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A Liouville theorem for plurisubharmonic currents

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Abstract. — The goal of this paper is to extend the concepts of algebraic and Liouville currents, previously defined for positive closed currents by M. Blel, S. Mimouni and G. Raby, to psh currents on $\mathbb{C}^n$. Thus, we study the growth of the projective mass of positive currents on $\mathbb{C}^n$ whose support is contained in a tubular neighborhood of an algebraic subvariety. We also give a sufficient condition guaranteeing that a negative psh current is Liouville. Moreover, we prove that every negative psh algebraic current is Liouville. For the particular case of closed currents, under adequate support conditions, we obtain a structure theorem.

Résumé. — Le but de ces papiers est d'étendre les concepts de courants algébrique et Liouville précédemment définis pour les courants positifs fermés par M. Blel, S. Mimouni et G. Raby aux courants psh sur $\mathbb{C}^n$. Nous étudions alors la croissance de la masse projective des courants positifs définis sur $\mathbb{C}^n$ dont le support est contenu dans un voisinage tubulaire d'une sous-variété algébrique. Ensuite, nous donnons une condition suffisante, garantissant qu'un courant négatif et psh soit Liouville. De plus, on montre que tout courant négative psh et algébrique est Liouville. Dans le cas particulier des courants fermés, et sous des conditions adéquates sur le support, nous obtenons un théorème de structure.

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1. Introduction

The class of positive, or negative plurisubharmonic (psh for short) currents appears today as a tool for the study of analytic objects and as a natural extension of plurisubharmonic functions [Ga], [D-L], [D-E-E]... In [D-S], the authors used negative psh currents to describe polynomial hulls of compact sets in $\mathbb{C}^n$. In particular, positive pluriharmonic currents have so many important application in non Kähler geometry [H-L],[A-B] and also for the study of laminations with singularities of a compact set in $\mathbb{P}^2$ [F-S]. In the present work, we deal with the *algebraic* and *Liouville* properties of this class of currents related with certain support conditions. The following definition will be useful.

Definition 1.1. — Let T be a current of order zero and of bidimension (p,p) on $\mathbb{C}^n$. One says that T is algebraic if there exists a current $\widetilde{T}$ of order zero on $\mathbb{P}^n$ such that $\widetilde{T}=T$ on $\mathbb{C}^n$ and $\widetilde{T}=0$ on the hyperplane at infinity H_∞.

Let ω_{FS} be the Fubini-Study Kähler form on $\mathbb{P}^n$, its restriction to $\mathbb{C}^n$ is given by $\omega_{FS}=dd^c\log(1+|z|^2)$ up to a constant. The topic of our paper is *positive algebraic* currents, i.e. currents T on $\mathbb{C}^n\subset\mathbb{P}^n$ that have finite mass locally near the hyperplane at infinity. This is equivalently formulated by saying that the *projective mass* $\|T\|_{p.m}=\displaystyle\int_{\mathbb{C}^n}T\wedge\omega_{FS}^p$ is finite, (p,p) denoting the bidimension of T. Note that $T\wedge\omega_{FS}^p$ is the trace measure of T with respect to the Fubini-Study Kähler form ω_{FS}. On the other hand if we extend the concept of the *degree* for positive currents, then it is clear that there is a one-to-one correspondence between the class of positive algebraic currents and those of finite projective mass or equivalently of finite degrees.

Thanks to the Demailly-Lelong-Jensen formula [De], a positive pluriharmonic current T is algebraic if and only if the quantity $\nu_T(r):=\frac{\sigma_T(\{|z|<r\})}{\tau_p p! r^{2p}}$ is bounded independently of r, where $\sigma_T:=T\wedge\beta^p$ is the trace of T with respect to the flat metric $\beta=dd^c|z|^2$. Then, one recovers the definition given by [B-M-R] in case when T is closed.

Example 1.2 of positive algebraic currents. —

1. Let $\mathcal{L}=\{v\in psh(\mathbb{C}^2),\ v(z,w)\leqslant\log^+||(z,w)||+\mathcal{O}(1)$ at infinity$\}$. By [Le], for all $v\in\mathcal{L}$ the current dd^cv is algebraic. Conversely, all algebraic closed positive current of bidegree $(1,1)$ on $\mathbb{C}^2$ can be written cdd^cv with $c\geqslant 0$ and $v\in\mathcal{L}$.

2. Let g be the Hénon map defined by $g(z,w)=(z^2+c+aw,z)$, with $(a,c)\in\mathbb{C}^*\times\mathbb{C}$. We denote by $G^+(z,w)=\lim_{n\to\infty}(1/2^n)\log^+||g^n(z,w)||$ and $G^-(z,w)=\lim_{n\to\infty}(1/2^n)\log^+||g^{-n}(z,w)||$. By [B-S], we have $G^\pm\in\mathcal{L}$. It follows that the currents $T^\pm=dd^cG^\pm$ are algebraic on $\mathbb{C}^2$. These currents, which are also called Green currents, play a central role in the theory of complex dynamics. Let be $\widetilde{T}^-$ the trivial extension of T^- on $\mathbb{P}^2$ and φ a negative quasi-psh function on $\mathbb{P}^2$. By [C-G], one has $\varphi\in L^1(\widetilde{T}^-\wedge\omega_{FS})$. It follows that the current $\varphi_{|\mathbb{C}^2}T^-$ is negative and algebraic on $\mathbb{C}^2$.

3. Let $\chi\in\mathscr{D}([0,1])$, $\psi(z,w)=\chi(|z|^2)+i\chi(|w|^2)$, with $(z,w)\in\mathbb{C}^2$, and $T=i\partial\psi\wedge\overline{\partial\psi}$. Then, T is a positive pluriharmonic current on $\mathbb{C}^2$. Moreover, it is not hard to see that T has a total finite projective mass on $\mathbb{C}^2$, therefore it is algebraic and the trivial extension $\widetilde{T}$ is a positive pluriharmonic current on $\mathbb{P}^2$ (see [D-E-E]). Let us note here that in [F-S], the authors give explicitly the current $\widetilde{T}$ and used it for the evaluation of the infimum of the energy (i.e. $\inf\{\int_{\mathbb{P}^2}T\wedge T,\ T\geqslant 0 \text{ on } \mathbb{P}^2,\ dd^cT=0, \int_{\mathbb{P}^2}T\wedge\omega_{FS}=1\}$).

By [B-M-R], a closed positive current of bidegree $(1,1)$ and with tubular support (i.e. included in $\{|P|\leqslant c^{te}\}$ where P is a non constant polynomial in $\mathbb{C}[z_1,...,z_n]$) is shown to be an algebraic current. In the first section of the present work, we will consider positive currents T whose support is contained in the tube $\{|P_1|+\ldots+|P_s|\leqslant c^{te}\}$. With adequate conditions on the polynomials P_j, $j=1,...,s$, we study the growth of the projective mass of T and the quantity ν_T according to whether T or $-T$ is psh. In particular, we show that if $dd^cT=0$, then T is algebraic. More precisely we prove :

Theorem 1.3. — *Let $P_1,...,P_s$, $(s+k=n)$ be polynomials in $\mathbb{C}[z_1,...,z_n]$ having the same degree δ. Suppose that the intersection of the zeros of their homogeneous parts of top degrees form an algebraic subset of codimension s in $\mathbb{P}^{n-1}$. Let T be a positive current with bidimension (p,p) on $\mathbb{C}^n$ such that $p\geqslant k$. Assume that dd^cT is negative and that* $\operatorname{Supp}T\subset\{|P_1|+...+|P_s|\leqslant 1\}$. *Then, there exists a constant $c>0$ such that for all $r\geqslant 1$ we have : $\nu_T(r)\leqslant c$. If furthermore T is also pluriharmonic, then T is an algebraic current.*

In the special case where T is a negative plurisubharmonic function, Theorem 1.3 is classical without any assumption on the support. Notice that in [E-M], the authors establish Theorem 1.3 in the case when $P_1=$

$z_{k+1}^{\delta}, ..., P_s = z_n^{\delta}$ (i.e. the support of T is contained in a strip). In the context of dynamics, this class of currents is interesting and many of them can be constructed as the invariant currents of certain polynomial endomorphisms [Du],... Furthermore, S. Giret shows in his thesis [Gi] that the class of positive closed currents with support in a strip are well preserved by pulling-back by a blow up with smooth center.

On the other hand, it is important to point out that Theorem 1.3 deals with a much larger class of currents than the class of closed currents. In fact when T is $d-$closed, Theorem 1.3 is an immediate consequence of Theorem 2.4 in [B-M-R], by almost the same proof as in corollary 2.5 of that paper. Indeed, the condition that the algebraic hypersurfaces $\{P_i = 0\}$ intersect properly at infinity is clearly preserved by taking the intersection with a general hyperplane H of $\mathbb{C}^n$.

