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Nonlinear Maps between Besov and Sobolev spaces

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Résumé. — Notre résultat principal est que pour une grande famille d'applications non linéaires entre espaces de Besov et de Sobolev, l'interpolation est un phénomène propre aux petites dimensions. Ceci prolonge des résultats obtenus précédemment par Kumlin [13] pour des applications analytiques au cas d'applications Hölder continues ou encore Lipschitz (Corollaires 1 and 2), et qui remontent aux idées de B.E.J. Dahlberg [8].

Abstract. — Our main result shows that for a large class of nonlinear local mappings between Besov and Sobolev space, interpolation is an exceptional low dimensional phenomenon. This extends previous results by Kumlin [13] from the case of analytic mappings to Lipschitz and Hölder continuous maps (Corollaries 1 and 2), and which go back to ideas of the late B.E.J. Dahlberg [8].

1. Main result

Our main result shows that for a large class of nonlinear local mappings between Besov and Sobolev space, interpolation is an exceptional low dimensional phenomenon. We give results which are extensions of previous results by Kumlin [13] from the case of analytic mappings to Lipschitz and Hölder continuous maps (Corollaries 1 and 2), and which go back to ideas of the late B.E.J. Dahlberg [8].

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First an important definition in this context: In the formulation of the Main Theorem we use a notion of "a set of mappings $\mathcal{F}$ admitting interpolation on scales of Banach spaces" which is given by

Definition 1.1. — *Let* $\mathcal{A} = \{A_\theta\}_{\theta\in\Theta=[s_0,s_1]}$ *and* $\mathcal{A}' = \{A'_{\theta'}\}_{\theta'\in\Theta'=[s'_0,s'_1]}$ *be ordered scales of Banach spaces, i.e.* $A_{\theta_1} \subset A_{\theta_2}$ *whenever* $\theta_1 > \theta_2$ *and* $A'_{\theta'_1} \subset A'_{\theta'_2}$ *whenever* $\theta'_1 > \theta'_2$ *respectively. We say that a family* $\mathcal{F}$ *admits interpolation on the ordered scales* $(\mathcal{A}, \mathcal{A}')$ *if for every* $F \in \mathcal{F}$

1. $F(u) \in A'_{s'_0}$ *for all* $u \in A_{s_0}$,
2. $F(u) \in A'_{s'_1}$ *for all* $u \in A_{s_1}$, *and*
3. *the increasing function* $s'_F(s) = \sup\{t \in [s'_0, s'_1] : F(u) \in A'_t \ \text{ all } u \in A_s\}$ *is a mapping of* (s_0, s_1) *onto* (s'_0, s'_1).

In the following we look at scales of interpolation spaces as the scales of Banach spaces and, in particular, we consider scales of Sobolev and Besov spaces. Here and in the following we assume that $1 \leqslant p \leqslant 2 \leqslant p'$ are dual exponents, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$, and that $1 \leqslant r \leqslant \infty$. The Besov space (we refer to section 3 for those unfamiliar with these spaces) $B_p^{s,q}(\Omega)$ will be written simply $B_p^{s,q}$ in case $\Omega = \mathbf{R}^n$. The same convention will be used for the Sobolev spaces H_2^s discussed below.

Notice that the space dimension n **will play an important role in the results**. All functions u below are defined on $\mathbf{R}^n$.

Definition 1.2. — *A family* $\mathcal{F}$ *of mappings is said to admit interpolation on the scale of Besov spaces* $(B_p^{s+1,r}, B_{p'}^{s',r})$, $0 \leqslant s \leqslant \sigma$, $0 \leqslant s' \leqslant \sigma'$ *if for every* $F \in \mathcal{F}$

1. $F(u) \in L_{p'}$ *for all* $u \in B_p^{1,r}$,
2. $F(u) \in B_{p'}^{\sigma',r}$ *for all* $u \in B_p^{\sigma+1,r}$, *and*
3. *there exists a realvalued function* $r' = r'_F(s)$, $r'_F(s) \geqslant r$, *such that the the increasing function*

$$s'_F(s) = \sup\{t \in [0, \sigma'] : F(u) \in B_{p'}^{t,r'} \ \text{ all } \ u \in B_p^{s+1,r}\}$$

is a mapping of $(0, \sigma)$ *onto* $(0, \sigma')$.

In the linear case, familiar interpolation methods give $r = r'$ and with $s = \theta\sigma$ that $s' = \theta\sigma'$ for $0 < \theta < 1$. In case of nonlinear mappings, the interpolation may result in $s'_T(s) < s$ even when $\sigma = \sigma'$ as we will see in Theorem 2.1 below. We will (as in the references above) study the case when $\mathcal{F} = \{F\}$ is a singleton set. Here F is a local mapping consisting of the composition $u \mapsto f(u)$ with a reasonably smooth function f. If as above $\mathcal{F}$ admits interpolation we say, for short, that *the mapping F admits interpolation.* Notice that in the case of compositions, necessarily $\sigma' \leqslant \sigma+1$. Here, and in the following $[x]$ denotes the integer part of $x \in \mathbf{R}$.

