ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

Aubin Arroyo, Adolfo Guillot

A rigidity phenomenon for germs of actions of $\mathbf{R}^2$

Tome XIX, nº 3-4 (2010), p. 539-565.

<http://afst.cedram.org/item?id=AFST_2010_6_19_3-4_539_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIX, n° 3-4, 2010
pp. 539–565

A rigidity phenomenon for germs of actions of $\mathbf{R}^2$

Aubin Arroyo, Adolfo Guillot[1]

Abstract. — We study germs of Lie algebras generated by two commuting vector fields in manifolds that are *maximal* in the sense of Palais (those which do not present any evident obstruction to be the local model of an action of $\mathbf{R}^2$). We study three particular pairs of homogeneous quadratic commuting vector fields (in $\mathbf{R}^2$, $\mathbf{R}^3$ and $\mathbf{R}^4$) and study the maximal Lie algebras generated by commuting vector fields whose 2-jets at the origin are the given homogeneous ones. In the first case we prove that the quadratic algebra is a smooth normal form. In the second and third ones, we prove that the orbit structure is, from a topological viewpoint, the one of the quadratic part.

Résumé. — On étudie les germes d'algèbres de Lie de champs de vecteurs engendrées par deux champs de vecteurs commutants sur une variété qui sont *maximales* au sens de Palais (qui ne présentent aucune obstruction évidente pour être le modèle local d'une action de $\mathbf{R}^2$). On étudie trois couples particuliers de champs de vecteurs commutants quadratiques et homogènes (sur $\mathbf{R}^2$, $\mathbf{R}^3$ et $\mathbf{R}^4$) et on étudie les algèbres de Lie maximales qui sont engendrées par des champs commutants dont le deuxième jet à l'origine est donné par les champs homogènes. Dans le premier cas on prouve que l'algèbre quadratique est une forme normale lisse pour l'algèbre. Dans les deux derniers, on prouve que la structure des orbites est, du point de vue topologique, celle de la partie quadratique.

(*) Reçu le 19/06/2008, accepté le 19/05/2009

Partially supported by CONACyT grant 58354 and PAPIIT-UNAM grant IN102307.

(1) Instituto de Matemáticas, Unidad Cuernavaca Universidad Nacional Autónoma de México, A.P. 273-3 Admon. 3, Cuernavaca, Morelos, 62251 México
aubin@matcuer.unam.mx, adolfo@matcuer.unam.mx

1. Introduction

The study of flows on manifolds passes naturally through the study of vector fields. Understanding the singularities of vector fields has been a central problem in the theory of differential equations. Several theorems prove that the linearization of a vector field at a singular point preserves, under certain circumstances, most of the properties of the original vector field. For example, Hartman-Grobman's Theorem states that any smooth vector field is topologically conjugate to its linear part in a neighborhood of an equilibrium point whenever the linear part is hyperbolic. The regularity of the conjugacy can be improved under extra assumptions. Poincaré's linearization theorem asserts that analytic vector fields with eigenvalues in the Poincaré domain and without resonances are analytically linearizable. Sternberg proved [11] that in the presence of resonances the conjugacy can be as smooth as the order to which the linearization approximates the original vector field. If, at an equilibrium point, a vector field is seen as a perturbation of its linear part, these theorems can be seen as local rigidity results.

The correspondence flows/vector fields is almost perfect: every flow is given by integration of a vector field and every germ of smooth vector field is the local model of some flow. The local theory of flows and the local theory of vector fields coincide.

The infinitesimal counterpart of an action of $\mathbf{R}^2$ on a manifold is a pair of commuting vector fields. Not every pair of germs of commuting vector fields is the local model of an $\mathbf{R}^2$-action: the local theory of actions of $\mathbf{R}^2$ is different from the local theory of pairs of commuting vector fields.

To be precise, let $\mathfrak{A}$ be the subspace of the vector space of vector fields in the manifold M generated by the commuting vector fields X_1 and X_2. We say that $\mathfrak{A}$ is a *(marked, commutative, two-dimensional) Lie algebra of vector fields.* A *solution* of $\mathfrak{A}$ with initial condition $p \in M$ is a mapping $\Phi : U \to M$, defined on an open subset U of $\mathbf{R}^2$ containing 0 such that $\Phi(0) = p$ and for every $s \in U$ and every i, $X_i(\Phi(s)) = d/dt\, \Phi(s + te_i)|_{t=0}$, where (e_1, e_2) is the usual basis of $\mathbf{R}^2$.

The existence of solutions and the uniqueness of their germs at 0 is guaranteed by the fact that the local flows of commuting vector fields commute. However (unlike the case of flows) these local solutions cannot be, in general, glued into a maximal solution. Palais proposes the following definition:

DEFINITION 1.1 (PALAIS [4]). — *A commutative Lie algebra of vector fields on M is* maximal *if for every $p \in M$ there exists a solution $\Phi : U \to M$ with initial condition p such that for every sequence $s_i \in U$ such that $\lim_{i\to\infty} s_i \in \partial U$, the sequence $\Phi(s_i)$ leaves every compact subset of M.*

In other words, in a maximal commutative Lie algebra of vector fields, the local solutions can be extended in a unique way (as it is the case for the maximal solutions of a differential equation) to obtain a *maximal* solution.

For an action $\Psi : \mathbf{R}^2 \times M \to M$, the induced commutative Lie algebra of vector fields in M is maximal since the function $\Phi_p : \mathbf{R}^2 \to M$ defined by $\Phi_p(s) = \Psi(s,p)$ is a solution that trivially satisfies Palais' condition since $\partial\mathbf{R}^2 = \emptyset$.

The restriction of a maximal commutative Lie algebra of vector fields in M (in particular, of the Lie algebra generating a given action) to an open subset gives still a maximal Lie algebra. We can thus, as remarked by Rebelo [6], speak of *germs* of maximal Lie algebras. Actions of $\mathbf{R}^2$ on manifolds are locally modeled by germs of maximal Lie algebras and not by arbitrary ones.

It appears thus as necessary to understand the obstructions that a germ of Lie algebra of commuting vector fields must overcome in order to be maximal.

To our knowledge, the property of maximality has not been studied in the real setting, though the analogue property of *semi-completeness* of holomorphic vector fields has been successfully dealt with in the last decade. A holomorphic vector field X in the complex manifold M is said to be *semi-complete* if its real and imaginary parts, as real commutative vector fields on M, generate a maximal Lie algebra (Rebelo's original definition [6] is equivalent to this one).

Germs of singular holomorphic vector fields with non-degenerate linear part are, at least if the dimension of the ambient space is small, essentially semi-complete [7], [8]. As the linear part becomes more degenerate, germs of semi-complete holomorphic vector fields become scarce and, in dimension two, the rareness of these objects can be seen through the following rigidity result of Ghys and Rebelo [2]: *if X is a semi-complete holomorphic vector field in a neighborhood of the origin in $\mathbf{C}^2$ with an isolated singularity at the origin and vanishing first jet and if X_2 is the quadratic homogenous vector field given by its second jet, then X_2 is a semi-complete vector field and there exists a holomorphic function f such that the germs at the origin X and fX_2 are holomorphically conjugate.*

Returning to the real setting, there exist some linearization results — analogue to the ones for vector fields — concerning germs of commutative Lie algebras of vector fields. Dumortier and Roussarie [1] proved that a pair of germs of C^∞ commuting vector fields in $\mathbf{R}^n$ in a neighborhood of a common singular point are C^∞-linearizable if the linear part (of the pair) is hyperbolic and does not satisfy any resonance condition of finite order. They also prove a result about C^k-linearization in the presence of certain resonances. A corollary of the results of Dumortier and Roussarie is that the pairs of germs of commuting vector fields at the origin of $\mathbf{R}^n$ satisfying the hypotheses of their theorems generate a maximal Lie algebra, since they are equivalent to their linear part (maximality is invariant under changes of coordinates). In some sense, almost every pair of commuting vector fields with non-degenerate linear part at a common singular point is maximal.

This article is about the maximality of Lie algebras generated by commuting vector fields without linear part. One cannot expect to have a unified treatment of these algebras. We can, however, restrict this universe to those commuting vector fields whose Taylor development starts with a *given* pair of polynomial homogenous commuting vector fields. Pairs of (nonlinear) homogenous commuting vector fields do not abound. A generic non-linear homogenous polynomial vector field will only commute with its multiples and only some, very special (or very trivial), vector fields will have a bigger center.

We restrict our interest to three special maximal Lie algebras generated by commuting quadratic homogenous vector fields (in $\mathbf{R}^2$, $\mathbf{R}^3$ and $\mathbf{R}^4$) and study the smooth commuting vector fields whose Taylor development starts with these homogenous ones and which, moreover, generate a maximal Lie algebra. Our results are the following:

Theorem A. — *Consider in $\mathbf{R}^2$ the maximal Lie algebra generated by the commuting vector fields*

$$X_0 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y},\ Y_0 = -2xy\frac{\partial}{\partial x} + (x^2 - y^2)\frac{\partial}{\partial y}. \qquad (1.1)$$

Let X_1 and Y_1 be $C^{n+\alpha}$, ($n \geqslant 2$, $0 < \alpha < 1$), commuting vector fields defined in a neighborhood of the origin of $\mathbf{R}^2$ and generating a Lie algebra whose germ at the origin is maximal. Suppose that the n-jets of X_1 and Y_1 at the origin are X_0 and Y_0. Then there exists a $C^{(n-1)+\alpha}$ map from $(\mathbf{R}^2, 0)$ to $(\mathbf{R}^2, 0)$ that redresses X_1 and Y_1 into X_0 and Y_0.

THEOREM B. — *Consider in $\mathbf{R}^3$ the maximal Lie algebra generated by the commuting vector fields*

$$X_0 = (-3x^2 + y^2 + 2xz)\frac{\partial}{\partial x} + 2y(-3x + 2z)\frac{\partial}{\partial y} + 2z(3x - z)\frac{\partial}{\partial z}, \quad (1.2)$$

$$Y_0 = 2y(-x + z)\frac{\partial}{\partial x} + (3x^2 - y^2)\frac{\partial}{\partial y} + 2zy\frac{\partial}{\partial z}. \quad (1.3)$$

Let X_1 and Y_1 be C^3 commuting vector fields in a neighborhood of the origin of $\mathbf{R}^3$, generating a maximal Lie algebra in some neighborhood of the origin and such that their second jets at the origin are, respectively, X_0 and Y_0. Then there exists an homeomorphism that redresses the foliation generated by X_1 and Y_1 into the foliation generated by X_0 and Y_0.