Denote by $\|T\|_{p.m}(r) = \int_{\{|z|\leqslant r\}} T \wedge w_{FS}^p$ the projective mass of T carried by $\{|z| \leqslant r\}$ and by $N_T(r) = \int_1^r \nu_T(t)/t\ dt$ the counting map associated to T. As indicated in the introduction if T is positive and pluriharmonic the quantity $\nu_T(r)$ coincides with $\|T\|_{p.m}(r)$, hence a direct computation shows that T is algebraic if and only if $\nu_T(r) = \mathcal{O}(1)$ or, equivalently, $N_T(r) = \mathcal{O}(log\ r)$ (this equivalence will be proved later). In the general situation, we obtain the following estimates:

Proposition 1.4. — *Let T be a positive current of bidimension (p, p) on $\mathbb{C}^n$.*

1. *In both cases when T is psh or dd^cT is negative, we have the growth estimate $\nu_{dd^cT}(r) = \mathcal{O}(\nu_T(\sqrt{2}r))$. In particular, if ν_T is bounded then dd^cT has finite total projective mass i.e. dd^cT is algebraic.*
2. *If T is psh then $\|T\|_{p.m}(r) = \mathcal{O}(\nu_T(r))$. In particular, when ν_T is bounded or equivalently, N_T has logarithmic growth, then T is algebraic. If dd^cT is negative, then there exists $c, c' > 0$ such that for every $r \geqslant 2$, we have : $\|T\|_{p.m}(r) \leqslant c + c'(\nu_T(r) + N_T(\sqrt{2}r))$. In particular if ν_T is bounded, then the projective mass of T carried by $\{|z| \leqslant r\}$ has at most logarithmic growth i.e. $\|T\|_{p.m}(r) = \mathcal{O}(log\ r)$.*

Let T be a positive current on $\mathbb{C}^n$, we say that T is *Liouville* if for every holomorphic function f on $\mathbb{C}^n$, bounded on the support of T one has: $T \wedge dd^c|f|^2 = 0$. The previous definition coincides then with the definition given in [B-M-R] when T is closed. In the same paper, the authors prove that a closed positive algebraic current is a Liouville current. For negative psh currents we obtain our second main result:

Theorem 1.5. — *Let T be a negative psh current of bidimension (p,p) on $\mathbb{C}^n$. Let u be a $\mathcal{C}^2$ plurisubharmonic function on $\mathbb{C}^n$ and bounded on the support of T. Assume that we have the following growth condition: $\|T\|_{p.m}(r) = \mathcal{O}((\log\log r)^s)$ for some $s \geqslant 0$, then the currents $T \wedge du \wedge d^c u$ and $T \wedge dd^c u$ vanish, and therefore T is Liouville. In particular, every algebraic negative psh current is Liouville.*

Notice that Theorem 1.5 asserts the following elementary statement: there are no negative psh compactly supported currents of bi-dimension (p,p), if $p > 0$ (we will take $u = |z|^2$ and remark that T has a globally finite projective mass).

Another immediate consequence of Theorem 1.5 is the following: let P is a non constant polynomial in $\mathbb{C}[z_1, ..., z_n]$ and T be a negative psh current of bi-dimension (p,p) on $\mathbb{C}^n$ with support contained in $\{|P| \leqslant 1\}$. Then the current $T \wedge dP \wedge d\overline{P}$ vanishes when T is algebraic (Theorem 1.5 for $u = |P|^2$). This allows us to prove a structure theorem: if $F : \mathbb{C}^n \to \mathbb{C}^k$ is a equi-dimensional polynomial map and T is a *closed* positive algebraic current of bidegree (k,k) on $\mathbb{C}^n$ supported on the inverse image by F of a compact subset of $\mathbb{C}^k$, then T can be split into two currents, the first of which can be written as an average of integration currents on components of fibres of F and the other is supported by an algebraic set containing the critical points of F. More precisely, we prove the following theorem:

Theorem 1.6. — *Let $F = (P_1, ..., P_k) : \mathbb{C}^n \to \mathbb{C}^k$ be a polynomial mapping such that for all $t \in \mathbb{C}^k$ the codimension of the fiber $F^{-1}(t)$ in $\mathbb{C}^n$ is k. Let T be a positive closed algebraic current of bidegree (k,k) on $\mathbb{C}^n$ such that* $\operatorname{Supp} T \subset \{|P_1| + ... + |P_k| \leqslant 1\}$. *Denote by V the space of connected components of different fibers $P^{-1}(t)$ in $\mathbb{C}^n$, $t \in \mathbb{C}^k \smallsetminus F(X)$, where X is the set of critical values of F, then there exists a unique positive measure μ on V and a positive closed algebraic current R supported by an algebraic set containing the critical points of F such that $T = \int_{v \in V} [P^{-1}(t)]_v d\mu(v) + R$.*

In the following result, we give conditions on T weaker than those on the support, guaranteeing that T is a Liouville current. Let $\pi : \mathbb{C}^n \to \mathbb{C}$ be the orthogonal projection. By [B-E], the slice $\langle T, \pi, z' \rangle$ exists outside of a pluripolar set in $\mathbb{C}$. Let us denote by $\beta' = dd^c|z'|^2$, $\beta'' = dd^c|z''|^2$ and $v_\varepsilon(z', z'') = |z'|^2 + \varepsilon|z''|^2$ for $\varepsilon > 0$.

Theorem 1.7. — *Let T be a closed positive current of bidegree $(1,1)$ on $\mathbb{C}^n$. Assume that for all $R > 0$ there exists a function $\varepsilon(R)$ such that*

$0 < \varepsilon(R) \longrightarrow 0$ if $R \longrightarrow \infty$, with $\int_{\{v_{\varepsilon(R)}<R\}} T \wedge \beta''^{n-1} = o(\frac{R^{n-1}}{\varepsilon(R)})$ and a constant $\gamma > 0$ such that for almost all z', the mass $||\langle T, \pi, z'\rangle||(\{v_{\varepsilon(R)} < R\}) = \mathcal{O}(R^{n-2-\gamma})$. Then, for all positive psh function u on $\mathbb{C}^n$ and bounded on the support of T, we have $T \wedge dd^c u = 0$. In particular, T is a Liouville current.

Example 1.8. — In $\mathbb{C}^2$, the class of closed positive currents (and in the same way negative psh) of bidegree $(1,1)$ and having a support in a strip $\{|z''| \leqslant 1\}$ satisfies the hypothesis of theorem 1.7.

2. Preliminaries

Let Ω be an open set of $\mathbb{C}^n$. As usual $\mathscr{D}_{p,q}(\Omega)$ denotes the space of smooth and compactly supported $(p,q)-$form on Ω. The dual $\mathscr{D}'_{p,q}(\Omega)$ is the space of currents of bidimension (p,q) or of bidegree $(n-p, n-q)$. For $T \in \mathscr{D}'_{p,p}(\Omega)$, one says that T is positive if for all $\alpha_1, ..., \alpha_p$ in $\mathscr{D}_{1,0}(\Omega)$, the distribution $T \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge ... \wedge i\alpha_p \wedge \overline{\alpha}_p$ determines a positive measure on Ω. We say that T is plurisubharmonic (psh for short) if the current dd^cT is positive, and pluriharmonic if $dd^cT = 0$. Let $\beta = dd^c|z|^2$, (where $d = \partial + \overline{\partial}, d^c = (i/2)(\overline{\partial} - \partial)$), be the Kähler form on $\mathbb{C}^n$. Let $T \in \mathscr{D}'_{p,p}(\Omega)$ be of order zero on Ω. Then

$$T = i^{(n-p)^2} \sum_{|I|=|J|=n-p} T_{IJ} dz_I \wedge d\bar{z}_J,$$

with T_{IJ} are complex measures. One defines the mass measure of the current T by $\|T\| = \sum_{|I|=|J|=n-p} |T_{IJ}|$, and the trace measure by:

$$\sigma_T = \frac{1}{4^p p!} T \wedge \beta^p = (2^{-p} \sum_{|I|=n-p} T_{II})\tau_n.$$

Let A be a closed subset in Ω and T a current of order zero on $\Omega \smallsetminus A$. Let $\widetilde{T}$ be the trivial extension of T by zero across A. We say that $\widetilde{T}$ exists if T has locally finite mass on Ω. In the remaining part of this paper, we denote by $\|T\|_{p.m}(r) = \int_{\{|z|\leqslant r\}} T \wedge w_{FS}^p$ the projective mass of T carried by $\{|z| \leqslant r\}$ and $\|T\|_{p.m} = \int_{\mathbb{C}^n} T \wedge w_{FS}^p$ the total projective mass on $\mathbb{C}^n$. Let T be a positive current of bidimension (p,p) such that the measure $dd^cT \wedge \beta^{p-1}$ is positive on Ω. Let φ be a $\mathcal{C}^2$ function on Ω such that $\log \varphi$ is plurisubharmonic on $\{z \in \Omega;\ \varphi(z) > 0\}$. Let

$$B(r) = \{z \in \Omega;\ \varphi(z) < r\}, \qquad w = dd^c\varphi \qquad \text{and} \quad \alpha = dd^c \log \varphi.$$

For $0 < r_1 < r_2$ such that $\operatorname{Supp} T \cap B(r_2)$ is relatively compact in Ω, one has the following Lelong-Jensen formula [De] which is our basic tool in this paper:

$$\begin{aligned}\frac{1}{r_2^p}\int_{B(r_2)} T\wedge w^p - \frac{1}{r_1^p}\int_{B(r_1)} T\wedge w^p = &\int_{B(r_1,r_2)} T\wedge \alpha^p \\ &+ \int_{r_1}^{r_2}\Big(\frac{1}{t^p}-\frac{1}{r_2^p}\Big)\int_{B(t)} dd^cT\wedge w^{p-1} \\ &+ \Big(\frac{1}{r_1^p}-\frac{1}{r_2^p}\Big)\int_0^{r_1} dt\int_{B(t)} dd^cT\wedge w^{p-1}.\end{aligned}$$

Recall that a homogeneous polynomial of degree δ in n variables depends on $(\delta+n-1)!/\delta!(n-1)!)$ coefficients. Hence, a polynomial homogeneous map $F = (F_1, ..., F_n)$ can be identified with an an element of $\mathbb{C}^N$, where $N = n(\delta+n-1)!/\delta!(n-1)!$. Moreover, by [G-K-Z] p.427, there exists a unique polynomial $Res(F_1, ..., F_n)$ in the coefficients of $F_1, ..., F_n$, such that $Res(F_1, ..., F_n) = 0$ if and only if the map F in degenerate. With the last identification, the space of all homogeneous, non degenerate, polynomial maps of degree δ on $\mathbb{C}^n$ is an open subset of $\mathbb{C}^N$.

