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The polar curve of a foliation on $\mathbb{P}^2$

Rogério S. Mol[(1)]

Abstract. — We study some properties of the polar curve $P_l^{\mathcal{F}}$ associated to a singular holomorphic foliation $\mathcal{F}$ on the complex projective plane $\mathbb{P}^2$. We prove that, for a generic center $l \in \mathbb{P}^2$, the curve $P_l^{\mathcal{F}}$ is irreducible and its singular points are exactly the singular points of $\mathcal{F}$ with vanishing linear part. We also obtain upper bounds for the algebraic multiplicities of the singularities of $\mathcal{F}$ and for its number of radial singularities.

Résumé. — On étudie dans cet article quelques propriétés de la courbe polaire $P_l^{\mathcal{F}}$ associée à un feuilletage holomorphe singulier $\mathcal{F}$ dans le plan projectif complexe $\mathbb{P}^2$. On démontre que, pour un centre $l \in \mathbb{P}^2$ générique, la courbe $P_l^{\mathcal{F}}$ est irréductible et ses points singuliers sont précisément les points singuliers de $\mathcal{F}$ avec partie linéaire nulle. On obtient aussi des bornes supérieurs pour la multiplicité algébrique des singularités de $\mathcal{F}$ et pour son nombre de singularités radiales.

1. Basic definitions

A foliation of degree $d \geqslant 0$ on the complex projective plane $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$ is a non trivial morphism of vector bundles $\Phi : H^{\otimes(1-d)} \to T\mathbb{P}^2$, where H stands for the hyperplane bundle. Two such maps define the same foliation if one is multiple of the other by a non zero complex number. We denote by $\mathcal{F}$ the foliation and the bundle $T\mathcal{F} := H^{\otimes(1-d)}$ is called the tangent bundle of $\mathcal{F}$. The singular set of $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is formed by the points

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of $\mathbb{P}^2$ over which Φ fails to be injective. Equivalently, if $\Theta_{\mathbb{P}^2} = \mathcal{O}(T\mathbb{P}^2)$ is the tangent sheaf of $\mathbb{P}^2$ and $\mathcal{H} = \mathcal{O}(H)$ is the sheaf of sections of H, defining $\mathcal{F}$ is equivalent to giving a morphism of locally free analytic sheaves $\mathcal{H}^{\otimes(1-d)} \to \Theta_{\mathbb{P}^2}$, two such morphisms defining the same foliation if and only if they are multiple from each other by a non zero complex number. Thus, the space of foliations of degree d, denoted by $\mathcal{F}\mathrm{ol}(d)$, can be identified with the projectivization of $H^0(\mathbb{P}^2, \mathrm{Hom}(\mathcal{H}^{\otimes(1-d)}, \Theta_{\mathbb{P}^2})) \simeq H^0(\mathbb{P}^2, \Theta^2_{\mathbb{P}}(d-1))$. Shortly

$$\mathcal{F}\mathrm{ol}(d) \;=\; \mathbb{P}(H^0(\mathbb{P}^2, \Theta^2_{\mathbb{P}}(d-1))).$$

Geometrically, the map Φ that defines $\mathcal{F}$ establishes, for every $p \in \mathbb{P}^2$ outside $\mathrm{Sing}(\mathcal{F})$, a direction in $T_p\mathbb{P}^2$ which will be denoted by $T_p\mathcal{F}$. The distribution of tangent directions $p \mapsto T_p\mathcal{F}$, where $p \in \mathbb{P}^2 \setminus \mathrm{Sing}(\mathcal{F})$, defines, on this set, a non singular holomorphic foliation. This distribution of directions corresponds, in affine coordinates $(x, y) \in \mathbb{P}^2$, to the one defined by a polynomial vector field of the form

$$\mathbf{v} \;=\; P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{1.1}$$

To be more precise, this vector field can be written in the form

$$\mathbf{v} \;=\; G(x,y)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \widetilde{P}(x,y)\frac{\partial}{\partial x} + \widetilde{Q}(x,y)\frac{\partial}{\partial y}, \tag{1.2}$$

where $G(x, y)$, when non-zero, is a homogeneous polynomial of degree d, while $\widetilde{P}(x, y)$ and $\widetilde{Q}(x, y)$ have degrees at most d. The line at infinity is invariant by $\mathcal{F}$ if and only if $G(x, y) \equiv 0$, in which case at least one of the degrees of $\widetilde{P}(x, y)$ or $\widetilde{Q}(x, y)$ is actually d. This writing for $\mathbf{v}$ shows that, considered as a meromorphic vector field on $\mathbb{P}^2$, it has a pole of order $d-1$ in the line at infinity L_∞. The degree d is interpreted geometrically as the number of tangencies, with multiplicities counted, between $\mathcal{F}$ and a line $L \subset \mathbb{P}^2$ non invariant by $\mathcal{F}$. The degree of a foliation $\mathcal{F}$ will be denoted by $\deg(\mathcal{F})$.

It is evident that in the affine coordinates considered, $\mathrm{Sing}(\mathcal{F})$ is the set of common zeroes of $P(x, y)$ and $Q(x, y)$. If this set has a component of codimension one, it means that $P(x, y)$ and $Q(x, y)$ have a common factor. So, by canceling this factor, we can always suppose that $\mathrm{Sing}(\mathcal{F})$ is a finite set of points (evidently, in this case, the actual degree of $\mathcal{F}$ is less than the number d we started with). Furthermore, by a proper choice of the affine plane, we can always suppose that the line at infinity L_∞ is not invariant by $\mathcal{F}$ and that it contains no points in $\mathrm{Sing}(\mathcal{F})$.