THEOREM C. — *Consider in $\mathbf{R}^4$ the maximal Lie algebra generated by the commuting vector fields*

$$X_0 = (x_1^2 - x_2^2)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2} + (x_3^2 - x_4^2)\frac{\partial}{\partial x_3} + 2x_3x_4\frac{\partial}{\partial x_4}, \quad (1.4)$$

$$Y_0 = -2x_1x_2\frac{\partial}{\partial x_1} + (x_1^2 - x_2^2)\frac{\partial}{\partial x_2} - 2x_3x_4\frac{\partial}{\partial x_3} + (x_3^2 - x_4^2)\frac{\partial}{\partial x_4}. \quad (1.5)$$

Let X_1 and Y_1 be C^3 commuting vector fields in a neighborhood of the origin of $\mathbf{R}^4$, generating a maximal Lie algebra in some neighborhood of the origin and such that their second jets at the origin are, respectively, X_0 and Y_0. Then there exists an homeomorphism that redresses the foliation generated by X_1 and Y_1 into the foliation generated by X_0 and Y_0.

The commuting vector fields of the first two theorems are *isolated* in the sense that any pair of quadratic commuting vector fields close to the ones under consideration can be obtained from the original one by linear combinations and linear changes of coordinates. The ones in the third example come from a holomorphic vector field in $\mathbf{C}^2$ that is isolated as a semi-complete quadratic homogenous one.

We thank Santiago López de Medrano. Conversations with him led us to Example 3.6. We also thank Adriana Ortiz and Alberto Verjovsky.

2. An example in dimension 2

The holomorphic vector field $z^2\partial/\partial z$ in $\mathbf{C}$ is semi-complete since we have a maximal local action given by $\Phi(z_0, t) = z_0/(1 - tz_0)$. We owe the following observation to Rebelo: *a holomorphic semi-complete vector field*

$X = (z^2+\cdots)\partial/\partial z$ in a neighborhood of the origin of $\mathbf{C}$ can be redressed, by a holomorphic change of coordinates, to the vector field $z^2\partial/\partial z$. In fact, the unique 1-form ω such that $\omega \cdot X = 1$ has a pole of order two at the origin, and, up to biholomorphisms, its sole invariant is its residue at the origin. Hence X can be redressed to one of the vector fields $z^2/(1+az)\partial/\partial z,\ a \in \mathbf{C}$ in a neighborhood of the origin. Integration by parts shows that the vector field is semi-complete only in the case $a = 0$. The real and imaginary parts of $z^2\partial/\partial z$ are generated by the commuting vector fields (1.1). Theorem A extends Rebelo's observation to the real setting.

Proof of Theorem A. — Let $\mathfrak{A}_0$ be the Lie algebra generated by (1.1). A maximal solution of $\mathfrak{A}_0$ is given $\Phi : \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}^2 \setminus \{0\}$

$$(u,v) \overset{\Phi}{\mapsto} \left(-\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right). \tag{2.1}$$

The determinant of X_0 and Y_0 is given by $(x^2+y^2)^2$, and vanishes exclusively at the origin. Let D be a disk around the origin such that X_1 and Y_1 are linearly independent in $D^* = D \setminus \{0\}$. For $c \in \mathbf{R}^*$, let $h^c : \mathbf{R}^2 \to \mathbf{R}^2$ be the homothety given by $(x,y) \mapsto (cx, cy)$. Consider the family of vector fields in D parametrized by $c \in (-1,1)$ and given by

$$X_c = \begin{cases} \frac{1}{c}Dh^c(X_1), & \text{if } c \neq 0; \\ X_0, & \text{if } c = 0. \end{cases} \tag{2.2}$$

This family of vector fields depends continuously on c, since if $X_c = f_c\partial/\partial x + g_c\partial/\partial y$ then $f_c = c^{-2}f_1(cx,cy)$, which is continuous at $c = 0$. Define Y_c similarly and note that $[X_c, Y_c] = 0$. Let $\mathfrak{A}_c$ be the Lie algebra generated by X_c and Y_c. For any $c \neq 0$, $\mathfrak{A}_c$ is equivalent to and $\mathfrak{A}_1$ and is thus maximal. For each c, consider the pair of 1-forms (ω_c^X, ω_c^Y) which are dual to X_c and Y_c, this is,

$$\omega_c^X \cdot X_c \equiv 1,\ \omega_c^X \cdot Y_c \equiv 0,\ \omega_c^Y \cdot X_c \equiv 0,\ \omega_c^Y \cdot Y_c \equiv 1.$$

This pair defines an $\mathbf{R}^2$-valued differential 1-form. The condition of maximality implies that integration of this form along an open path is never zero. We will now prove that this integral vanishes along a closed path.

Let $\gamma_0 : [0,1] \to \mathbf{R}^2 \setminus \{0\}$ be a smooth closed curve that generates the fundamental group of D^*. Let $\beta^X(t)$ and $\beta^Y(t)$ be the functions such that

$$\gamma_0'(t) = \beta^X(t)X_0|_{\gamma_0(t)} + \beta^Y(t)Y_0|_{\gamma_0(t)}.$$

The absence of periods in the solution (2.1) implies that

$$\int_0^1 \beta^X(t)dt = 0,\ \int_0^1 \beta^Y(t)dt = 0. \tag{2.3}$$

For each c, let $\gamma_c(t)$ be the solution to the differential equation

$$\gamma_c'(t) = \beta^X(t)X_c|_{\gamma_c(t)} + \beta^Y(t)Y_c|_{\gamma_c(t)}$$

with initial condition $\gamma_c(0) = \gamma_0(0)$. These differential equations depend continuously on the parameter c and thus, by the continuity of the solutions of a differential equation with respect to parameters, for a sufficiently small c, say $-c_0 \leqslant c \leqslant c_0$, we have that $\gamma_c(t)$ is defined in the interval $[0,1]$. Notice that the identities (2.3) imply that the integral of the form (ω_c^X, ω_c^Y) vanishes along the curve $\gamma_c([0,1])$. By the maximality of $\mathfrak{A}_c$, the curve is closed. The mapping $H : [0,1] \times [0,c_0] \to D^*$ given by $H(t,c) = \gamma_c(t)$ gives an homotopy between $\gamma_0([0,1])$ and $\gamma_c([0,1])$. This proves that the forms ω_c^X and ω_c^Y are exact. Let $\widetilde{u}$ and $\widetilde{v}$ be the functions such that $d\widetilde{u} = \omega_1^X$ and $d\widetilde{v} = \omega_1^Y$. Consider the mapping

$$\Phi \circ (\widetilde{u}, \widetilde{v}), \tag{2.4}$$

where Φ is the solution (2.1). Notice that, by construction, it maps the vector fields X_1 and Y_1 to X_0 and Y_0. It is of class $C^{(n+1)+\alpha}$ in D^* and extends continuously to the origin. Let M_i be the unique Riemannian metric on D^* that makes X_i and Y_i orthonormal. The redressing map (2.4) is an isometry with respect to these metrics and, in particular, is conformal with respect to the conformal structure which lies underneath. Let

$$X_1 = X_0 + p_1\frac{\partial}{\partial x} + p_2\frac{\partial}{\partial y}, \quad Y_1 = Y_0 + q_1\frac{\partial}{\partial x} + q_2\frac{\partial}{\partial y}.$$

The functions p_i and q_i are of class $C^{n+\alpha}$ and have vanishing n-jets. Let

$$\Delta = (x^2+y^2)^2 + (p_1+q_2)x^2 + 2(p_2-q_2)yx - (p_2+q_2)y^2 + (p_1q_2 - p_2q_1).$$

Consider in D^* the metric $M_1' = \Delta^2(x^2+y^2)^{-2}M_1$, conformally equivalent to M_1 on D^*. A straightforward computation shows that M_1' is explicitly given by

$$dx^2 + dy^2 + \frac{1}{(x^2+y^2)^2}\left[Edx^2 + 2Fdxdy + Gdy^2\right], \tag{2.5}$$

for

$$\begin{aligned}
E &= 2x^2q_2 + 4p_2xy + p_2^2 - 2y^2q_2 + q_2^2,\\
F &= (p_2+q_1)y^2 - (p_2+q_1)x^2 + 2(q_2-p_1)xy - (p_2p_1 + q_2q_1),\\
G &= 2p_1x^2 - 4xyq_1 - 2p_1y^2 + p_1^2 + q_1^2.
\end{aligned}$$

We affirm that this $C^{n+\alpha}$ metric in D^* extends as a $C^{(n-2)+\alpha}$ metric on D. It is sufficient to prove the following claims:

1. If f is a $C^{n+\alpha}$ function with vanishing n-jet and P is a homogenous polynomial of degree two, then $(x^2+y^2)^{-2}Pf$ is $C^{(n-2)+\alpha}$.

2. If f and g are a $C^{n+\alpha}$ functions with vanishing n-jet, $(x^2+y^2)^{-2}fg$ is $C^{(n-2)+\alpha}$.

It can be easily seen inductively that if f is a $C^{n+\alpha}$ function with vanishing n-jet and P is a polynomial of degree two, then every partial derivative of $(x^2+y^2)^{-2}Pf$ of the m^{th} $(0 \leqslant m \leqslant n-2)$ order is the sum of terms of the form

$$\frac{Q_{2+i}f_{m-i}}{(x^2+y^2)^{2+i}},\ 0 \leqslant i \leqslant m, \tag{2.6}$$

where Q_{2+i} is a homogenous polynomial of degree $2+i$ and f_{m-i} is a C^{n-m+i} function with vanishing $n-m+i$ jet. On the other hand, by Taylor's Theorem (with integral form of the remainder), every $C^{k+\alpha}$ $(k \geqslant 1)$ function with vanishing k-jet can be written as the sum of terms of the form Sh, where S is a homogenous polynomial of degree k and h a $C^{0+\alpha}$ function vanishing at the origin. In this way, a term of the form (2.6) can be written as a sum of terms of the form $(x^2+y^2)^{-(2+i)}QSh$, where, Q is a homogenous polynomial of degree $2+i$, S is a homogenous polynomial of degree $n-m+i$ and h is a $C^{0+\alpha}$ function vanishing at the origin. Since $(x^2+y^2)^{-(2+i)}QS$ is homogenous of degree $n-2-m \geqslant 0$ (it is, in particular, bounded, though not necessarily continuous at the origin), and the product of a $C^{0+\alpha}$ function times a bounded function is $C^{0+\alpha}$, we conclude that *if f is a $C^{n+\alpha}$ function with vanishing n jet and P is a polynomial of degree two, then $(x^2+y^2)^{-2}Pf$ is in $C^{(n-2)+\alpha}$*. In the same spirit, if f and g are $C^{n+\alpha}$ functions with vanishing n jet, then every partial derivative of $(x^2+y^2)^{-2}fg$ of the m^{th} $(0 \leqslant m \leqslant n)$ order is the sum of terms of the form

$$\frac{Q_i f_j g_k}{(x^2+y^2)^{2+i}},\ i+j+k=m, \tag{2.7}$$

where Q_i is a homogenous polynomial of degree i, f_j is a C^{n-j} function with vanishing $(n-j)$-jet and g_k is a C^{n-k} function with vanishing $(n-k)$-jet. By the same argument as before, any term of the form (2.7) can be written as sums of terms of the form $(x^2+y^2)^{2+i}Q_iS_jS_kr_jr_k$ where Q_i is a homogenous polynomial of degree i, S_j is a homogenous polynomial of degree $n-j$, S_k is a homogenous polynomial of degree $n-k$ and r_j and r_k are $C^{0+\alpha}$ functions vanishing at the origin. The function $(x^2+y^2)^{2+i}Q_iS_jS_k$ is homogenous of degree $2(n-2)-m \geqslant n-2 \geqslant 0$. We thus have that *if f and g are a $C^{n+\alpha}$ functions with vanishing n-jet, $(x^2+y^2)^{-2}fg$ is in $C^{(n-2)+\alpha}$*. We conclude that the metric (2.5) is $C^{(n-2)+\alpha}$. According to the Theorem on the existence of isothermal coordinates [10, Chapter 9, Addendum 1], *any* system of isothermal coordinates for a $C^{(n-2)+\alpha}$ metric is of class $C^{(n-1)+\alpha}$. Since the conformal structure induced by M_0 is the one induced by the metric dx^2+dy^2, the map (2.4) gives isothermal coordinates for M_1' and has thus the sought regularity. $\square$