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3. Proof of Theorem 1.3

The proof of Theorem 1.3 is divided into two steps :

Proof. — **Step 1: case when $P_1 = z_{k+1}^\delta, ..., P_s = z_n^\delta$.**
First let us suppose that $p = k$, otherwise T vanishes. In fact, the current T is $\mathbb{C}-$flat, therefore if π is the projection on $\mathbb{C}^k$, the slice $\langle T, \pi, z'\rangle$ exists for almost all $z' \in \mathbb{C}^k$ and it is a positive current having a negative dd^c and a compact support in $\mathbb{C}^n$. By [D-E-E], we have $\langle T, \pi, z'\rangle = 0$ for almost all z' and also $T = 0$ by applying the slicing formula for the $\mathbb{C}-$flat currents. Let us now continue the proof for the interesting case $p = k$. For instance assume that T is smooth. Let $\chi \in \mathscr{D}(\mathbb{R})$, $\chi(t) = 1$ if $|t| \leqslant 1$ and $\chi = 0$ if $|t| > 2$. Let $\beta' = dd^c|z'|^2$, and for $a = (a_1, ..., a_p) \in \mathbb{C}^p$, let us denote $g(a) = \displaystyle\int_{\mathbb{C}^n} T\wedge\chi(|z'+a|^2)\beta'^p$. Then

$$\begin{aligned}
2i\frac{\partial^2 g}{\partial a_1\overline{\partial} a_1} &= \int_{\mathbb{C}^n} T\wedge \frac{2i\partial^2}{\partial a_1\overline{\partial} a_1}\chi(|z'+a|^2)\beta'^p\\
&= \int_{\mathbb{C}^n} T\wedge \frac{2i\partial^2}{\partial z_1\overline{\partial} z_1}\chi(|z'+a|^2)\beta'^p\\
&= \int_{\mathbb{C}^n} T\wedge dd^c\Big(\chi(|z'+a|^2)\frac{i}{2}dz_2\wedge d\overline{z}_2\wedge ...\wedge \frac{i}{2}dz_p\wedge d\overline{z}_p\Big)\\
&= \int_{\mathbb{C}^n} \chi(|z'+a|^2)dd^cT\wedge \frac{i}{2}dz_2\wedge d\overline{z}_2\wedge ...\wedge \frac{i}{2}dz_p\wedge d\overline{z}_p
\end{aligned}$$

Taking into account the fact that dd^cT is negative, the last integral is negative. Hence the function $a_1\mapsto -g(a_1,a_2,...,a_p)$ is negative and subharmonic on $\mathbb{C}$, therefore it is constant with respect to a_1 and is equal to $g(0,a_2,...,a_p)$. By iteration, one shows that g is independent from the variables $a_1,...,a_p$. Then, $g(a)=g(0)=\displaystyle\int_{\mathbb{C}^n} T\wedge\chi(|z'|^2)\beta'^p$. Thus, there exists a constant $C>0$ such that $\int_{|z'|\leqslant 1,z''} T\wedge\beta'^p\leqslant C$. let be $j\in\{p+1,...,n\}$, then

$$\begin{aligned}
\int_{\mathbb{C}^n} T\wedge\chi^2(|z'|^2)dd^c|z_j|^2\wedge\beta'^{p-1} &= \int_{\mathbb{C}^n} T\wedge dd^c(|z_j|^2\chi^2(|z'|^2)\beta'^{p-1})\\
&\quad -\int_{\mathbb{C}^n} |z_j|^2T\wedge dd^c(\chi^2(|z'|^2))\wedge\beta'^{p-1}\\
&\quad -\int_{\mathbb{C}^n} T\wedge d\chi^2(|z'|^2)\wedge d^c|z_j|^2\wedge\beta'^{p-1}\\
&\quad -\int_{\mathbb{C}^n} T\wedge d|z_j|^2\wedge d^c\chi^2(|z'|^2)\wedge\beta'^{p-1}\\
&= (1)+(2)+(3)+\overline{(3)}.
\end{aligned}$$

On the other hand, using Stokes's theorem and the fact that $|z_j|^2\chi^2(|z'|^2)\beta'^{p-1}$ has a compact support relatively to T, we find

$$(1)=\int_{\mathbb{C}^n} T\wedge dd^c\Big(|z_j|^2\chi^2(|z'|^2)\beta'^{p-1}\Big)=\int_{\mathbb{C}^n} dd^cT\wedge|z_j|^2\chi^2(|z'|^2)\beta'^{p-1}\leqslant 0.$$

Hence, we get the inequality

$$\int_{\mathbb{C}^n} T\wedge\chi^2(|z'|^2)dd^c|z_j|^2\wedge\beta'^{p-1}\leqslant (2)+(3)+\overline{(3)}. \tag{3.1}$$

The term numbered (2) satisfies:

$$(2) = -\int_{\mathbb{C}^n} |z_j|^2 T \wedge dd^c(\chi^2(|z'|^2)) \wedge \beta'^{p-1} \leqslant C \int_{1\leqslant|z'|\leqslant 2} T \wedge \beta'^p \leqslant C_1.$$

The existence of constant C follows from the fact that $|z_j|$ is bounded on the support of T, observing that $|\chi|$, $|\chi'|$ and $|\chi''|$ are bounded. To obtain C_1, we may slightly modify χ by taking $\chi(t) = 1$ if $|t| \leqslant 2$ and $\chi = 0$ if $|t| > 3$ and repeat the above argument. Let $\varphi \in \mathscr{D}(\mathbb{R})$, $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on $\operatorname{Supp}\chi$. According to the Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
\Big|(3)\Big| &\leqslant \Big|\int_{\mathbb{C}^n} T \wedge 2\chi(|z'|^2)\varphi(|z'|^2) d\chi(|z'|^2) \wedge d^c|z_j|^2 \wedge \beta'^{p-1}\Big| \\
&\leqslant (1/\varepsilon)\int_{\mathbb{C}^n} T \wedge 2\varphi^2(|z'|^2) d\chi(|z'|^2) \wedge d^c\chi(|z'|^2) \wedge \beta'^{p-1} \\
&\qquad\qquad + \varepsilon \int_{\mathbb{C}^n} T \wedge 2\chi^2(|z'|^2) d|z_j|^2 \wedge d^c|z_j|^2 \wedge \beta'^{p-1} \\
&\leqslant (C_2/\varepsilon)\int_{1\leqslant|z'|\leqslant 2} T \wedge \beta'^p + 4\varepsilon \int_{\mathbb{C}^n} T \wedge \chi^2(|z'|^2) dz_j \wedge d\overline{z}_j \wedge \beta'^{p-1}.
\end{aligned}$$

Choosing $\varepsilon = 1/8$ and by (3.1), we get:

$$\int_{\mathbb{C}^n} T\wedge\chi^2(|z'|^2)dd^c|z_j|^2\wedge\beta'^{p-1} \leqslant C_1+8C_2+\frac{1}{2}\int_{\mathbb{C}^n} T\wedge\chi^2(|z'|^2)dd^c|z_j|^2\wedge\beta'^{p-1}.$$

Put $C_3 = 2(C_1 + 8C_2)$. As $dd^c|z''|^2 = \sum_{j=p+1}^n dd^c|z_j|^2$, we have:

$$\int_{\mathbb{C}^n} T \wedge \chi^2(|z'|^2) dd^c|z''|^2 \wedge \beta'^{p-1} \leqslant (n-p)C_3.$$

In order to show that the integral $\int_{\mathbb{C}^n} T \wedge \chi^2(|z'|^2)(dd^c|z''|^2)^2 \wedge \beta'^{p-2}$ is finite, we use the last inequality and we rewrite the previous proof with β'^{p-1} replaced by $dd^c|z''|^2 \wedge \beta'^{p-2}$. While proceeding by induction, we show that there exists a constant $C_4 > 0$ such that for $1 \leqslant s \leqslant p$ we have:

$$\int_{\mathbb{C}^n} T \wedge \chi^2(|z'|^2)(dd^c|z''|^2)^s \wedge \beta'^{p-s} \leqslant C_4.$$

It follows that there exists $C_5 > 0$ such that:

$$\int_{|z'|\leqslant 1, |z''|\leqslant 1} T \wedge \beta^p \leqslant \int_{\mathbb{C}^n} T \wedge \chi^2(|z'|^2)\beta^p \leqslant C_5.$$

By the above induction argument, in order to prove the last inequality for T not smooth, we get the following $\displaystyle\int_{|z'|\leqslant 1, z''} T \wedge \beta'^p \leqslant C$. In fact let T_ε

be a regularization of T and let g_ε be the function associated with T_ε. The sequence T_ε converges weakly to T, and it is not hard to see that the sequence $g_\varepsilon(a)$ tends to $g(a)$ (we may replace $\chi(|z'+a|^2)\beta'^p$ by $\chi(|z''|)\chi(|z'+a|^2)\beta'^p$ to get a compactly supported test form, and we observe that the integral is unchanged since T has support in a strip). By the above argument we find that g_ε is constant with respect to a, thus g is constant as well and therefore the desired inequality follows. For $r > 1$, one can cover $\{z, |z'| < r\}$ by at most $([r]+1)^{2p}$ unit cubes, where $[r]$ denotes the integer part of r. Therefore $\int_{B(0,r)} T \wedge \beta^p \leqslant C_5([r]+1)^{2p}$.

