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Ahlfors’ currents in higher dimension


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0. Introduction

Let $f : V \mapsto X$ be a nondegenerate holomorphic map between an open connected complex manifold $V$ (non-compact) of dimension $k$ and a compact Hermitian manifold $(X, \omega)$ of dimension larger than or equal to $k$. We consider an exhaustion function $	au$ on $V$. This means that (see [14]):

(i) $\tau : V \mapsto [0, +\infty]$ is $C^1$.

(ii) $\tau$ is proper (i.e. $\tau^{-1}(\text{compact}) = \text{compact}$).

(iii) There exists $r_0 > 0$ such that $\tau$ has only isolated critical points in $\tau^{-1}([r_0, +\infty])$.

In this article we will employ the notation $V(r) = \tau^{-1}([0, r])$.

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The first important example is $V = \mathbb{C}^k$ and $\tau = \|z\|^2$. When $k = 1$ we are studying entire curves in $X$. Another example is that of a pseudoconvex domain $V$ in $\mathbb{C}^k$. If $\tau_0$ is its exhaustion function, we can easily transform $\tau_0$ into a function $\tau$ which satisfies the previous hypothesis (see [11] p. 63-65).

The goal of this article is to construct Ahlfors’ currents in $X$ starting from $V$ and $f$. By definition, an Ahlfors’ current is a \textbf{closed} positive current of bidimension $(k,k)$ which is the limit of a sequence $\frac{f_*[V(r_n)]}{\text{volume}(f(V(r_n)))}$ (here $r_n \to +\infty$ and $\text{volume}(f(V(r_n))) := \int_{V(r_n)} f^* \omega^k$ is the volume of $f(V(r_n))$ counted with multiplicity). When $V = \mathbb{C}$ and $\tau = \|z\|^2$, M. McQuillan constructed such currents in [10] (see [1] too). These currents are fundamental tools in the study of the hyperbolicity of $X$ (see for example [6]). When the dimension of $V$ is larger than or equal to 2 it is not always possible to produce Ahlfors’ currents. Indeed, for example, there exist domains $\Omega$ in $\mathbb{C}^2$ which are biholomorphic to $\mathbb{C}^2$ and such that $\Omega \neq \mathbb{C}^2$ (Fatou-Bieberbach domains). As a consequence, to produce Ahlfors’ currents it is necessary to add a hypothesis on $f$.

When the dimension of $X$ is equal to $k$, there exist criteria which imply that $f(V)$ is dense in $X$ (see [3], [13], [14], [8], [7], [2] and [12]). These criteria use the degrees of $f$ (see [3]) or the growth of the function $f$.

Our goal is to give criteria which use these degrees in order to produce Ahlfors’ currents in $X$. Of course, in the case where the dimension of $X$ is equal to $k$, the existence of such currents will automatically imply that $f(V)$ is dense in $X$. Indeed, $[X]$ is the only positive closed current of bidimension $(k,k)$ in $X$ (up to normalization).

In this article, we will use the following degrees ($t_{k-1}$ will be slightly different from Chern’s one):

$$t_k(r) = \int_{V(r)} f^* \omega^k,$$

which is the volume of $f(V(r))$ counted with multiplicity, and

$$t_{k-1}(r) = \int_{V(r)} i \partial \tau \wedge \overline{\partial} \tau \wedge f^* \omega^{k-1}.$$

Let $C$ be the set of critical values of $\tau$ in $[r_0, +\infty[$. $V$ is connected and non-compact so we can suppose that $[r_0, +\infty[ \subset \tau(V)$.

The criteria that we will give on $t_k$ and $t_{k-1}$ will strongly use the following inequality:
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**Theorem 0.1.** — The functions $t_k$ and $t_{k-1}$ are $C^1$ on $]r_0, +\infty[\setminus \mathcal{C}$ and $C^0$ on $]r_0, +\infty[\setminus \mathcal{C}$ then

$$
\| \partial f_\ast [V(r)] \|^2 \leq K(X) t'_{k-1}(r) t'_k(r).
$$

Here $K(X)$ is a constant which depends only on $(X, \omega)$ and

$$
\| \partial f_\ast [V(r)] \| := \sup_{\Psi \in \mathcal{F}(k-1, k)} \| \langle \partial f_\ast [V(r)], \Psi \rangle \|
$$

where $\mathcal{F}(k-1, k)$ is the set of smooth $(k-1, k)$ forms $\Psi$ with $\| \Psi \| := \max_{x \in X} \| \Psi(x) \| \leq 1$.