THEOREM 1.3 (MAIN THEOREM). — *Let $f \in C^{[\sigma']+1}$ where $\sigma' > \frac{1+2p}{p'}$ and $1 \leqslant p \leqslant 2 \leqslant p'$ with dual exponents p, p'. Assume that the mapping*

$$F: \ u \mapsto f(u)$$

admits interpolation on the scale of Besov spaces $(B_p^{s+1,r}, B_{p'}^{s',r})$, $0 \leqslant s \leqslant \sigma$, $0 \leqslant s' \leqslant \sigma'$. Moreover assume that there exists a $\beta > 0$ such that

$$s'_F(s) \geqslant \beta s \ \text{ for } \ 0 < s < \sigma,$$

where $s'_F(s)$ is defined above. Set $\phi(p) = 2(p-1)(1+2p)$ and let $n(p, \beta) = \max(\frac{\phi(p)}{\beta}, 2+2p) - 1$. Then either the space dimension $n \leqslant n(p, \beta)$ or else $f(z) = Dz$ for some constant D.

Notice that the result is independent of the interpolation method used.

Remark 1.4. — The proof of the Main Theorem provides more detailed information: There is a strictly increasing function $p(\beta)$ of β , $1 < p(\beta) \leqslant \frac{1+\sqrt{5}}{2}$ such that

$$n(p, \beta) = 1 + 2p \ \textit{ for } \ 1 \leqslant p < p(\beta)$$

and

$$n(p, \beta) = \frac{\phi(p)}{\beta} - 1 \ \textit{ for } \ p(\beta) \leqslant p \leqslant 2.$$

In addition $p(\beta) = 1 + O(\beta)$ as $\beta \to 0$.

In particular $n(2, \beta) = \frac{10}{\beta} - 1$ and we get the following important special case:

THEOREM 1.5 (MAIN THEOREM L_2 CASE). — *Let $f \in C^{[\sigma']+1}$, with $\sigma' > \frac{5}{2}$, be a function such that the mapping*

$$F: \ u \mapsto f(u)$$

admits interpolation on the scale of Sobolev spaces $(H_2^{s+1}, H_2^{s'})$, $0 \leqslant s \leqslant \sigma$, $0 \leqslant s' \leqslant \sigma'$. Assume that there is a $\beta > 0$ such that s'_F, the function defined above, satisfies the inequality

$$s'_F(s) \geqslant \beta s \ \text{ for } \ 0 < s < \sigma.$$

Then either the space dimension $n \leqslant \frac{10}{\beta} - 1$ *or else* $f(z) = Dz$ *for some constant* D*.*

Remark 1.6. — In the Main Theorem "L_2 case", the existence of a $\beta > 0$ such that

$$s_F'(s) \geqslant \beta s \ \text{ for } \ 0 < s < \sigma$$

follows for, say, f sufficiently smooth satisfying $|f(x)| \leqslant C|x|$, $|f'(x)| \leqslant C$ for all $x \in \mathbf{R}^n$. Here clearly $s_F'(s) \geqslant \frac{1}{\sigma}s$ holds for $0 < s < \sigma$.

For local mappings, the Main Theorem applies to the interpolation results of Heintz and von Wahl for analytic mappings [11], those of Peetre [18] for Lipschitz mappings and of Maligranda [16] for Hölder continuous mappings. In all cases, the Main theorem tells that for a large class of nonlinear mappings, interpolation is an exceptional, low dimensional phenomenon. We demonstrate this in two corollaries, which are consequences of the main theorem and Theorem 2.1 in section 2.

We say that the mapping $H_2^1 \ni u \mapsto f(u) \in L_2$ is *Hölder continuous of order* α*, Lipschitz continuous if* $\alpha = 1$, if

$$\|f(u) - f(v)\|_{L_2} \leqslant g(\|u\|_{H_2^1}, \|v\|_{H_2^1})\|u - v\|_{H_2^1}^{\alpha} \ \text{ for } \ u, v \in H_2^1, \qquad (1.1)$$

where $g(.,.)$ is a locally bounded function on $\mathbf{R}_+^2$, increasing in each of its arguments.

Corollary 1.7. — *Let* $\sigma > 0$, $\sigma' > \frac{5}{2}$ *and let* $f \in C^{[\sigma']+1}$*. Assume that*

$$H_2^1 \ni u \mapsto f(u) \in L_2$$

is Lipschitz continuous and that

$$\|f(u)\|_{H_2^{\sigma'}} \leqslant h(\|u\|_{H_2^1})\|u\|_{H_2^{\sigma+1}} \ \text{ for } \ u \in H_2^{\sigma+1}, \qquad (1.2)$$

where $h(\cdot)$ *is a locally bounded increasing function on* $\mathbf{R}_+$*. Then either* $n < 10$ *or else* $f(z) = Dz$ *for some constant* D*.*

We may weaken the assumptions on the mapping properties of f and still get results that are consequences of the Main Theorem, as we will prove below.

Corollary 1.8. — *Let* $\sigma > 0$, $\sigma' > \frac{5}{2}$ *and let* $f \in C^{[\sigma']+1}$*. Assume that*

$$H_2^1 \ni u \mapsto f(u) \in L_2$$

is Hölder continuous of order α, $0 < \alpha < 1$ and that the mapping

$$H_2^{\sigma+1} \ni u \mapsto f(u) \in H_2^{\sigma'}$$

has at most power growth, i.e.

$$\|f(u)\|_{H_2^{\sigma'}} \leqslant h(\|u\|_{H_2^1})\|u\|^{\mu}_{H_2^{\sigma+1}}, \text{ for } u \in H_2^{\sigma+1}, \tag{1.3}$$

where $h(\cdot)$ is a locally bounded increasing function on $\mathbf{R}_+$. Assume that $\mu \geqslant \alpha$. Then there exists an integer $n(\alpha)$ such that either $n < n(\alpha)$ or else $f(z) = Dz$ for some constant D. Moreover $n(\alpha) \leqslant O(\frac{1}{\alpha})$ as $\alpha \to 0$.