Example 2.1. — The loss of regularity in the above Theorem is not (entirely) an artifact of the proof, as this example will show. Consider the commuting vector fields

$$\begin{aligned} X_1 &= X_0 + \frac{y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}-1}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right), \\ Y_1 &= Y_0 + \frac{x\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}-1}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right), \end{aligned}$$

defined in a neighborhood of $0 \in \mathbf{R}^2$. They are higher order perturbations of X_0 and Y_0 and are of class $C^{2+\alpha}$ for every $\alpha \in (0,1)$ — the second partial derivatives of the coefficients are in fact Lipschitz —. The algebra is maximal since the map

$$(x,y) \mapsto \left(\frac{1}{1-\sqrt{x^2+y^2}}\right)(x,y)$$

redresses X_1 and Y_1 onto the vector fields X_0 and Y_0. This change of coordinates is, by the above Theorem, of class $C^{1+\alpha}$ for every α (its partial derivatives are in fact Lipschitz) but is *not* C^2.

Remark 2.2. — We can use the previous example to construct an action of $\mathbf{R}^2$ on the sphere S^2 having only one fixed point. Consider a sufficiently small neighborhood of the origin U together with the vector fields X_1 and Y_1. Identify $U\setminus\{0\}$ with the coordinate vector fields of $\mathbf{R}^2$. The resulting surface is S^2 together with an action of $\mathbf{R}^2$. This action is C^3 and is C^1-conjugate to the standard action of $\mathbf{R}^2$ on S^2. However, it is *not* C^2-conjugate to it.

We should not expect analogue rigidity results to hold for other pairs of quadratic homogenous commuting vector fields. Consider the Lie algebra $\mathfrak{A}_c$ generated by the commuting vector fields

$$X_c = x^2\frac{\partial}{\partial x},\ Y_c = cx^2y\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \tag{2.8}$$

The algebra is maximal since the solution with initial condition (x_0, y_0) is

$$\Phi(x_0,y_0;u,v) = \left(\frac{x_0}{1-ux_0+cx_0\log[1-vy_0]}, \frac{y_0}{1-vy_0}\right)$$

Notice that if $x_0y_0 \neq 0$, the restriction of the above solution to the one parameter subgroup $u = 0$ converges, as v goes to infinity, to $(x_0, 0)$ if $c = 0$ and to $(0, 0)$ if $c \neq 0$. The solutions of an algebra $\mathfrak{A}_c$ with $c \neq 0$ cannot even be topologically conjugate to the solutions of $\mathfrak{A}_0$.

3. An example in dimension 3

Let $\mathfrak{A}_0$ be the commutative two-dimensional Lie algebra of quadratic homogenous vector fields in $\mathbf{R}^3$ generated by (1.2) and (1.3). We will prove in §3.1 that this algebra is maximal. Although this follows from the results in [3] (where it is shown that the complexifications of these fields integrate into a maximal local holomorphic action of $\mathbf{C}^2$ on $\mathbf{C}^3$), we will give a full proof of this fact (following the previous reference to some extent) in order to get a deeper understanding of the real geometry of the action and, in particular, of the nature of the maximal domains of definition of the solutions. Theorem B will be proved in §3.2. For its proof, we will use standard tools from the theory of hyperbolic vector fields, such as the Invariant Manifold Theorem and the Grobman-Hartman Theorem (see [5]), as well as the following well-known principle:

Proposition 3.1 (Kupka Phenomenon). — *Let $\mathcal{F}$ be a singular foliation in $(\mathbf{R}^{n+1}, 0)$ generated by two C^k vector fields and suppose that their rank is 1 at the origin. Then the foliation is a product: there exist C^k coordinates $\{x_i\}$ where the foliation is generated by two vector fields of the forms $\partial/\partial x_{n+1}$ and $\sum_{i=1}^{n} f_i(x_1, \ldots, x_n)\partial/\partial x_i$.*

Proof of the Kupka Phenomenon. — Let X and Y be the vector fields generating $\mathcal{F}$ and suppose that X does not vanish at the origin. Choose C^k coordinates, $\{x_i\}$, where $X = \partial/\partial x_{n+1}$. If, in these coordinates, $Y = \sum_{i=1}^{n+1} f_i \partial/\partial x_i$, let $Y_1 = Y - f_{n+1}X$. Since Y_1 and X generate a foliation, there exists a function A such that $[X, Y_1] = AY_1$. If we set

$$g(x_1, \ldots, x_n, x_{n+1}) = \exp\left(-\int_0^{x_{n+1}} A(x_1, \ldots, x_n, y)dy\right),$$

we have $[X, gY_1] = g[X, Y_1] + (X \cdot g)Y_1 = (Ag + X \cdot g)Y_1 = 0$. The vector fields X and gY_1 are the required ones. $\square$

3.1. Integrating the action

Let

$$Q = 9x^4 + y^4 - 4x^3z + 6x^2y^2 + 4y^2z^2 - 12xy^2z, \quad g = -z^2Q \tag{3.1}$$

and notice that g is a homogenous first integral common to X_0 and Y_0. The leaves of the induced foliation are contained in its level surfaces. The zero locus of Q is a cone over an irreducible rational quartic having three cusps (the *deltoid* or *tricuspid*) that can be projectively parametrized by

$$\kappa \mapsto [4\kappa^2(1+\kappa^2) : -8\kappa^3 : (3\kappa^2+1)^2]. \tag{3.2}$$

The three cusps appear at the directions $[1:1:2]$, $[1:-1:2]$, $[0:0:1]$, when $\kappa \in \{-1,0,1\}$. The Lie algebra has rank two in the complement of the union of these lines. Since $Q(x,y,0) = (3x^2+y^2)^2$, the line $\{z=0\}$ is a bitangent of the quartic having contact at two imaginary points. The linear transformations

$$\begin{aligned}
\sigma(x,y,z) &= (x,-y,z), & (3.3)\\
\rho(x,y,z) &= -(x,y,z), & (3.4)\\
\eta(x,y,z) &= \tfrac{1}{2}(-x-y+z, 3x-y-z, 2z), & (3.5)
\end{aligned}$$

preserve the first integral g and act upon the Lie algebra by

$$\begin{array}{rclrcl}
\eta_* X_0 &=& -\frac{1}{2}X_0 + \frac{3}{2}X_1, & \eta_* Y_0 &=& -\frac{1}{2}X_0 - \frac{1}{2}Y_0,\\
\sigma_* X_0 &=& X_0, & \sigma_* Y_0 &=& -Y_0,\\
\rho_* X_0 &=& -X_0, & \rho_* Y_0 &=& -Y_0.
\end{array}$$

We will now investigate the nature of the domains where the solutions are defined:

The plane $\{z=0\}$

The algebra is maximal in restriction to this plane since the restriction of the algebra is linearly equivalent to the algebra generated by (1.1). The maximal solutions are defined in the complement of a point in $\mathbf{R}^2$.

The cone $\{Q=0\}$

The mapping $\Phi : \mathbf{R}^2 \setminus \{u(u+v)(u-v)=0\} \to \mathbf{R}^3$ given by

$$(u,v) \mapsto \left(-\frac{2u[u^2+v^2]}{[3u^2+v^2][u^2-v^2]}, \frac{4u^2 v}{[3u^2+v^2][u^2-v^2]}, -\frac{[3u^2+v^2]}{2u[u^2-v^2]} \right),$$

maps each connected component of its definition domain to each one of the six two-dimensional orbits contained in the cone $\{Q=0\}$, mapping, respectively, the vector fields $\partial/\partial u$ and $\partial/\partial v$ to X_0 and Y_0. In restriction to this cone, $\mathfrak{A}_0$ is thus maximal.

The generic orbits

Let $Z_0 = \frac{1}{2}(X_0+Y_0)$ and $Z_1 = \frac{1}{2}(X_0-Y_0)$. These vector fields are completely integrable, for the polynomial functions given by

$$\begin{array}{ll} p_0 = z(x+y), & q_0 = z(3x^2+y^2+2yz), \\ p_1 = z(x-y), & q_1 = z(3x^2+y^2-2yz), \end{array} \tag{3.6}$$

satisfy, for $j = 2, 3$,

$$\begin{array}{rclcrcl} Z_j \cdot p_{1-j} & = & 0, & & Z_j \cdot p_j & = & q_j, \\ Z_j \cdot q_{1-j} & = & 0, & & Z_j \cdot q_j & = & 6p_j^2. \end{array} \tag{3.7}$$

Moreover,

$$4p_j^3 - q_j^2 = -z^2 Q \tag{3.8}$$

is independent of j. Hence, when restricted to the parametrized orbits of Z_j, p_j satisfies the Weierstrass differential equation $(d\xi/dt)^2 = 4\xi^3 - g$. Define the function $\phi : \mathbf{R} \times \mathbf{R}^* \to \mathbf{R}$ by

$$\phi(x,c) = \frac{\sqrt[3]{2}}{3}\int_0^x \frac{d\xi}{(\xi^2+c)^{2/3}}.$$

For $c \in \mathbf{R}^*$, let $k_c = \lim_{x\to\infty}\phi(x,c)$. Observe that $k_c < \infty$ and that since

$$\phi(x,\beta^2 c) = \frac{\sqrt[3]{2}}{3}\int_0^{x/\beta} \frac{d(\beta\xi)}{([\beta\xi]^2+\beta^2 c)^{2/3}} = \beta^{-1/3}\phi\left(\frac{x}{\beta},c\right),$$