By using the previous inequality we can prove some criteria which imply the existence of Ahlfors’ currents. Indeed, the difficulty for the construction of Ahlfors’ currents is the closedness of a limit of $t'_{k-1}(r) r t'_k(r)$ volume($f(V(r)))$ and the previous Theorem gives an estimate for $\| \partial f_\ast [V(r)] \|$. Here we give the following two criteria:

**Theorem 0.2.** — We suppose that $f$ is nondegenerate and of finite-type (i.e. there exist $C_1, C_2, r_1 > 0$ such that volume($f(V(r)))$ $\leq C_1 r^{C_2}$ for $r \geq r_1$).

If

$$
\limsup_{r \to +\infty} \frac{t_{k-1}(r)}{r^2 t_k(r)} = 0
$$

then there exists a sequence $r_n$ which goes to infinity such that $\frac{f_\ast [V(r_n)]}{\text{volume}(f(V(r_n)))}$ converges to a closed positive current with bidimension $(k, k)$ and mass equal to 1.

When $V = \mathbb{C}$ and $\tau = \| z \|^2$, the finite-type hypothesis holds modulo a Brody renormalization (see for example [9]).

We now give one criterion which does not use this hypothesis.

**Theorem 0.3.** — If $f$ is nondegenerate and if there exist $\varepsilon > 0$ and $L > 0$ such that:

$$
\limsup_{r \not\in \mathcal{C}, r \to +\infty} \frac{t'_{k-1}(r)}{r t_k(r)^{1-\varepsilon}} \leq L
$$

then there exists a sequence $r_n$ which goes to infinity such that $\frac{f_\ast [V(r_n)]}{\text{volume}(f(V(r_n)))}$ converges to a closed positive current with bidimension $(k, k)$ and mass equal to 1.
The plan of this article is the following: in the first part we prove the inequality (Theorem 0.1), in the second one we give the proof of both criteria (Theorems 0.2 and 0.3). In the third part, we give a new formulation of the criteria in the special case where $V = C^k$.

1. Proof of the inequality

Let $\mathcal{C}$ be the set of critical values of $\tau$ in $[r_0, +\infty[. We recall that we can suppose $[r_0, +\infty[ \subset \tau(V)$. Notice that point (iii) in the hypothesis on $\tau$ implies that $\mathcal{C}$ is discrete. When $r \in ]r_0, +\infty[ and r \notin \mathcal{C}$ then $\tau : \tau^{-1}(]r - \varepsilon, r + \varepsilon[) \mapsto ]r - \varepsilon, r + \varepsilon[ is a submersion for $\varepsilon > 0$ small enough. In particular, $\tau^{-1}(r)$ is a submanifold of $V$ and $\partial V(r) = \tau^{-1}(r)$. When $r \in \mathcal{C}$, then $\tau^{-1}(r)$ is a compact set which is a submanifold of $V$ outside a neighbourhood of a finite number of points.

We begin now with the following lemma:

**Lemma 1.1.** — The functions $t_k$ and $t_{k-1}$ are $C^1$ on $]r_0, +\infty[ \mathcal{C}$ and $C^0$ on $]r_0, +\infty[.$

**Proof.** — The form $f^*\omega^k$ is positive and smooth and $i\partial\tau \wedge \overline{\partial}\tau \wedge f^*\omega^{k-1}$ is positive and continuous ($\tau$ is $C^1$) so it is enough to show that $t(r) = \int_{V(r)} \Phi$ is $C^1$ on $]r_0, +\infty[ \mathcal{C}$ and $C^0$ on $]r_0, +\infty[ with \Phi$ a positive continuous form of bidegree $(k, k)$.