Alternatively, we can represent a foliation $\mathcal{F}$ on $\mathbb{P}^2$ in homogeneous coordinates $(X : Y : Z) \in \mathbb{P}^2$. Suppose that the affine plane considered above is $Z = 1$, so that $x = X/Z$ and $y = Y/Z$. Let $\eta = -Q(x, y)dx + P(x, y)dy$ be the dual form of $\mathbf{v}$. By taking the natural projection $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$, we have that $\pi^*\eta$ is a meromorphic 1-form in $\mathbb{C}^3$ with a pole of order $d + 1$ along $Z = 0$. By canceling this pole, we get a polynomial 1-form

$$\omega \;=\; A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ, \tag{1.3}$$

where A, B and C are homogeneous polynomials of degree $d + 1$. Evidently, the rays from the origin of $\mathbb{C}^3$ must be contained in the leaves of the codimension one foliation defined by ω. This property is expressed as the Euler condition

$$XA(X, Y, Z) + YB(X, Y, Z) + ZC(X, Y, Z) = 0. \tag{1.4}$$

We remark that, in this case, $\mathrm{Sing}(\mathcal{F})$ is the set of common zeroes of A, B and C and the hypothesis that $\mathrm{Sing}(\mathcal{F})$ has codimension at least two is equivalent to the fact that A, B and C have no common factor. In the present text we will make use of both representations for $\mathcal{F}$, in affine and in homogeneous coordinates.

We now proceed to the definition of polar curve of a foliation. Let us denote by $T_p^{\mathbb{P}}\mathcal{F}$ the the line through p with direction $T_p\mathcal{F}$. We call it the *projective tangent line* of $\mathcal{F}$ at $p \in \mathbb{P}^2 \setminus \mathrm{Sing}(\mathcal{F})$. The *polar locus* of $\mathcal{F}$ with *center* at $l \in \mathbb{P}^2$ is the closure of the set of points $p \in \mathbb{P}^2 \setminus \mathrm{Sing}(\mathcal{F})$ such that $T_p^{\mathbb{P}}\mathcal{F}$ passes through l:

$$P_l^{\mathcal{F}} \;=\; \overline{\{p \in \mathbb{P}^2 \setminus \mathrm{Sing}(\mathcal{F});\ l \in T_p^{\mathbb{P}}\mathcal{F}\}}.$$

In affine coordinates $(x, y) \in \mathbb{P}^2$ as above, taking $l = (x_0, y_0)$, $P_l^{\mathcal{F}}$ has as equation

$$(y - y_0)P(x, y) - (x - x_0)Q(x, y) = 0. \tag{1.5}$$

In homogeneous coordinates $(X : Y : Z) \in \mathbb{P}^2$, when $\mathcal{F}$ is induced by a 1-form as in (1.3), the polar locus with center $l = (\alpha : \beta : \gamma)$ is given by the equation

$$\alpha A(X, Y, Z) + \beta B(X, Y, Z) + \gamma C(X, Y, Z) = 0. \tag{1.6}$$

Anyone of the two previous equations shows easily that the center l always belongs to the polar curve $P_l^{\mathcal{F}}$.

Recall that the *radial foliation* with center $l \in \mathbb{P}^2$ is the one given in affine coordinates $(x, y) \in \mathbb{P}^2$ by the vector field

$$\mathbf{v} \;=\; (x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y},$$

where $l = (x_0, y_0)$. An easy calculation shows that in homogeneous coordinates $(X : Y : Z) \in \mathbb{P}^2$ the radial foliation centered at $l = (\alpha : \beta : \gamma)$ is given by the 1-form

$$\eta = (\beta Z - \gamma Y)dX + (\gamma X - \alpha Z)dY + (\alpha Y - \beta X)dZ.$$

We have the following:

Proposition 1.1. — *The polar locus of a foliation $\mathcal{F}$ on $\mathbb{P}^2$ with center $l \in \mathbb{P}^2$ is the whole $\mathbb{P}^2$ if and only if $\mathcal{F}$ is the radial foliation with center l. In all other cases the polar locus is a curve of degree $d+1$, where d is the degree of $\mathcal{F}$.*

Proof. — Let us suppose that $\alpha A(X,Y,Z) + \beta B(X,Y,Z) + \gamma C(X,Y,Z) \equiv 0$. We assume that $\gamma = 1$, so that

$$C = -(\alpha A + \beta B). \tag{1.7}$$

Putting this in Euler's relation (1.4) we get

$$XA + YB - Z(\alpha A + \beta B) = 0 \;\Leftrightarrow\; (X - \alpha Z)A = -(Y - \beta Z)B.$$

If $d_0 = \deg(A) = \deg(B) = d + 1 \geqslant 2$, we would find a common factor for A and B. By (1.7), this would also be a factor for C, which contradicts the fact that the points in $\mathrm{Sing}(\mathcal{F})$ are isolated. Thus, $d_0 = 1$ and, modulo multiplication by a non-zero constant, we have

$$A = -(Y - \beta Z)\,, \quad B = X - \alpha Z,$$

and, by Euler's relation, $C = \alpha Y - \beta X$. We conclude that $\mathcal{F}$ is induced by the 1-form

$$\omega = -(Y - \beta Z)dX + (X - \alpha Z)dY + (\alpha Y - \beta X)dZ,$$

that is, $\mathcal{F}$ is the radial foliation with center $(\alpha : \beta : 1) \in \mathbb{P}^2$. □

By varying the center $l = (\alpha : \beta : \gamma)$ of the polar curve, we produce a linear system in $\mathbb{P}^2$ generated by the divisors $A = 0, B = 0$ and $C = 0$. We remark that, if $d = \deg(\mathcal{F}) \geqslant 1$, this linear system has dimension two. In fact, if this dimension were less than two, then one of the polynomials A, B or C would be written as a linear combination of the others, say $C = \alpha A + \beta B$ with $\alpha, \beta \in \mathbb{C}$. We are then in the situation of the proof of the above proposition, and a contradiction is reached. Thus, for a foliation $\mathcal{F}$ of degree at least 1 we have a net of polar curves, the so-called *polar net*. Its base points are exactly the singular points of $\mathcal{F}$. The polar net of a foliation was thoroughly studied in [4], where it is used to prove that, for $d \geqslant 2$, the scheme $\mathrm{Sing}(\mathcal{F})$ determines $\mathcal{F}$.

