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Almost sure Weyl asymptotics for non-self-adjoint elliptic operators on compact manifolds

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ABSTRACT. — In this paper, we consider elliptic differential operators on compact manifolds with a random perturbation in the 0th order term and show under fairly weak additional assumptions that the large eigenvalues almost surely distribute according to the Weyl law, well-known in the self-adjoint case.

RÉSUMÉ. — Dans ce travail, nos considérons des opérateurs différentiels elliptiques sur des variétés compactes avec une perturbation aléatoire dans le terme d'orde 0. Sous des hypothèses supplémentaires assez faibles, nous montrons que les grandes valeurs propres se distribuent selon la loi de Weyl, bien connue dans le cas auto-adjoint.

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1. Introduction

This work is a continuation of a series of works concerning the asymptotic distribution of eigenvalues for non-self-adjoint (pseudo-)differential operators with random perturbations. Since the works of L.N. Trefethen [11], E.B. Davies [2], M. Zworski [12] and many others (see for instance [5] for further references) we know that the resolvents of such operators tend to have very large norms when the spectral parameter is in the range of the symbol, and consequently, the eigenvalues are unstable under small perturbations of the operator. It is therefore quite natural to study the effect of random perturbations. Mildred Hager [5] studied quite general classes of non-self-adjoint h-pseudodifferential operators on the real line with a suitable random potential added, and she showed that the eigenvalues distribute according to the natural Weyl law with a probability very close to 1 in the semi-classical limit ($h \to 0$). Due to the method, this result was restricted to the interior of the range of the leading symbol p of the operator and with a non-vanishing assumption on the Poisson bracket $\{p, \overline{p}\}$.

In [6] the results were generalized to higher dimension and the boundary of the range of p could be included, but the perturbations where no more multiplicative. In [9, 10] further improvements of the method were introduced and the case of multiplicative perturbations was handled in all dimensions.

W. Bordeaux Montrieux [1] studied elliptic systems of differential operators on S^1 with random perturbations of the coefficients, and under some additional assumptions, he showed that the large eigenvalues obey the Weyl law *almost surely*. His analysis was based on a reduction to the semi-classical case (using essentially the Borel-Cantelli lemma), where he could use and extend the methods of Hager [5].

The purpose of the present work is to extend the results of [1] to the case of elliptic operators on compact manifolds by replacing the one dimensional semi-classical techniques by the more recent result of [10]. For simplicity, we treat only the scalar case and the random perturbation is a potential.

Let X be a smooth compact manifold of dimension n. Let P^0 be an elliptic differential operator on X of order $m \geqslant 2$ with smooth coefficients and with principal symbol $p(x, \xi)$. In local coordinates we get, using standard multi-index notation,

$$P^0 = \sum_{|\alpha| \leqslant m} a_\alpha^0(x) D^\alpha, \quad p(x, \xi) = \sum_{|\alpha| = m} a_\alpha^0(x) \xi^\alpha. \tag{1.1}$$

Recall that the ellipticity of P^0 means that $p(x, \xi) \neq 0$ for $\xi \neq 0$. We assume that

$$p(T^*X) \neq \mathbf{C}. \tag{1.2}$$

Fix a strictly positive smooth density of integration dx on X, so that the L^2 norm $\| \cdot \|$ and inner product $(\cdot | \cdot)$ are unambiguously defined. Let $\Gamma : L^2(X) \to L^2(X)$ be the antilinear operator of complex conjugation, given by $\Gamma u = \overline{u}$. We need the symmetry assumption

$$(P^0)^* = \Gamma P^0 \Gamma, \tag{1.3}$$

where $(P^0)^*$ is the formal complex adjoint of P^0. As in [10] we observe that the property (1.3) implies that

$$p(x, -\xi) = p(x, \xi), \tag{1.4}$$

and conversely, if (1.4) holds, then the operator $\frac{1}{2}(P^0 + \Gamma(P^0)^*\Gamma)$ has the same principal symbol p and satisfies (1.3).

Let $\widetilde{R}$ be an elliptic second order differential operator on X with smooth coefficients, which is self-adjoint and strictly positive. Let $\epsilon_0, \epsilon_1, \ldots$ be an orthonormal basis of eigenfunctions of $\widetilde{R}$ so that

$$\widetilde{R}\epsilon_j = (\mu_j^0)^2 \epsilon_j, \quad 0 < \mu_0^0 < \mu_1^0 \leqslant \mu_2^0 \leqslant \ldots \tag{1.5}$$

Our randomly perturbed operator is

$$P_\omega^0 = P^0 + q_\omega^0(x), \tag{1.6}$$

where ω is the random parameter and

$$q_\omega^0(x) = \sum_0^\infty \alpha_j^0(\omega) \epsilon_j. \tag{1.7}$$

Here we assume that $\alpha_j^0(\omega)$ are independent complex Gaussian random variables of variance σ_j^2 and mean value 0:

$$\alpha_j^0 \sim \mathcal{N}(0, \sigma_j^2), \tag{1.8}$$

where

$$(\mu_j^0)^{-\rho} e^{-(\mu_j^0)^{\frac{\beta}{M+1}}} \lesssim \sigma_j \lesssim (\mu_j^0)^{-\rho}, \tag{1.9}$$

$$M = \frac{3n - \frac{1}{2}}{s - \frac{n}{2} - \epsilon}, \; 0 \leqslant \beta < \frac{1}{2}, \; \rho > n, \tag{1.10}$$

where s, ρ, ϵ are fixed constants such that

$$\frac{n}{2} < s < \rho - \frac{n}{2}, \; 0 < \epsilon < s - \frac{n}{2}.$$

Let $H^s(X)$ be the standard Sobolev space of order s. As will follow from considerations below, we have $q_\omega^0 \in H^s(X)$ almost surely since $s < \rho - \frac{n}{2}$. Hence $q_\omega^0 \in L^\infty$ almost surely, implying that P_ω^0 has purely discrete spectrum.

Consider the function $F(w) = \arg p(w)$ on S^*X. For given $\theta_0 \in S^1 \simeq \mathbf{R}/(2\pi\mathbf{Z})$, $N_0 \in \dot{\mathbf{N}} := \mathbf{N} \setminus \{0\}$, we introduce the property $P(\theta_0, N_0)$:

$$\sum_1^{N_0} |\nabla^k F(w)| \neq 0 \text{ on } \{w \in S^*X; F(w) = \theta_0\}. \tag{1.11}$$

Notice that if $P(\theta_0, N_0)$ holds, then $P(\theta, N_0)$ holds for all θ in some neighborhood of θ_0. Also notice that if X is connected and X, p are analytic and the analytic function F is non constant, then $\exists N_0 \in \dot{\mathbf{N}}$ such that $P(\theta_0, N_0)$ holds for all θ_0.

We can now state our main result.