We take $r \in ]r_0, +\infty[ \mathcal{C}$ and $\varepsilon > 0$ such that $\tau : \tau^{-1}(]r - \varepsilon, r + \varepsilon[) \mapsto ]r - \varepsilon, r + \varepsilon[ is a submersion. Now, if $r' \in ]r - \varepsilon, r[$, we have:

$$\frac{t(r) - t(r')}{r - r'} = \frac{1}{r - r'} \int_{\tau^{-1}(]r', r[)} \Phi = \frac{1}{r - r'} \int_{]r', r[} \tau_* \Phi.$$

The form $\tau_* \Phi$ is continuous so it is equal to $\alpha(s)ds$ with $\alpha$ in $C^0(]r - \varepsilon, r + \varepsilon[). We obtain:

$$\frac{t(r) - t(r')}{r - r'} = \frac{1}{r - r'} \int_{r'}^r \alpha(s)ds$$

which converges to $\alpha(r)$ when $r' \to r$. The same thing happens when we consider $r' \in ]r, r + \varepsilon[$, so the function $t$ is differentiable at $r$ and $t'(r) = \alpha(r). In particular $t is $C^1$ on $]r_0, +\infty[ \mathcal{C}.$

**Remark 1.2.** — Notice that here we did not use that $\Phi$ is positive. We will use this remark in the proof of Theorem 0.1.
Now, consider $r \in \mathcal{C}$. If we take $\varepsilon > 0$, then we can find two neighbourhoods $W_\varepsilon \subset W_{2\varepsilon}$ of the (finite) number of the critical points in $\{ \tau = r \}$ such that $\int_{W_{2\varepsilon}} \Phi \leq \varepsilon$ (because $\Phi$ is continuous). Now, let $\psi$ be a $C^\infty$ function which is equal to 1 in a neighbourhood of $W_\varepsilon$ and to 0 outside $W_{2\varepsilon}$ ($0 \leq \psi \leq 1$). Then, if $r' < r$,

$$t(r) - t(r') = \int_{V(r) \setminus V(r')} \psi \Phi + \int_{V(r) \setminus V(r')} (1 - \psi) \Phi \leq \varepsilon + \int_{V(r) \setminus V(r')} (1 - \psi) \Phi.$$ 

If $\alpha > 0$ is small then $\tau$ is a submersion on $\tau^{-1}([r - \alpha, r + \alpha[) \cap (V \setminus W_\varepsilon)$. In particular the function

$$r' \mapsto \int_{V(r) \setminus V(r')} (1 - \psi) \Phi = \int_{r'}^{r} \tau_* ((1 - \psi) \Phi)$$

goes to 0 when $r' \to r$. The same thing happens when we take $r' > r$. As a consequence, there exists $\delta > 0$ such that if $|r - r'| < \delta$ then $|t(r) - t(r')| \leq 2\varepsilon$, i.e. $t$ is continuous at $r$. □

We give now the proof of Theorem 0.1.

We take $r \in ]r_0, +\infty[ \setminus \mathcal{C}$. We have:

$$\| \partial f_* [V(r)] \| = \sup_{\Psi \in \mathcal{F}(k-1, k)} | \langle \partial f_* [V(r)], \Psi \rangle |$$

where $\mathcal{F}(k-1, k)$ is the set of smooth $(k-1, k)$ forms $\Psi$ with $\| \Psi \| = \max_{x \in X} \| \Psi(x) \| \leq 1$. If $\Psi \in \mathcal{F}(k-1, k)$ then we can write (see for example [5] chapter III Lemma 1.4)

$$\Psi = \sum_{i=1}^{K(X)} \theta_i \wedge \Omega_i$$

where $K(X)$ is a constant which depends only on $X$, the $\theta_i$ are smooth forms of bidegree $(0, 1)$ with $\| \theta_i \| \leq 1$ and the $\Omega_i$ are (strongly) positive smooth forms of bidegree $(k-1, k-1)$ with $\| \Omega_i \| \leq K(X)$. So, to prove the inequality it is sufficient to bound from above $| \langle \partial f_* [V(r)], \theta \wedge \Omega \rangle |^2$ by $K'(X) t_{k-1}^\epsilon (r) t_k (r)$ with $\theta$ a smooth form of bidegree $(0, 1)$ with $\| \theta \| \leq 1$, $\Omega$ a positive smooth form of bidegree $(k-1, k-1)$ with $\| \Omega \| \leq 1$ and $K'(X)$ a constant which depends only on $(X, \omega)$.