$k_c = c^{-1/6}k_1$, $k_{-c} = c^{-1/6}k_{-1}$ (for $c > 0$). For $c \in \mathbf{R}^*$ fixed, the function $\phi(\cdot,c) : \mathbf{R} \to (-k_c,k_c)$ is odd and is a diffeomorphism but for the points x where $x^2 + c = 0$ (if any). Let $F : \mathbf{R}^3 \setminus \{g = 0\} \to \mathbf{R}^3$ be given by $F(x,y,z) = (\phi(q_0,g),\phi(q_1,g),g)$ and let (u,v,w) be the coordinates of the target space. The identities (3.7) and the relation (3.8), imply that the images of the vector fields Z_0 and Z_1 under F are given by the restriction to the image (of the domain where F is differentiable) of the vector fields $\partial/\partial u$, $\partial/\partial v$. We will now investigate the nature of the image. The mapping F can be decomposed as:

$$(x,y,z) \overset{F_1}{\mapsto} (q_0,q_1,g) \overset{F_2}{\mapsto} (\phi(q_0,g),\phi(q_1,g),g).$$

The images of Z_0 and Z_1 under F_1 are respectively given by the restriction to the image of the vector fields

$$\frac{3}{\sqrt[3]{2}}(q_0^2+g)^{2/3}\frac{\partial}{\partial q_0}, \quad \frac{3}{\sqrt[3]{2}}(q_1^2+g)^{2/3}\frac{\partial}{\partial q_1}. \tag{3.9}$$

In $\mathbf{RP}^2$, the conics

$$C_\tau = \{(\tau q_0 - q_1)/z = 0; \tau \in \mathbf{RP}^1,\ \tau^2 \neq 1\}$$

foliate the complement of $\{yz = 0\}$. If we set $\tau = \tan(\theta)$ then, for $\tau^2 \neq 1$, $C_\tau \subset \mathbf{RP}^2$ can be parametrized by its intersection with the line $x = \alpha y$ for $\alpha \in \mathbf{R}$:

$$\alpha \overset{\gamma}{\mapsto} \left[2\alpha[\sin(2\theta) + 1] : 2[\sin(2\theta) + 1] : (3\alpha^2 + 1)\cos(2\theta)\right].$$

Consider a lift $\widetilde{\gamma} : \mathbf{R} \to \mathbf{R}^3 \setminus \{0\}$ of γ lying in the level surface $\{g = g_0\}$. The image under F_1 is contained in the line $q_1 = \tau q_0$. Moreover, for $r^2 = (q_0^2 + q_1^2) \circ \widetilde{\gamma}$ we have

$$r^2(\theta, \alpha) = \frac{2g_0(3\alpha^2 + 1)}{\cos(2\theta)\alpha^3 - 3\alpha^2 + 3\cos(2\theta)\alpha - 1},$$

which is a strictly monotone function of α since

$$\frac{d(r^{-2})}{d\alpha} = \frac{3\cos(2\theta)(\alpha^2 - 1)^2}{2g_0(3\alpha^2 + 1)^2}.$$

In the same fashion, when $\{y = 0\}$ is parametrized by $[\beta : 0 : 1]$, we find $r^2(\beta) = 18\beta g_0/(4 - 9\beta)$ and hence the value $r^2 = -2g_0$ (corresponding to $\beta = \infty$) is unattained. Since, in restriction to $\{z > 0\}$, $q_0 + q_1 > 0$, the image of $\{g = g_0\} \cap \{z > 0\}$ under F_1 is given by

- $\{(q_0, q_1, g);\ q_0 + q_1 > 0, g = g_0\} \setminus \{(\sqrt{-g_0}, \sqrt{-g_0}, g_0)\}$ if $g_0 < 0$; by
- $\{(q_0, q_1, g);\ q_0 + q_1 > 0, g = g_0\}$ if $g_0 > 0$.

The images of the vector fields (3.9) under F_2 are, respectively, the vector fields $\partial/\partial u$ and $\partial/\partial v$. We thus have

Proposition 3.2. — *The solution of $\mathfrak{A}_0$ with initial condition (x_0, y_0, z_0) with $z_0 > 0$ is defined in an open subset of $\mathbf{R}^2$ which is, if $g(x_0, y_0, z_0) > 0$, the interior of a triangle; if $g(x_0, y_0, z_0) < 0$, the complement of the centroid within the interior of a triangle.*

These results are summarized in Figure 1.

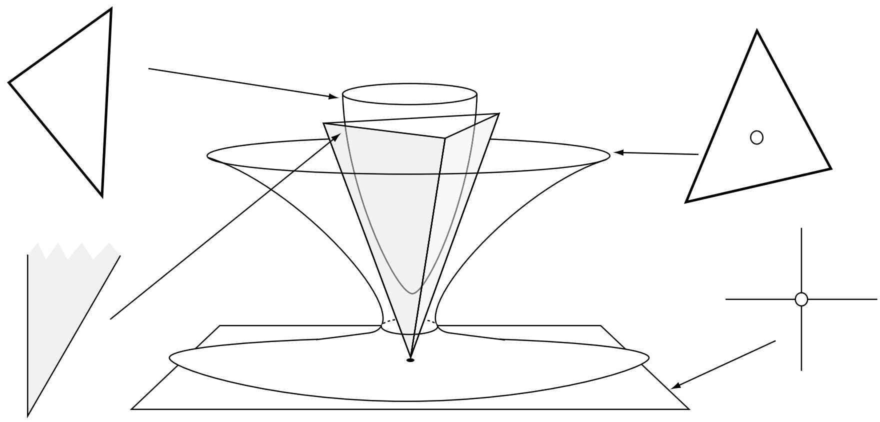

Figure 1. — Domains of definition of the solutions of $\mathfrak{A}_0$

3.2. Proof of Theorem B

Let X_1 and Y_1 be vector fields defined in a neighborhood of the origin of $\mathbf{R}^3$ satisfying the hypothesis of Theorem B and let $\mathfrak{A}_1$ be the maximal Lie algebra they generate.

3.2.1. On the holonomy of $\mathcal{F}_1$

We will blowup the origin of $\mathbf{R}^3$, $(\widetilde{\mathbf{R}}^3, \Delta) \to (\mathbf{R}^3, 0)$ in order to understand better the foliations. Consider the coordinates

$$(\zeta_1, \zeta_2, \zeta_3) = (x, y/x, z/x) \tag{3.10}$$

for the blow-up. Observe that, by Taylor's Theorem (in its integral remainder version), a C^r vector field in the origin of $\mathbf{R}^3$ with vanishing k-jet ($k \leqslant r-1$) becomes, after the blow-up, a C^{r-1} vector field of the form

$$\zeta_1^k \left[\zeta_1 f_1(\zeta_1,\zeta_2,\zeta_3)\frac{\partial}{\partial\zeta_1} + f_2(\zeta_1,\zeta_2,\zeta_3)\frac{\partial}{\partial\zeta_2} + f_3(\zeta_1,\zeta_2,\zeta_3)\frac{\partial}{\partial\zeta_3}\right], \tag{3.11}$$

where the functions f_i are C^{r-1-k}.

In this chart, Δ is given by $\{\zeta_1 = 0\}$. Let $\widetilde{X}_i$ and $\widetilde{Y}_i$ be the transforms of the vector fields X_i and Y_i. We have

$$\widetilde{X}_0 = \zeta_1\left[\zeta_1(\zeta_2^2+2\zeta_3-3)\frac{\partial}{\partial\zeta_1} - \zeta_2(3-2\zeta_3+\zeta_2^2)\frac{\partial}{\partial\zeta_2} - \zeta_3(4\zeta_3+\zeta_2^2-9)\frac{\partial}{\partial\zeta_3}\right],$$

$$\widetilde{Y}_0 = \zeta_1\left[2\zeta_1\zeta_2(\zeta_3-1)\frac{\partial}{\partial\zeta_1} - (3+\zeta_2^2-2\zeta_2^2\zeta_3)\frac{\partial}{\partial\zeta_2} + 2\zeta_3\zeta_2(\zeta_3-2)\frac{\partial}{\partial\zeta_3}\right].$$

The transform of the radial vector field E becomes $\widetilde{E} = \zeta_1\partial/\partial\zeta_1$. The foliation generated by $\zeta_1^{-1}\widetilde{X}_i$ and $\zeta_1^{-1}\widetilde{Y}_i$ extends to a foliation $\mathcal{F}_i$ in $\widetilde{\mathbf{R}}^3$. Since $X_1 - X_0$ is a C^3 vector field with vanishing second jet, from the observation preceding formula (3.11), $\widetilde{X}_1 - \widetilde{X}_0$ is a C^2 vector field of the form $\zeta_1^2[\zeta_1 g_1\partial/\partial\zeta_1 + g_2\partial/\partial\zeta_2 + g_3\partial/\partial\zeta_3]$. Hence, the restrictions of $\zeta_1^{-1}\widetilde{X}_0$ and $\zeta_1^{-1}\widetilde{X}_1$ to the exceptional divisor agree (the same happens for $\zeta_1^{-1}\widetilde{Y}_0$ and $\zeta_1^{-1}\widetilde{Y}_1$).

The singularities of $\mathcal{F}_0$ (and hence those of $\mathcal{F}_1$) within Δ are given by its intersection with the plane $\{z = 0\}$ and with the cone $\{Q = 0\}$ (the vector fields $\zeta_1^{-1}X_0$ and $\zeta_1^{-1}Y_0$ are not linearly independent along these curves). On the complement of these curves, $\mathcal{F}_i$ has two two-dimensional leaves. One of them, L_a, is bounded by $\{z = 0\}$ and by $\{Q = 0\}$ (it is topologically an annulus). The other, L_b, is bounded by $\{Q = 0\}$ and is simply connected. From our previous results, the holonomy of $\mathcal{F}_0$ along L_a is trivial.

PROPOSITION 3.3. — *The holonomy of $\mathcal{F}_1$ along L_a is trivial.*

Proof. — We will use Rebelo's *renormalization* principle [7]. On every two-dimensional orbit of $\mathcal{F}_i$ not contained in Δ we have the natural parametrization given by a solution of $\mathfrak{A}_i$. Through the solutions of $\mathfrak{A}_i$, these two-dimensional leaves are locally modeled in $\mathbf{R}^2$ with changes of coordinates in the group of translations of $\mathbf{R}^2$: in Thurston's terminology [12], the leaves have an $(\mathbf{R}^2, \mathbf{R}^2)$ or *translation* structure. Since the group of translations of $\mathbf{R}^2$ is naturally contained in the group $\mathsf{Sim}(\mathbf{R}^2)$ of similarities (homotheties and isometries) of $\mathbf{R}^2$, these leaves are also naturally endowed with a a $(\mathsf{Sim}(\mathbf{R}^2), \mathbf{R}^2)$ or *similarity* structure. The translation structure does not extend to the leaves of $\mathcal{F}_i$ contained in Δ, where the vector fields $\widetilde{X}_i$ and $\widetilde{Y}_i$ vanish. However, the similarity structure does. Let $\Pi : \widetilde{\mathbf{R}}^3 \to \Delta$ be the natural projection, whose fibers contain the orbits of $\widetilde{E}$. By the homogenous nature of $\mathfrak{A}_0$, two solutions of $\mathfrak{A}_0$ induce, through Π, parametrizations of L that differ by precomposition with an homothety and a translation: the similarity structure of the leaves is invariant by the homotheties of $\mathbf{R}^3$ and, in consequence, it extends to the two-dimensional orbits of $\mathcal{F}_0$ within Δ. By Proposition 3.2, the component L_a is, as a surface with a similarity structure, equivalent to the complement of the centroid in a triangle; the component L_b, the interior of a triangle.