Step 2: general case. The hypothesis implies that there exists an homogeneous polynomial system $(Q_{\alpha_1}, ..., Q_{\alpha_p})$ of $\mathbb{C}[z_1, ..., z_n]$ such that each polynomial Q_{α_j} is of degree δ and so that the homogeneous parts of higher degrees of the polynomials $Q_{\alpha_1}, ..., Q_{\alpha_p}, P_1, ..., P_s$ vanish simultaneously at the *single* point 0. Therefore, the map f_α defined on $\mathbb{C}^n$ by

$$f_\alpha(z) := (Q_{\alpha_1}(z), ..., Q_{\alpha_p}(z), P_1(z), ..., P_s(z)),$$

is proper and finite. The current $(f_\alpha)_*T$ is positive of bidimension (k,k) with negative dd^c on $\mathbb{C}^n$. Moreover, $\mathrm{Supp}(f_\alpha)_*T \subset \{|z_{p+1}| + ... + |z_n| \leqslant 1\}$. Let $|\alpha| = |(\alpha_1, ..., \alpha_p)| = \alpha_1 + ... + \alpha_p$. We claim that: *There exists a different system of homogeneous polynomial $(Q_{\alpha_1}, ..., Q_{\alpha_k})_{1\leqslant|\alpha|\leqslant\mu}$, that each of one satisfy the above condition of the map f_α and such that :*

$$|z|^{2p\delta-2p}(dd^c|z|^2)^p \leqslant \sum_{1\leqslant|\alpha|\leqslant\mu} dQ_{\alpha_1} \wedge \overline{dQ}_{\alpha_1} \wedge ... \wedge dQ_{\alpha_p} \wedge \overline{dQ}_{\alpha_p}.$$

In fact, in view of the characterization of the non degenerate polynomial homogeneous maps (see the end of section 2), one can makes an appropriate large choice of different system $(Q_{\alpha_1}, ..., Q_{\alpha_k})_{1\leqslant|\alpha|\leqslant\mu}$ (μ is big enough) so that the homogeneous part of degree δ of the map f_α is non degenerate and all the monomials of degree δ appear in the decomposition of the product $dQ_{\alpha_1} \wedge \overline{dQ}_{\alpha_1} \wedge ... \wedge dQ_{\alpha_p} \wedge \overline{dQ}_{\alpha_p}$. More precisely, for a sufficiently large selection of different coefficients of Q_{α_1}, we can obtain the inequality: $\sum_{\alpha_1} dQ_{\alpha_1} \wedge \overline{dQ}_{\alpha_1} \geqslant |z|^{2\delta-2} dd^c|z|^2$, and similarly for the other α_j. Let us now continue the proof of step 2. Let $r \geqslant 1$, we have :

$$\begin{aligned}\int_{B(r)} T \wedge |z|^{2p\delta-2p}(dd^c|z|^2)^p &\leqslant \int_{B(r)} T \wedge \sum_{1\leqslant|\alpha|\leqslant\mu} dQ_{\alpha_1} \wedge \overline{dQ}_{\alpha_1} \wedge ... \wedge dQ_{\alpha_p} \wedge \overline{dQ}_{\alpha_p} \\ &\leqslant \int_{B(r)} T \wedge \sum_{1\leqslant|\alpha|\leqslant\mu} f_\alpha^*(dd^c|w|^2)^p \\ &= \sum_{1\leqslant|\alpha|\leqslant\mu} \int_{f_\alpha(B(r))} (f_\alpha)_*T \wedge (dd^c|w|^2)^p.\end{aligned}$$

Let

$$|f_\alpha| = \sum_{j=1}^{s} |P_j| + \sum_{l=1}^{p} |Q_{\alpha_l}|.$$

So, we have $|f_\alpha(z)| \leqslant c_\alpha(1+|z|^2)^\delta$ for suitable constants $c_\alpha > 0$. This implies that $B(r) \subset K_r = \bigcap_{1\leqslant|\alpha|\leqslant\mu} f_\alpha^{-1}(B(c_\alpha^1 r^\delta))$, where c_α^1 are positive constants. By replacing r with $c_\alpha^1 r^\delta$ in the previous inequality, we obtain:

$$\begin{aligned}
\int_{B(r)} T\wedge|z|^{2p\delta-2p}(dd^c|z|^2)^p &\leqslant \int_{K_r} T\wedge|z|^{2p\delta-2p}(dd^c|z|^2)^p \\
&\leqslant \sum_{1\leqslant|\alpha|\leqslant\mu}\int_{f_\alpha(K_r)} (f_\alpha)_*T\wedge(dd^c|w|^2)^p \\
&\leqslant \sum_{1\leqslant|\alpha|\leqslant\mu}\int_{B(c_\alpha^1 r^\delta)} (f_\alpha)_*T\wedge(dd^c|w|^2)^p.
\end{aligned}$$

Let be $j_0 \in \mathbb{N}^*$ such that $r/2^{j_0} \leqslant 1$ and for $j = 1, ..., j_0$, setting $B(r/2^j, r/2^{j-1}) = B(r/2^{j-1}) \smallsetminus B(r/2^j)$. Then,

$$\begin{aligned}
&(r/2^j)^{2p\delta-2p}\int_{B(r/2^j,r/2^{j-1})} T\wedge(dd^c|z|^2)^p \\
&\qquad\leqslant \int_{B(r/2^j,r/2^{j-1})} T\wedge|z|^{2p\delta-2p}(dd^c|z|^2)^p \\
&\qquad\leqslant \sum_{1\leqslant|\alpha|\leqslant\mu}\int_{B(c_\alpha^1(r/2^{j-1})^\delta)} (f_\alpha)_*T\wedge(dd^c|w|^2)^p \\
&\qquad\leqslant (r/2^{j-1})^{2p\delta} c \sum_{1\leqslant|\alpha|\leqslant\mu}(c_\alpha^1)^{2p}.
\end{aligned}$$

The last inequality is a consequence of **step 1** (one can choose c_α^1 big enough so that $c_\alpha^1\left(r/2^{j_0-1}\right)^\delta \geqslant 1$). We put $c_1 = c\sum_{1\leqslant|\alpha|\leqslant\mu}(c_\alpha^1)^{2p}$, therefore

$$\begin{aligned}
\textstyle\int_{B(r/2^{j-1},r/2^j)} T\wedge(dd^c|z|^2)^p &\leqslant c_1\left(r/2^j\right)^{2p-2p\delta}\left(r/2^{j-1}\right)^{2p\delta} \\
&= c_1 2^{2p-2p\delta} r^{2p}\left(1/2^{j-1}\right)^{2p}.
\end{aligned}$$

As $B(r, r/2^{j_0}) = \cup_{j=1}^{j_0} B(r/2^{j-1}, r/2^j)$ and $r/2^{j_0} \leqslant 1$, it is easy to see that :

$$\int_{B(r)\smallsetminus B(1)} T\wedge(dd^c|z|^2)^p \leqslant \int_{B(r,r/2^{j_0})} T\wedge(dd^c|z|^2)^p \leqslant c_1 r^{2p}.$$

Now, we conclude the proof by an enough perturbation of the center so that we cover all the balls $B(r)$. In particular, if T is pluriharmonic, then according to the Lelong-Jensen formula, it is easy to see that T has a finite total projective mass, therefore it is algebraic. $\square$

Remark 3.1. — In Theorem 1.3, the hypothesis that $dd^cT \leqslant 0$ is necessary as the following example shows: let $D(0,1)$ be the unit disk in $\mathbb{C}$ and let h be a positive subharmonic function on $\mathbb{C}$. Pick $f, g \in \mathscr{D}(D(0,1)) \geqslant 0$ such that $g(z_2)dd^c|z_2|^2 \geqslant -dd^c f(z_2)$, and

$$T = f(z_2)dd^c|z_1|^2 + g(z_2)(h(z_1) + |z_1|^2)dd^c|z_2|^2.$$

Then T is a positive psh current of bidegree $(1,1)$ having a support in the strip $\{(z_1,z_2) \in \mathbb{C}^2, |z_2| < 1\}$, but $\nu_T(r)$ is not bounded.

As mentioned above, the logarithmic growth of the counting function $N_T(r)$ characterizes algebraic positive pluriharmonic currents on $\mathbb{C}^n$. The following result clarifies the relation between the growths of the projective mass of T on $\{|z| < r\}$ and the quantity $\nu_T(r)$, when T is a positive or negative psh current.