If $\varepsilon > 0$ is small then $\tau : \tau^{-1}([r - \varepsilon, r + \varepsilon[) \mapsto ]r - \varepsilon, r + \varepsilon[)$ is a submersion. Now, if we take $r' \in ]r - \varepsilon, r[\,$, we have:
$$A(r', r) := \left| \frac{1}{r - r'} \int_{r'}^{r} \langle \partial f^*[V(s)], \theta \wedge \Omega \rangle ds \right|$$

$$= \left| \frac{1}{r - r'} \int_{r'}^{r} \langle \partial[V(s)], f^*\theta \wedge f^*\Omega \rangle ds \right|.$$ 

If we use the Stokes’ Theorem, we have:

$$A(r', r) = \left| \frac{1}{r - r'} \int_{r'}^{r} \langle \partial[V(s)], f^*\theta \wedge f^*\Omega \rangle ds \right|$$

$$= \left| \frac{1}{r - r'} \int_{r'}^{r} \langle [\tau = s], f^*\theta \wedge f^*\Omega \rangle ds \right|,$$

because for $s \in [r - \varepsilon, r + \varepsilon]$ the boundary of $V(s)$ is $\{ \tau = s \}$.

We obtain:

$$A(r', r) = \left| \frac{1}{r - r'} \int_{r'}^{r} \left( \int_{\tau = s} f^*\theta \wedge f^*\Omega \right) ds \right|.$$ 

Now $\tau : \tau^{-1}([r - \varepsilon, r + \varepsilon]) \mapsto [r - \varepsilon, r + \varepsilon]$ is a submersion, so by using Fubini’s Theorem (see [4] p. 334), we have:

$$A(r', r) = \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} d\tau \wedge f^*\theta \wedge f^*\Omega \right|$$

$$= \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} \partial\tau \wedge f^*\theta \wedge f^*\Omega \right|.$$ 

Now, if we consider,

$$\{\phi, \psi\} := \int_{V(r) \setminus V(r')} i\phi \wedge \overline{\psi} \wedge f^*\Omega$$

where $\phi$ and $\psi$ are continuous forms of bidegree $(1, 0)$, then $\{\phi, \phi\} \geq 0$ (because $\Omega$ is positive) and so by using the proof of the Cauchy-Schwarz’s inequality we obtain that:

$$|\{\phi, \psi\}| \leq (\{\phi, \phi\})^{1/2}(\{\psi, \psi\})^{1/2}.$$
In particular,

\[
A(r', r)^2 \leq \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} i\partial \tau \wedge \overline{\partial} \tau \wedge f^*\Omega \right| \times \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} i\overline{f^*\theta} \wedge f^*\theta \wedge f^*\Omega \right|.
\]

Now \( i\overline{f^*\theta} \wedge f^*\theta \wedge f^*\Omega \) is equal to \( f^*(i\overline{\theta} \wedge \theta \wedge \Omega) \) and \( i\overline{\theta} \wedge \theta \wedge \Omega \leq K'(X)\omega^k \) (which means that \( K'(X)\omega^k - i\overline{\theta} \wedge \theta \wedge \Omega \) is a (strongly) positive form). Here \( K'(X) \) depends only on \((X, \omega)\) because \( \|\theta\| \leq 1 \) and \( \|\Omega\| \leq 1 \).

As a consequence, we have:

\[
\left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} i\partial \tau \wedge \overline{\partial} \tau \wedge f^*\Omega \right| \leq K'(X) \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} f^*\omega^k \right| = K'(X) \left( \frac{t_{k-1}(r) - t_{k}(r')}{r - r'} \right).
\]

On the other hand, there exists a constant \( K''(X) \) with \( \Omega \leq K''(X)\omega^{k-1} \) (we use \( \|\Omega\| \leq 1 \)). So, we have

\[
\left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} i\partial \tau \wedge \overline{\partial} \tau \wedge f^*\Omega \right| \leq K''(X) \left( \frac{t_{k-1}(r) - t_{k-1}(r')}{r - r'} \right).
\]

We obtain:

\[
A(r', r)^2 \leq K(X) \left( \frac{t_{k-1}(r) - t_{k-1}(r')}{r - r'} \right) \left( \frac{t_k(r) - t_k(r')}{r - r'} \right).
\]

Now, when \( r' \to r \)

\[
A(r', r)^2 \to |\langle \partial f_s[V(r)], \theta \wedge \Omega \rangle|^2
\]

because the function \( s \mapsto \langle \partial f_s[V(s)], \theta \wedge \Omega \rangle = -\int_{V(s)} \partial f^*(\theta \wedge \Omega) \) is continuous on \([r - \varepsilon, r + \varepsilon] \) (see remark 1.2).