In many cases, the *growth condition* (1.3) can be derived in low dimensions from (a possibly local version of) the inequality

$$\sup_{|\gamma|=[s']} \|\partial_x^\gamma f(u)\|_{L_2} \leqslant \\ C \max(\|f^{(l)}(u)\|_{L_r}\|u\|^{(l-1)}_{L_\infty} : 1 \leqslant l \leqslant [s']) \sup_{|\gamma|=[s']+1} \|\partial_x^\gamma u\|_{L_2}, \tag{1.4}$$

valid for $r > n$. Here we denote $\partial_x^\gamma = \partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\ldots\partial_{x_n}^{\gamma_n}$ with $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ and $|\gamma| = \Sigma_{i=1}^n \gamma_i$. The inequality (1.4) follows from the Sobolev lemma and the Gagliardo-Nirenberg inequality (see [9], [17], and also e.g. Hörmander [12], Corollary 6.4.5). Conditions under which $u \mapsto f(u)$ is bounded as a mapping between Besov spaces (or between Lizorkin-Triebel spaces) are given e.g. in Bourdaud et al. [4] (see also Kumlin [13] and Dahlberg [8], and the references given in [4]).

Examples of nonlinear, non-local mappings between Besov and Sobolev spaces generated by local nonlinear maps have been extensively studied in the context of initial value problems for nonlinear Klein-Gordon and Wave equations:

$$\partial_t^2 u - \Delta_x u + m^2 u + f(u) = 0,\, t > 0,\, x \in \mathbf{R}^n.$$

In the case of not necessarily small initial data, f is usually assumed to have at most polynomial growth of the form $f(u) \simeq |u|^{\rho-1}u$. In the subcritical and critical case (i.e. $\rho \leqslant 1 + \frac{4}{n-2}$), the solution operators are Lipschitz continuous mappings from H_2^1 to L_2 (cf. [10], [5] and [1]). Using constructions and ideas similar to those of Theorem 2.6 in the present paper, the authors proved in [6] that the solution operators for these nonlinear equations are not Lipschitz continuous as mappings from H_2^1 to L_2 for supercritical exponents ρ ($\rho > 1 + \frac{4}{n-2}$). Related results for polynomial nonlinearities f have been obtained by Lebeau [14], [15].

2. Nonlinear maps and nonlinear interpolation

In order to prove our results for $\alpha \leqslant 1$ we have to introduce and use the Besov spaces $B_p^{s,q}$, $B_{p'}^{s',q'}$ and real interpolation based on Peetre's K-function (see [2], [3] and [7]), in fact mainly for $p = p' = 2$. In general p and p' are assumed to be dual exponents, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leqslant p \leqslant 2$ and the standard inclusions (again see [3] pp. 142 and 152-153) between Besov and Sobolev spaces

$$\begin{array}{lll} B_p^{s,p} & \subset H_p^s & \subset B_p^{s,2} \\ B_{p'}^{s,p'} & \supset H_{p'}^s & \supset B_{p'}^{s,2} \end{array}$$

make the L_2-results below to be consequences of the corresponding Besov space results.

The following is a variation of results by Peetre [18] (for $\alpha = 1$) and Maligranda [16].

Theorem 2.1. — *Let $0 < \alpha \leqslant 1$, and let f satisfy (1.1) and also the conditions of either Corollary 1.7 or Corollary 1.8, so that*

$$H_2^{\sigma+1}(\mathbf{R}^n) \ni u \mapsto f(u) \in H_2^{\sigma'}(\mathbf{R}^n)$$

with the estimate (1.2) for $\alpha = 1$, i.e.

$$\|f(u)\|_{H_2^{\sigma'}} \leqslant h(\|u\|_{H_2^1})\|u\|_{H_2^{\sigma+1}} \text{ for } u \in H_2^{\sigma+1},$$

or the estimate (1.3) in case $0 < \alpha < 1$, i.e.

$$\|f(u)\|_{H_2^{\sigma'}} \leqslant h(\|u\|_{H_2^1})\|u\|^{\mu}_{H_2^{\sigma+1}} \text{ for } u \in H_2^{\sigma+1},$$

with $\mu \geqslant \alpha$. Here $h(\cdot)$ denotes a locally bounded increasing function on $\mathbf{R}_+$. For $\theta \in (0,1)$ let $s = \sigma\theta$, $s' = \sigma'\theta$ and let $r \geqslant 1$. Then the inclusion

$$B_2^{s+1,r}(\mathbf{R}^n) \ni u \mapsto f(u) \in B_2^{\frac{\alpha}{\mu}s', \frac{r}{\alpha}}(\mathbf{R}^n),$$

where $\mu = 1$ if $\alpha = 1$, holds.

Corollary 2.2. — *Under the assumptions of Theorem 3, with $\sigma = \sigma'$, the inclusion*

$$H_2^{s+1}(\mathbf{R}^n) \ni u \mapsto f(u) \in H_2^{s'}(\mathbf{R}^n) \tag{2.1}$$

holds for $0 < s < s_0 = \frac{n}{2} - 1$ and any $s' < \frac{\alpha}{\mu}s$ for $\alpha < 1$, and with $s' = s$ if $\alpha = 1$.

The corollary follows as mentioned directly from Theorem 2.1 and the inclusions between Besov spaces above. Notice that in general, $\sigma' \leqslant \sigma + 1$, and scaling gives the result in this slightly more general case.