THEOREM 1.1. — *Assume that $m \geqslant 2$. Let $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi$ and assume that $P(\theta_1, N_0)$ and $P(\theta_2, N_0)$ hold for some $N_0 \in \dot{\mathbf{N}}$. Let $g \in C^\infty([\theta_1, \theta_2];]0, \infty[)$ and put*

$$\Gamma_{\theta_1, \theta_2; 0, \lambda}^g = \{re^{i\theta}; \theta_1 \leqslant \theta \leqslant \theta_2, \; 0 \leqslant r \leqslant \lambda g(\theta)\}.$$

Then for every $\delta \in]0, \frac{1}{2} - \beta[$ there exists $C > 0$ such that almost surely: $\exists C(\omega) < \infty$ such that for all $\lambda \in [1, \infty[$:

$$|\#(\sigma(P_\omega^0) \cap \Gamma_{\theta_1, \theta_2; 0, \lambda}^g) - \frac{1}{(2\pi)^n} \text{vol}\, p^{-1}(\Gamma_{\theta_1, \theta_2; 0, \lambda}^g)| \tag{1.12}$$

$$\leqslant C(\omega) + C\lambda^{\frac{n}{m} - \frac{1}{m}(\frac{1}{2} - \beta - \delta)\frac{1}{N_0 + 1}}.$$

Here $\sigma(P_\omega^0)$ denotes the spectrum and $\#(A)$ denotes the number of elements in the set A. In (1.12) the eigenvalues are counted with their algebraic multiplicity.

The proof actually allows to have almost surely a simultaneous conclusion for a whole family of θ_1, θ_2, g:

THEOREM 1.2. — *Assume that $m \geqslant 2$. Let Θ be a compact subset of $[0, 2\pi]$. Let $N_0 \in \mathbf{N}$ and assume that $P(\theta, N_0)$ holds uniformly for $\theta \in \Theta$. Let $\mathcal{G}$ be a subset of $\{(g, \theta_1, \theta_2); \ \theta_j \in \Theta, \theta_1 \leqslant \theta_2, \ g \in C^\infty([\theta_1, \theta_2];]0, \infty[)\}$ with the property that g and $1/g$ are uniformly bounded in $C^\infty([\theta_1, \theta_2];]0, \infty[)$ when (g, θ_1, θ_2) varies in $\mathcal{G}$. Then for every $\delta \in]0, \frac{1}{2} - \beta[$ there exists $C > 0$ such that almost surely: $\exists C(\omega) < \infty$ such that for all $\lambda \in [1, \infty[$ and all $(g, \theta_1, \theta_2) \in \mathcal{G}$, we have the estimate (1.12).*

The condition (1.9) allows us to choose σ_j decaying faster than any negative power of μ_j^0. Then from the discussion below, it will follow that $q_\omega(x)$ is almost surely a smooth function. A rough and somewhat intuitive interpretation of Theorem 1.2 is then that for almost every elliptic operator of order $\geqslant 2$ with smooth coefficients on a compact manifold which satisfies the conditions (1.2), (1.3), the large eigenvalues distribute according to Weyl's law in sectors with limiting directions that satisfy a weak non-degeneracy condition.

2. Some examples

Let $f \in C^\infty(S^1)$ be non-vanishing and take its values in a closed sector $\Gamma \subset \mathbf{C}$ of angle $< \pi$. Thus there exist $\theta_0 \in \mathbf{R}$, $\alpha \in [0, \pi/2[$ such that

$$\arg f(S^1) = [\theta_0 - \alpha, \theta_0 + \alpha]. \tag{2.1}$$

Assume for simplicity that $\theta_0 = 0$. Then (see [1, 7]) the spectrum of $f(x)D$ can be computed directly and we see that it is constituted by the simple eigenvalues

$$\lambda_k = \frac{k}{\langle 1/f \rangle}, \ k \in \mathbf{Z}, \tag{2.2}$$

where $\langle 1/f \rangle$ denotes the mean-value of the function $1/f$. Since $1/f$ is non-vanishing with values in the sector $\overline{\Gamma}$, the same holds for $\langle 1/f \rangle$.

The antisymmetric operator $f^{1/2}Df^{1/2} = f^{-1/2}(fD)f^{1/2}$ has the same spectrum and the elliptic symmetric operator

$$P^0 = (f^{1/2}Df^{1/2})^2 = Df^2D - \frac{1}{4}(f')^2 - \frac{1}{2}ff'' \tag{2.3}$$

therefore has the spectrum

$$\{\mu_0, \mu_1, \mu_2, ...\}, \ \mu_k = \lambda_k^2 = \frac{k^2}{\langle 1/f \rangle^2}, \tag{2.4}$$

where μ_0 is a simple eigenvalue and μ_1, μ_2, ... are double. The principal symbol of P^0 is given by

$$p(x, \xi) = f(x)^2 \xi^2 \tag{2.5}$$

and its range is the sector

$$[0, \infty[e^{i[-2\alpha, 2\alpha]} \tag{2.6}$$

(having chosen $\theta_0 = 0$) which does not intersect the open negative half axis. The eigenvalues μ_k are situated on a half axis inside the range (2.6), and unless $\arg f = $ Const., we see that Weyl asymptotics does not hold for P^0. On the other hand, if we add the non-degeneracy assumption,

$$\sum_1^{N_0} |(\frac{d}{dx})^k \arg f(x)| \neq 0, \ x \in S^1, \quad \text{for some } N_0 \in \mathbf{N} \setminus \{0\}, \tag{2.7}$$

then the property $P(\theta, N_0)$ holds for all θ and we know from the Theorems 1.1, 1.2 that Weyl asymptotics holds almost surely for the random perturbations $P_\omega^0 = P^0 + q_\omega^0$ if q_ω^0 is given in (1.7)–(1.10).

Despite the fact that (in some sense and with the additional conditions in our main theorems) almost all symmetric elliptic differential operators obey Weyl asymptotics, it is probably a difficult task to find explicit operators with this property outside the class of normal operators and operators with principal symbol having constant argument. To find such examples one would probably like to assume the coefficients to be analytic but in that case Weyl asymptotics is unlikely to hold. Indeed, in the analytic case there is the possibility to make an analytic distorsion (for instance by replacing the underlying compact analytic manifold by a small deformation) which will not change the spectrum (by ellipticity and analyticity) but which will replace the given real phase space by a deformation, likely to change the Weyl law. In one and two dimensions analytic distorsions have been used to determine the spectrum (by making the operator more normal) in the two-dimensional semi-classical case this was done by one of the authors, first with A. Melin, and in [8] it was shown that the resulting law is in general different from the Weyl law (naively because a complex Bohr-Sommerfeld law relies on going out in the complex domain while the Weyl law only uses the real cotangent space).