Proposition 3.2. — *Let T be a positive current of bidimension (p,p) on $\mathbb{C}^n$.*

1. *In both cases when T is psh or dd^cT is negative, we have the growth estimate $\nu_{dd^cT}(r) = \mathcal{O}(\nu_T(\sqrt{2}r))$. In particular, if ν_T is bounded then dd^cT has finite total projective mass i.e. dd^cT is algebraic.*
2. *If T is psh then $\|T\|_{p.m}(r) = \mathcal{O}(\nu_T(r))$. In particular, when ν_T is bounded or equivalently, N_T has logarithmic growth, then T is algebraic. If dd^cT is negative, then there exists $c, c' > 0$ such that for every $r \geqslant 2$, we have : $\|T\|_{p.m}(r) \leqslant c + c'(\nu_T(r) + N_T(\sqrt{2}r))$. In particular if ν_T is bounded, then the projective mass of T carried by $\{|z| \leqslant r\}$ has at most logarithmic growth, i.e. $\|T\|_{p.m}(r) = \mathcal{O}(log\ r)$.*

Proof. — (1) Assume that dd^cT is negative and consider a function $\chi \in \mathscr{D}(\mathbb{R})$ such that $\chi(t) = 1$ if $|t| \leqslant 1$, and $\chi = 0$ if $|t| > 2$. Let be $\nu_{dd^cT}(r) = \frac{1}{r^{2p-2}} \int_{B(0,r)} dd^cT \wedge \beta^{p-1}$. Thanks to Stokes' theorem we have:

$$\begin{aligned}
\nu_{dd^cT}(r) &\geqslant \frac{1}{r^{2p-2}} \int_{B(0,\sqrt{2}r)} dd^cT \wedge \chi(\frac{|z|^2}{r^2})\beta^{p-1} \\
&= \frac{1}{r^{2p-2}} \int_{B(0,\sqrt{2}r)} T \wedge dd^c\chi(\frac{|z|^2}{r^2}) \wedge \beta^{p-1} \\
&= \frac{1}{r^{2p-2}} \int_{B(0,\sqrt{2}r)} T \wedge \chi'(\frac{|z|^2}{r^2}) \frac{\beta^p}{r^2} \\
&\quad + \frac{1}{r^{2p-2}} \int_{B(0,\sqrt{2}r)} T \wedge \chi''(\frac{|z|^2}{r^2}) \frac{d|z|^2 \wedge d^c|z|^2}{r^4} \wedge \beta^{p-1}
\end{aligned}$$

$$= \frac{1}{r^{2p}} \int_{B(0,\sqrt{2}r)} T \wedge \chi'(\frac{|z|^2}{r^2})\beta^p$$
$$+\frac{1}{r^{2p}} \int_{B(0,\sqrt{2}r)} T \wedge \chi''(\frac{|z|^2}{r^2})\frac{d|z|^2 \wedge d^c|z|^2}{r^2} \wedge \beta^{p-1}.$$

As $|\chi'|$ and $|\chi''|$ are bounded, and $d|z|^2 \wedge d^c|z|^2 \leqslant |z|^2 dd^c|z|^2$, then

$$\nu_{dd^cT}(r) \geqslant -c\nu_T(\sqrt{2}r) - c'\nu_T(\sqrt{2}r) \geqslant -c''\nu_T(\sqrt{2}r).$$

In the case of a psh current, we can reverse the above inequalities. Thus the desired estimate follows. In particular, if ν_T is bounded then the current dd^cT is algebraic (since it is closed). Observe that the fact that ν_T is bounded is equivalent to N_T having logarithmic growth. Indeed, since T is psh, then ν_T is increasing, so we have $N_T(r) \leqslant \nu_T(r)\log r \leqslant \int_r^{r^2} \nu_T(t)/t\, dt \leqslant N_T(r^2)$.

(2) Assume that T is psh. By applying the Lelong-Jensen formula [De] to the function $\varphi(z) = 1 + |z|^2$, one easily shows that the projective mass of T growth at most as ν_T (since the quantities involving dd^cT are positive). Assume now that dd^cT is negative and choose $c > 0$ such that $\nu_{dd^cT}(r) \geqslant -c\nu_T(\sqrt{2}r)$, for all $r > 0$. Let $2 \leqslant r_1 \leqslant t \leqslant r_2$, we denote by $B(t) = \{\varphi(z) < t\}$ and $B(r_1, r_2) = \{r_1 < \varphi(z) < r_2\}$. Using the negativity of the measure $dd^cT \wedge \beta^{p-1}$, a direct computation gives :

$$\int_{r_1}^{r_2} \left(\frac{1}{t^p} - \frac{1}{r_2^p}\right) dt \int_{B(t)} dd^cT \wedge \beta^{p-1}$$
$$\geqslant -c\int_2^{r_2} \frac{(t-1)^{p-1}}{t^p} \nu_T(\sqrt{2t-2})dt$$
$$\geqslant -2c\int_1^{\sqrt{2r_2-2}} \nu_T(t)/tdt = -2cN_T(\sqrt{2r_2-2}).$$

$$\left(\frac{1}{r_1^p} - \frac{1}{r_2^p}\right) \int_1^{r_1} dt \int_{B(t)} dd^cT \wedge \beta^{p-1}$$
$$\geqslant c_1 - \frac{c}{r_1^p}\int_2^{r_1} (t-1)^{p-1}\nu_T(\sqrt{2t-2})dt$$
$$\geqslant c_1 - cN_T(\sqrt{2r_1-2}) \geqslant c_1 - cN_T(\sqrt{2r_2-2}),$$

where $c_1 = \frac{1}{2^p}\int_1^2 dt \int_{B(t)} dd^cT \wedge \beta^{p-1}$. Thus using the Lelong-Jensen formula, we deduce the following estimate:

$$\int_{B(r_1,r_2)} T \wedge (dd^c \log(1+|z|^2))^p \leqslant -c_1 + \nu_T(\sqrt{r_2-1}) + 3cN_T(\sqrt{2r_2-2}).$$

For $r_1 = 2$ and $r_2 = r^2 + 1$, we have $\|T\|_{p.m}(r) \leqslant c_2 + c_3(\nu_T(r) + N_T(\sqrt{2}r))$. In particular if ν_T is bounded, it is clear that $N_T(r) = \mathcal{O}(\log r)$. Thus, the projective mass has at most logarithmic growth. $\square$

It is a classical fact that closed positive currents on $\mathbb{P}^n$ can be described as conical currents on $\mathbb{C}^{n+1}$. For negative psh currents, the following result is an immediate consequence of proposition 3.2:

Corollary 3.3. — *Let T be an algebraic negative psh current of bidimension (p,p) on $\mathbb{C}^n$, then there exists a positive pluriharmonic current Θ of bidimension $(p+1,p+1)$ and conical on $\mathbb{C}^{n+1}$, which is the trivial extension of $\pi^*(\widetilde{T})$ across 0, π is canonic projection $\pi : \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$.*

Proof. — By considering the current $-T$ instead, we may assume that T is positive. Since T is algebraic, by the Lelong-Jensen formula ν_T must be bounded then according to proposition 3.2, the current dd^cT is algebraic. By [D-E-E], the residual current $S = \widetilde{dd^cT} - dd^c\widetilde{T}$ is positive and closed, and supported by H_∞. This implies that the current $dd^c\widetilde{T}$ is negative on $\mathbb{P}^n$. Thanks to Stokes' theorem, we have $0 = \int_{\mathbb{P}^n} dd^c\widetilde{T} \wedge \omega_{FS}^{p-1} \leqslant 0$. Then the current $\widetilde{T}$ is positive and pluriharmonic on $\mathbb{P}^n$. Let $\pi : \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$ be the canonical submersion. Then the current $\pi^*(\widetilde{T})$ is positive pluriharmonic of bidimension $(p+1, p+1)$ on $\mathbb{C}^{n+1} \smallsetminus \{0\}$, and by [D-E-E] it can be extended to a positive pluriharmonic current Θ on $\mathbb{C}^{n+1}$. It is clear that the current Θ is conical, i.e. $h_r^*\Theta = \Theta$, for every $r \in \mathbb{C}^*$, with $h_r(z) = rz$. It follows that $\nu_\Theta(r) \equiv$ is constant for all $r > 0$. The Lelong-Jensen formula implies therefore $\Theta \wedge (dd^c \log|z|)^{p+1} = 0$ on $\mathbb{C}^{n+1} \smallsetminus \{0\}$. $\square$

Remark 3.4. — Let T be a positive psh current of bidimension (p,p) on $\mathbb{C}^n$. Assume that $\nu_T(r)$ is bounded. Then by proposition 3.2. the currents T and dd^cT are algebraic. As a consequence, the trivial extension currents $\widetilde{T}$ and $\widetilde{dd^cT}$ are of order zero on $\mathbb{P}^n$. By [D-E-E], there exists a closed positive current S supported by H_∞ such that $dd^c\widetilde{T} = \widetilde{dd^cT} - S$. Then, $\widetilde{T}$ is a *dsh* current (a current T is said *dsh* if $T = T_1 - T_2$ and $dd^cT_i = \Omega_i^+ - \Omega_i^-$, $i = 1, 2$. with T_i negative and $\Omega_i^\pm$ are positive closed). The class of *dsh* currents recently introduced in complex dynamics turns out to be very useful. It is easy to see that the current $\pi^*(\widetilde{T})$ can be extended to a *dsh* and conical current on $\mathbb{C}^{n+1}$.

4. Proof of Theorem 1.5

Before we give the proof of Theorem 1.5, let us state the following remark:

Remark 4.1. — (1) Notice that the problem of defining the wedge product $T\wedge dd^c u$ for a locally bounded psh function u and a negative (or positive) psh current T is still open. For this raison we add the assumption that u is $\mathcal{C}^2$ in Theorem 1.5. However, by a regularization argument we could replace this hypothesis by assuming merely that u is $\mathcal{C}^1$ psh function on $\mathbb{C}^n$ (resp. that u is locally bounded psh function on $\mathbb{C}^n$), under the extra assumption that T is normal, i.e. dT is of order zero (resp. that T is closed).

