag{9.15}$$

$$\frac{\mathcal{O}(1)}{h^n} \left(\frac{\widetilde{\epsilon}}{r} + \mathcal{O}_{\widehat{M}}(1)(r^{\widehat{M}} + \ln(\frac{1}{r}))\mu(\partial\Gamma + D(0, r)) \right),$$

where μ denotes the measure $\Delta\phi L(dz)$. Choose $\widehat{M} = 1$.

According to Section 10 in [7], we know that near Γ, the measure $\frac{1}{2\pi}\Delta\phi L(dz)$ is the push forward under p of $(2\pi)^{-n}$ times the symplectic volume element, and we can replace $\frac{1}{2\pi}\Delta\phi L(dz)$ by this push forward in (9.15). Moreover $u^{-1}(0)$ is the set of eigenvalues of P_δ so we can rephrase (9.15) (with $\widehat{M} = 1$) as

$$|\#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma))| \leqslant \tag{9.16}$$

$$\frac{\mathcal{O}(1)}{h^n} \left(\frac{\widetilde{\epsilon}}{r} + \mathcal{O}(1)(r + \ln(\frac{1}{r}))\mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0, r))) \right).$$

This concludes the proof of Theorem 1.1, with P replaced by the slightly more general operator P_0.

10. Appendix: Review of some h-pseudodifferential calculus

We recall some basic h-pseudodifferential calculus on compact manifolds, including some fractional powers in the spirit of R. Seeley [10]. Recall from [8] that if $X \subset \mathbf{R}^n$ is open, $0 < \rho \leqslant 1$, $m \in \mathbf{R}$, then $S_\rho^m(X \times \mathbf{R}^n) = S_{\rho,1-\rho}^m(X \times \mathbf{R}^n)$ is defined to be the space of all $a \in C^\infty(X \times \mathbf{R}^n)$ such that $\forall K \Subset X$, $\alpha, \beta \in \mathbf{N}^n$, there exists a constant $C = C(K, \alpha, \beta)$, such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C\langle\xi\rangle^{m-\rho|\beta|+(1-\rho)|\alpha|}, \ (x,\xi) \in K \times \mathbf{R}^n. \tag{10.1}$$

When $a(x,\xi) = a(x,\xi;h)$ depends on the additional parameter $h \in]0,h_0]$ for some $h_0 > 0$, we say that $a \in S_\rho^m(X \times \mathbf{R}^n)$, if (10.1) holds uniformly with respect to h. For h-dependent symbols, we introduce $S_\rho^{m,k} = h^{-k}S_\rho^m$. When $\rho = 1$ it is customary to suppress the subscript ρ.

Let now X be a compact n-dimensional manifold. We say that $R = R_h : \mathcal{D}'(X) \to C^\infty(X)$ is negligible, and write $R \equiv 0$, if the distribution-kernel K_R satisfies $\partial_x^\alpha \partial_y^\beta K_R(x,y) = \mathcal{O}(h^\infty)$ for all $\alpha, \beta \in \mathbf{N}^n$ (when expressed in local coordinates).

We say that an operator $P = P_h : C^\infty(X) \to \mathcal{D}'(X)$ belongs to the space $L^{m,k}(X)$ if $\phi P_h \psi$ is negligible for all $\phi, \psi \in C^\infty(X)$ with disjoint supports and if for every choice of local coordinates $x_1, ..., x_n$, defined on the open subset $\widetilde{X} \subset X$ (that we view as a subset of $\mathbf{R}^n$), we have on $\widetilde{X}$ for every $u \in C_0^\infty(\widetilde{X})$:

$$Pu(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(x-y)\cdot\theta} a(x,\theta;h)u(y)dyd\theta + Ku(x), \tag{10.2}$$

where $a \in S^{m,k}(\widetilde{X} \times \mathbf{R}^n)$ and K is negligible.

The correspondence $P \mapsto a$ is not globally well-defined, but the various local maps give rise to a bijection

$$L^{m,k}(X)/L^{m-1,k-1}(X) \to S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X), \tag{10.3}$$

where we notice that $S^{m,k}(T^*X)$ is well-defined in the natural way. The image $\sigma_P(x,\xi)$ of $P \in L^{m,k}(X)$ is called the principal symbol.

Pseudodifferential operators in the above classes map C^∞ to C^∞ and extend to well-defined operators $\mathcal{D}'(X) \to \mathcal{D}'(X)$. They can therefore be

composed with each other: If $P_j \in L^{m_j,k_j}(X)$, for $j = 1, 2$, then $P_1 \circ P_2 \in L^{m_1+m_2,k_1+k_2}$. Moreover $\sigma_{P_1 \circ P_2}(x,\xi) = \sigma_{P_2}(x,\xi)\sigma_{P_1}(x,\xi)$.