To illustrate this, let us consider the second order differential operator on S^1,

$$P^0 = a(x)D^2 + b(x) + c(x), \tag{2.8}$$

where the coefficients a, b, c are smooth (and 2π-periodic when considered as functions on $\mathbf{R}$). We assume that P^0 is elliptic, so that $a(x) \neq 0$ and even that the range of $\arg a$ is the interval $[-2\alpha, 2\alpha]$ for some $\alpha \in [0, \frac{\pi}{2}[$. Then $a(x) = f(x)^2$, where f is smooth, non-vanishing and the range of $\arg f$ is $[-\alpha, \alpha]$. The Bohr-Sommerfeld quantization condition, which correctly describes the large eigenvalues when P^0 is self-adjoint and more generally when $a > 0$, would predict that the large eigenvalues μ_k^2 are determined by the condition

$$I(\mu_k) = 2\pi k + \mathcal{O}(1), \ k \in \mathbf{Z}, \ |k| \gg 1, \tag{2.9}$$

where $I(\mu)$ is the action, defined by $I(\mu) = \int_0^{2\pi} \xi(x, \mu) dx$, $\xi(x, \mu) = \mu/f(x)$, so that $p(x, \xi(x, \mu)) = \mu^2$, where $p(x, \xi) = f(x)^2 \xi^2$ is the principal symbol of P^0. Notice that this simplifies to

$$\mu_k = \frac{k}{\langle 1/f \rangle} + \mathcal{O}(1). \tag{2.10}$$

We also recall that the remainder has a complete asymptotic expansion in negative powers of k. As we have seen, this rule is correct in the special case of the operator (2.3) and as we have noticed it becomes almost surely false if we add a random smooth zero order term (at least in the symmetric case).

However, the Bohr-Sommerfeld rule is correct under suitable analyticity assumptions, as we shall now review: Look for a complex change of variables $x = x(t)$ with $0 = x(0)$ so that $f(x)D_x = \kappa D_t$, for a suitable $\kappa \in \mathbf{C} \setminus \{0\}$. We get

$$\frac{dt}{dx} = \frac{\kappa}{f(x)},$$

so the inverse $t(x)$ is given by

$$t = \kappa \int_0^x \frac{dy}{f(y)}. \tag{2.11}$$

if f is merely smooth we can only define a complex curve $t(x)$ by (2.11) for real x. We now determine κ by the condition that $t(2\pi) = 2\pi$, i.e.

$$\kappa = \frac{1}{\langle 1/f \rangle}. \tag{2.12}$$

Now assume that f extends to a holomorphic non-vanishing function in a 2π-periodic simply connected neighborhood Ω of $\mathbf{R}$ in $\mathbf{C}$. Then $t(x)$ extends

to a holomorphic function on Ω, and we assume that the set $\{x \in \Omega; t(x) \in \mathbf{R}\}$ contains (the image of) a smooth 2π-periodic curve $\gamma : \mathbf{R} \to \Omega$ such that $\gamma(0) = 0$, $\gamma(2\pi) = 2\pi$. Also assume that b, c extend to holomorphic functions on Ω. Notice that if $f_0 > 0$ is an analytic 2π-periodic function and if f is a small perturbation of f_0 in a fixed neighborhood of $\mathbf{R}$, then f fulfills the assumptions above. In a small neighborhood of γ we can replace the variable x by t and we get the operator

$$\widetilde{P} = \kappa^2 D_t^2 + \widetilde{b}(t)D_t + \widetilde{c}(t), \qquad (2.13)$$

well-defined in a small neighborhood of $\mathbf{R}_t$. For this operator it is quite easy to justify the Bohr–Sommerfeld rule by some version of the complex WKB-method (cf [4]). Now the Bohr-Sommerfeld rule is clearly invariant under the change of variables above. Moreover, eigenfunctions of $\widetilde{P}$ defined near $\mathbf{R}_t$ are also eigenfunctions of P^0 with respect to the x-variables in a neighborhood of γ and since P^0 is elliptic in Ω, they extend to holomorphic functions in Ω and by restriction become eigenfunctions on $\mathbf{R}_x$. The same remark holds for generalized eigenfunctions. Hence the eigenvalues of $\widetilde{P}$ are also eigenvalues of P^0. This argument works equally well in the other direction so we can identify completely the spectra of $\widetilde{P}$ and of P^0 and this completes the (review of the) justification of the Bohr-Sommerfeld rule (and hence of the non-validity of Weyl asymptotics when $\arg f$ is non-constant) for the operator (2.8) in the analytic case.

3. Volume considerations

In the next section we shall perform a reduction to a semi-classical situation and work with $h^m P^0$ which has the semi-classical principal symbol p in (1.1). As in [6, 9, 10], we introduce

$$V_z(t) = \mathrm{vol}\,\{\rho \in T^*X;\, |p(\rho) - z|^2 \leqslant t\},\ t \geqslant 0. \qquad (3.1)$$

PROPOSITION 3.1. — *For any compact set $K \subset \dot{\mathbf{C}} = \mathbf{C} \setminus \{0\}$, we have*

$$V_z(t) = \mathcal{O}(t^\kappa),\ \ \text{uniformly for } z \in K,\ 0 \leqslant t \ll 1, \qquad (3.2)$$

with $\kappa = 1/2$.

The property (3.2) for some $\kappa \in]0,1[$ is required in [6, 9, 10] near the boundary of the set Γ, where we count the eigenvalues. Another important quantity appearing there was

$$\mathrm{vol}\, p^{-1}(\gamma + D(0,t)), \qquad (3.3)$$

where $D(0,t) = \{z \in \mathbf{C}; |z| < t\}$, $\gamma = \partial\Gamma$ and $\Gamma \Subset \dot{\mathbf{C}}$ is assumed to have piecewise smooth boundary. From (3.2) with general κ it follows that the volume (3.3) is $\mathcal{O}(t^{2\kappa-1})$, which is of interest when $\kappa > 1/2$. In our case, we shall therefore investigate $\mathrm{vol}\,(\gamma + B(0,t))$ more directly, when γ is (the image of) a smooth curve. The following result implies Proposition 3.1:

PROPOSITION 3.2. — *Let γ be the curve $\{re^{i\theta} \in \mathbf{C}; r = g(\theta),\ \theta \in S^1\}$, where $0 < g \in C^1(S^1)$. Then*

$$\mathrm{vol}\,(p^{-1}(\gamma + D(0,t))) = \mathcal{O}(t),\ t \to 0.$$

Proof. — This follows from the fact that the radial derivative of p is $\neq 0$. More precisely, write $T^*X \setminus 0 \ni \rho = rw$, $w \in S^*X$, $r > 0$, so that $p(\rho) = r^m p(w)$, $p(w) \neq 0$. If $\rho \in p^{-1}(\gamma + D(0,t))$, we have for some $C \geqslant 1$, independent of t,

$$g(\arg p(w)) - Ct \leqslant r^m |p(w)| \leqslant g(\arg p(w)) + Ct,$$

$$\left(\frac{g(\arg p(w)) - Ct}{|p(w)|} \right)^{\frac{1}{m}} \leqslant r \leqslant \left(\frac{g(\arg p(w)) + Ct}{|p(w)|} \right)^{\frac{1}{m}},$$

so for every $w \in S^*X$, r has to belong to an interval of length $\mathcal{O}(t)$. $\qquad\square$

We next study the volume in (3.3) when γ is a radial segment of the form $[r_1, r_2]e^{i\theta_0}$, where $0 < r_1 < r_2$ and $\theta_0 \in S^1$.