(2) Thanks to proposition 1.4, if $\nu_T(r)+N_T(\sqrt{2}r)=\mathcal{O}((\log\log r)^s)$ for some $s\geqslant 0$, then T is Liouville. Moreover, If T is pluriharmonic, it suffice to assume that $N_T(r)=\mathcal{O}((\log\log r)^s)$ for some $s\geqslant 0$. Indeed, since ν_T is positive and increasing, we have $(\log r)\nu_T(r)\leqslant N_T(r^2)$.

Proof of Theorem 1.5. — There is no restriction in assuming that T is positive and dd^cT is negative and that $u\in[0,1/2]$ on support of T. Let $\chi\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, such that $\chi=1$ on $]-\infty,1.1]$, $\chi=0$ on $[1.9,+\infty[$ and $0\leqslant-\chi'\leqslant 2$. We consider the sequence $(\psi_j)_{j\in\mathbb{N}}$ defined by $\psi_j(z)=\chi(2^{-j-1}\psi(z))$ with $\psi(z)=\log(1+|z|^2)$ and $K_j=\{z\in\mathbb{C}^n;\ \psi(z)\leqslant 2^{j+1}\}$. Then $\bigcup K_j=\mathbb{C}^n$ and K_j is contained in $\overset{\circ}{K}_{j+1}=$ the interior of K_{j+1} for all j. Since $2^{-j-1}\psi(z)$ is bounded above by 1 on K_j and bounded below by 2 on $\mathbb{C}^n\smallsetminus K_{j+1}$, one has $0\leqslant\psi_j\leqslant 1$, $\psi_j=1$ on a neighborhood of K_j included in $\overset{\circ}{K}_{j+1}$. It is easy to see that there exists a constant $c>0$ depending only on $|\chi'|$ and $|\chi''|$ such that $-dd^c\psi_j\leqslant c\big(\frac{1}{2^{2(j+1)}}d\psi\wedge d^c\psi+\frac{1}{2^{j+1}}dd^c\psi\big)$. Since $d\psi_j\wedge d^c\psi_j$ is positive, we infer

$$\begin{aligned}-dd^c\psi_j^2 &= -2\psi_j dd^c\psi_j-2d\psi_j\wedge d^c\psi_j\leqslant-2\psi_j dd^c\psi_j\\ &\leqslant 2c\Big(\frac{1}{2^{2(j+1)}}d\psi\wedge d^c\psi+\frac{1}{2^{j+1}}dd^c\psi\Big).\end{aligned}$$

Putting $\alpha=dd^c\psi$, we find

$$\int_{\mathbb{C}^n}dd^c\log(1+u^2)\wedge T\wedge\alpha^{p-1}=\lim_{j\to\infty}\int_{\mathbb{C}^n}\psi_j^2 dd^c\log(1+u^2)\wedge T\wedge\alpha^{p-1}.\quad(4.1)$$

To simplify the notation, we denote by $B_j=\overset{\circ}{K}_{j+1}\smallsetminus\overset{\circ}{K}_j=\{2^{j+1}\leqslant\psi<2^{j+2}\}$. Since $\psi_j=0$ on $\mathbb{C}^n\smallsetminus\overset{\circ}{K}_{j+1}$, and $\psi_j=1$ on $\overset{\circ}{K}_j$, a simple computation yields

$$\begin{aligned}\int_{\mathbb{C}^n} dd^c(\psi_j^2\log(1+u^2))\wedge T\wedge\alpha^{p-1} \\ &= \int_{\overset{\circ}{K}_{j+1}} \psi_j^2 dd^c\log(1+u^2)\wedge T\wedge\alpha^{p-1} \\ &+ \int_{B_j}\log(1+u^2)dd^c\psi_j^2\wedge T\wedge\alpha^{p-1} \\ &- 2\int_{B_j}\psi_j d^c\log(1+u^2)\wedge d\psi_j\wedge T\wedge\alpha^{p-1} \\ &+ 2\int_{B_j}\psi_j d\log(1+u^2)\wedge d^c\psi_j\wedge T\wedge\alpha^{p-1}.\end{aligned}$$

Observe that the form $\psi_j^2\log(1+u^2)$ has a compact support. Thus by Stokes's theorem, we get

$$\int_{\mathbb{C}^n} dd^c(\psi_j^2\log(1+u^2))\wedge T\wedge\alpha^{p-1} = \int_{\mathbb{C}^n}\psi_j^2\log(1+u^2)\wedge dd^cT\wedge\alpha^{p-1}\leqslant 0.$$

It follows that

$$\begin{aligned}\int_{\overset{\circ}{K}_{j+1}} \psi_j^2 dd^c\log(1+u^2)\wedge T\wedge\alpha^{p-1} \\ &\leqslant \int_{B_j} -\log(1+u^2)dd^c\psi_j^2\wedge T\wedge\alpha^{p-1} \\ &+ 2\int_{B_j}\psi_j d^c\log(1+u^2)\wedge d\psi_j\wedge T\wedge\alpha^{p-1} \\ &- 2\int_{B_j}\psi_j d\log(1+u^2)\wedge d^c\psi_j\wedge T\wedge\alpha^{p-1}.\end{aligned} \tag{4.2}$$

Since $\log(1+u^2)$ is bounded on SuppT, according to the domination of $-dd^c\psi_j^2$, there exists a constant $c_1>0$ such that :

$$\begin{aligned}\int_{B_j} -\log(1+u^2)dd^c\psi_j^2\wedge T\wedge\alpha^{p-1} &\leqslant \tfrac{c_1}{2^{2(j+1)}}\int_{B_j} d\psi\wedge d^c\psi\wedge T\wedge\alpha^{p-1} \\ &+ \tfrac{c_1}{2^{j+1}}\int_{B_j} T\wedge\alpha^p.\end{aligned} \tag{4.3}$$

Assume for instance that T is smooth. Then applying Stokes theorem and using the equality $-d^c\psi\wedge dT\wedge\alpha^{p-1}=d\psi\wedge d^cT\wedge\alpha^{p-1}$, we obtain

$$\begin{aligned}-\int_{\{\psi<t\}} d^c\psi\wedge dT\wedge\alpha^{p-1} &= \int_{\{\psi<t\}} d(\psi-t)\wedge d^cT\wedge\alpha^{p-1}\\ &= \int_{\{\psi<t\}} (t-\psi)\wedge dd^cT\wedge\alpha^{p-1}.\end{aligned}$$

The form dd^cT is negative, so the last integral is also nonpositive. Hence, for $0<r_1<r_2$, we deduce

$$\begin{aligned}\int_{\{r_1\leqslant\psi<r_2\}} d\psi\wedge d^c\psi\wedge T\wedge\alpha^{p-1} &= \int_{r_1}^{r_2} dt\int_{\{\psi=t\}} d^c\psi\wedge T\wedge\alpha^{p-1}\\ &= \int_{r_1}^{r_2} dt\int_{\{\psi<t\}} T\wedge\alpha^p - \int_{r_1}^{r_2} dt\int_{\{\psi<t\}} d^c\psi\wedge dT\wedge\alpha^{p-1}\\ &\leqslant \int_{r_1}^{r_2} dt\int_{\{\psi<t\}} T\wedge\alpha^p \leqslant (r_2-r_1)\int_{\{\psi<r_2\}} T\wedge\alpha^p.\end{aligned}\tag{4.4}$$

If T is not smooth, we consider a regularization T_ε of T and we use the classical fact that the sequence $\int_{\{\psi<r\}} T_\varepsilon\wedge\varphi$ tends to $\int_{\{\psi<r\}} T\wedge\varphi$ for every smooth differential form φ on $\mathbb{C}^n$ and for every r such that $\{\psi=r\}$ is not charged by the mass of T. Observe that the r's not satisfying the previous condition are at most a countable set. Moreover, the last integrals in (4.4) are left continuous with respect to r_1 and r_2. Therefore, we obtain the desired statement by considering two sequences r_1^k, r_2^k converging respectively to r_1, r_2 and passing to the limit. Let us continue the proof : by hypothesis, there exists $c>0$ such that $\int_{|z|<t} T\wedge\alpha^p\leqslant c(\log(\log t))^s$. Then, we get the inequality

$$(\star)\qquad \int_{\{\psi<t\}} T\wedge\alpha^p = \int_{\{|z|<e^t-1\}} T\wedge\alpha^p \leqslant c(\log t)^s.$$

Taking account the above inequalities, we then have

$$\int_{B_j} d\psi\wedge d^c\psi\wedge T\wedge\alpha^{p-1}\leqslant 2^{j+1}(c(j+2)^s(\log 2)^s).\tag{4.5}$$

Moreover, we have

$$\frac{c_1}{2^{j+1}}\int_{B_j} T\wedge\alpha^p \leqslant \frac{c_1}{2^{j+1}}\int_{\{\psi<2^{j+2}\}} T\wedge\alpha^p \leqslant \frac{c_1 c(j+2)^s(\log 2)^s}{2^{j+1}}.$$

Summing up (4.3) and (4.5), we can find $c_2 > 0$ such that

$$\int_{B_j} -\log(1+u^2)dd^c\psi_j^2 \wedge T \wedge \alpha^{p-1} \leqslant \frac{c_2 j^s}{2^{j+1}}. \tag{4.6}$$