Remark 2.3. — The result of Theorem 2.1 is sufficient for our purposes, places in our context natural restrictions on f, and allows a simple proof. The results of Maligranda will in our context replace $\frac{\alpha}{\mu}\bar{s}$ with $\tilde{\alpha}\bar{s}$ where

$$\tilde{\alpha} = \alpha(\mu - \frac{s}{s_0}(\mu - \alpha))^{-1} \geqslant \frac{\alpha}{\mu}$$

again by assumption with $\mu = 1$ if $\alpha = 1$.

Correspondingly, in Corollary 2.2 $s' < \frac{\alpha}{\mu}s$ can be replaced by $s' < \tilde{\alpha}s$. This can be used to give more detailed asymptotic estimates of $n(\alpha)$ as $\alpha \to 0$ in Corollary 1.8.

The L_p-version of the following result due to Kumlin [13] is the main ingredient in the proof of the Main Theorem.

THEOREM 2.4 (KUMLIN [13]). — *Assume that n is a positive integer and $s, s' \geqslant 0$ satisfy*

1. $0 < s + 1 < \frac{n}{2}$,
2. $\frac{3}{2} < s' < \frac{n}{2}$,
3. $n > \frac{4ss'-2s-2}{2s'-3}$, *and*
4. *there exists an integer $k \geqslant 2$ such that* $n > 2s + 2 + 2\frac{s-s'+1}{k-1}$.

If under these assumptions $f \in C^{[s']+1}$ and

$$H_2^{s+1}(\mathbf{R}^n) \ni u \mapsto f(u) \in H_2^{s'}(\mathbf{R}^n) \tag{2.2}$$

then f is a polynomial of degree at most min$([s'], k-1)$ *and* $f(0) = 0$.

Condition 4. in Theorem 2.4 is motivated by the following observation:

PROPOSITION 2.5. — *Let $\Phi \in C_o^\infty(\mathbf{R}^n)$, $\Phi(0) \neq 0$ and define $H_\tau(x) = |x|^\tau \Phi(x)$. Then $H_\tau \in H_2^{\bar{s}}$ if and only if $\bar{s} < \tau + \frac{n}{2}$.*

We refer the reader to [7] (Proposition 4.2) for the straightforward proof of this proposition.

Next we give a sketch of the proof of Theorem 2.4. For a complete proof we refer to the Appendix.

Proof. — (of Theorem 2.4): Assume that $f(z)$ is a polynomial of degree k or higher in z. Since 4) holds for $\bar{k} \geqslant k$ if it holds for k, we may assume that the coefficient of z^k is nonzero, and that $f(z)$ is of degree k. Take $z = H_\tau \in H_2^{s+1}$ such that $H_\tau(x)^k \sim |x|^{\tau k}\Phi(x)^k \notin H_2^{s'}$. By the proposition this is the case if

$$\tau k + \frac{n}{2} - s' < 0 < \tau + \frac{n}{2} - s - 1$$

which follows from 4) with a suitable choice of τ. It remains to prove that 1) through 3) imply that f is a polynomial of degree at most $[s']$. We construct a function v (to use as a counterexample) as follows. Compare Dahlberg [8]. Let $y^j = (10j, 0, \ldots, 0) \in \mathbf{R}^n$ and let $u \in C_0^\infty$ with support in $\{|x| \leqslant 2\}$ and such that $u(x) = x_1$ in $\{|x| \leqslant 1\}$. Define

$$v(x) = \sum_{j=1}^{\infty} A_j u(\frac{x - y^j}{\epsilon_j}), \quad x \in \mathbf{R}^n, \tag{2.3}$$

where $0 < A_j \uparrow \infty$ and $\epsilon_j = A_j^{-\lambda}$ with $\lambda > 0$ to be chosen later. We note that $v \in C^\infty$. If

$$\sum_j A_j^2 \epsilon_j^{n-2(s+1)} < \infty \tag{2.4}$$

then a straightforward computation (at least for integers s and for fractional s see the Appendix) shows that $v \in H_2^{s+1}(\mathbf{R}^n)$. If f is not a polynomial of degree at most $[s']$, then there is an interval $[a, b]$, $a < b$, such that $|f^{([s']+1)}(t)| > 0$ for $a \leqslant t \leqslant b$. If we use the special form of our function u, we find that

$$\|f(v)\|^2_{H_2^{s'}} \geqslant C \sum_j A_j^{2s'-1} \epsilon_j^{n-2s'}.$$

See *Claim 2* in the Appendix. If then

$$\sum_j A_j^{2s'-1} \epsilon_j^{n-2s'} = \infty \tag{2.5}$$

this contradicts the mapping property (2.2), i.e.

$$f(H_2^{s+1}(\mathbf{R}^n)) \subset H_2^{s'}(\mathbf{R}^n)$$

and our Theorem will be proved. Assumptions 1) through 3) imply that it is possible to choose $\{A_j\}$ and λ so that the properties (2.4) and (2.5) are satisfied. This completes the proof of Theorem 2.4. $\square$

In the proof of Theorem 2.4 above, it is easy to see what happens if we replace the H_2-spaces by H_p, $H_{p'}$-spaces. Using the definition of the Besov spaces in terms of moduli of continuity below (again, see [3],[2] and section 3), the construction and the proof via the L_p versions of (2.4) and (2.5) is essentially the same, although more technical (see Appendix or [13]), as that already given of Theorem 2.4.