PROPOSITION 3.3. — *Let $\theta_0 \in S^1$, $N_0 \in \dot{\mathbf{N}}$ and assume that $P(\theta_0, N_0)$ holds. Then if $0 < r_1 < r_2$ and γ is the radial segment $[r_1, r_2]e^{i\theta_0}$, we have*

$$\mathrm{vol}\,(p^{-1}(\gamma + D(0,t))) = \mathcal{O}(t^{1/N_0}),\ t \to 0.$$

Proof. — We first observe that it suffices to show that

$$\mathrm{vol}_{S^*X}\, F^{-1}([\theta_0 - t, \theta_0 + t]) = \mathcal{O}(t^{1/N_0}).$$

This in turn follows for instance from the Malgrange preparation theorem: At every point $w_0 \in F^{-1}(\theta_0)$ we can choose coordinates $w_1, ..., w_{2n-1}$, centered at w_0, such that for some $k \in \{1, ..., N_0\}$, we have that $\partial_{w_1}^j (F - \theta_0)(w_0)$ is $= 0$ when $0 \leqslant j \leqslant k-1$ and $\neq 0$ when $j = k$. Then by Malgrange's preparation theorem, we have

$$F(w) - \theta_0 = G(w)(w_1^k + a_1(w_2, ..., w_{2n-1})w_1^{k-1} + ... + a_k(w_2, ..., w_{2n-1})),$$

where G, a_j are real and smooth, $G(w_0) \neq 0$, and it follows that

$$\operatorname{vol}\left(F^{-1}([\theta_0 - t, \theta_0 + t]) \cap \operatorname{neigh}(w_0)\right) = \mathcal{O}(t^{1/k}).$$

It then suffices to use a simple compactness argument. □

Now, let $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$, $g \in C^\infty([\theta_1, \theta_2];]0, \infty[)$ and put

$$\Gamma^g_{\theta_1, \theta_2; r_1, r_2} = \{re^{i\theta};\ \theta_1 \leqslant \theta \leqslant \theta_2,\ r_1 g(\theta) \leqslant r \leqslant r_2 g(\theta)\}, \qquad (3.4)$$

for $0 \leqslant r_1 \leqslant r_2 < \infty$. If $0 < r_1 < r_2 < +\infty$ and $P(\theta_j, N_0)$ hold for $j = 1, 2$, then the last two propositions imply that

$$\operatorname{vol} p^{-1}(\partial \Gamma^g_{\theta_1, \theta_2; r_1, r_2} + D(0, t)) = \mathcal{O}(t^{1/N_0}),\ t \to 0. \qquad (3.5)$$

4. Semiclassical reduction

We are interested in the distribution of large eigenvalues ζ of P^0_ω, so we make a standard reduction to a semi-classical problem by letting $0 < h \ll 1$ satisfy

$$\zeta = \frac{z}{h^m},\ |z| \asymp 1,\ h \asymp |\zeta|^{-1/m}, \qquad (4.1)$$

and write

$$h^m(P^0_\omega - \zeta) = h^m P^0_\omega - z =: P + h^m q^0_\omega - z, \qquad (4.2)$$

where

$$P = h^m P^0 = \sum_{|\alpha| \leqslant m} a_\alpha(x; h)(hD)^\alpha. \qquad (4.3)$$

Here

$$
\begin{aligned}
a_\alpha(x; h) &= \mathcal{O}(h^{m-|\alpha|}) \text{ in } C^\infty, \\
a_\alpha(x; h) &= a^0_\alpha(x) \text{ when } |\alpha| = m.
\end{aligned} \qquad (4.4)
$$

So P is a standard semi-classical differential operator with semi-classical principal symbol $p(x, \xi)$.

Our strategy will be to decompose the random perturbation

$$h^m q^0_\omega = \delta Q_\omega + k_\omega(x),$$

where the two terms are independent, and with probability very close to 1, δQ_ω will be a semi-classical random perturbation as in [10] while

$$\|k_\omega\|_{H^s} \leqslant h, \qquad (4.5)$$

and

$$s \in]\frac{n}{2}, \rho - \frac{n}{2}[\tag{4.6}$$

is fixed. Then $h^m P_\omega^0$ will be viewed as a random perturbation of $h^m P^0 + k_\omega$ and we will apply the main result of [10] that we recall in the appendix for the convenience of the reader. In order to achieve this without extra assumptions on the order m, we will also have to represent some of our coefficients $\alpha_j^0(\omega)$ as sums of two independent Gaussian random variables.

We start by examining when

$$\|h^m q_\omega^0\|_{H^s} \leqslant h. \tag{4.7}$$

PROPOSITION 4.1. — *There is a constant $C > 0$ such that (4.7) holds with probability*

$$\geqslant 1 - \exp(C - \frac{1}{2Ch^{2(m-1)}}).$$

Proof. — We have

$$h^m q_\omega^0 = \sum_0^\infty \alpha_j(\omega)\epsilon_j, \quad \alpha_j = h^m \alpha_j^0 \sim \mathcal{N}(0, (h^m \sigma_j)^2), \tag{4.8}$$

and the α_j are independent. Now, using standard functional calculus for $\widetilde{R}$ as in [9, 10], we see that

$$\|h^m q_\omega^0\|_{H^s}^2 \asymp \sum_0^\infty |(\mu_j^0)^s \alpha_j(\omega)|^2, \tag{4.9}$$

where $(\mu_j^0)^s \alpha_j \sim \mathcal{N}(0, (\widetilde{\sigma}_j)^2)$ are independent random variables and $\widetilde{\sigma}_j = (\mu_j^0)^s h^m \sigma_j$.

Now recall the following fact, established by Bordeaux Montrieux [1], improving and simplifying a similar result in [6]: Let $d_0, d_1, \dots$ be a finite or infinite family of independent complex Gaussian random variables, $d_j \sim \mathcal{N}(0, (\widehat{\sigma}_j)^2)$, $0 < \widehat{\sigma}_j < \infty$, and assume that $\sum \widehat{\sigma}_j^2 < \infty$. Then for every $t > 0$,

$$\mathbf{P}(\sum |d_j|^2 \geqslant t) \leqslant \exp(\frac{-1}{2\max \widehat{\sigma}_j^2}(t - C_0 \sum \widehat{\sigma}_j^2)). \tag{4.10}$$

Here $\mathbf{P}(A)$ denotes the probability of the event A and $C_0 > 0$ is a universal constant. The estimate is interesting only when $t > C_0 \sum \widehat{\sigma}_j^2$ and for such

values of t it improves if we replace $\{d_0, d_1, ...\}$ by a subfamily. Indeed, $\sum \widehat{\sigma}_j^2$ will then decrease and so will $\max \widehat{\sigma}_j^2$.