Finally, if we take \( r' \to r \) in the inequality (1.1), we have:

\[
|\langle \partial f_s[V(r)], \theta \wedge \Omega \rangle|^2 \leq K(X)t'_{k-1}(r)t'_{k}(r)
\]

which gives the desired inequality.
2. Proof of Theorems 0.2 and 0.3

2.1. Proof of the first criterion

We begin with this lemma:

**Lemma 2.1.** — If \( f \) is nondegenerate and of finite-type then there exists a constant \( K > 0 \) such that:

\[
\forall r_2 > 0 \ \exists r' \geq r_2 \text{ with } \operatorname{vol}(f(V(2r))) \leq K \operatorname{vol}(f(V(r))).
\]

**Proof.** — The hypothesis implies that there exist \( C_1, C_2, r_1 > 0 \) such that \( \operatorname{vol}(f(V(r))) \leq C_1 r C_2 \) for \( r \geq r_1 \).

If the conclusion of the lemma fails then for all \( K > 0 \) there exists \( r_2 > 0 \) such that for all \( r \geq r_2 \) we have \( \operatorname{vol}(f(V(2r))) \geq K \operatorname{vol}(f(V(r))) \).

So, if we take \( K >> 2C_2 \) then we obtain (if \( l \) is large enough):

\[
C_1 (2^l r_2) C_2 \geq \operatorname{vol}(f(V(2^l r_2))) \geq K^l \operatorname{vol}(f(V(r))).
\]

As a consequence we have

\[
\operatorname{vol}(f(V(r_2))) \leq C_1 r_2 C_2 \left( \frac{2C_2}{K} \right)^l,
\]

which implies that \( \operatorname{vol}(f(V(r_2))) = 0 \) when we take \( l \to \infty \). It contradicts the fact that \( f \) is nondegenerate. \( \square \)

By using this lemma, we can find a sequence \( R_n \to +\infty \) which satisfies

\[
\operatorname{vol}(f(V(2R_n))) \leq K \operatorname{vol}(f(V(R_n))).
\]

Theorem 0.1 gives now that:

\[
\int_{R_n}^{2R_n} \| \partial f_s[V(r)] \| \, dr \leq K(X) \int_{R_n}^{2R_n} \sqrt{t'_{k-1}(r)} \sqrt{t'_k(r)} \, dr.
\]

We give the following sense to the integrals: for example, if there is one point \( a_n \) of \( \mathcal{C} \) in \([R_n, 2R_n]\), we consider \( \int_{R_n}^{2R_n} = \lim_{\epsilon \to 0} \int_{[R_n, a_n-\epsilon] \cup (a_n+\epsilon, 2R_n]} \).

All the functions that we consider are non negative, so the limit exists in \([0, +\infty]\).
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Now, by using the Cauchy-Schwarz’s inequality, the last integral is smaller than

$$K(X) \left( \int_{R_n}^{2R_n} t'_{k-1}(r)dr \right)^{1/2} \left( \int_{R_n}^{2R_n} t'_k(r)dr \right)^{1/2} \leq K(X) \sqrt{t_{k-1}(2R_n) \sqrt{t_k(2R_n)}}.$$

For the last inequality it is important to use that $t_{k-1}$ and $t_k$ are continuous on $]r_0, +\infty[$ (see Theorem 0.1).

It implies that there exists a sequence $r_n \in [R_n, 2R_n]$ such that:

$$\|\partial f^*[V(r_n)]\| \leq \frac{K(X)}{R_n} \sqrt{t_{k-1}(2R_n) \sqrt{t_k(2R_n)}},$$

i.e.

$$\frac{\|\partial f^*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \leq 2K(X) \sqrt{\frac{t_{k-1}(2R_n)}{(2R_n)^2 t_k(2R_n)} \times \frac{t_k(2R_n)}{t_k(r_n)}}$$

because $\text{volume}(f(V(r_n))) = t_k(r_n)$.