By (4.5) and the inequality of Cauchy-Schwarz, for $\varepsilon > 0$, we can estimate the second term of the right member of the inequality (4.2) as follows :

$$\begin{aligned}
&\Big|\int_{B_j} \psi_j d^c \log(1+u^2) \wedge d\psi_j \wedge T \wedge \alpha^{p-1}\Big| \\
\leqslant\ & \varepsilon \int_{B_j} \psi_j^2 d\log(1+u^2) \wedge d^c \log(1+u^2) \wedge T \wedge \alpha^{p-1} \\
&\quad + \frac{1}{\varepsilon}\int_{B_j} d\psi_j \wedge d^c\psi_j \wedge T \wedge \alpha^{p-1} \\
\leqslant\ & \varepsilon \int_{B_j} \psi_j^2 du \wedge d^c u \wedge T \wedge \alpha^{p-1} \\
&\quad + \frac{1}{2^{2(j+1)}\varepsilon}\int_{B_j} \chi'^2(2^{-j-1}\psi(z)) d\psi \wedge d^c\psi \wedge T \wedge \alpha^{p-1} \\
\leqslant\ & \varepsilon \int_{B_j} \psi_j^2 du \wedge d^c u \wedge T \wedge \alpha^{p-1} + \frac{c_3 j^s}{2^{j+1}\varepsilon}.
\end{aligned} \tag{4.7}$$

The latter inequality is derived from (4.5) and the fact that χ'^2 is bounded. The second and the third terms of the second member of (4.2) are conjugate. Then by using (4.2),(4.6) and (4.7) we get :

$$\begin{aligned}
&\int_{\overset{\circ}{K}_{j+1}} \psi_j^2 dd^c \log(1+u^2) \wedge T \wedge \alpha^{p-1} \\
\leqslant\ & 2\varepsilon \int_{B_j} \psi_j^2 du \wedge d^c u \wedge T \wedge \alpha^{p-1} + \frac{2c_3 j^s}{2^{j+1}\varepsilon} + \frac{c_2 j^s}{2^{j+1}}
\end{aligned} \tag{4.8}$$

A simple computation shows that $dd^c \log(1+u^2) \geqslant \frac{2(1-u^2)du\wedge d^c u}{(1+u^2)^2} \geqslant (24/25) du \wedge d^c u$ on $\operatorname{Supp} T$ (since $u \in [0, 1/2]$). Observe that the last integral on B_j is bounded from above by the same integral on $\overset{\circ}{K}_{j+1}$. Consider $\varepsilon = 23/100$, then

$$\tfrac{1}{2}\int_{\overset{\circ}{K}_{j+1}} \psi_j^2 du \wedge d^c u \wedge T \wedge \alpha^{p-1} \leqslant \frac{c_4 j^s}{2^{j+1}}. \tag{4.9}$$

As ψ_j is equal to 1 on K_j, it is clear that $\frac{1}{2}\int_{K_j} du \wedge d^c u \wedge T \wedge \alpha^{p-1} \leqslant \frac{c_4 j^s}{2^{j+1}}$.

By letting j go to infinity, one gets the equality $\int_{\mathbb{C}^n} du \wedge d^c u \wedge T \wedge \alpha^{p-1} = 0$.

We repeat the same argument and take into account that the integral in the right hand side of (4.8) vanishes, and in this way we obtain $dd^c\log(1+u^2)\wedge T=0$. On the other hand, by the hypotheses on u, it is easy to see that $(8/5)u\,dd^cu\leqslant dd^c\log(1+u^2)$ on $\operatorname{Supp}T$. Hence, we conclude that $u\,dd^cu\wedge T=0$ and also $dd^cu\wedge T=0$ (replace u by $(u+1/2)/2$). This implies in particular that T is Liouville. When T is algebraic we may take $s=0$ in the inequality $(\star)$, since T has a globally finite projective mass. □

Remark 4.2. — Observe that in the proof of Theorem 1.5 we essentially need the fact $\|T\|_{p.m}(e^{2^j})=o(2^j)$. Such an hypothesis is obviously true when the projective mass of T has growth $\|T\|_{p.m}(r)=\mathcal{O}(\log(\log r))^s$.

5. Case of closed currents

This section is reserved to the case of *closed* positive currents, and we prove here Theorems 1.6 and 1.7. For the proof of Theorem 1.6, we need the following lemma cf. [De].

LEMMA 5.1. — *Let T be a closed positive current of bidimension (p,p) on an open subset U of $\mathbb{C}^n$, such that $dz_j\wedge d\overline{z}_j\wedge T=0$ for $j=p+1,...,n$. Let $\pi:z\mapsto z''=(z_{p+1},...,z_n)$ be the canonical projection on $\mathbb{C}^{n-p}$. Assume that the fibers $\pi^{-1}(t)$ are connected. Then $T=\int_{\pi(U)}[z''=t]d\mu_U(t)$ where μ_U is a positive Radon measure on $\pi(U)$.*

Proof. — Let $T=i^{(n-p)^2}\sum_{|I|=|J|=n-p}T_{IJ}dz_I\wedge d\overline{z}_J$. For $j=p+1,...,n$, one has $dz_j\wedge d\overline{z}_j\wedge T=0$. This imply that $T_{IJ}=0$ for all I,J such that it exists $j\in\{p+1,...,n\}$ satisfying $j\notin(I\cup J)$. Therefore, if $\{p+1,...,n\}\not\subset(I\cup J)$, we have $T_{IJ}=0$. We suppose that I,J verify $(I\cup J)\cap\{1,...,p\}\neq\emptyset$. For $\lambda=(\lambda_1,...,\lambda_n)\in\mathbb{C}^n$ and $S=(s_{p+1},...,s_n)$, let $\lambda_S=\lambda_{s_{p+1}}...\lambda_{s_n}$ and consider $I_0=(p+1,...,n)$. Thanks to a standard inequality (see e.g. Demailly [De]), we get:

$$\begin{aligned}\lambda_I\lambda_J|T_{IJ}| &\leqslant 2^p\sum_{I\cap J\subset M\subset I\cup J}\lambda_M^2T_{MM}\\ &= 2^p\lambda_{I_0}^2T_{I_0I_0}+2^p\sum_{M,M\cap\{1,...,p\}\neq\emptyset}\lambda_M^2T_{MM}\\ &= 2^p\lambda_{I_0}^2T_{I_0I_0}.\end{aligned}\tag{5.1}$$

Indeed, to show the second equality, we consider the set M such that $M\cap\{1,...,p\}\neq\emptyset$. As $|M|=n-p$, there exists necessarily $j\in\{p+1,...,n\}\smallsetminus M$.

According to the above, we have $T_{MM}=0$. Then, $\lambda_I\lambda_J|T_{IJ}|\leqslant 2^p\lambda_{I_0}^2T_{I_0I_0}$. Let $s\in(I\cup J)\cap\{1,...,p\}$, $\lambda_s=1/\varepsilon$ and $\lambda_m=1$ for $m\neq s$. By using (5.1), we find

$$|T_{IJ}|\leqslant\begin{cases}2^p\varepsilon T_{I_0I_0} & \text{si} \quad s\notin I\cap J\\ 2^p\varepsilon^2T_{I_0I_0} & \text{si} \quad s\in I\cap J.\end{cases}$$

It follows that the only non vanishing coefficients of T are the T_{IJ} satisfying $(I\cup J)\cap\{1,...,p\}=\emptyset$. As $|I|=|J|=n-p$, one necessarily has $I=J=I_0$, therefore

$$T=i^{(n-p)^2}T_{I_0I_0}dz_{I_0}\wedge d\overline{z}_{I_0}=T_{I_0I_0}idz_{p+1}\wedge d\overline{z}_{p+1}\wedge...\wedge idz_n\wedge d\overline{z}_n.$$

Since T is $d-$closed, for $j=1,...,p$, we obtain $\partial T_{I_0I_0}/\partial z_j=\partial T_{I_0I_0}/\partial\overline{z}_j=0$, thus $T_{I_0I_0}$ is independent of the variables $z_1,...,z_p$. Therefore, for $\psi\in D_{p,p}(U)$, we have:

$$\begin{aligned}\langle T,\psi\rangle &= \int_U T_{I_0I_0}idz_{p+1}\wedge d\overline{z}_{p+1}\wedge...\wedge idz_n\wedge d\overline{z}_n\\ &= \int_{\pi(U)}\Big(\int_{\mathbb{C}^p\times\{z''\}}\psi\Big)T_{I_0I_0}idz_{p+1}\wedge d\overline{z}_{p+1}\wedge...\wedge idz_n\wedge d\overline{z}_n\\ &= \int_{\pi(U)}\Big(\int_{\{z''=t\}}\psi\Big)d\mu(t).\end{aligned}$$

□

Proof of Theorem 1.6. — Since T is algebraic, by theorem 1.5 we have $dP_j\wedge d\overline{P}_j\wedge T=0$ for all $j\in\{1,...,k\}$. Let $a\in\mathbb{C}^n$ be such that $\operatorname{rank}(dF(a))=k$. Then one can suppose that

$$\Big|\Big(\frac{\partial P_i}{\partial z_{n+j-k}}(a)\Big)_{1\leqslant i,j\leqslant k}\Big|\neq 0.$$

The property remains true on a neighborhood U of a such that the map $f(z)=\big(z_1,...,z_{n-k},P(z)\big)$ is biholomorphic on U. Then, for all $j\in\{1,...,k\}$,

$$f^*(dz_{n+j-k}\wedge\overline{dz}_{n+j-k}\wedge f_*T)=dP_j\wedge\overline{dP}_j\wedge T=0 \quad \text{on} \quad U.$$