Apply this to (4.9) with $d_j = (\mu_j^0)^s \alpha_j$, $t = h^2$. Here, we recall that $\widetilde{\sigma}_j = (\mu_j^0)^s h^m \sigma_j$, and get from (1.9), (4.6) that

$$\max \widetilde{\sigma}_j^2 \asymp h^{2m}, \tag{4.11}$$

while

$$\sum_0^\infty \widetilde{\sigma}_j^2 \lesssim h^{2m} \sum_0^\infty (\mu_j^0)^{2(s-\rho)}. \tag{4.12}$$

Let $N(\mu) = \#(\sigma(\sqrt{\widetilde{R}}) \cap]0, \mu])$ be the number of eigenvalues of $\sqrt{\widetilde{R}}$ in $]0, \mu]$, so that $N(\mu) \asymp \mu^n$ by the standard Weyl asymptotics for positive elliptic operators on compact manifolds. The last sum in (4.12) is equal to

$$\int_0^\infty \mu^{2(s-\rho)} dN(\mu) = \int_0^\infty 2(\rho - s)\mu^{2(s-\rho)-1} N(\mu) d\mu,$$

which is finite since $2(s - \rho) + n < 0$ by (4.6). Thus

$$\sum_0^\infty \widetilde{\sigma}_j^2 \lesssim h^{2m}, \tag{4.13}$$

and the proposition follows from applying (4.9), (4.11), (4.12) to (4.10) with $t = h^2$. $\square$

We next review the choice of parameters for the random perturbation in [10] (see also the appendix and [9]). This perturbation is of the form δQ_ω,

$$Q_\omega = h^{N_1} q_\omega, \ \delta = \tau_0 h^{N_1 + n}, \ 0 < \tau_0 \leqslant \sqrt{h}, \tag{4.14}$$

where

$$q_\omega(x) = \sum_{0 < h\mu_k^0 \leqslant L} \alpha_k(\omega) \epsilon_k(x), \ |\alpha|_{\mathbf{C}^D} \leqslant R, \tag{4.15}$$

and a possible choice of L, R is

$$L = Ch^{-M}, \quad R = Ch^{-\widetilde{M}}, \tag{4.16}$$

with

$$M = \frac{3n - \kappa}{s - \frac{n}{2} - \epsilon}, \quad \widetilde{M} = \frac{3n}{2} - \kappa + (\frac{n}{2} + \epsilon)M. \tag{4.17}$$

Here $\epsilon > 0$ is any fixed parameter in $]0, s - \frac{n}{2}[$ and $\kappa \in]0, 1]$ is the geometric exponent appearing in (3.2), in our case equal to $1/2$.

The exponent N_1 is given by

$$N_1 = \widetilde{M} + sM + \frac{n}{2}, \tag{4.18}$$

and q_ω should be subject to a probability density on $B_{\mathbf{C}^D}(0, R)$ of the form $C(h)e^{\Phi(\alpha;h)}L(d\alpha)$, where

$$|\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_4}), \tag{4.19}$$

for some constant $N_4 \geqslant 0$.

Write

$$q_\omega^0 = q_\omega^1 + q_\omega^2, \tag{4.20}$$

$$q_\omega^1 = \sum_{0 < h\mu_j^0 \leqslant L} \alpha_j^0(\omega)\epsilon_j, \ q_\omega^2 = \sum_{h\mu_j^0 > L} \alpha_j^0(\omega)\epsilon_j. \tag{4.21}$$

From Proposition 4.1 and its proof, especially the observation after (4.10), we know that

$$\|h^m q_\omega^2\|_{H^s} \leqslant h \text{ with probability } \geqslant 1 - \exp(C_0 - \frac{1}{2Ch^{2(m-1)}}). \tag{4.22}$$

We write

$$P + h^m q_\omega^0 = (P + h^m q_\omega^2) + h^m q_\omega^1,$$

and recall that the main result in [10] is valid also when P is replaced by the perturbation $P + h^m q_\omega^2$, provided that $\|h^m q_\omega^2\|_{H^s} \leqslant h$. (See the comment after Theorem A.1.)

The next question is then wether $h^m q_\omega^1$ can be written as $\tau_0 h^{2N_1+n} q_\omega$ where $q_\omega = \sum_{0 < h\mu_j^0 \leqslant L} \alpha_j \epsilon_j$ and $|\alpha|_{\mathbf{C}^D} \leqslant R$ with probability close to 1. We get

$$\alpha_j = \frac{1}{\tau_0} h^{m-2N_1-n} \alpha_j^0(\omega) \sim \mathcal{N}(0, \hat{\sigma}_j^2),$$

$$\frac{1}{\tau_0} h^{m-2N_1-n}(\mu_j^0)^{-\rho} e^{-(\mu_j^0)^{\frac{\beta}{M+1}}} \lesssim \hat{\sigma}_j \lesssim \frac{1}{\tau_0} h^{m-2N_1-n}(\mu_j^0)^{-\rho}.$$

Applying (4.10), we get

$$\mathbf{P}(|\alpha|_{\mathbf{C}^D}^2 \geqslant R^2) \leqslant \exp(C - \frac{R^2 \tau_0^2}{Ch^{2(m-2N_1-n)}}), \tag{4.23}$$

which is $\mathcal{O}(1)\exp(-h^{-\delta})$ provided that

$$-2\widetilde{M} + 2\frac{\ln(1/\tau_0)}{\ln(1/h)} + 2(2N_1 + n - m) \leqslant -\delta. \tag{4.24}$$

Here $\tau_0 \leqslant \sqrt{h}$ and if we choose $\tau_0 = \sqrt{h}$ or more generally bounded from below by some power of h, we see that (4.24) holds for any fixed δ, provided that m is sufficiently large.

In order to avoid such an extra assumption, we shall now represent α_j^0 for $h\mu_j^0 \leqslant L$ as the sum of two independent Gaussian random variables. Let $j_0 = j_0(h)$ be the largest j for which $h\mu_j^0 \leqslant L$. Put

$$\sigma' = \frac{1}{C}h^K e^{-Ch^{-\beta}}, \text{ where } K \geqslant \rho(M+1), \ C \gg 1 \qquad (4.25)$$

so that $\sigma' \leqslant \frac{1}{2}\sigma_j$ for $1 \leqslant j \leqslant j_0(h)$. The factor h^K is needed only when $\beta = 0$.