Now we have

$$\frac{t_k(2R_n)}{t_k(r_n)} \leq K$$

and by using the hypothesis,

$$\sqrt{\frac{t_{k-1}(2R_n)}{(2R_n)^2 t_k(2R_n)}} \to 0.$$

So, we obtain that

$$\frac{\|\partial f^*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \to 0.$$

The current $T_n := \frac{f^*[V(r_n)]}{\text{volume}(f(V(r_n)))}$ is positive with bidimension $(k,k)$ and mass equal to 1, so there exists a subsequence of $(T_n)$ which converges to a positive current $T$ with bidimension $(k,k)$ and mass 1. Moreover,

$$\|\partial T_n\| = \frac{\|\partial f^*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \to 0,$$

so the limit current $T$ is closed. This proves the first criterion.
2.2. Proof of the second criterion

Take $\varepsilon > 0$ and $L > 0$ such that
\[
\limsup_{r \in \mathcal{C}, \, r \to +\infty} \frac{t_{k-1}^r(r)}{rt_k(r)^{1-\varepsilon}} \leq L.
\]

Let $R_n$ be a sequence of positive reals which goes to $+\infty$. By using Theorem 0.1, we have (see the proof of the last criterion for the definition of the integrals):
\[
\int_{r_0+1}^{R_n} \frac{\|\partial f[V(r)]\|^2}{t_{k-1}^r(r)t_k(r)^{1+\varepsilon}} dr \leq K(X) \int_{r_0+1}^{R_n} \frac{t_k^r(r)}{t_k(r)^{1+\varepsilon}} dr.
\]

This last integral is smaller than $K'(X,f)$ (here we use the fact that $\frac{1}{t_k(r)}$ is continuous on $[r_0, +\infty[$).

So, we have
\[
\int_{r_0+1}^{+\infty} \frac{1}{r} \left( \frac{r\|\partial f[V(r)]\|^2}{t_{k-1}^r(r)t_k(r)^{1+\varepsilon}} \right) dr \leq K'(X,f),
\]
and $\int_{r_0+1}^{+\infty} \frac{1}{r} \, dr = +\infty$ implies that there exists a sequence $r_n \to +\infty$ such that $r_n \notin \mathcal{C}$ and:
\[
\varepsilon(n) := \frac{r_n\|\partial f[V(r_n)]\|^2}{t_{k-1}^r(r_n)t_k(r_n)^{1+\varepsilon}} \to 0.
\]

We obtain
\[
\left( \frac{\|\partial f[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \right)^2 = \frac{\varepsilon(n)}{r_n} \frac{t_{k-1}^r(r_n)}{t_k(r_n)^{1-\varepsilon}} \leq (L + 1)\varepsilon(n),
\]
by hypothesis (for $n$ large enough).

So,
\[
\frac{\|\partial f[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \to 0.
\]

Now, by using exactly the same argument as in the proof of the previous criterion, we obtain that there exists a subsequence of $T_n := \frac{f[V(r_n)]}{\text{volume}(f(V(r_n)))}$ which converges to a closed positive current of bidimension $(k,k)$ and with mass equal to 1.
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3. The special case $V = \mathbb{C}^k$

In this paragraph we consider the special case where $V = \mathbb{C}^k$.

Let $\beta$ be the standard Kähler form in $\mathbb{C}^k$. We want to transform our previous criteria by using $\beta$ instead of $i\partial \tau \wedge \bar{\partial} \tau$. More precisely, we consider:

$$a_k(r) = \int_{B(0,r)} f^* \omega^k$$

and

$$a_{k-1}(r) = \int_{B(0,r)} \beta \wedge f^* \omega^{k-1}.$$ 

Then we can prove a new formulation of our three Theorems:

**Theorem 3.1.** — The functions $a_k$ and $a_{k-1}$ are $C^1$ on $]0, +\infty[$ and for $r > 0$ we have

$$\|\partial f_*[B(0, r)]\|_2 \leq K(X)a'_{k-1}(r)a'_k(r).$$

Here $\|\cdot\|$ is the norm in the sense of currents and $K(X)$ is a constant which depends only on $(X, \omega)$.