THEOREM 2.6 (KUMLIN [13]). — *Assume that n is a positive integer, $1 \leqslant p \leqslant 2 \leqslant p'$ where $\frac{1}{p} + \frac{1}{p'} = 1$, $q \geqslant 1$ and $s, s' \geqslant 0$ satisfy*

1. $0 < s + 1 < \frac{n}{p}$,
2. $\frac{1+p}{p'} < s' < \frac{n}{p'}$,
3. $n > (p-1)\frac{ss'p'-s-1}{s'-\frac{1+p}{p'}}$, *and*
4. *there exists an integer $k \geqslant 2$ such that $n(\frac{1}{p} - \frac{1}{k}\frac{1}{p'}) > s + 1 - \frac{s'}{k}$.*

If under these assumptions $f \in C^{[s']+1}$ and

$$B_p^{s+1,q}(\mathbf{R}^n) \ni u \mapsto f(u) \in B_{p'}^{s',q}(\mathbf{R}^n) \tag{2.6}$$

then f is a polynomial of degree at most min$([s'], k-1)$ and $f(0) = 0$.

We are now in position to give the proof of the Main Theorem.

Proof. — Since conditions in Theorem 2.6 are all strict inequalities, it is enough to prove the result with s' replaced by βs, with $0 < \beta \leqslant 1$, in 1) through 4). Take $s = \frac{1+2p}{p'\beta}$ so that $s' = \frac{1+2p}{p'}$, which is allowed since by assumption $\sigma' > \frac{1+2p}{p'}$.
Let us now refer to Theorem 2.6 conditions 1) through 4). With our choice of s', and with $\phi(p) = 2(p-1)(1+2p)$ and $k = 2$, by elementary computations these are satisfied if

1. $n > \phi_1(p, \beta) \equiv \frac{\phi(p)}{2\beta} + p$
2. $n > \phi_2(p) \equiv 1 + 2p$
3. $n > \phi_3(p, \beta) \equiv \frac{\phi(p)}{\beta} - 1$
4. $n > \phi_4(p, \beta) \equiv \frac{\phi(p)}{(3-p)\beta} + (p - \frac{p^2+2p-1}{3-p})$

Here 1),2) and 4) are exact reformulations of the corresponding inequalities in Theorem 2.6, while in 3) the exact expression $\frac{\phi(p)}{\beta} - \frac{1}{p-1}$ has been replaced by $\phi_3(p,\beta) = \frac{\phi(p)}{\beta} - 1$. This more restrictive choice will simplify the computations and results below.

Notice that ϕ, and so ϕ_1, ϕ_2 and ϕ_3 are strictly increasing functions of p. We first concentrate on the inequalities 1) through 3). Straightforward computations show that $\phi_1 = \phi_2 = \phi_3$ for $p = p(\beta)$, $1 < p(\beta) < 2$ the solution of

$$p^2 = (1+p)\frac{1+\beta}{2}$$

Differentiating, we see that $p(\beta)$ is strictly increasing, and so

$$1 < p(\beta) \leqslant p(1) = \frac{1+\sqrt{5}}{2}.$$

We also have $p(\beta) = 1 + O(\beta)$ as $\beta \to 0$. Since $\phi(p)$ is a second order polynomial in p with zeros $-\frac{1}{2}$ and 1, we get

$$\phi_3 \geqslant \phi_1 \geqslant \phi_2 \text{ for } p(\beta) \leqslant p \leqslant 2,$$
$$\phi_3 \leqslant \phi_1 \leqslant \phi_2 \text{ for } p(\beta) \geqslant p \geqslant 1.$$

Thus 1) through 3) hold if $n > \max(\phi_3, \phi_2)$. We next show that $\phi_4 \leqslant \max(\phi_3, \phi_2)$, and hence also 4) holds if $n > \max(\phi_3, \phi_2)$. If we use that

$$\frac{\phi(p)}{\beta} \leqslant 2 + 2p \text{ for } 1 \leqslant p \leqslant p(\beta),$$
$$\frac{\phi(p)}{\beta} \geqslant 2 + 2p \text{ for } p(\beta) \leqslant p \leqslant 2,$$

straightforward computations show that

$$\phi_4 = \tfrac{\phi(p)}{(3-p)\beta} + (p - \tfrac{p^2+2p-1}{3-p}) \leqslant 1 + 2p = \phi_2 \quad \text{for } 1 \leqslant p \leqslant p(\beta),$$
$$\phi_4 = \tfrac{\phi(p)}{(3-p)\beta} + (p - \tfrac{p^2+2p-1}{3-p}) \leqslant \tfrac{\phi(p)}{\beta} - 1 = \phi_3 \quad \text{for } p(\beta) \leqslant p \leqslant 2.$$

Thus 1) through 4) hold for

$$n > \max(\phi_3, \phi_2) = \max(\tfrac{\phi(p)}{\beta} - 1, 1 + 2p) = \max(\tfrac{\phi(p)}{\beta}, 2 + 2p) - 1.$$

Since we have assumed that the mapping $u \mapsto f(u)$ admits interpolation, the condition (2.6) in Theorem 2.6 is also satisfied, and the main theorem is proved. $\square$

For the convenience of the reader we will give a proof of Theorem 2.1 in next section and in that context also a very short introduction to the necessary concepts of real interpolation and Besov spaces.

3. Besov spaces, real interpolation and the proof of Theorem 2.1

In this section we will shortly remind of the basic definitions and properties of real interpolation and Besov spaces. The basic references are [2], [3] and [7], to which we refer the reader for additional information.