For $j \leqslant j_0$, we may assume that $\alpha_j^0(\omega) = \alpha_j'(\omega) + \alpha_j''(\omega)$, where $\alpha_j' \sim \mathcal{N}(0, (\sigma')^2)$, $\alpha_j'' \sim \mathcal{N}(0, (\sigma_j'')^2)$ are independent random variables and

$$\sigma_j^2 = (\sigma')^2 + (\sigma_j'')^2,$$

so that

$$\sigma_j'' = \sqrt{\sigma_j^2 - (\sigma')^2} \asymp \sigma_j.$$

Put $q_\omega^1 = q_\omega' + q_\omega''$, where

$$q_\omega' = \sum_{h\mu_j^0 \leqslant L} \alpha_j'(\omega)\epsilon_j, \ q_\omega'' = \sum_{h\mu_j^0 \leqslant L} \alpha_j''(\omega)\epsilon_j.$$

Now (cf (4.20)) we write

$$P + h^m q_\omega^0 = (P + h^m(q_\omega'' + q_\omega^2)) + h^m q_\omega'.$$

The main result of [10] is valid for random perturbations of

$$P_0 := P + h^m(q_\omega'' + q_\omega^2),$$

provided that $\|h^m(q_\omega'' + q_\omega^2)\|_{H^s} \leqslant h$, which again holds with a probability as in (4.22). The new random perturbation is now $h^m q_\omega'$ which we write as $\tau_0 h^{2N_1+n}\widetilde{q}_\omega$, where $\widetilde{q}_\omega$ takes the form

$$\widetilde{q}_\omega(x) = \sum_{0 < h\mu_j^0 \leqslant L} \vartheta_j(\omega)\epsilon_j, \qquad (4.26)$$

with new independent random variables

$$\vartheta_j = \frac{1}{\tau_0}h^{m-2N_1-n}\alpha_j'(\omega) \sim \mathcal{N}(0, (\frac{1}{\tau_0}h^{m-2N_1-n}\sigma'(h))^2). \qquad (4.27)$$

Now, by (4.10),

$$\mathbf{P}(|\vartheta|^2_{\mathbf{C}^D} > R^2) \leqslant \exp(\mathcal{O}(1)D - \frac{R^2\tau_0^2}{\mathcal{O}(1)(h^{m-2N_1-n}\sigma'(h))^2}).$$

Here by Weyl's law for the distribution of eigenvalues of elliptic self-adjoint differential operators, we have $D \asymp (L/h)^n$. Moreover, L, R behave like certain powers of h.

- In the case when $\beta = 0$, we choose $\tau_0 = h^{1/2}$. Then for any $a > 0$ we get

$$\mathbf{P}(|\vartheta|_{\mathbf{C}^D} > R) \leqslant C \exp(-\frac{1}{Ch^a})$$

  for any given fixed a, provided we choose K large enough in (4.25).

- In the case $\beta > 0$ we get the same conclusion with $\tau_0 = h^{-K}\sigma'$ if K is large enough.

In both cases, we see that the independent random variables ϑ_j in (4.26), (4.27) have a joint probability density $C(h)e^{\Phi(\alpha;h)}L(d\alpha)$, satisfying (4.19) for some N_4 depending on K.

With $\kappa = 1/2$, we put

$$\epsilon_0(h) = h^\kappa((\ln\frac{1}{h})^2 + \ln\frac{1}{\tau_0}),$$

where τ_0 is chosen as above. Notice that $\epsilon_0(h)$ is of the order of magnitude $h^{\kappa-\beta}$ up to a power of $\ln\frac{1}{h}$. Then Theorem 1.1 in [10] (recalled in the appendix) gives:

PROPOSITION 4.2. — *There exists a constant $N_4 > 0$ depending on ρ, n, m such that the following holds: Let $\Gamma \Subset \dot{\mathbf{C}}$ have piecewise smooth boundary. Then $\exists C > 0$ such that for $0 < r \leqslant 1/C$, $\widetilde{\epsilon} \geqslant C\epsilon_0(h)$, we have with probability*

$$\geqslant 1 - \frac{C\epsilon_0(h)}{rh^{n+\max(n(M+1),N_4+\widetilde{M})}}e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}}, \tag{4.28}$$

that

$$|\#(h^m P_\omega^0) \cap \Gamma) - \frac{1}{(2\pi h)^n}\text{vol}(p^{-1}(\Gamma))| \leqslant \tag{4.29}$$

$$\frac{C}{h^n}(\frac{\widetilde{\epsilon}}{r} + C(r + \ln(\frac{1}{r})\text{vol}(p^{-1}(\partial\Gamma + D(0,r)))))).$$

As noted in [9] this gives Weyl asymptotics provided that

$$\ln(\frac{1}{r})\operatorname{vol} p^{-1}(\partial\Gamma + D(0,r)) = \mathcal{O}(r^{\alpha}), \qquad (4.30)$$

for some $\alpha \in]0,1]$ (which would automatically be the case if κ had been larger than $1/2$ instead of being equal to $1/2$), and we can then choose $r = \widetilde{\epsilon}^{1/(1+\alpha)}$, so that the right hand side of (4.29) becomes $\leqslant C\widetilde{\epsilon}^{\frac{\alpha}{1+\alpha}}h^{-n}$.

As in [9, 10] we also observe that if Γ belongs to a family $\mathcal{G}$ of domains satisfying the assumptions of the proposition uniformly, then with probability

$$\geqslant 1 - \frac{C\epsilon_0(h)}{r^2 h^{n+\max(n(M+1),N_4+\widetilde{M})}}e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}}, \qquad (4.31)$$

the estimate (4.29) holds uniformly and simultaneously for all $\Gamma \in \mathcal{G}$.

5. End of the proof

Let θ_1, θ_2, N_0 be as in Theorem 1.1, so that $P(\theta_1, N_0)$ and $P(\theta_2, N_0)$ hold. Combining the propositions 3.1, 3.2, 3.3, we see that (4.30) holds for every $\alpha < 1/N_0$ when $\Gamma = \Gamma^g_{\theta_1,\theta_2;1,\lambda}$, $\lambda > 0$ fixed, and Proposition 4.2 gives:

PROPOSITION 5.1. — *With the parameters as in Proposition 4.2 and for every $\alpha \in]0, \frac{1}{N_0}[$, we have with probability*

$$\geqslant 1 - \frac{C\epsilon_0(h)}{\widetilde{\epsilon}^{\frac{1}{1+\alpha}} h^{n+\max(n(M+1),N_4+\widetilde{M})}}e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}} \qquad (5.1)$$

that

$$|\#(\sigma(h^m P_\omega) \cap \Gamma^g_{\theta_1,\theta_2;1,\lambda}) - \frac{1}{(2\pi h)^n}\operatorname{vol}(p^{-1}(\Gamma^g_{\theta_1,\theta_2;1,\lambda}))| \leqslant C\frac{\widetilde{\epsilon}^{\frac{\alpha}{1+\alpha}}}{h^n}. \quad (5.2)$$

Moreover, the conclusion (5.2) is valid simultaneously for all $\lambda \in [1,2]$ and all (θ_1,θ_2) in a set where $P(\theta_1, N_0)$, $P(\theta_2, N_0)$ hold uniformly, with probability

$$\geqslant 1 - \frac{C\epsilon_0(h)}{\widetilde{\epsilon}^{\frac{2}{1+\alpha}} h^{n+\max(n(M+1),N_4+\widetilde{M})}}e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}}. \qquad (5.3)$$