Polar curves are a particular case of more general objects, polar varieties associated to distributions in projective manifolds. These objects were studied in [10] and a short description is given below. First, a *singular holomorphic distribution* $\mathcal{F}$ of dimension r on a complex manifold M of dimension m is a coherent analytic subsheaf $\mathcal{T}$ of rank r of the tangent sheaf $\Theta_M = \mathcal{O}(TM)$ of M. We call $\mathcal{T}$ the *tangent sheaf* of $\mathcal{F}$. The singular set of $\mathcal{F}$, denoted $\mathrm{Sing}(\mathcal{F})$, is defined as the singular set of the sheaf $\Theta_M/\mathcal{T}$, that is, the set of points where the stalks are not free modules over $\mathcal{O}_M$. When $\mathcal{T}$ is involutive, that is, when its stalks are invariant by the Lie bracket,

$$[\mathcal{T}_p, \mathcal{T}_p] \subset \mathcal{T}_p \ \forall\ p \in M,$$

$\mathcal{T}$ actually determines a (non singular) foliation on $M \setminus \mathrm{Sing}(\mathcal{F})$. This is a consequence of Frobenius theorem. However, integrability is not necessary for the forthcoming definition.

Let us now suppose that the ambient manifold M is projective, say $M \subset \mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$. If $\mathcal{F}$ is a distribution with tangent sheaf $\mathcal{T}$ defined on M, then for each $p \in M \setminus \mathrm{Sing}(\mathcal{F})$, there is a unique r-dimensional plane $T^{\mathbb{P}}_p\mathcal{F} \subset \mathbb{P}^n$ passing through p with direction determined by the stalk $\mathcal{T}_p$ in $T_pM \subset T_p\mathbb{P}^n$. We fix a flag of linear subspaces on $\mathbb{P}^n$

$$\mathcal{D} : L_{r+1} \subset L_r \subset \cdots \subset L_2 \subset \mathbb{P}^n, \tag{1.8}$$

where $\mathrm{codim} L_j = j$. For $k = 1, \ldots, r$, the k-th *polar locus* of $\mathcal{F}$ with respect to $\mathcal{D}$ is defined as

$$P^{\mathcal{F}}_k \ = \ Cl\{p \in M \setminus \mathrm{Sing}(\mathcal{F});\ \dim(T^{\mathbb{P}}_p\mathcal{F} \cap L_{r-k+2}) \geqslant k-1\},$$

where Cl stands for the closure in M. We remark that $p \in M \setminus \mathrm{Sing}(\mathcal{F})$ belongs to $P^{\mathcal{F}}_k$ if and only if the subspaces of $\mathbb{C}^{n+1}$ corresponding to $T^{\mathbb{P}}_p\mathcal{F}$ and to L_{r-k+2} fail to span $\mathbb{C}^{n+1}$. For a generic choice of $\mathcal{D}$ and for $k = 1, \ldots, r$, $P^{\mathcal{F}}_k$ is empty or is a reduced analytic variety of pure codimension k whose class $\left[P^{\mathcal{F}}_k\right] \in A_{m-k}(M)$ is independent of the flag, $A_{m-k}(M)$ standing for the Chow group of M of complex dimension $m-k$. When $\mathcal{F}$ is a foliation on $\mathbb{P}^2$, then the polar curve with center l is the $P^{\mathcal{F}}_1$ defined above by taking $L_2 = \{l\}$ as the element of codimension two in the flag.

2. Some properties of the polar curve

From this point to the end of the text we will fix our attention on foliations on $\mathbb{P}^2$ with isolated singularities. We will start by a definition:

Definition 2.1. — Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be foliations on $\mathbb{P}^2$ of the same degree d. We say that $\widetilde{\mathcal{F}}$ is a *radial modification* of $\mathcal{F}$ with center at $l \in \mathbb{P}^2$ if, in homogeneous coordinates $(X : Y : Z) \in \mathbb{P}^2$, there are polynomial 1-forms ω and $\tilde{\omega}$ that induce $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ and a homogeneous polynomial $\Phi(X, Y, Z)$ of degree d such that

$$\tilde{\omega} \;=\; \omega + \Phi\eta,$$

where $\eta = (\beta Z - \gamma Y)dX + (\gamma X - \alpha Z)dY + (\alpha Y - \beta X)dZ$ is the 1-form that induces the radial foliation centered at $l = (\alpha : \beta : \gamma)$.