We can invert elliptic operators: If $P_h \in L^{m,k}$ is elliptic in the sense that $|\sigma_P(x,\xi)| \geqslant \frac{1}{C}h^{-k}\langle\xi\rangle^m$, then P_h is invertible (either as a map on C^∞ or on $\mathcal{D}'$) for $h > 0$ small enough, and the inverse Q belongs to $L^{-m,-k}$. (If we assume invertibility in the full range $0 < h \leqslant h_0$ then the conclusion holds in that range.) Notice that $\sigma_Q(x,\xi) = 1/\sigma_P(x,\xi) \in S^{-m,-k}/S^{-m-1,-k-1}$.

The proof of these facts is a routine application of the method of stationary phase, following for instance the presentation in [4].

Let $\widetilde{R}$ be a positive elliptic 2nd order differential operator with smooth coefficients on X, self-adjoint with respect to some smooth positive density on X. Let $r(x,\xi)$ be the principal symbol of $\widetilde{R}$ in the classical sense, so that $r(x,\xi)$ is a homogeneous polynomial in ξ with $r(x,\xi) \asymp |\xi|^2$. Then $P := h^2\widetilde{R}$ belongs to $L^{2,0}(X)$ and $\sigma_{h^2\widetilde{R}} = r$.

PROPOSITION 10.1. — *For every $s \in \mathbf{R}$, we have $(1 + h^2\widetilde{R})^{2s} \in L^{2s,0}$ and the principal symbol is given by $(1 + r(x,\xi))^s$.*

Proof. — It suffices to show this for s sufficiently large negative. In that case we have

$$(1 + h^2\widetilde{R})^s = \frac{1}{2\pi i}\int_\gamma (1+z)^s(z - h^2\widetilde{R})^{-1}dz, \qquad (10.4)$$

where γ is the oriented boundary of the sector $\arg(z + \frac{1}{2}) < \pi/4$. For $z \in \gamma$, we write

$$(z - h^2\widetilde{R}) = |z|(\frac{z}{|z|} - \widetilde{h}^2\widetilde{R}), \ \widetilde{h} = \frac{h}{|z|^{1/2}},$$

and notice that $\frac{z}{|z|} - \widetilde{h}^2\widetilde{R} \in L^{2,0}$ is elliptic when we regard $\widetilde{h}$ as the new semi-classical parameter. By self-adjointness and positivity we know that this operator is invertible, so $(\frac{z}{|z|} - \widetilde{h}^2\widetilde{R})^{-1} \in L^{-2,0}$, and for every system of local coordinates the symbol (in the sense of $\widetilde{h}$-pseudodifferential operators) is

$$\frac{1}{\frac{z}{|z|} - r(x,\xi)} + a, \quad a \in S^{-3,-1}. \qquad (10.5)$$

The symbol of $(z - h^2\widetilde{R})^{-1}$ as an h-pseudodifferential operator is therefore

$$\frac{1}{|z|(\frac{z}{|z|} - r(x, \frac{\xi}{|z|^{1/2}}))} + \frac{1}{|z|}a(x, \frac{\xi}{|z|^{1/2}}). \qquad (10.6)$$

Here the first term simplifies to $(z - r(x, \xi))^{-1}$ and the corresponding contribution to (10.4) has the symbol $(1 + r(x, \xi))^s$.

The contribution from the remainder in (10.6) to the symbol in (10.4) is

$$b(x, \xi) := \frac{1}{2\pi i} \int_\gamma \frac{(1+z)^s}{|z|} a(x, \frac{\xi}{|z|^{1/2}}) dz,$$

where we will use the estimate

$$\partial_x^\alpha \partial_\xi^\beta \frac{1}{|z|} a(x, \frac{\xi}{|z|^{1/2}}) = \mathcal{O}(\frac{h}{|z|^{(3+|\beta|)/2}} \langle \frac{\xi}{|z|^{1/2}} \rangle^{-3-|\beta|}) = \mathcal{O}(h)(|z|+|\xi|^2)^{-\frac{1}{2}(3+|\beta|)}. \tag{10.7}$$

Thus,

$$\partial_x^\alpha \partial_\xi^\beta b = \mathcal{O}(1) h \int_\gamma |z|^s (|z| + |\xi|^2)^{-\frac{1}{2}(3+|\beta|)} |dz|. \tag{10.8}$$

In a region $|\xi| = \mathcal{O}(1)$, we get

$$\partial_x^\alpha \partial_\xi^\beta b = \mathcal{O}(1).$$

In the region $|\xi| \gg 1$ shift the contour γ in (10.4) to the oriented boundary of the sector $\arg(z + \frac{1}{2}|\xi|^2) < \frac{\pi}{4}$. Then we get (10.8) for the shifted contour and the integral can now be estimated by

$$\mathcal{O}(h) \int_{|\xi|^2/C}^\infty t^{s - \frac{3}{2} - \frac{|\beta|}{2}} dt = \mathcal{O}(h|\xi|^{2s-1-|\beta|}).$$

The proposition follows. $\quad \square$

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