For $0 < \delta \ll 1$, choose $\widetilde{\epsilon} = h^{-\delta}\epsilon_0 \leqslant Ch^{\frac{1}{2}-\beta-\delta}(\ln\frac{1}{h})^2$, so that $\widetilde{\epsilon}/\epsilon_0 = h^{-\delta}$. Then for some N_5 we have for every $\alpha \in]0, 1/N_0[$ that

$$|\#(\sigma(h^m P_\omega)\cap\Gamma^g_{\theta_1,\theta_2;1,\lambda}) - \frac{1}{(2\pi h)^n}\operatorname{vol}(p^{-1}(\Gamma^g_{\theta_1,\theta_2;1,\lambda}))| \leqslant \frac{C_\alpha}{h^n}(h^{\frac{1}{2}-\delta-\beta}(\ln\frac{1}{h})^2)^{\frac{\alpha}{1+\alpha}},$$
$$(5.4)$$

simultaneously for $1 \leqslant \lambda \leqslant 2$ and all (θ_1, θ_2) in a set where $P(\theta_1, N_0)$, $P(\theta_2, N_0)$ hold uniformly, with probability

$$\geqslant 1 - \frac{C}{h^{N_5}} e^{-\frac{1}{Ch^\delta}}. \tag{5.5}$$

Here $\alpha/(1+\alpha) \nearrow 1/(N_0+1)$ when $\alpha \nearrow 1/N_0$, so the upper bound in (5.4) can be replaced by

$$\frac{C_\delta}{h^n} h^{(\frac{1}{2} - \beta - 2\delta)/(N_0+1)}.$$

Assuming $P(\theta_1, N_0)$, $P(\theta_2, N_0)$, we want to count the number of eigenvalues of P_ω^0 in

$$\Gamma_{1,\lambda} = \Gamma_{\theta_1, \theta_2; 1, \lambda}^g$$

when $\lambda \to \infty$. Let $k(\lambda)$ be the largest integer k for which $2^k \leqslant \lambda$ and decompose

$$\Gamma_{1,\lambda} = \left(\bigcup_0^{k(\lambda)-1} \Gamma_{2^k, 2^{k+1}} \right) \cup \Gamma_{2^{k(\lambda)}, \lambda}.$$

In order to count the eigenvalues of P_ω^0 in $\Gamma_{2^k, 2^{k+1}}$ we define h by $h^m 2^k = 1$, $h = 2^{-k/m}$, so that

$$\#(\sigma(P_\omega^0) \cap \Gamma_{2^k, 2^{k+1}}) = \#(\sigma(h^m P_\omega^0) \cap \Gamma_{1,2}),$$
$$\frac{1}{(2\pi)^n} \mathrm{vol}\,(p^{-1}(\Gamma_{2^k, 2^{k+1}})) = \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma_{1,2})).$$

Thus, with probability $\geqslant 1 - C2^{\frac{N_5 k}{m}} e^{-2^{\frac{\delta k}{m}}/C}$ we have

$$|\#(\sigma(P_\omega^0) \cap \Gamma_{2^k, 2^{k+1}}) - \frac{1}{(2\pi)^n} \mathrm{vol}\, p^{-1}(\Gamma_{2^k, 2^{k+1}})| \leqslant C_\delta 2^{\frac{kn}{m}} 2^{-\frac{k}{m}(\frac{1}{2} - \beta - 2\delta)\frac{1}{N_0+1}}. \tag{5.6}$$

Similarly, with probability $\geqslant 1 - C2^{N_5 k(\lambda)/m} e^{-2^{\delta k(\lambda)/m}/C}$, we have

$$|\#(\sigma(P_\omega^0) \cap \Gamma_{2^{k(\lambda)}, \widetilde{\lambda}}) - \frac{1}{(2\pi)^n} \mathrm{vol}\, p^{-1}(\Gamma_{2^{k(\lambda)}, \widetilde{\lambda}})| \leqslant C_\delta \lambda^{\frac{n}{m}} \lambda^{-\frac{1}{m}(\frac{1}{2} - \beta - 2\delta)\frac{1}{N_0+1}}, \tag{5.7}$$

simultaneously for all $\widetilde{\lambda} \in [\lambda, 2\lambda[$.

Now, we proceed as in [1], using essentially the Borel–Cantelli lemma. Use that

$$\sum_\ell^\infty 2^{N_5 \frac{k}{m}} e^{-2^{\delta \frac{k}{m}}/C} = \mathcal{O}(1) 2^{N_5 \frac{\ell}{m}} e^{-2^{\delta \frac{\ell}{m}}/C},$$

$$\sum_{2^k \leqslant \lambda} 2^{k\frac{n}{m}} 2^{-\frac{k}{m}(\frac{1}{2} - \beta - 2\delta)\frac{1}{N_0+1}} = \mathcal{O}(1) \lambda^{\frac{n}{m} - \frac{1}{m}(\frac{1}{2} - \beta - 2\delta)\frac{1}{N_0+1}},$$

to conclude that with probability $\geqslant 1 - C2^{N_5 \frac{\ell}{m}} e^{-2^{\delta \frac{\ell}{m}}/C}$, we have

$$|\#(\sigma(P_\omega^0) \cap \Gamma_{2^\ell, \lambda})| \leqslant C_\delta \lambda^{\frac{n}{m} - \frac{1}{m}(\frac{1}{2} - \beta - \delta)\frac{1}{N_0 + 1}} + C(\omega)$$

for all $\lambda \geqslant 2^\ell$. This statement implies Theorem 1.1. □

Proof of Theorem 1.2. — This is just a minor modification of the proof of Theorem 1.1. Indeed, we already used the second part of Proposition 4.2, to get (5.7) with the probability indicated there. In that estimate we are free to vary (g, θ_1, θ_2) in $\mathcal{G}$ and the same holds for the estimate (5.6). With these modifications, the same proof gives Theorem 1.2. □

A. Review of the main result of [10]

As before we let X be a compact smooth manifold of dimension n. On X we consider an h-differential operator P which in local coordinates takes the form,

$$P = \sum_{|\alpha| \leqslant m} a_\alpha(x; h)(hD)^\alpha, \tag{A.1}$$

where we use standard multiindex notation and let $D = D_x = \frac{1}{i}\frac{\partial}{\partial x}$. We assume that the coefficients a_α are uniformly bounded in C^∞ for $h \in]0, h_0]$, $0 < h_0 \ll 1$. (We will also discuss the case when we only have some Sobolev space control of $a_0(x)$.) Assume

$$a_\alpha(x; h) = a_\alpha^0(x) + \mathcal{O}(h) \text{ in } C^\infty, \tag{A.2}$$
$$a_\alpha(x; h) = a_\alpha(x) \text{ is independent of } h \text{ for } |\alpha| = m.$$

Notice that this assumption is invariant under changes of local coordinates.