The notion of radial modification defines an equivalence relation in the space of foliations on $\mathbb{P}^2$ and polar curves classify the equivalence classes under this relation as shown in the following proposition:

Proposition 2.2. — *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be foliations on $\mathbb{P}^2$ of the same degree $d \geqslant 1$. Denote by $P_l^{\mathcal{F}}$ and $P_l^{\widetilde{\mathcal{F}}}$ the polar loci of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ centered at $l \in \mathbb{P}^2$. Then $P_l^{\mathcal{F}} = P_l^{\widetilde{\mathcal{F}}}$ if and only if $\widetilde{\mathcal{F}}$ is a radial modification of $\mathcal{F}$ with center at l.*

Proof. — Let ω and $\tilde{\omega}$ be 1-forms defining $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. We first suppose that $P_l^{\mathcal{F}} = P_l^{\widetilde{\mathcal{F}}}$. By multiplying ω and $\tilde{\omega}$ by non zero constants, we may suppose that that $P_l^{\mathcal{F}}$ and $P_l^{\widetilde{\mathcal{F}}}$ have the same equation. We keep the notation ω and $\tilde{\omega}$ for these new 1-forms. Thus, the foliation induced by $\nu = \tilde{\omega} - \omega$ has $\mathbb{P}^2$ as polar locus with center l. By proposition 1.1, this foliation must be the radial foliation with center l, which is induced by the 1-form η as above. In order to have compatible degrees, we find $\Phi(X, Y, Z)$ homogeneous of degree d such that $\nu = \tilde{\omega} - \omega = \Phi\eta$. Thus $\tilde{\omega} = \omega + \Phi\eta$, which gives the result. The proof of the converse is straightforward. □

Let us now fix affine coordinates $(x, y) \in \mathbb{P}^2$. Suppose that $l = (0, 0)$ is the center of the polar curve $P_l^{\mathcal{F}}$, which has $F(x, y) = yP(x, y) - xQ(x, y) = 0$ as equation. Remark that $P_l^{\mathcal{F}}$ is invariant by $\mathcal{F}$ if and only if the tangent lines to $P_l^{\mathcal{F}}$ and to $\mathcal{F}$ coincide wherever both are defined. It follows from the very definition of $P_l^{\mathcal{F}}$ that this is equivalent to the fact that the tangent lines to $P_l^{\mathcal{F}}$ pass through $l = (0, 0)$. This is true if and only if

$$\mathbf{r}(F(x, y)) \;=\; xF_x(x, y) + yF_y(x, y) \;\equiv\; 0$$

on $P_l^{\mathcal{F}}$, where $\mathbf{r} = x\partial/\partial x + y\partial/\partial y$ is the radial vector field centered at $l = (0,0)$. Notice that when $F(x,y)$ is homogeneous of degree d, by Euler's formula we have

$$\mathbf{r}(F(x,y)) \;=\; dF(x,y),$$

which evidently vanishes on $P_l^{\mathcal{F}}$. This motivates the following definition:

Definition 2.3. — We say that a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is *homogeneous* with center at $l \in \mathbb{P}^2$ if $\mathcal{F}$ is induced in affine coordinates $(x,y) \in \mathbb{P}^2$ centered at $l = (0,0)$ by a polynomial vector field

$$\mathbf{v} \;=\; P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

such that $P(x,y)$ and $Q(x,y)$ are homogeneous polynomials of the same degree.

Let $\mathcal{F}$ be a foliation as in the above definition, where P and Q are homogeneous polynomials of degree d_0. In view of expression (1.2), if $G \not\equiv 0$, we would have $\deg(\mathcal{F}) = d_0 - 1$ and $\tilde{P} = \tilde{Q} = 0$, so that $\mathcal{F}$ would be the radial foliation centered at $l = (0,0)$ having $G = 0$ as a curve of singularities. So, excluding the possibility of non isolated singularities, we must have $G \equiv 0$, giving $P = \tilde{P}$, $Q = \tilde{Q}$ and $\deg(\mathcal{F}) = d_0$. We remark that, in this case, the line at infinity is $\mathcal{F}$-invariant.

If $\mathcal{F}$ is a homogeneous foliation centered at $l = (0,0)$, then its polar curve $P_l^{\mathcal{F}}$ is defined in affine coordinates as the set of zeroes of a homogeneous polynomial and, as we have seen above, it is $\mathcal{F}$-invariant. More generally, we have:

Proposition 2.4. — *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$. Suppose that its polar curve $P_l^{\mathcal{F}}$ centered at $l \in \mathbb{P}^2$ is reduced. Then $P_l^{\mathcal{F}}$ is $\mathcal{F}$-invariant if and only if $\mathcal{F}$ is a radial modification of a homogeneous foliation centered at l.*

Proof. — We have already proved the "if" part. Let us then suppose that $P_l^{\mathcal{F}}$ is invariant by $\mathcal{F}$. We fix affine coordinates $(x,y) \in \mathbb{P}^2$ for which $l = (0,0)$. We have

$$xF_x(x,y) + yF_y(x,y) \;\equiv\; 0$$

on $P_l^{\mathcal{F}}$. Since $P_l^{\mathcal{F}}$ is reduced with equation $F(x,y) = 0$, it follows that $F(x,y)$ divides $xF_x(x,y) + yF_y(x,y)$. Since the degree of the latter is less than or equal to the degree of the former, we conclude that both polynomials have the same degree and, thus, there exists a non-zero constant α such that

$$\alpha F(x,y) \;=\; xF_x(x,y) + yF_y(x,y).$$

But this happens if and only if $F(x,y)$ is homogeneous, in which case α is the degree of $F(x,y)$. Let us now look at the expression

$$F(x,y) = yP(x,y) - xQ(x,y).$$

We decompose $P(x,y) = P_0(x,y) + P_1(x,y)$ in the following way: $P_0(x,y)$ involves all monomials which vanish when we produce $F(x,y)$ and $P_1(x,y)$ comprises the ones which do not vanish. We do the same for $Q(x,y)$, getting $Q(x,y) = Q_0(x,y) + Q_1(x,y)$. On the one hand, we have