Let $C_1 \subset C_0$ be a Banach space couple. Then the K-functional $K(t, \phi; C_0, C_1)$ is defined by

$$K(t, \phi; C_0, C_1) = \inf\{\|\phi_0\|_{C_0} + t\|\phi_1\|_{C_1} : \phi = \phi_0 + \phi_1, \phi_i \in C_i\},$$

where $\phi \in C_1$ and $t \geqslant 0$. Notice that

$$K(t, \phi; C_o, C_1) \leqslant \|\phi\|_{C_0} \text{ for } t \geqslant 1. \tag{3.1}$$

We define $C_{\theta,q} = (C_0, C_1)_{\theta,q}$ as the completion of C_1 in the norm

$$\|\phi\|_{C_{\theta,q}} = (\int_0^\infty (t^{-\theta} K(t, \phi; C_0, C_1))^q \frac{dt}{t})^{1/q}.$$

If $C_1 = H_p^{s_1}, C_0 = H_p^{s_0}, s_1 > s_0$, then $C_{\theta,q}$ defines the Besov space $B_p^{s,q}$, where $s = (1-\theta)s_0 + \theta s_1$. By definition, the family of Besov spaces have a number of natural convexity and inclusion properties, as mentioned earlier, for which we refer the reader to the already given basic references, and in particular to [3]. In this context let us remind of the following well-known interpolation result: This is important in the proof of Theorems 2.4 and 2.6, when we want to translate the effect of a lower bound of the derivatives of f in terms of bounds on Besov space norms. $B_p^{s,q}$ has the intrinsic norm (among many)

$$\|v\|_{B_p^{s,q}} \simeq \sum_{|\alpha|=[s]} \left(\int_0^\infty (t^{-s+[s]} \omega_p^{(r)}(t, \partial^\alpha v))^q \frac{dt}{t} \right)^{\frac{1}{q}} + \|v\|_{L_p}, \tag{3.2}$$

where $[s] = \sup\{m \in \mathbf{Z} : m \leqslant s\}$ and $r \geqslant 1$ and we let $\omega_p^{(r)}(t, v)$ denote the r-th modulus of continuity of v in L_p, i.e.

$$\omega_p^{(r)}(t, v) = \sup_{|h| \leqslant t} \| \sum_{k=0}^r rk(-1)^{r-k} v(\cdot + kh)\|_{L_p}. \tag{3.3}$$

We may now begin the proof of Theorem 2.1.

Proof. — In the following we let c denote a locally bounded function of the H_2^1-norm of u, or a constant, where c may be different at each occurence.

Similarily we let C denote constants that may vary from line to line. Set $K(t,v) \equiv K(t,v;H_2^1,H_2^{\sigma+1})$ and $K'(t,v) \equiv K(t,v;L_2,H_2^{\sigma'})$. Choose $e(t) \in H_2^{\sigma+1}$ such that

$$\|u-e(t)\|_{H_2^1} + t\|e(t)\|_{H_2^{\sigma+1}} \leqslant 2K(t,u) \tag{3.4}$$

In particular, by (3.4) we have

$$\|e(t)\|_{H_2^1} \leqslant \|u-e(t)\|_{H_2^1} + \|u\|_{H_2^1} \leqslant 3\|u\|_{H_2^1}. \tag{3.5}$$

By (1.1) and (1.3) we get

$$\begin{aligned} K'(t^\mu, f(u)) &\leqslant \|f(u)-f(e)\|_{L_2} + t^\mu \|f(e)\|_{H_2^{\sigma'}} \\ &\leqslant g(\|u\|_{H_2^1}, \|e(t)\|_{H_2^1})\|u-e\|_{H_2^1}^\alpha + h(\|e(t)\|_{H_2^1})t^\mu(\|e\|_{H_2^{\sigma+1}})^\mu \end{aligned}$$

and by the bound (3.5) and since by assumption, g is a locally bounded function increasing in both variables and h is a locally bounded increasing function

$$g(\|u\|_{H_2^1}, \|e(t)\|_{H_2^1}) \leqslant c, \quad h(\|e(t)\|_{H_2^1}) \leqslant c,$$

where c, as remarked above, denotes a locally bounded function of $\|u\|_{H_2^1}$. Then by the choice of $e(t)$,

$$K'(t^\mu, f(u)) \leqslant cK(t,u)^\alpha + cK(t,u)^\mu$$

Thus we get

$$\begin{aligned} \int_0^\infty (t^{-\theta\alpha}K'(t^\mu, f(u)))^{\frac{r}{\alpha}} \frac{dt}{t} &\leqslant c\int_0^\infty (t^{-\theta}K(t,u))^r \frac{dt}{t} \\ &+ c\int_0^\infty (t^{-\theta\frac{\alpha}{\mu}}K(t,u))^{\frac{r\mu}{\alpha}} \frac{dt}{t} \end{aligned}$$

A change of variable in the first integral yields

$$\int_0^\infty (t^{-\theta\alpha}K'(t^\mu, f(u)))^{\frac{r}{\alpha}} \frac{dt}{t} = \frac{1}{\mu}\int_0^\infty (t^{-\theta\frac{\alpha}{\mu}}K'(t, f(u)))^{\frac{r}{\alpha}} \frac{dt}{t}$$

and hence we obtain

$$\begin{aligned} \int_0^\infty (t^{-\theta\frac{\alpha}{\mu}}K'(t, f(u)))^{\frac{r}{\alpha}} \frac{dt}{t} &\leqslant c\int_0^\infty (t^{-\theta}K(t,u))^r \frac{dt}{t} \\ &+ c\int_0^\infty (t^{-\theta\frac{\alpha}{\mu}}K(t,u))^{\frac{r\mu}{\alpha}} \frac{dt}{t} \end{aligned}$$

By the definition and the inclusions between the Besov spaces, noting that $\frac{\alpha}{\mu} \leqslant 1$ with equality only if $\alpha = \mu = 1$, this finally ends up as

$$\begin{aligned}\|f(u)\|_{B_2^{s'\frac{\alpha}{\mu},\frac{r}{\alpha}}} &\leqslant c(\|u\|_{B_2^{s+1,r}})^{\alpha} + c(\|u\|_{B_2^{s\frac{\alpha}{\mu}+1,r\frac{\mu}{\alpha}}})^{\mu} \\ &\leqslant c(\|u\|_{B_2^{s+1,r}})^{\alpha} + c(\|u\|_{B_2^{s+1,r}})^{\mu}.\end{aligned}$$

This completes the proof of Theorem 2.1. $\square$

Before we end, let us notice that we have wasted information in a number of places in the proof of Theorem 2.1, in order to avoid technical arguments involving advanced properties of real interpolation. For a more careful and complete discussion see Maligranda [16].