Also assume that P is elliptic in the classical sense, uniformly with respect to h:

$$|p_m(x, \xi)| \geqslant \frac{1}{C}|\xi|^m, \tag{A.3}$$

for some positive constant C, where

$$p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha \tag{A.4}$$

is invariantly defined as a function on T^*X. (In the main text, p_m coincides with p and also with p below.) It follows that $p_m(T^*X)$ is a closed cone in $\mathbf{C}$ and we assume (as in (1.2)) that

$$p_m(T^*X) \neq \mathbf{C}. \tag{A.5}$$

If $z_0 \in \mathbf{C} \setminus p_m(T^*X)$, we see that $\lambda z_0 \notin \Sigma(p)$ if $\lambda \geqslant 1$ is sufficiently large and fixed, where $\Sigma(p) := p(T^*X)$ and p is the semiclassical principal symbol

$$p(x, \xi) = \sum_{|\alpha| \leqslant m} a_\alpha^0(x) \xi^\alpha. \tag{A.6}$$

Actually, (A.5) can be replaced by the weaker condition that $\Sigma(p) \neq \mathbf{C}$.

Standard elliptic theory and analytic Fredholm theory now show that if we consider P as an unbounded operator: $L^2(X) \to L^2(X)$ with domain $\mathcal{D}(P) = H^m(X)$ (the Sobolev space of order m), then P has purely discrete spectrum.

We will need the symmetry assumption (cf (1.3))

$$P^* = \Gamma P \Gamma, \tag{A.7}$$

which implies that

$$p(x, -\xi) = p(x, \xi). \tag{A.8}$$

As before, let $V_z(t) := \mathrm{vol}\,(\{\rho \in \mathbf{R}^{2n};\ |p(\rho) - z|^2 \leqslant t\})$. For $\kappa \in]0, 1]$, $z \in \Omega$, we consider the property (cf (3.2)) that

$$V_z(t) = \mathcal{O}(t^\kappa),\ 0 \leqslant t \ll 1. \tag{A.9}$$

Since $r \mapsto p(x, r\xi)$ is a polynomial of degree m in r with non-vanishing leading coefficient, we see that (A.9) holds with $\kappa = 1/(2m)$.

The random potential will be of the form

$$q_\omega(x) = \sum_{0 < \mu_k \leqslant L} \alpha_k(\omega) \epsilon_k(x),\ |\alpha|_{\mathbf{C}^D} \leqslant R, \tag{A.10}$$

where ϵ_k is the orthonormal basis of eigenfunctions of $h^2 \widetilde{R}$, where $\widetilde{R}$ is as in the introduction. Moreover, $h^2 \widetilde{R} \epsilon_k = \mu_k^2 \epsilon_k$, $\mu_k > 0$ (so that $\mu_k = h\mu_k^0$, cf (1.5)). We choose $L = L(h)$, $R = R(h)$ in the interval

$$h^{\frac{\kappa - 3n}{s - \frac{n}{2} - \epsilon}} \ll L \leqslant Ch^{-M}, \qquad M \geqslant \frac{3n - \kappa}{s - \frac{n}{2} - \epsilon}, \tag{A.11}$$

$$\frac{1}{C} h^{-(\frac{n}{2} + \epsilon)M + \kappa - \frac{3n}{2}} \leqslant R \leqslant Ch^{-\widetilde{M}}, \qquad \widetilde{M} \geqslant \frac{3n}{2} - \kappa + (\frac{n}{2} + \epsilon)M,$$

for some $\epsilon \in]0, s - \frac{n}{2}[$, $s > \frac{n}{2}$, so by Weyl's law for the large eigenvalues of elliptic self-adjoint operators, the dimension D is of the order of magnitude

$(L/h)^n$. We introduce the small parameter $\delta = \tau_0 h^{N_1+n}$, $0 < \tau_0 \leqslant \sqrt{h}$, where

$$N_1 := \widetilde{M} + sM + \frac{n}{2}. \tag{A.12}$$

The randomly perturbed operator is

$$P_\delta = P + \delta h^{N_1} q_\omega =: P + \delta Q_\omega. \tag{A.13}$$

The random variables $\alpha_j(\omega)$ will have a joint probability distribution

$$P(d\alpha) = C(h)e^{\Phi(\alpha;h)} L(d\alpha), \tag{A.14}$$

where for some $N_4 > 0$,

$$|\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_4}), \tag{A.15}$$

and $L(d\alpha)$ is the Lebesgue measure. ($C(h)$ is the normalizing constant, assuring that the probability of $B_{\mathbf{C}^D}(0, R)$ is equal to 1.)

We also need the parameter

$$\epsilon_0(h) = (h^\kappa + h^n \ln \frac{1}{h})(\ln \frac{1}{\tau_0} + (\ln \frac{1}{h})^2), \tag{A.16}$$

which is of the same order of magnitude as the one in Section 4 when $\kappa = 1/2$, and assume that $\tau_0 = \tau_0(h)$ is not too small, so that $\epsilon_0(h)$ is small. Let $\Omega \Subset \mathbf{C}$ be open, simply connected not entirely contained in $\Sigma(p)$. The main result of [10] is:

THEOREM A.1. — *Under the assumptions above, let $\Gamma \Subset \Omega$ have (piecewise) smooth boundary, let $\kappa \in]0,1]$ be the parameter in (A.10), (A.11), (A.16) and assume that (A.9) holds uniformly for z in a neighborhood of $\partial\Gamma$. Then there exists a constant $C > 0$ such that for $C^{-1} \geqslant r > 0$, $\widetilde{\epsilon} \geqslant C\epsilon_0(h)$ we have with probability*

$$\geqslant 1 - \frac{C\epsilon_0(h)}{rh^{n+\max(n(M+1), N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} \tag{A.17}$$

that:

$$|\#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{(2\pi h)^n} \text{vol}\,(p^{-1}(\Gamma))| \leqslant \tag{A.18}$$

$$\frac{C}{h^n}\left(\frac{\widetilde{\epsilon}}{r} + C(r + \ln(\frac{1}{r})\text{vol}\,(p^{-1}(\partial\Gamma + D(0,r))))\right).$$

Here $\#(\sigma(P_\delta) \cap \Gamma)$ denotes the number of eigenvalues of P_δ in Γ, counted with their algebraic multiplicity.

Actually, the theorem holds for the slightly more general operators, obtained by replacing P by $P_0 = P + \delta_0(h^{\frac{n}{2}}q_1^0 + q_2^0)$, where $\|q_1^0\|_{H_h^s} \leqslant 1$, $\|q_2\|_{H^s} \leqslant 1$, $0 \leqslant \delta_0 \leqslant h$. Here, H^s is the standard Sobolev space and H_h^s is the same space with the natural semiclassical h-dependent norm.

We also have a result valid simultaneously for a family $\mathcal{C}$ of domains $\Gamma \subset \Omega$ satisfying the assumptions of Theorem A.1 uniformly in the natural sense: With a probability

$$\geqslant 1 - \frac{\mathcal{O}(1)\epsilon_0(h)}{r^2 h^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}}, \qquad (A.19)$$

the estimate (A.18) holds simultaneously for all $\Gamma \in \mathcal{C}$.

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