$$F(x,y) \;=\; yP_1(x,y) - xQ_1(x,y)$$

and, since $F(x,y)$ is homogeneous, all monomials in $P_1(x,y)$ and in $Q_1(x,y)$ have the same degree, so that the foliation induced by the vector field

$$P_1(x,y)\frac{\partial}{\partial x} + Q_1(x,y)\frac{\partial}{\partial y}$$

is homogeneous. On the other hand, we have

$$yP_0(x,y) - xQ_0(x,y) \;\equiv\; 0,$$

so that there exists a polynomial $\Phi(x,y)$ such that $P_0(x,y) = x\Phi(x,y)$ and $Q_0(x,y) = y\Phi(x,y)$. Therefore we conclude that

$$P(x,y)\frac{\partial}{\partial x}+Q(x,y)\frac{\partial}{\partial y} \;=\; P_1(x,y)\frac{\partial}{\partial x}+Q_1(x,y)\frac{\partial}{\partial y}+\Phi(x,y)\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right),$$

meaning that $\mathcal{F}$ is a radial modification of a homogeneous foliation. □

3. Singularities of the polar curve

We recall that as l runs over $\mathbb{P}^2$, the curves $P_l^{\mathcal{F}}$ describe a linear system of dimension two with base $\mathrm{Sing}(\mathcal{F})$. By Bertini's theorem (see [9]), the generic element of a linear system is smooth outside the base locus. So, for generic $l \in \mathbb{P}^2$, the singular set of $P_l^{\mathcal{F}}$ is contained in $\mathrm{Sing}(\mathcal{F})$.

Suppose that $\mathcal{F}$ is induced in affine coordinates by a vector field as in (1.1), so that its polar curve centered at $l = (x_0, y_0)$ is given by the equation

$$F(x,y) \;=\; (y-y_0)P(x,y) - (x-x_0)Q(x,y) = 0.$$

For future reference, we calculate the derivatives of F with respect to x and y:

$$\begin{cases} F_x(x,y) \;=\; (y-y_0)P_x(x,y) - (x-x_0)Q_x(x,y) - Q(x,y) \\ F_y(x,y) \;=\; P(x,y) + (y-y_0)P_y(x,y) - (x-x_0)Q_y(x,y) \end{cases} \qquad (3.1)$$

These equations give promptly that the center l is a regular point for $P_l^{\mathcal{F}}$ whenever it lies outside $\mathrm{Sing}(\mathcal{F})$.

Fix now $p \in \mathrm{Sing}(\mathcal{F})$. We may suppose that $p = (0,0)$, and thus $P(0,0) = Q(0,0) = 0$. Let $k = m_p(\mathcal{F})$ be the algebraic multiplicity of $\mathcal{F}$ at p, that is, the smallest of the orders of P and Q. We write

$$P(x,y) = P_k(x,y) + \tilde{P}(x,y) \text{ and } Q(x,y) = Q_k(x,y) + \tilde{Q}(x,y),$$

where P_k and Q_k comprise all monomials of degree k, while $\tilde{P}$ and $\tilde{Q}$ assemble those of higher degree. We thus have

$$F(x,y) = -y_0 P_k(x,y) + x_0 Q_k(x,y) + S(x,y),$$

where all monomials in $S(x,y)$ have degree $k+1$ or greater. We conclude that, for generic $l = (x_0, y_0)$, the algebraic multiplicity of $P_l^{\mathcal{F}}$ at p is $k = m_p(\mathcal{F})$. Summarizing this discussion:

Proposition 3.1. — *For generic $l \in \mathbb{P}^2$ it holds* $\mathrm{Sing}(P_l^{\mathcal{F}}) \subset \mathrm{Sing}(\mathcal{F})$. *Furthermore if $p \in \mathrm{Sing}(P_l^{\mathcal{F}})$, then*

$$m_p(P_l^{\mathcal{F}}) \; = \; m_p(\mathcal{F}) \, ,$$

where m_p denotes the algebraic multiplicity.

Keeping the above notation, we say that $p \in \mathrm{Sing}(\mathcal{F})$ is a non-degenerate singularity if $\det D\mathbf{v}(0,0) \neq 0$. According to the above proposition, if p is a non-degenerate singularity for $\mathcal{F}$, then p is a regular point for $P_l^{\mathcal{F}}$. Recalling that foliations having only non-degenerate singularities are generic in the space of foliation of degree d, we have

Corollary 3.2. — *For generic $\mathcal{F} \in \mathcal{F}\mathrm{ol}(d)$, the polar curve $P_l^{\mathcal{F}}$ centered at a generic point $l \in \mathbb{P}^2$ is non-singular.*

4. Radial singularities

Let us denote by $\mathrm{Sing}_{1,dic}(\mathcal{F}) \subset \mathrm{Sing}(\mathcal{F})$ the set of dicritical singularities of algebraic multiplicity 1. We remark that $\mathrm{Sing}_{1,dic}(\mathcal{F})$ contains the set of *radial* singularities, consisting of those which become non singular after a single blow-up. Let us suppose that $p = (0,0) \in \mathrm{Sing}_{1,dic}(\mathcal{F})$. The polynomial vector field inducing $\mathcal{F}$ has the form

$$\mathbf{v} \; = \; (x + \tilde{P}(x,y))\frac{\partial}{\partial x} + (y + \tilde{Q}(x,y))\frac{\partial}{\partial y},$$

where $\tilde{P}$ and $\tilde{Q}$ refer to the terms of degree 2 or greater ($\tilde{P} = \tilde{Q}$ in the case of radial singularities).