4. Appendix

In the appendix we supply the full proof of Theorems 2.4 and 2.6, where we use the formulation of the Besov norm given in (3.2). Theorem 2.4 is a special case of Theorem 2.6 so it is enough to prove the later one. Moreover we use the embeddings

$$B_p^{s,q} \subset B_p^{\tilde{s},\tilde{q}}$$

for $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \tilde{q} \leqslant \infty$ and $s \geqslant \tilde{s}$. See [3].

Proof. — Since conditions 1)-4) in Theorem 2.6 only involve strict inequalities we can without loss of generality assume that $s, s' \in \mathbf{R}_+ \setminus \mathbf{N}$ due to the embeddings above.

Set (as already mentioned in section 3)

$$v(x) = \sum_{j=1}^{\infty} A_j u(\frac{x - y^j}{\epsilon_j}), \quad x \in \mathbf{R}^n,$$

where $u \in C_0^{\infty}$ with support in $\{|x| \leqslant 2\}$ such that $u(x) = x_1$ in $\{|x| \leqslant 1\}$, $y^j = (10j, 0, \ldots, 0) \in \mathbf{R}^n$ and with $0 < A_j \uparrow \infty$ and $\epsilon_j = A_j^{-\lambda}$, $\lambda > 0$, to be choosen later. We note that $v \in C^{\infty}$. It remains to prove that if f is **not** a polynomial of degree at most $[s']$ there exists a $\lambda > 0$ such that

$$\begin{cases} v \in C^{\infty}(\mathbf{R}^n) \bigcap B_p^{s+1,p}(\mathbf{R}^n) \\ f(v) \notin B_{p'}^{s',p'}(\mathbf{R}^n) \end{cases}$$

provided conditions 1)-3) in Theorem 2.6 are fullfilled. Note that we have used the first embedding result above here. The full statement of the theorem is then a direct consequence of Proposition 2.5.

Claim 1. — $\Sigma_{j=1}^{\infty} A_j^p \epsilon_j^{n-(s+1)p} < \infty$ implies that $v \in B_p^{s+1,p}(\mathbf{R}^n)$.

Proof. — Consider $\omega_p^{(1)}(t, \partial^\alpha v) \equiv \sup_{|\eta| \leqslant t} \|\partial^\alpha v(\cdot + \eta) - \partial^\alpha v(\cdot)\|_{L_p}$ where $|\alpha| = [s+1] = s + 1 - \sigma$.

For $0 < t < 1$ we have

$$(t^{-\sigma}\omega_p^{(1)}(t, \partial^\alpha v))^p = \sup_{|\eta|\leqslant t} \int_{\mathbf{R}^n} |\frac{\partial_x^\alpha v(x+\eta) - \partial_x^\alpha v(x)}{t^\sigma}|^p dx \leqslant$$

$$\leqslant \Sigma_{j=1}^{\infty} A_j^p \sup_{|\eta|\leqslant t} \int_{\mathbf{R}^n} |\frac{\partial_x^\alpha u(\frac{x+\eta-y^j}{\epsilon_j}) - \partial_x^\alpha u(\frac{x-y^j}{\epsilon_j})}{t^\sigma}|^p dx =$$

$$= \Sigma_{j=1}^{\infty} A_j^p \epsilon_j^{n-[s+1]p-\sigma p} \sup_{|\eta|\leqslant t} \int_{\mathbf{R}^n} |\frac{\partial_x^\alpha u(x - \frac{y^j-\eta}{\epsilon_j}) - \partial_x^\alpha u(x - \frac{y^j}{\epsilon_j})}{(\frac{t}{\epsilon_j})^\sigma}|^p dx$$

where

$$\sup_{|\eta|\leqslant t} \int_{\mathbf{R}^n} |\frac{\partial_x^\alpha u(x - \frac{y^j-\eta}{\epsilon_j}) - \partial_x^\alpha u(x - \frac{y^j}{\epsilon_j})}{(\frac{t}{\epsilon_j})^\sigma}|^p dx \leqslant C \min((\frac{t}{\epsilon_j})^{(1-\sigma)p}, (\frac{t}{\epsilon_j})^{-\sigma p})$$

by the mean value theorem. This yields

$$\int_0^1 (t^{-\sigma}\omega_p^{(1)}(t, \partial^\alpha v))^p \frac{dt}{t} \leqslant$$

$$\leqslant \Sigma_{j=1}^{\infty} A_j^p \epsilon_j^{n-(s+1)p} \int_0^1 \min((\tfrac{t}{\epsilon_j})^{(1-\sigma)p}, (\tfrac{t}{\epsilon_j})^{-\sigma p}) \frac{dt}{t} \leqslant$$

$$\leqslant C\Sigma_{j=1}^{\infty} A_j^p \epsilon_j^{n-(s+1)p}.$$

For $1 \leqslant t$ we have

$$\omega_p^{(1)}(t, \partial^\alpha v) \leqslant \Sigma_{j=1}^{\infty} 2A_j \epsilon_j^{\frac{n}{p}-[s+1]} (\int_{\mathbf{R}^n} |\partial_x^\alpha v(x)|^p \, dx)^{\frac{1}{p}} \leqslant C\Sigma_{j=1}^{\infty} A_j \epsilon_j^{\frac{n}{p}-[s+1]}.$$