We claim that the similarity structure of the leaves of $\mathcal{F}_1$ also extends continuously to L_a and L_b, inducing in them the very same similarity structure. Once again, let $h^c : \mathbf{R}^3 \to \mathbf{R}^3$ be the homothety given by $h^c(x) = cx$. Define X_c, Y_c and $\mathfrak{A}_c$ like in formula (2.2). Let $p \in \mathbf{R}^3$,

$p \notin \Delta$. Let $\phi_0 : D \to \mathbf{R}^3$ be a solution of $\mathfrak{A}_0$ with initial condition p for some disk $D \subset \mathbf{R}^2$. If D is small enough then, for small values of c, we have solutions $\phi_c : D \to \mathbf{R}^3$ of $\mathfrak{A}_c$ with the same initial condition. These solutions depend continuously on c. If $\phi_c : U \to \mathbf{R}^3$ is a solution of $\mathfrak{A}_c$ then the function $h^c \circ \phi_c(cs, ct)$ is a solution of $\mathfrak{A}_1$ through $h^c(p)$. We have: $\lim_{c\to 0} h^c \circ \phi_c(s,t) = \Pi \circ \phi_0(s,t)$. This implies the following: *If ψ_c is the solution of $\mathfrak{A}_1$ with initial condition $h^c(p)$ then, as $c \to 0$, $\psi_c(c^{-1}s, c^{-1}t)$ converges pointwise to* $\Pi \circ \phi_0(s,t)$. This proves our claim.

Let γ be a closed curve in L_a generating its fundamental group. Since, through the solutions of $\mathfrak{A}_0$, L_a is isometric, in this geometry, to the complement of the centroid in the interior of a triangle (the key point here being that it is an open *subset* of $\mathbf{R}^2$), the image of γ under the developing map of the similarity structure is a closed curve. In order to define the holonomy of $\mathcal{F}_1$ along γ, we can lift it to a neighboring leaf $L \notin \Delta$ of $\mathcal{F}_1$ by an isometry of similarity structures. The lift γ' of γ to L is such that the image of (a lift of) γ under the developing map of the similarity structure (which can be chosen to be the same as the developing map of the translation structure) is a closed curve. Hence, the integral of the natural $\mathbf{R}^2$-valued form in L vanishes along γ'. Since $\mathfrak{A}_1$ is maximal, γ' must be closed. This proves the Proposition. $\square$

Let $\mathcal{G}$ be foliation by curves transverse to Δ given by the field of directions induced by the radial vector field. The foliation $\mathcal{G}$ is transverse to $\mathcal{F}_0$ and $\mathcal{F}_1$ in L_a. Let ρ be a leaf of $\mathcal{G}$ intersecting L_a. Let Φ be a diffeomorphism unto its image that is the identity in restriction to Δ and ρ, that preserves $\mathcal{G}$, and that maps the leaves of $\mathcal{F}_0$ into the leaves of $\mathcal{F}_1$. Its germ at L_a is unique and it is well-defined since the holonomies of $\mathcal{F}_0$ and $\mathcal{F}_1$ along A are trivial. A similar construction can be done for L_b.

3.2.2. Deformations of the invariant plane

After blowing-up the origin, the invariant plane $\{z = 0\}$ of $\mathcal{F}_0$ intersects Δ along a circle ℓ. We will prove that there is an invariant surface for $\mathcal{F}_1$ intersecting Δ transversely along ℓ and that the conjugacy Φ can be extended to a neighborhood of ℓ.

We will now analyze $\mathcal{F}_i$ in a neighborhood of ℓ. Let $\theta = \arctan(y/x)$. It is a parameter for ℓ. Notice that along this curve, the vector fields $\zeta_1^{-1}\widetilde{X}_i$ and $\zeta_1^{-1}\widetilde{Y}_i$ have rank 1 and that the level surfaces of θ are planes that intersect ℓ transversely. By the Kupka Phenomenon, at every point of ℓ, $\mathcal{F}_1$ is generated by a vector field which is tangent to $d\theta$ and one which is transverse. Let Z_i be a vector field defined in a neighborhood of ℓ, tangent to $\mathcal{F}_i$, and transverse to $d\theta$ (such a vector field can be constructed, for example, by gluing the local ones by means of a partition of unity). Let T be the plane $\theta^{-1}(0) = \{\zeta_2 = 0\}$, $\mathcal{F}_i^T$ the foliation that $\mathcal{F}_i$ induces in T and $h_i : (T,p) \to (T,p)$ be the first return map of the flow of Z_i (when following θ positively). In a neighborhood of ℓ, $\mathcal{F}_i$ is the suspension of $\mathcal{F}_i^T$ by h_i.

The vector field $\zeta_1^{-1}\widetilde{X}_0$ is tangent to T and its restriction to the latter is given by $\zeta_1(2\zeta_3 - 3)\partial/\partial\zeta_1 - \zeta_3(4\zeta_3 - 9)\partial/\partial\zeta_3$. In particular, $\mathcal{F}_0^T$ is generated by this vector field. It is not difficult to see that $\mathcal{F}_1^T$ is generated by a vector field of the form $\zeta_1^{-1}\widetilde{X}_0 + \zeta_1(\zeta_1 f_1\partial/\partial\zeta_1 + f_3\partial/\partial\zeta_3)$ for some functions f_i. The linear part of this field at the origin is given by $-3\zeta_1\partial/\partial\zeta_1 + (f_3(0)\zeta_1 + 9\zeta_3)\partial/\partial\zeta_3$. Thus, by the Invariant Manifold Theorem for hyperbolic vector fields of saddle type, we have two invariant curves of $\mathcal{F}_i^T$. The first one is necessarily $T \cap \Delta$. The second one intersects Δ transversely (it corresponds to the invariant plane $\{z = 0\}$ for $i = 0$). These curves must be preserved by h_i and are associated to two invariant surfaces of $\mathcal{F}_i$ that intersect transversely along ℓ. The triviality of the holonomy of $\mathcal{F}_i$ along L_a implies that h_i not only preserves the foliation $\mathcal{F}_i$ but, in a neighborhood of the leaf $T \cap \Delta$, its square induces the identity map in the leaf space of $\mathcal{F}_i$ (a curve generating the fundamental group of A that is sufficiently close to ℓ intersects $T \cap \Delta$ at two points).

We have all the ingredients to extend Φ to a neighborhood of ℓ: the foliations that $\mathcal{F}_0$ and $\mathcal{F}_1$ induce in T are, by the Theorem of Grobman-Hartman, topologically equivalent at the origin and the maps h_i induce the same map at the level of the leaf spaces.

Let $H(x,y,z) = z^4/Q$. It is a homogenous function such that L_a is the subset of Δ where it is strictly positive and such that ℓ is the zero locus of H. Choose for Z_0 the oriented field of lines defined by the vector field $Z_0 = yX_0 + (z - 3x)Y_0$. It is transverse to the level sets of θ and, since $Z_0 \cdot H = 0$, it is tangent to the one-dimensional foliation given by the intersection of $\mathcal{F}_0$ and dH (whenever they are transverse). Let h_0 be the corresponding holonomy. Let

$$(\widehat{\zeta}_1, \widehat{\zeta}_2, \widehat{\zeta}_3)(\zeta_1, \zeta_2, \zeta_3) \tag{3.12}$$

be coordinates around 0 such that Z_0 is tangent to the vector field $\partial/\partial\widehat{\zeta}_2$ and such that $\widehat{\zeta}_i|_T = \zeta_i$. Let $U \subset T$ be a subset such that these coordinates are defined for all $(\zeta_1, \zeta_3) \in U$ and all $\zeta_2 \in (-\epsilon_3, \epsilon_3)$ for some $\epsilon_3 > 0$.

Let $\psi : (T, 0) \to (T, 0)$ be a germ of homeomorphism that maps $\mathcal{F}_1^T$ to $\mathcal{F}_0^T$. Suppose, without loss of generality, that it is the identity in restriction to Δ and choose a representative taking values in a subset of the form $\{|H|<\epsilon_2,|\zeta_1|<c_1\}$ for $\epsilon_2<\epsilon_3$. Let $R=H^{-1}(\epsilon_1)$ for some $\epsilon_1<\epsilon_2$. Let $R^T=R\cap T$ (its connected components are two arcs that intersect transversely $\Delta\cap T$). Modify ψ by post-composing it with a homeomorphism preserving $\mathcal{F}_1^T$ in such a way that it is the identity in restriction to R^T. This can be done by shearing along the orbits of $\mathcal{F}_1$ while preserving the arc length in each one of them.

We claim that the mapping Φ can be modified in such a way that, in a neighborhood of $R\cap\Delta$, $\theta\circ\Phi=\theta$ and $H\circ\Phi=H$, while still mapping the leaves of $\mathcal{F}_0$ into the leaves of $\mathcal{F}_1$. In fact, the foliations $\mathcal{F}_0$, dH and $d\theta$ have ℓ as an isolated component of their locus of non-transversality and thus, close to Δ but away from ℓ, θ and H are coordinates when restricted to a leaf of $\mathcal{F}_i$. For $a<b$, let $f_{a,b}:\mathbf{R}\to[0,1]$ be smooth function such that $f(s)=0$ if $s<a$ and $f(s)=1$ if $s>b$. Let $\Phi^\dagger$ be function that maps leaves of $\mathcal{F}_0$ to leaves of $\mathcal{F}_1$ and that is defined by the conditions $H\circ\Phi^\dagger=H\circ\Phi$ and

$$\theta\circ\Phi^\dagger=[1-f_{\epsilon_6,\epsilon_7}\circ H\circ\Phi]\theta+[f_{\epsilon_6,\epsilon_7}\circ H\circ\Phi]\theta\circ\Phi$$

for suitable ϵ_6 and ϵ_7, $\epsilon_7>\epsilon_6>\epsilon_3$. Notice that $\Phi^\dagger$ is the identity in restriction to Δ; that if $H(p)<\epsilon_4$, $\theta\circ\Phi^\dagger=\theta(p)$; and that, if $H(p)>\epsilon_5$, $\Phi^\dagger$ agrees with Φ. Modify Φ by declaring it equal to $\Phi^\dagger$ whenever Φ and $\Phi^\dagger$ do not agree. In an analogue way, we can define $\Phi^\ddagger$ in such a way that $\theta\circ\Phi^\ddagger=\theta\circ\Phi$ and

$$H\circ\Phi^\ddagger=[1-f_{\epsilon_4,\epsilon_5}\circ H\circ\Phi]H+[f_{\epsilon_4,\epsilon_5}\circ H\circ\Phi]H\circ\Phi,$$

for suitable ϵ_4 and ϵ_5, $\epsilon_i<\epsilon_{i+1}$. Once again, modify Φ by replacing it by $\Phi^\ddagger$ whenever they do not agree. In this way, in a neighborhood of $R\cap\Delta$, $\theta\circ\Phi=\theta$ and $H\circ\Phi=H$.