**Proof.** — We apply Theorem 0.1 with $V = \mathbb{C}^k$ and $\tau = \|z\|^2$ (here we have $\mathcal{C} = \{0\}$) and then for $r > 0$:

$$\|\partial f_*[V(r^2)]\|_2 \leq K'(X)t'_{k-1}(r^2)t'_k(r^2).$$

Now, $a_k(r) = t_k(r^2)$, so $a_k$ is $C^1$ in $]0, +\infty[$ and

$$t'_k(r^2) = \frac{a'_k(r)}{2r}.$$ 

The function $a_{k-1}(r) = t(r^2)$ with $t(r) = \int_{V(r^2)} \beta \wedge f^* \omega^{k-1}$ so $a_{k-1}$ is $C^1$ in $]0, +\infty[$ (see proof of Lemma 1.1).

Moreover,

$$t_{k-1}(r^2) = \int_{V(r^2)} i\partial \tau \wedge \bar{\partial} \tau \wedge f^* \omega^{k-1} = \int_{B(0,r)} i\partial \tau \wedge \bar{\partial} \tau \wedge f^* \omega^{k-1},$$

and

$$i\partial \tau \wedge \bar{\partial} \tau = i \sum_{i,j} z_i \bar{z}_j dz_i \wedge d\bar{z}_j.$$ 

On $B(0, r)$ this last form is smaller than $K(k)\beta r^2$.
If we take $0 < r' < r$ then
\[ t_{k-1}(r^2) - t_{k-1}(r'^2) = \int_{B(0,r) \setminus B(0,r')} i\partial\tau \wedge \overline{\partial\tau} \wedge f^*\omega^{k-1} \]
\[ \leq K(k)r^2 \int_{B(0,r) \setminus B(0,r')} \beta \wedge f^*\omega^{k-1}. \]

If we divide by $r - r'$ and take the limit $r' \to r$, we obtain:
\[ 2rt_{k-1}'(r^2) \leq K(k)r^2a_{k-1}'(r). \]

Finally, we have:
\[ \|\partial f_*[B(0,r)]\|_2 = \|\partial f_*[V(r^2)]\|_2 \leq K'(X)t'_{k-1}(r^2)t'_k(r^2) \leq K(X)a'_{k-1}(r)a'_k(r), \]
with $K(X) = K(k)K'(X)$ (we recall that the dimension of $X$ is larger than or equal to $k$). This is the inequality that we were looking for. \hfill \Box

Now if we replace in the proof of Theorems 0.2 and 0.3 the function $t_{k-1}$ by $a_{k-1}$, the function $t_k$ by $a_k$ and $V(r)$ by $B(0,r)$ then we obtain the two following criteria:

**Theorem 3.2.** — We suppose that $f$ is nondegenerate and with finite-type (i.e. there exist $C_1$, $C_2$, $r_1 > 0$ such that $\text{volume}(f(B(0,r))) \leq C_1r^{C_2}$ for $r \geq r_1$).

If
\[ \limsup_{r \to +\infty} \frac{a_{k-1}(r)}{r^2a_k(r)} = 0 \]
then there exists a sequence $r_n$ which goes to infinity such that $\frac{f_*[B(0,r_n)]}{\text{volume}(f(B(0,r_n)))}$ converges to a closed positive current with bidimension $(k,k)$ and mass equal to 1.

**Theorem 3.3.** — If $f$ is nondegenerate and if there exist $\varepsilon > 0$ and $L > 0$ such that:
\[ \limsup_{r \to +\infty} \frac{a'_{k-1}(r)}{rak(r)^{1-\varepsilon}} \leq L \]
then there exists a sequence $r_n$ which goes to infinity such that $\frac{f_*[B(0,r_n)]}{\text{volume}(f(B(0,r_n)))}$ converges to a closed positive current with bidimension $(k,k)$ and mass equal to 1.
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Notice that when $k = 1$ then $a_{k-1}(r) = \pi r^2$ and therefore, in this context, the hypothesis of this criterion is always fulfilled if $f$ is nondegenerate.

Bibliography