In this case, the polar curve $P_l^{\mathcal{F}}$ centered at $l = (x_0, y_0)$ is defined by the equation

$$F(x,y) \ = \ (y - y_0)(x + \tilde{P}(x,y)) - (x - x_0)(y + \tilde{Q}(x,y)) \ = \ 0.$$

We have

$$\begin{cases} F_x(x,y) \ = \ (y - y_0)(1 + \tilde{P}_x(x,y)) - (x - x_0)\tilde{Q}_x(x,y) - (y + \tilde{Q}(x,y)) \\ F_y(x,y) \ = \ (x + \tilde{P}(x,y)) + (y - y_0)\tilde{P}_y(x,y) - (x - x_0)(1 + \tilde{Q}_y(x,y)) \end{cases}$$

so that $F_x(0,0) = -y_0$ and $F_y(0,0) = x_0$. This implies that the tangent line to $P_l^{\mathcal{F}}$ at $p = (0,0)$ passes through $l = (x_0, y_0)$. Thus, $\mathrm{Sing}_{1,dic}(\mathcal{F})$ is contained in the set of tangencies between $P_l^{\mathcal{F}}$ and the pencil of lines with base l. The number of these tangencies, with multiplicities counted, is the degree of $\check{P}_l^{\mathcal{F}}$, the projective dual of $P_l^{\mathcal{F}}$ (see [12]). Noticing that l is in this set of tangencies but may be chosen outside $\mathrm{Sing}(\mathcal{F})$, we have the following:

Proposition 4.1. — *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$. Then*

$$\#\mathrm{Sing}_{1,dic}(\mathcal{F}) \ \leqslant \deg(\check{P}_l^{\mathcal{F}}) - 1 \ = \ d(d+1) - \sum_{p \in \mathrm{Sing}(P_l^{\mathcal{F}})} (\widetilde{m}_p + \widetilde{\mu}_p - 1) - 1, \quad (4.1)$$

where $\widetilde{m}_p$ and $\widetilde{\mu}_p$ refer to the algebraic multiplicity and the Milnor number of $P_l^{\mathcal{F}}$ at p. In particular, this gives un upper bound for the number of radial singularities of $\mathcal{F}$.

We remark that the upper bound in (4.1) is written in terms of algebraic multiplicities and Milnor numbers of the polar curve $P_l^{\mathcal{F}}$ and it works for any center $l \in \mathbb{P}^2$ outside $\mathrm{Sing}(\mathcal{F})$ and not specifically in the generic case. Nevertheless, by taking $P_l^{\mathcal{F}}$ as the generic polar curve, we can establish a bound in terms of data of the foliation $\mathcal{F}$. We proceed as follows. Let us denote by m_p and μ_p the algebraic multiplicity and the Milnor number of $\mathcal{F}$ at $p \in \mathrm{Sing}(\mathcal{F})$. We know that, in this case, $\widetilde{m}_p = m_p$. In what concerns Milnor numbers, it is easy to see from equations (3.1) that, for generic $l \in \mathbb{P}^2$, the vector field $F_x(x,y)\partial/\partial x + F_y(x,y)\partial/\partial y$ has algebraic multiplicity $m_p - 1$ at $p = (0,0) \in \mathrm{Sing}(\mathcal{F})$. Thus $\widetilde{\mu}_p \geqslant (m_p - 1)^2$ for every $p \in \mathrm{Sing}(\mathcal{F})$ (see [3]), and we get that the term $\widetilde{m}_p + \widetilde{\mu}_p - 1$ in the sum in (4.1) is greater than or equal to $m_p + (m_p - 1)^2 - 1 = m_p(m_p - 1)$. Substituting in (4.1), we have

$$\#\mathrm{Sing}_{1,dic}(\mathcal{F}) \ \leqslant \ d(d+1) - \sum_{p \in \mathrm{Sing}(\mathcal{F})} m_p(m_p - 1) - 1. \quad (4.2)$$

5. Irreducibility of the generic polar

The generic element of a linear system of curves in the projective plane, or, more generally, of projective divisors, need not be irreducible. However, projective linear systems whose generic element is irreducible are well characterized by the Bertini-Krull theorem (see [1] and theorem 37 in [11]):

THEOREM 5.1. — *Let K be an algebraically closed field of characteristic 0. Let $r \geqslant 1$ and $P_0, P_1, \ldots, P_r$ be non-zero distinct polynomials, not all of them constant, relatively prime in $K[\underline{x}]$, where $\underline{x} = (x_1, \ldots, x_n)$ are independent variables. Then the set of $\underline{\lambda} = (\lambda_0 : \lambda_1 : \ldots : \lambda_r) \in \mathbb{P}^r = \mathbb{P}^r_{\mathbb{C}}$ such that*

$$F = F(\underline{x}, \underline{\lambda}) = P_0 + \lambda_1 P_1 + \cdots + \lambda_r P_r$$

is reducible in $K[\underline{x}]$ is a Zariski dense set in $\mathbb{P}^r$ if and only if the following condition holds: there exist relatively prime polynomials $\Phi, \Psi \in K[\underline{x}]$, with $\deg_{\underline{x}}(F) > \max\{\deg(\Phi), \deg(\Psi)\}$, and $r+1$ polynomials $h_i(u, v) \in K[u, v]$ homogeneous of degree s, for some $s > 1$, such that

$$P_i = h_i(\Phi(\underline{x}), \Psi(\underline{x})) = \sum_{k=0}^{s} a_{ik} \Phi(\underline{x})^k \Psi(\underline{x})^{s-k} \ , \ i = 0, \ldots, r.$$

This means that, setting

$$H(u, v, \underline{\lambda}) = \sum_{i=0}^{r} \lambda_i h_i(u, v),$$

we have $F(\underline{x}, \underline{\lambda}) = H(\Phi(\underline{x}), \Psi(\underline{x}), \underline{\lambda})$.