Modify ϕ by post-composing it by an $\mathcal{F}_1$-preserving homeomorphism that preserves R^T and such that, along R, $\Phi=\phi$. This can be done by permuting the leaves of $\mathcal{F}_1$ as required while preserving R^T and mapping one leaf to another preserving arc-length.

Modify Z_1 in such a way that, in a neighborhood of $\Delta\cap R$, it induces the unique field of directions tangent to dH and $\mathcal{F}_1$. Let h_1 be the corresponding holonomy. In the coordinates (3.12), the holonomy transformation of $\mathcal{F}_0$ induced by Z_0 from $\theta^{-1}(0)$ to $\theta^{-1}(\pi-\delta)$ is exactly h_0 (since the holonomy from $\theta^{-1}(\pi-\delta)$ to $\theta^{-1}(\pi)$ is the identity). Let $\eta:[-\delta,0]\times T\to T$ be a homotopy (through homeomorphisms) connecting h_0 (at $t=-\delta$) to $\phi^{-1}\circ h_1\circ\phi$ (at $t=0$) and fixing R^T. This can be done by interpolating on each leaf h_0 and $\phi^{-1}\circ h_1\circ\phi$ by arc length. Modify Z_0 in the interval $\theta\in[-\delta,0]$ such a way that the holonomy from $\theta^{-1}(-\delta)$ to $\theta^{-1}(t)$ equals $\eta(t,\cdot)$. After this modification we obtain a new h_0 that, by definition, is conjugate to h_1 via ϕ.

Finally, extend Φ to a neighborhood of ℓ by declaring it, beyond R, equal to ϕ on $T=\theta^{-1}(0)$ and by defining it on $\{\theta=c\}$ by following positively Z_0 to T, applying ϕ and then following Z_1 negatively to $\{\theta=c\}$.

3.2.3. Deformations of the cone over the smooth part of the deltoid

We will now consider the chart $\{z\neq 0\}$ of the blow-up in the coordinates $(\xi_1,\xi_2,\xi_3)=(x/z,y/z,z)$. The vector fields are now

$$\begin{aligned}
\widetilde{X}_0 &= \xi_3\left[(4\xi_1-9\xi_1^2+\xi_2^2)\frac{\partial}{\partial\xi_1}+6\xi_2(1-2\xi_1)\frac{\partial}{\partial\xi_2}-2\xi_3(1-3\xi_1)\frac{\partial}{\partial\xi_3}\right],\\
\widetilde{Y}_0 &= \xi_3\left[2\xi_2(1-2\xi_1)\frac{\partial}{\partial\xi_1}+3(\xi_1^2-\xi_2^2)\frac{\partial}{\partial\xi_2}+2\xi_2\xi_3\frac{\partial}{\partial\xi_3}\right].
\end{aligned}$$

If we let $\widetilde{Q}(\xi_1,\xi_2)=6\xi_2^2\xi_1^2-12\xi_2^2\xi_1+4\xi_2^2+9\xi_1^4+\xi_2^4-4\xi_1^3$, then $\xi_3^6\widetilde{Q}$ is now a primitive common first integral of $\widetilde{X}_0$ and $\widetilde{Y}_0$. The foliation $\mathcal{F}_0$ is given by the contraction of $\bigwedge_i d\xi_i$ by $\xi_3^{-1}X_0$ and $\xi_3^{-1}Y_0$ which, up to a constant factor, is the form $\alpha_0=\xi_3\,d\widetilde{Q}+6\widetilde{Q}\,d\xi_3$. Its singularities are the sets $\Sigma=\{\widetilde{Q}=0\}\cap\{\xi_3=0\}$ — the deltoid within the exceptional divisor — and $\{\widetilde{Q}=0\}\cap\{d\widetilde{Q}=0\}$ — the three one-dimensional orbits corresponding to the cusps —. In an analogue way, $\mathcal{F}_1$ is given by a C^1 form

$$\alpha_1=6\widetilde{Q}d\xi_3+\xi_3 d\widetilde{Q}+\xi_3^2(g_1d\xi_1+g_2d\xi_2)+\xi_3g_3d\xi_3, \tag{3.13}$$

for some continuous functions g_i.

The deltoid Σ is contained in the singular locus of $\mathcal{F}_1$. Parametrize it by (3.2). Let σ_1, σ_2 and σ_3 be the arcs given, respectively, by $\kappa\in(0,1)$, $\kappa\in(1,-1)$ and $\kappa\in(-1,0)$. The restriction of the vector field $\xi_3^{-1}Y_1$ to Σ equals that of $\xi_3^{-1}Y_0$ and is $2\kappa^2(\kappa^2-1)(3\kappa^2+1)^{-1}\partial/\partial\kappa$, which does not vanish along the smooth arcs σ_i. By the Kupka Phenomenon, the foliation $\mathcal{F}_1$ along σ_2 may be studied through the foliation it induces, say, in the plane $P=\{\xi_2=0\}$, which intersects σ_2 transversely at the point $p=(4/9,0,0)$, corresponding to $\kappa=\infty$. The vector field $\xi_3^{-1}\widetilde{X}_0$ is tangent to P and in restriction to it has an equilibrium point at p, where its linear part is, up to a constant factor, $L=6\xi_1\partial/\partial\xi_1-\xi_3\partial/\partial\xi_3$. Arguing like before, the restriction of $\mathcal{F}_1$ to P has a singularity at p and is given by a C^1 vector field with linear part $L+c\xi_3\partial/\partial\xi_1$ (in particular, the foliations induced by $\mathcal{F}_1$ and $\mathcal{F}_0$ at (P,p) are topologically conjugate). By the Kupka Phenomenon and the Invariant Manifold Theorem there exists a smooth surface S_2, invariant by $\mathcal{F}_1$, intersecting transversely the exceptional divisor along σ_2, more or less in the same way the smooth part of the cone $\{\widetilde{Q}=0\}$ does for $\mathcal{F}_0$ (in particular, $\mathcal{F}_1$ and $\mathcal{F}_0$ are topologically conjugate in a neighborhood of any compact subset of σ_2). Thanks to the symmetries (3.4), the foliation $\mathcal{F}_1$ along the other arcs has an analogue description: there exist surfaces S_i intersecting Δ along σ_i etc.

3.2.4. Deformations of the cone over the cusps of the deltoid

We will study the foliation in a neighborhood of the point p where $(\xi_1,\xi_2,\xi_3)=(0,0,0)$, corresponding to the intersection of a one-dimensional orbit of $\mathfrak{A}_0$ with Δ. We will consider the family of two-dimensional foliations $\mathcal{F}_1^\epsilon$ given by the foliation (by curves, with singularities) that $\mathcal{F}_1$ induces in the plane $\{\xi_3=\epsilon\}$. A form giving this foliation is the C^3 form (varying continuously with ϵ)

$$\alpha_1^\epsilon = d\widetilde{Q} + \epsilon[g_1(\xi_1,\xi_2,\epsilon)d\xi_1 + g_2(\xi_1,\xi_2,\epsilon)d\xi_2],$$

for $$d\widetilde{Q} = 3(\xi_1\xi_2^2+3\xi_1^3-\xi_1^2-\xi_2^2)d\xi_1+\xi_2(3\xi_1^2-6\xi_1+2+\xi_2^2)d\xi_2.$$

Let $B(\epsilon_0,\delta_0,\eta_0)$ be the box — diffeomorphic to a cube — delimited

- at the top, by the plane $\{\xi_3=\epsilon_0\}$, at the bottom by Δ;
- at the left and right, by the planes $\{\xi_2=\eta_0\}$ and $\{\xi_2=-\eta_0\}$;
- at the front (resp. back), by the surface that contains the line ρ_- (resp. ρ_+) parametrized by $s\mapsto(-\delta_0,0,s)$ (resp. $s\mapsto(\delta_0,0,s)$) and whose intersection with the plane $\{\xi_3=\epsilon\}$ is an integral curve of $\mathcal{F}_1^\epsilon$.

The faces of the box other than Δ will be denoted by $\mathsf{F}_{\mathsf{top}}$, $\mathsf{F}_{\mathsf{left}}$, $\mathsf{F}_{\mathsf{right}}$, $\mathsf{F}_{\mathsf{front}}$ and $\mathsf{F}_{\mathsf{back}}$. The box is well-defined if ϵ_0, δ_0 and η_0 are small enough, for $d\widetilde{Q}\wedge d\xi_2$ has an isolated singularity at p and, if $\widetilde{X}_1=\widetilde{X}_0+\xi_3^2(f_1\partial/\partial\xi_1+f_2\partial/\partial\xi_2+\xi_3f_3\partial/\partial\xi_3)$, then $\xi_3^{-1}\widetilde{X}_1\cdot\xi_3=2\xi_3(3\xi_1-1)+\xi_3^3f_3$ which, after dividing by ξ_3, does not vanish at p. Up to shrinking the box, we may suppose that, in restriction to B, $d\xi_3(\widetilde{X}_1)=0$ only in Δ.

The surface S_1 constructed at the end of §3.2.3 intersects $\mathsf{F}_{\mathsf{left}}$ transversely. Up to shrinking the box, we can suppose that this intersection goes from $\mathsf{F}_{\mathsf{left}}\cap\Delta$ to $\mathsf{F}_{\mathsf{left}}\cap\mathsf{F}_{\mathsf{top}}$. Except for the curve induced by S_1, all the curves of the foliation that $\mathcal{F}_1$ induces in $\mathsf{F}_{\mathsf{left}}$ go from $\mathsf{F}_{\mathsf{left}}\cap\mathsf{F}_{\mathsf{top}}$ to either $\mathsf{F}_{\mathsf{left}}\cap\mathsf{F}_{\mathsf{front}}$ or $\mathsf{F}_{\mathsf{left}}\cap\mathsf{F}_{\mathsf{back}}$. We have an analogue situation in $\mathsf{F}_{\mathsf{right}}$ with respect to the surface S_3. On $\mathsf{F}_{\mathsf{front}}$ and $\mathsf{F}_{\mathsf{back}}$, by construction, $\mathcal{F}_1$ induces the same foliation as $d\xi_3$. Only two-dimensional orbits of $\mathcal{F}_1$ intersect the interiors of $\mathsf{F}_{\mathsf{left}}$, $\mathsf{F}_{\mathsf{right}}$, $\mathsf{F}_{\mathsf{front}}$ and $\mathsf{F}_{\mathsf{back}}$.