But $\Sigma_{j=1}^{\infty} A_j^p \epsilon_j^{n-(s+1)p} < \infty$ implies that $\Sigma_{j=1}^{\infty} A_j \epsilon_j^{\frac{n}{p}-[s+1]} < \infty$ assuming that $\epsilon_j = A_j^{-\lambda} = 2^{-\lambda j}$ for some $\lambda > 0$. Thus

$$\int_1^\infty (t^{-\sigma}\omega_p^{(1)}(t, \partial^\alpha v))^p \frac{dt}{t} < \infty.$$

This gives

$$\|v\|_{B_p^{s+1,p}} \simeq \|v\|_{L_p} + \Sigma_{|\alpha|=[s+1]} (\int_0^\infty (t^{-\sigma}\omega_p^{(1)}(t, \partial^\alpha v))^p \, \frac{dt}{t})^{\frac{1}{p}} < \infty.$$

Claim 2. — $\Sigma_{j=1}^{\infty} A_j^{s'p'-1} \epsilon_j^{n-s'p'} < \infty$ implies that f is a polynomial of degree at most $[s']$.

Proof. — Set $s' = [s'] + \sigma'$. Assume that f is not a polynomial of degree at most $[s']$. Then there exists a and b, $a < b$, such that

$$d(a,b) \equiv \inf_{t \in [a,b]} |f^{([s']+1)}(t)| > 0.$$

Set

$$S_j(\eta) = \{x \in \mathbf{R}^n : |x + \eta - y^j| < \epsilon_j \text{ and } a < \frac{A_j}{\epsilon_j}(x_1 + \eta_1 - y_1^j) < b\}.$$

Since $\frac{A_j}{\epsilon_j} \uparrow \infty$ as $j \to \infty$ it follows that the volume measure of $S_j(0) \bigcap S_j(\eta)$ is $\geqslant \frac{1}{2} \frac{\epsilon_j^n}{A_j}$, where $|\eta| \leqslant t_j \equiv \frac{\epsilon_j}{A_j} \frac{b-a}{10}$, for j large enough, say $j \geqslant j_0$. Then we get for $t_{j+1} \leqslant t \leqslant t_j$, $j \geqslant j_0$,

$$(t^{-\sigma'} \omega_{p'}^{(1)}(t, \partial^{([s'],0,0,\ldots,0)} f(v)))^{p'} =$$

$$= \sup_{|\eta| \leqslant t} \int_{\mathbf{R}^n} |\frac{\partial_{x_1}^{([s'],0,0,\ldots,0)} f(\Sigma_{j=1}^{\infty} A_j u(\frac{x+\eta-y^j}{\epsilon_j})) - \partial_{x_1}^{([s'],0,0,\ldots,0)} f(\Sigma_{j=1}^{\infty} A_j u(\frac{x-y^j}{\epsilon_j}))}{t^{\sigma'}}|^{p'} \, dx$$

$$\geqslant \sup_{|\eta| \leqslant t} \Sigma_{k=j_0}^{j} A_k^{[s']p'} \epsilon_k^{-[s']p'} \int_{S_k(0) \bigcap S_k(\eta)} |\frac{f^{([s'])}(A_k \frac{x_1+\eta_1-y_1^k}{\epsilon_k}) - f^{([s'])}(A_k \frac{x_1-y_1^k}{\epsilon_k})}{t^{\sigma'}}|^{p'} \, dx$$

$$\geqslant C \Sigma_{k=j_0}^{j} A_k^{[s']p'+p'} \epsilon_k^{-[s']p'-p'} t^{(1-\sigma')p'} \frac{\epsilon_k^n}{A_k}.$$

Thus we get

$$\|f(v)\|_{B_{p'}^{s',p'}} \geqslant (\int_0^{\infty} (t^{-\sigma'} \omega_{p'}^{(1)}(t, \partial^{([s'],0,0,\ldots,0)} f(v)))^{p'} \frac{dt}{t})^{\frac{1}{p'}}$$

$$\geqslant C(\Sigma_{j=1}^{\infty} \int_{t_{j+1}}^{t_j} (t^{-\sigma'} \omega_{p'}^{(1)}(t, \partial^{([s'],0,0,\ldots,0)} f(v)))^{p'} \frac{dt}{t})^{\frac{1}{p'}}$$

$$\geqslant C(\Sigma_{j=1}^{\infty} \{(\Sigma_{k=j_0}^{j} A_k^{p'([s']+1)-1} \epsilon_k^{n-p'([s']+1)})((\frac{\epsilon_j}{A_j})^{(1-\sigma')p'} - (\frac{\epsilon_{j+1}}{A_{j+1}})^{(1-\sigma')p'})\})^{\frac{1}{p'}}$$

$$\geqslant C(\Sigma_{j=j_0+1}^{\infty} A_j^{s'p'-1} \epsilon_j^{n-s'p'})^{\frac{1}{p'}} = \infty.$$

This yields a contradiction by the assumption

$$B_p^{s+1,p} \ni v \mapsto f(v) \in B_{p'}^{s',p'}$$

in the theorem. Hence $d(a,b) = 0$ for all $a < b$ and f is a polynomial of degree at most $[s']$. $\square$

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