We remark that the polar net of a foliation on $\mathbb{P}^2$ is not an arbitrary net, since its generators satisfy the Euler condition. This fact along with Bertini-Krull theorem allow us to prove the following:

PROPOSITION 5.2. — *The generic polar curve of a foliation on $\mathbb{P}^2$ is irreducible*

Proof. — We work in homogeneous coordinates. Suppose that the foliation $\mathcal{F}$ is induced by the 1-form

$$\omega \ = \ A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ,$$

where A, B and C are homogeneous polynomials of degree $d+1 = \deg(\mathcal{F})+1$, satisfying the Euler condition $XA+YB+ZC \equiv 0$. Suppose that the generic element of the polar net of $\mathcal{F}$,

$$\alpha A(X, Y, Z) + \beta B(X, Y, Z) + \gamma C(X, Y, Z) = 0 \ ; \ (\alpha : \beta : \gamma) \in \mathbb{P}^2,$$

is reducible. Then, by the Bertini-Krull theorem, there exists relatively prime polynomials $\Phi, \Psi \in \mathbb{C}[X,Y,Z]$, with $\max\{\deg(\Phi), \deg(\Psi)\} < d+1$, and homogeneous polynomials $h_0(u,v), h_1(u,v), h_2(u,v) \in \mathbb{C}[u,v]$ of degree s such that

$$A = h_0(\Phi,\Psi)\,,\ B = h_1(\Phi,\Psi)\,,\ C = h_2(\Phi,\Psi).$$

Let us write

$$h_i(u,v) = \sum_{k=0}^{s} a_{ik} u^k v^{s-k}\ ,\ i = 0,1,2.$$

Since $XA+YB+ZC \equiv 0$, putting all terms in Φ^s in the same side, we get

$$(Xa_{0s} + Ya_{1s} + Za_{2s})\Phi^s = -\sum_{k=0}^{s-1}(Xa_{0k} + Ya_{1k} + Za_{2k})\Phi^k\Psi^{s-k}$$

We first remark that $Xa_{0s}+Ya_{1s}+Za_{2s}$ is non zero for, otherwise, we would have $a_{0s} = a_{1s} = a_{2s} = 0$ and Ψ would be a common factor for A, B and C. Thus, since Φ and Ψ are relatively prime, the above expression gives that Ψ divides $Xa_{0s}+Ya_{1s}+Za_{2s}$. But this is possible if and only if $\deg(\Psi) = 1$ and $\Psi = \lambda(Xa_{0s}+Ya_{1s}+Za_{2s})$ for some $\lambda \in \mathbf{C}\setminus\{0\}$. By arguing in a symmetric manner, we also have that $\deg(\Phi) = 1$ and $\Phi = \nu(Xa_{00} + Ya_{10} + Za_{20})$ for some $\nu \in \mathbf{C}\setminus\{0\}$. To show that this leads to a contradiction, let us make the following linear change of coordinates:

$$\begin{cases} u = \Phi(X,Y,Z) = \nu(Xa_{00} + Ya_{10} + Za_{20}) \\ v = \Psi(X,Y,Z) = \lambda(Xa_{0s} + Ya_{1s} + Za_{2s}) \\ w = \Lambda(X,Y,Z) = Xa + Yb + Zc \end{cases}$$

where Λ is any linear form such that Φ, Ψ and Λ are linearly independent. If $(u,v,w) = F(X,Y,Z)$ denotes this linear change of coordinates, then $\mathcal{F}$ is induced in the new coordinates by a 1-form of the kind

$$\tilde{\omega} = F^*\omega = \tilde{A}(u,v)du + \tilde{B}(u,v)dv + \tilde{C}(u,v)dw,$$

where $\tilde{A}, \tilde{B}, \tilde{C}$ are homogeneous polynomial of degree $d+1$, in the variables u and v only. Euler condition gives $-w\tilde{C}(u,v) = u\tilde{A}(u,v) + v\tilde{B}(u,v)$, which implies that $\tilde{C}(u,v) = 0$. This is absurd, since, when $d = \deg(\mathcal{F}) \geqslant 1$, $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are linearly independent, as already proved in the text. $\square$

6. The genus of the polar curve

We summarize here the properties satisfied by the polar curve $P_l^{\mathcal{F}}$ for generic $l \in \mathbb{P}^2$:

(i) $P_l^{\mathcal{F}}$ is irreducible;

(ii) the degrees satisfy $d(P_l^{\mathcal{F}}) = d(\mathcal{F}) + 1$;

(iii) $\mathrm{Sing}(P_l^{\mathcal{F}}) \subset \mathrm{Sing}(\mathcal{F})$;

(iv) $m_p(\mathcal{F}) \ = m_p(P_l^{\mathcal{F}}) \ \forall\, p \in \mathrm{Sing}(\mathcal{F})$.

We remark that the genus, since it is calculated by a finite combinatorial process, is constant for the generic element of the polar net of $\mathcal{F}$. Property (i) allows us to apply Noether's formula for the genus of a plane curve (see [2]), yielding

$$\begin{aligned} g &= \frac{(d(P_l^{\mathcal{F}})-1)(d(P_l^{\mathcal{F}})-2)}{2} - \frac{1}{2}\sum_{p\in\tilde{S}} m_p(P_l^{\mathcal{F}})(m_p(P_l^{\mathcal{F}})-1) \\ &= \frac{d(d-1)}{2} - \frac{1}{2}\sum_{p\in\tilde{S}} m_p(P_l^{\mathcal{F}})(m_p(P_l^{\mathcal{F}})-1)\,, \end{aligned}$$

where $d = \deg(\mathcal{F})$ and $\tilde{S}$ denotes all points infinitely near $\mathrm{Sing}(P_l^{\mathcal{F}})$. In particular, when $m_p(\mathcal{F}) = 1$ for all points in $\mathrm{Sing}(\mathcal{F})$, we have that the the generic polar curve is smooth and its genus is $g = d(d-1)/2$, where $d = \deg(\mathcal{F})$.