The linear part of $\frac{1}{2}\xi_3^{-1}X_0$ at p is $2\xi_1\partial/\partial\xi_1 + 3\xi_2\partial/\partial\xi_2 - \xi_3\partial/\partial\xi_3$. The linear part of $\frac{1}{2}\xi_3^{-1}X_1$ at p is that of $\frac{1}{2}\xi_3^{-1}X_0$ plus $\xi_3(c_1\partial/\partial\xi_1 + c_2\partial/\partial\xi_2)$ for some constants c_i. By the Invariant Manifold Theorem there exists a unique C^1 curve ω containing p, transverse to Δ and invariant by $\widetilde{X}_1$. Since $\widetilde{Y}_1$ commutes with $\widetilde{X}_1$, its flow preserves the foliation induced by $\widetilde{X}_1$ and preserves thus the curve ω. We conclude that ω is a one-dimensional leaf of $\mathcal{F}_1$. We can suppose, up to shrinking the box B, that ω is transverse to $\mathsf{F}_{\mathsf{top}}$ and does not intersect the other faces.

For small ϵ, the intersection of B with the plane $\{\xi_3 = \epsilon\}$ is a rectangle B^ϵ bounded by two integral curves of $\mathcal{F}_1^\epsilon$ (the intersections with $\mathsf{F}_{\mathsf{front}}$ and $\mathsf{F}_{\mathsf{back}}$) and by two lines (the intersections of $\mathsf{F}_{\mathsf{left}}$ and $\mathsf{F}_{\mathsf{right}}$ with $\{\xi_3 = \epsilon\}$, denoted by $\mathsf{F}^\epsilon_{\mathsf{left}}$ and $\mathsf{F}^\epsilon_{\mathsf{right}}$) which are transverse to $\mathcal{F}_1^\epsilon$. In the interior of B^ϵ, $\mathcal{F}_1^\epsilon$ has a singularity corresponding to its intersection with ω. A partial holonomy relation is induced by $\mathcal{F}_1^\epsilon$ between $\mathsf{F}^\epsilon_{\mathsf{left}}$ and $\mathsf{F}^\epsilon_{\mathsf{right}}$. By our description of the foliation induced in $\mathsf{F}_{\mathsf{left}}$ and $\mathsf{F}_{\mathsf{right}}$, the holonomy relation identifies in a one-to-one way the complement of $S_1 \cap \mathsf{F}_{\mathsf{left}}$ in $\mathsf{F}_{\mathsf{left}}$ with the complement of $S_3 \cap \mathsf{F}_{\mathsf{right}}$ in $\mathsf{F}_{\mathsf{right}}$.

The situation is portrayed in Figure 2.

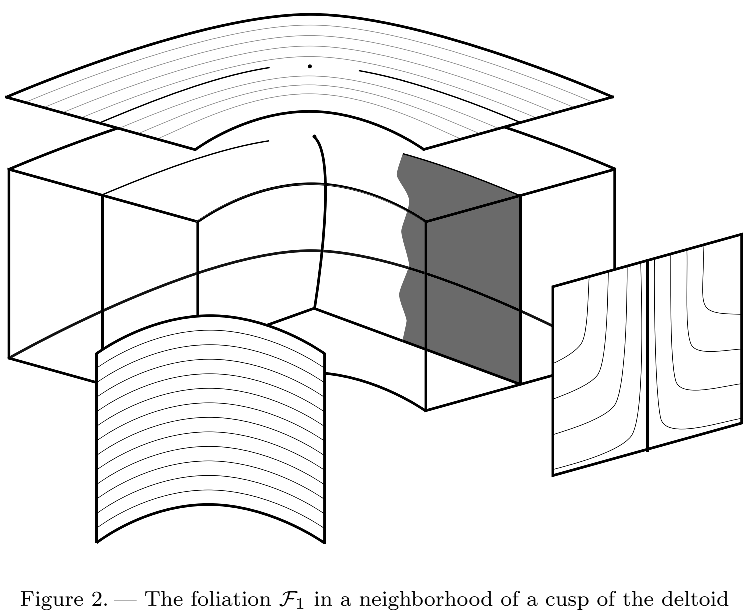

Figure 2. — The foliation $\mathcal{F}_1$ in a neighborhood of a cusp of the deltoid

Let q be a point in the interior of B^ϵ such that the orbit of $\mathcal{F}_1^\epsilon$ through q does not intersect $\mathsf{F}^\epsilon_{\mathsf{left}}$ and $\mathsf{F}^\epsilon_{\mathsf{right}}$. Let γ_q be the orbit of $\widetilde{X}_1$ through q parametrized by ϵ. Follow $\gamma_q(\epsilon)$ towards Δ. It cannot leave B, since the leaves of $\mathcal{F}_1$ that intersect $\mathsf{F}_{\mathsf{left}}$, $\mathsf{F}_{\mathsf{right}}$, $\mathsf{F}_{\mathsf{front}}$ and $\mathsf{F}_{\mathsf{back}}$ intersect either $\mathsf{F}^\epsilon_{\mathsf{left}}$ or $\mathsf{F}^\epsilon_{\mathsf{right}}$. This remains true if we shrink B: we must conclude that $\gamma_q(\epsilon)$ tends to p as ϵ tends to 0 and thus $\gamma_q = \omega$. In this way, within B_ϵ, the curves of $\mathcal{F}_1^\epsilon$ corresponding to S_1 and S_3 meet at the singularity induced by ω. Besides the orbits corresponding to S_1 and S_3, and the singularity induced by ω, all the orbits of $\mathcal{F}_1^\epsilon$ go from $\mathsf{F}^\epsilon_{\mathsf{left}}$ to $\mathsf{F}^\epsilon_{\mathsf{right}}$.

As a consequence, $\mathcal{F}_0$ and $\mathcal{F}_1$ are topologically equivalent in a neighborhood of p (without resorting to the maximality assumption concerning $\mathcal{F}_1$). This, together with our previous results (equivalence of $\mathcal{F}_1$ and $\mathcal{F}_0$ in a neighborhood of ℓ and of the arcs σ_i), finishes the Proof of Theorem B.

As a by-product of the proof, we obtain:

Corollary 3.4. — *Let (X_1, Y_1) and (X_2, Y_2) be two pairs of C^3 commuting vector fields defined in a neighborhood of the origin of $\mathbf{R}^3$ whose 2-jet at the origin is given by the vector fields (1.2)–(1.3). For each pair there is a C^2 surface tangent to $\{z = 0\}$ at the origin and invariant by the corresponding foliation. The corresponding foliations are topologically conjugate if and only if the holonomies around these surfaces are.*

3.3. Other maximal algebras in $\mathbf{R}^3$

We will hereby present other examples of maximal commutative Lie algebras of quadratic homogenous vector fields in $\mathbf{R}^3$. Apart from the one we already studied, they are the only interesting examples we know of. We ignore if they present some kind of rigidity.

Example 3.5. — The vector space underlying the Lie algebra generated by (1.2)–(1.3) can be characterized in the following way: It is formed by those quadratic homogenous vector fields that have in the sextic polynomial g a first integral. This first integral is the product of the equation of the deltoid and the square of its unique bitangent. Other real forms of the deltoid will yield other commutative Lie algebras.

From a complex viewpoint, there is, up to projective equivalence, a unique plane quartic having three cusps. It is rational and has only one bitangent. Its dual is the unique rational cubic having one node (arising from the bitangent) and has three inflection points coming from the cusps.

Following [9], there are two real forms of this cubic, according to the real nature of the singular point:

- if the node is a conjugate point, the rational cubic is the dual of the deltoid;
- if the node is real, the rational cubic is the dual of the cardioid.

The present example will come from the second case. The cardioid may be written, in homogenous coordinates, as the zero locus of the polynomial

$$Q = 3x^2y^2 - 18xyz^2 - 6x^2yz + 6xy^2z - 9x^2z^2 - 9y^2z^2 - 4x^3z + 4y^3z,$$

its bitangent being the line $\{z = 0\}$. The vector space of quadratic vector fields having in z^2Q a first integral is spanned by the vector fields

$$(x^2 + 2yz + 3xz)\frac{\partial}{\partial x} - xz\frac{\partial}{\partial y} - z(x+z)\frac{\partial}{\partial z},$$

$$yz\frac{\partial}{\partial x} + (y^2 - 2xz - 3yz)\frac{\partial}{\partial y} + z(z-y)\frac{\partial}{\partial z},$$

which commute. This Lie algebra is maximal by the results in [3].

Example 3.6. — Consider in $\mathbf{R}^3$ the commuting vector fields

$$X = y\left[x\frac{\partial}{\partial x} + (x-z)\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}\right], Y = x\left[(y-z)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}\right].$$

They are linearly independent in the complement of $\{xyz = 0\}$ and have the common first integral $Q = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. The surfaces $\{Q = -4\}$, $\{Q = 4\}$ and $\{Q = 0\}$ can be respectively parametrized by

$$\begin{aligned}
(u,v) &\mapsto \left(\frac{\sin(u)}{\sin(v)\sin(u+v)}, \frac{\sin(v)}{\sin(u)\sin(u+v)}, \frac{\sin(u+v)}{\sin(u)\sin(v)}\right),\\
(u,v) &\mapsto \left(\frac{\sinh(u)}{\sinh(v)\sinh(u+v)}, \frac{\sinh(v)}{\sinh(u)\sinh(u+v)}, \frac{\sinh(u+v)}{\sinh(u)\sinh(v)}\right),\\
(u,v) &\mapsto \left(\frac{u}{v(u+v)}, \frac{v}{u(u+v)}, \frac{u+v}{uv}\right).
\end{aligned}$$

In all cases, the parametrization maps the vector fields $\partial/\partial u$ and $\partial/\partial v$ to the restriction to the corresponding surface of the vector fields X and Y. This algebra is thus maximal.

Example 3.7. — Another pair of quadratic commuting vector fields in $\mathbf{R}^3$ is given by

$$(x^2-y^2-z^2)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y}+2xz\frac{\partial}{\partial z},$$

$$2xy\frac{\partial}{\partial x}-(x^2-y^2+z^2)\frac{\partial}{\partial y}+2yz\frac{\partial}{\partial z}.$$

Through stereographic projection, $\mathbf{R}^3$ acts in S^3. Restricting this action to $\mathbf{R}^2$ yields, in a neighborhood of the point at infinity, the above pair. It is naturally maximal.