Hence, for all $j\in\{1,...,k\}$, we have: $dz_{n+j-k}\wedge\overline{dz}_{n+j-k}\wedge f_*T=0$ on $f(U)$. On the other hand for $\psi\in D_{p,p}(U)$, one has $\langle T,\psi\rangle=\langle T,f^*f_*\psi\rangle=\langle f_*T,f_*\psi\rangle$. Therefore, according to lemma 5.1, there exists a unique positive Radon measure μ_U on $\pi(f(U))=F(U)$ such that :

$$\begin{aligned}\langle T,\psi\rangle=\int_{F(U)}\langle[z''=t],f_*\psi\rangle d\mu_U(t) &= \int_{F(U)}\langle f^*[z''=t],\psi\rangle d\mu_U(t)\\ &= \int_{P(U)}\langle[F=t],\psi\rangle d\mu_U(t).\end{aligned} \tag{5.2}$$

Let $X = \{a \in \mathbb{C}^n, \quad \mathrm{rank(dF(a))} \leqslant k-1\}$. We endow the set $\mathbb{C}^n \smallsetminus F^{-1}(F(X))$ with the relation $\sim$ such that $z \sim z'$ if and only if z and z' are in the same connected component of $F^{-1}(F(z))$. The quotient space $(\mathbb{C}^n \smallsetminus F^{-1}(F(X)))/\sim$ endowed with the induced topology can be identified with the space V of connected components of the fibers $F^{-1}(t)$, $t \in \mathbb{C}^k \smallsetminus F(X)$. Let us consider the covering $F_{|V} : V \to \mathbb{C}^k \smallsetminus F(X)$. When we let a vary on the same connected component v of $F^{-1}(t)$, and then let t vary on a neighborhood of $F(a)$, the uniqueness of the measures μ_U [where U covers a sufficiently small saturated neighborhood of v in $\mathbb{C}^n \smallsetminus F^{-1}(F(X))$] implies that the family $(\mu_U)_U$ match together and define a single measure on this neighborhood. This allows us to define a unique measure ν on the connected component of V containing v. By using (5.2), it follows that

$$T_{|\mathbb{C}^n \smallsetminus F^{-1}(F(X))} = \int_{v \in V} [F^{-1}(t)]_v d\nu(v) \ .$$

As the subset X is algebraic, $F(X)$ is contained in a union of at most countably many algebraic subsets, each of which are of codimension $\geqslant 1$ in $\mathbb{C}^k$ [Ch, p.41]. Then $F^{-1}(F(X)) \subset Z = \cup_j Z_j$, where Z_j is an algebraic subset of codimension $\geqslant 1$. Since T is of locally finite mass in $\mathbb{C}^n$, then by the El Mir's Theorem [El] the trivial extension $\widetilde{T}$ of $T_{|\mathbb{C}^n \smallsetminus Z}$ exists and is a closed positive current. Let $R = T - \widetilde{T}$, then R is also a positive and closed current on $\mathbb{C}^n$ of dimension $n-k$, and supported by the algebraic set Z. Moreover, since $\nu_R(r) \leqslant \nu_T(r)$ for $r > 0$, it is clear that R is algebraic. Finally, we have : $T = \widetilde{T} + R = \int_{v \in V} [F^{-1}(t)]_v d\nu(v) + R$. $\square$

Remark 5.2. — If $k = 1$, using the support theorem of Federer, the current R is proportional to the current of integration on the algebraic set Z and then we recover a result of [B-M-R].

Let $z = (z', z'') \in \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$. We denote by $\beta' = dd^c|z'|^2$, $\beta'' = dd^c|z''|^2$ and $v_\varepsilon(z', z'') = |z'|^2 + \varepsilon|z''|^2$ for $\varepsilon > 0$. Replacing the support condition by another one related to the growth of the trace measure of the slices carried by a cylinder, we obtain then Theorem 1.7.

Proof of Theorem 1.7. — By applying the formula of Lelong-Jensen [De] to the positive psh current uT and to the exhaustive function $v_\varepsilon = v_{\varepsilon(R)}$, and by using the fact that $dd^c(uT) = T \wedge dd^c u \geqslant 0$, for $0 < r < R$, we obtain the inequalities

$$\begin{aligned}\int_{\{v_\varepsilon<r\}} T\wedge dd^cu\wedge (dd^cv_\varepsilon)^{n-2}\int_r^R\left(\frac{1}{t^{n-1}}-\frac{1}{R^{n-1}}\right)dt \\ \leqslant \int_r^R\left(\frac{1}{t^{n-1}}-\frac{1}{R^{n-1}}\right)dt\int_{\{v_\varepsilon<t\}} T\wedge dd^cu\wedge (dd^cv_\varepsilon)^{n-2} \\ \leqslant \frac{1}{R^{n-1}}\int_{\{v_\varepsilon<R\}} uT\wedge (dd^cv_\varepsilon)^{n-1}.\end{aligned}$$

Let us put $A_n(r,R)=\int_r^R(\frac{1}{t^{n-1}}-\frac{1}{R^{n-1}})dt$, for $n>2$. We get

$$A_n(r,R)\int_{\{v_\varepsilon<r\}} T\wedge dd^cu\wedge (dd^cv_\varepsilon)^{n-2}\leqslant \frac{1}{R^{n-1}}\int_{\{v_\varepsilon<R\}} uT\wedge (dd^cv_\varepsilon)^{n-1}.$$

Since $(dd^cv_\varepsilon)^{n-2}=(n-2)\varepsilon^{n-3}\beta'\wedge\beta''^{n-3}+\varepsilon^{n-2}\beta''^{n-2}$ and $(dd^cv_\varepsilon)^{n-1}=\varepsilon^{n-2}\beta'\wedge\beta''^{n-2}+\varepsilon^{n-1}\beta''^{n-1}$, it follows that

$$\begin{aligned}A_n(r,R)\varepsilon^{n-2}\int_{\{v_\varepsilon<r\}} T\wedge dd^cu\wedge\beta''^{n-2} &\leqslant \frac{1}{R^{n-1}}\int_{\{v_\varepsilon<R\}} uT\wedge (dd^cv_\varepsilon)^{n-1}\\ &\leqslant \frac{\varepsilon^{n-1}}{R^{n-1}}\int_{\{v_\varepsilon<R\}} uT\wedge\beta''^{n-1}\\ &\quad+\frac{\varepsilon^{n-2}}{R^{n-1}}\int_{\{v_\varepsilon<R\}} uT\wedge\beta'\wedge\beta''^{n-2}.\end{aligned}$$

Let us consider $M=\sup\{u(z),\ z\in\operatorname{Supp}T\}$. Then we get

$$\begin{aligned}&A_n(r,R)\int_{\{v_\varepsilon<r\}} T\wedge dd^cu\wedge\beta''^{n-2}\\ &\leqslant \frac{M\varepsilon}{R^{n-1}}\int_{\{v_\varepsilon<R\}} T\wedge\beta''^{n-1}+\frac{M}{R^{n-1}}\int_{\{v_\varepsilon<R\}} T\wedge\beta'\wedge\beta''^{n-2}\\ &\leqslant \frac{M\varepsilon}{R^{n-1}}\int_{\{v_\varepsilon<R\}} T\wedge\beta''^{n-1}+\\ &\qquad+\frac{M}{R^{n-1}}\int_{\{|z'|^2<R\}}\langle T,\pi,z'\rangle(1\!\!1_{\{v_\varepsilon<R\}}\beta''^{n-2})\beta'\\ &\leqslant \frac{M\varepsilon}{R^{n-1}}\int_{\{v_\varepsilon<R\}} T\wedge\beta''^{n-1}+M\pi R^{-\gamma}.\end{aligned}\tag{5.3}$$

For $n>2$, we have $A_n(r,R)=\frac{1}{(n-2)}(\frac{1-n}{R^{n-2}}+\frac{1}{r^{n-2}})+\frac{r}{R^{n-1}}$. By considering the hypothesis on T and letting R tend to $+\infty$, we obtain

$$\int_{\{|z'|^2<r\}} T\wedge dd^cu\wedge\beta''^{n-2}=0\quad\forall\ r>0.$$

We obtain the equality $T \wedge dd^c u \wedge \beta''^{n-2} = 0$. Now, we use the first term in the expression of $(dd^c v_\varepsilon)^{n-2}$ and get

$$(n-2)A_n(r,R)\varepsilon^{n-3}\int_{\{v_\varepsilon<r\}} T\wedge dd^c u\wedge\beta'\wedge\beta''^{n-3} \leqslant \frac{1}{R^{n-1}}\int_{\{v_\varepsilon<R\}} uT\wedge(dd^c v_\varepsilon)^{n-1}.$$

Replacing β''^{n-2} by $\beta' \wedge \beta''^{n-3}$ in the inequalities (5.3) we get

$$A_n(r,R)\int_{\{v_\varepsilon<r\}} T\wedge dd^c u\wedge\beta'\wedge\beta''^{n-3} \leqslant \frac{M\varepsilon^2}{R^{n-1}}\int_{\{v_\varepsilon<R\}} T\wedge\beta''^{n-1} + M\varepsilon\pi R^{-\gamma}.$$

It follows also that $T\wedge dd^c u\wedge\beta'\wedge\beta''^{n-3} \equiv 0$. Since $\beta = \beta' + \beta''$, it is clear that $\beta^{n-2} = (n-2)\,\beta'\wedge\beta''^{n-3} + \beta''^{n-2}$, and this implies that $T\wedge dd^c u\wedge\beta^{n-2} \equiv 0$. The case $n = 2$ can be proved in the same way. □

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