Now, we consider the fact that $g \geqslant 0$ and, in Noether's formula, we neglect the points in $\tilde{S}$ which appear in the first blow up and after. By taking into account properties (iii) and (iv), we get the following bound for the algebraic multiplicities of the singularities of $\mathcal{F}$ in terms of $d = \deg(\mathcal{F})$:

Proposition 6.1. — *For a foliation $\mathcal{F}$ of degree $d \geqslant 1$ on $\mathbb{P}^2$ it holds*

$$\sum_{p\in\mathrm{Sing}(\mathcal{F})} m_p(m_p-1) \leqslant d(d-1)\,,$$

where m_p denotes the algebraic multiplicity of $p \in \mathrm{Sing}(\mathcal{F})$.

We say that two foliations $\mathcal{F}$ and $\tilde{\mathcal{F}}$ defined on surfaces M and $\tilde{M}$ are *topologically equivalent* if there is a homeomorphism $h : M \to \tilde{M}$ mapping

$\mathrm{Sing}(\mathcal{F})$ into $\mathrm{Sing}(\tilde{\mathcal{F}})$ and taking leaves of $\mathcal{F}$ into leaves of $\tilde{\mathcal{F}}$. Local topological equivalences preserve Milnor number (see [3]). Thus, if $p \in M$ is non-degenerate, which is equivalent to the fact that its Milnor number is one, then $h(p)$ is also non-degenerate.

Two foliations on $\mathbb{P}^2$ which are topological equivalent have the same degree. In fact, the first Chern class of the tangent bundle of a foliation is invariant by topological equivalences (see [8]). If $\mathcal{F}$ is a foliation on $\mathbb{P}^2$, then its tangent bundle $T\mathcal{F}$ is $H^{\otimes(d-1)}$, where $d = \deg(\mathcal{F})$. Another way to see the topological invariance of the degree is by considering Baum-Bott's theorem, which, for a foliation $\mathcal{F}$ of degree d on $\mathbb{P}^2$, reads

$$\sum_{p\in \mathrm{Sing}(\mathcal{F})} \mu_p \;=\; d^2 + d + 1,$$

where μ_p denotes the Milnor number at $p \in \mathrm{Sing}(\mathcal{F})$ (see, for instance, proposition 1.1 in [4]). The topological invariance of d is then a consequence of the topological invariance of Milnor numbers.

The above considerations give the following:

Proposition 6.2. — *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ of degree $d \geqslant 1$ with only non-degenerate singularities. Then the genus of its generic polar curve is $g = d(d-1)/2$. Furthermore, this genus is a topological invariant, meaning that if $\tilde{\mathcal{F}}$ is a foliation on $\mathbb{P}^2$ topologically equivalent to $\mathcal{F}$, then the genus of the generic polar curve of $\tilde{\mathcal{F}}$ is the same.*

This result leads to the following question:

Question. — Is the genus of the generic polar curve of a foliation $\mathcal{F}$ on $\mathbb{P}^2$ a topological invariant of the foliation?

The difficulty for answering this question is the fact that, in general, the desingularization data of the generic polar is not read in the desingularization data of the foliation. N. Corral has extensively worked on this subject, comparing, for a singularity p of a foliation $\mathcal{F}$ on $\mathbb{P}^2$ which is a generalized curve (see [3] for the definition), the desingularization of $\mathcal{F}$, which is equivalent to the desingularization of the set of its separatrices at p, with the desingularization of the generic polar curve (see [5], [7] and [6]).

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Bibliography

[1] Bodin (A.), Débes (P.), and Najib (S.). — Irreducibility of hypersurfaces. Comm. Algebra, 37(6):1884-1900, (2009).

[2] Brieskorn (E.) and Knörrer (H.). — Plane algebraic curves. Birkhäuser Verlag, Basel, (1986).

[3] Camacho (C.), Lins Neto (A.), and Sad (P.). — Topological invariants and equidesingularization for holomorphic vector fields. J. Differential Geom., 20(1):143-174, (1984).

[4] Campillo (A.) and Olivares (J.). — Polarity with respect to a foliation and Cayley- Bacharach theorems. J. Reine Angew. Math., 534:95-118, (2001).

[5] Corral (N.). — Sur la topologie des courbes polaires de certains feuilletages singuliers. Ann. Inst. Fourier (Grenoble), 53(3):787-814, (2003).

[6] Corral (N.). — Infinitesimal adjunction and polar curves. Bull. Braz. Math. Soc. (N.S.), 40(2):181-224, (2009).

[7] Corral (N.). — Polar pencil of curves and foliations. Astérisque, (323):161-179, (2009).

[8] Gómez-Mont (X.), Seade (J.), and Verjovsky (A.). — The index of a holomorphic ow with an isolated singularity. Math. Ann., 291(4):737-751, (1991).

[9] Griffiths (P.) and Harris (J.). — Principles of algebraic geometry. Wiley Classics Library. John Wiley Sons Inc., New York, (1994).

[10] Mol (R.S.). — Classes polaires associees aux distributions holomorphes de sous-espaces tangents. Bull. Braz. Math. Soc. (N.S.), 37(1):29-48, (2006).

[11] Schinzel (A.). — Polynomials with special regard to reducibility, volume 77 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, (2000).

[12] Wall (C.T.C.). — Singular points of plane curves, volume 63 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, (2004).