4. An example in $\mathbf{R}^4$

We will now investigate the perturbations of the algebra generated by the commuting vector fields (1.4) and (1.5) in $\mathbf{R}^4$, in order to prove Theorem C. If we set $u = x_1 + \sqrt{-1}\,x_2$ and $v = x_3 + \sqrt{-1}\,x_4$, these vector fields are the real and imaginary parts of the semicomplete holomorphic vector field

$$Z = u^2\frac{\partial}{\partial u}+v^2\frac{\partial}{\partial v}$$

in $\mathbf{C}^2$. Let us briefly describe the complex geometry of this vector field. Semicompleteness of the latter is a consequence of the fact that the vector field extends holomorphically when we compactify $\mathbf{C}\times\mathbf{C}$ into $\mathbf{CP}^1\times\mathbf{CP}^1$. The solution with initial condition (u_0, v_0) is

$$\tau \mapsto \left(\frac{u_0}{1-\tau u_0}, \frac{v_0}{1-\tau v_0}\right). \tag{4.1}$$

In the complement of the cone $\{uv(u-v)=0\}$, the vector field is transverse to the foliation by complex lines $\mathcal{G}$ given by the level surfaces of the meromorphic function $G(u,v) = v/u$. The vector field has the meromorphic first integral

$$\frac{uv}{u-v}. \tag{4.2}$$

The solutions are defined in the complement of two points in $\mathbf{C}$. As we approach these points, a solution goes to infinity but the function G evaluated in the solution tends either to 0 or ∞. As the complex time goes to infinity, the solution converges to the origin of $\mathbf{C}^2$ and G tends to 1. The orbits of Z define a global holonomy function between the level curves of G. Since the restriction of the first integral (4.2) to the line $v = \alpha u$ is $\alpha u/(1-\alpha)$, this holonomy is trivial.

The proof of Theorem C follows the same strategy used in §3.2 to prove part of Theorem B.

In the coordinates $\{x_i\}$ of $\mathbf{R}^4$, the foliation $\mathcal{G}$ is generated by the vector fields

$$E=\sum_{i=1}^{4} x_i\frac{\partial}{\partial x_i},\ R=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}+x_4\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_4}.$$

From (4.2), we obtain the homogenous (of degree 0) function

$$J=\frac{(x_1^2+x_2^2)x_3-(x_3^2+x_4^2)x_1}{(x_1^2+x_2^2)x_4-(x_3^2+x_4^2)x_2}$$

which is a common first integral of X_0 and Y_0. Let $\widetilde{\mathbf{R}}^4$ be the blow-up of the origin of $\mathbf{R}^4$ and denote the exceptional divisor by Δ. Let $\ell_z = G^{-1}(z)\cap\Delta$. In the coordinates $y_1 = x_1$ and $y_i = x_i/x_1$ for $i > 1$, the vector fields become

$$\widetilde{X}_0 = y_1\Big[y_1(1-y_2^2)\frac{\partial}{\partial y_1}+y_2(y_2^2+1)\frac{\partial}{\partial y_2}+ \\ +(y_3^2-y_4^2-y_3+y_3y_2^2)\frac{\partial}{\partial y_3}+y_4(2y_3-1+y_2^2)\frac{\partial}{\partial y_4}\Big],$$

$$\widetilde{Y}_0 = y_1\Big[-2y_1y_2\frac{\partial}{\partial y_1}+(y_2^2+1)\frac{\partial}{\partial y_2}+2y_3(y_2-y_4)\frac{\partial}{\partial y_3} \\ +(y_3^2-y_4^2+2y_4y_2)\frac{\partial}{\partial y_4}\Big].$$

For $i=0,1$, let $\mathcal{F}_i$ be the foliation in $\widetilde{\mathbf{R}}^4$ generated in this chart by $y_1^{-1}\widetilde{X}_i$ and $y_1^{-1}\widetilde{Y}_i$. The foliation $\mathcal{F}_0$ has no singularities away from Δ. In restriction to Δ, $\mathcal{F}_0$ has no singularities away from the circles ℓ_0, ℓ_1 and ℓ_∞. The vector fields $y_1^{-1}\widetilde{X}_0$ and $y_1^{-1}\widetilde{Y}_0$ do not vanish simultaneously along the lines ℓ_0 — parametrized in this chart by $s\mapsto(0,s,0,0)$ — and ℓ_1 — parametrized in this chart by $s\mapsto(0,s,1,s)$ —. In restriction to Δ, $\mathcal{F}_1$ and $\mathcal{F}_0$ are identical.

In restriction to $\Delta\approx\mathbf{RP}^3$, the level surfaces of J form a pencil of cubic surfaces whose base locus is given by ℓ_0, ℓ_1 and ℓ_∞. Each two-dimensional leaf of $\mathcal{F}_0$ is the complement, within one of these level surfaces, of the lines of the base locus of $\widetilde{J}$ — and is homeomorphic to a thrice punctured sphere —. The vector field R becomes

$$\widetilde{R}=y_1y_2\frac{\partial}{\partial y_1}-(1+y_2^2)\frac{\partial}{\partial y_2}+(y_4-y_3y_2)\frac{\partial}{\partial y_3}-(y_3+y_4y_2)\frac{\partial}{\partial y_4}.$$

Its solutions with initial condition in Δ have period π and correspond to the intersection of the leaves of $\mathcal{G}$ with Δ. Since $\widetilde{R} \cdot J = J^2 + 1$ (and $\tan'(z) = \tan(z)^2 + 1$), all the surfaces of the pencil intersect once, transversely, each one of these circles.

The solution (4.1) gives L the similarity structure of the complement of two points in $\mathbf{R}^2$ (in particular, of an open *subset* of $\mathbf{R}^2$). By following an argument completely analogous to that of Proposition 3.3, we conclude that for every two-dimensional leaf $L \subset \Delta$ of $\mathcal{F}_1$, the holonomy of $\mathcal{F}_1$ along L is trivial.

We have a projection $\widetilde{G} : \widetilde{\mathbf{R}}^4 \to \mathbf{CP}^1$ realizing the leaf space of $\mathcal{G}$. The values $\{0, 1, \infty\} \subset \mathbf{CP}^1$ correspond to three planes — intersecting Δ along ℓ_0, ℓ_1 and ℓ_∞ — in the complement of which $\mathcal{F}_0$ and $\mathcal{G}$ are transversal. Let $\rho = G^{-1}(1)$. Define Φ as the identity in an open subset of ρ containing ℓ_{-1}. Extend Φ by holonomy: $\Phi(p)$ is the unique point in $G^{-1}(G(p))$ such that the leaf of $\mathcal{F}_1$ through $\Phi(p)$ intersects ρ at the point that the orbit of $\mathcal{F}_0$ through p intersects ρ.

In order to extend this identification to neighborhoods of ℓ_0, ℓ_1 and ℓ_∞, we need to study $\mathcal{F}_1$ in a neighborhood of these curves. We will do it for ℓ_0 and ℓ_1. By the symmetry of the original system, the study of $\mathcal{F}_1$ in a neighborhood of ℓ_∞ is completely analogous to that of ℓ_0.

Let $\theta = \arctan(x_2/x_1) = \arctan(y_2)$. Let $T = \theta^{-1}(0)$. It its transverse to both ℓ_0 and ℓ_1. It intersects the first at $p_0 = (0, 0, 0, 0)$ and the latter at $p_1 = (0, 0, 1, 0)$. Let Z_i^j be an oriented field of lines tangent to $\mathcal{F}_i$ defined in a neighborhood of ℓ_j and transverse to $d\theta$. Let $h_i^j : (T, p_j) \to (T, p_j)$ be first return map induced by Z_i^j. It preserves the foliation induced by $\mathcal{F}_i$ in a neighborhood of p_j within T.

The vector field $y_1^{-1}\widetilde{X}_0$ is tangent to T. The linear part of its restriction to T at p_j is $y_1\partial/\partial y_1 - (-1)^j y_3\partial/\partial y_3 - (-1)^j y_4\partial/\partial y_4$. The foliation by curves induced by $\mathcal{F}_1$ in a neighborhood of p_j in T is induced by a vector field whose linear part is the latter plus a vector field of the form $y_1(a_j\partial/\partial y_3 + b_j\partial/\partial y_4)$.

The germs of foliations that $\mathcal{F}_0$ and $\mathcal{F}_1$ induce in (T, p_j) are, by the Grobman-Hartman Theorem, topologically conjugate. The maps h_0^j and h_1^j induce the same map at the level of the leaf spaces since the holonomy is trivial. As we did in §3.2.2, this allows us to extend Φ to a neighborhood of ℓ_j. This finishes the proof of Theorem C.

Remark 4.1. — The holomorphic vector field studied in this section belongs to the family of semi-complete vector fields $x^2\partial/\partial x + y(y+nx)\partial/\partial x$ (with $n \in \mathbf{Z}$) appearing in the work of Ghys and Rebelo [2] and accounting for almost every quadratic homogenous semi-complete vector field in $\mathbf{C}^2$. All these vector fields share essentially the same properties and analogue arguments to the ones given here can be given to extend our rigidity results to every vector field in this family.

Bibliography

[1] Dumortier (F.) and Roussarie (R.). — Smooth linearization of germs of $\mathbf{R}^2$-actions and holomorphic vector fields. Ann. Inst. Fourier (Grenoble), Vol. 30 no. 1, p. 31-64 (1980).

[2] Ghys (E.) and Rebelo (J.-C.). — Singularités des flots holomorphes. II. Ann. Inst. Fourier (Grenoble), Vol. 47 no. 4, p. 1117-1174 (1997).

[3] Guillot (A.). — Sur les exemples de Lins Neto de feuilletages algébriques. C. R. Math. Acad. Sci. Paris, Vol. 334 no. 9, p. 747-750 (2002).

[4] Palais (R. S.). — A global formulation of the Lie theory of transformation groups. Mem. Amer. Math. Soc., 22 (1957).

[5] Palis (J.) and de Melo (W.). — Introdução aos sistemas dinâmicos, volume 6 of Projeto Euclides. Instituto de Matemática Pura e Aplicada, Rio de Janeiro (1978).

[6] Rebelo (J. C.). — Singularités des flots holomorphes. Ann. Inst. Fourier (Grenoble), Vol. 46 no. 2, p. 411-428 (1996).

[7] Rebelo (J. C.). — Réalisation de germes de feuilletages holomorphes par des champs semi-complets en dimension 2. Ann. Fac. Sci. Toulouse Math. (6), Vol. 9 no. 4, p. 735-763 (2000).

[8] Reis (H.). — Equivalence and semi-completude of foliations. Nonlinear Anal., Vol. 64 no. 8, p. 1654-1665 (2006).

[9] Salmon (G.). — A treatise on the higher plane curves. Hodges and Smith, Dublin (1852).

[10] Spivak (M.). — A comprehensive introduction to differential geometry. Vol. IV. Publish or Perish Inc., Wilmington, Del., second edition (1979).

[11] Sternberg (S.). — Local contractions and a theorem of Poincaré. Amer. J. Math., Vol. 79: p. 809-824 (1957).

[12] Thurston (W. P.). — Three-dimensional geometry and topology. Vol. 1. Princeton University Press, Princeton, NJ (1997).