ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

Bernhard Banaschewski, Anthony Hager

The HSP-Classes of Archimedean l-groups with Weak Unit

Tome XIX, nº S1 (2010), p. 13-24.

<http://afst.cedram.org/item?id=AFST_2010_6_19_S1_13_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIX, n° Spécial, 2010
pp. 13–23

The HSP-Classes of Archimedean l-groups with Weak Unit

Bernhard Banaschewski[(1)], Anthony Hager[(2)]

To Mel Henriksen for his 80 th Birthday

Abstract. — W denotes the class of abstract algebras of the title (with homomorphisms preserving unit). The familiar H, S, and P from universal algebra are here meant in W. $\mathbb{Z}$ and $\mathbb{R}$ denote the integers and the reals, with unit 1, *qua* W-objects. V denotes a non-void finite set of positive integers. Let $\mathcal{G} \subseteq W$ be non-void and not $\{\{0\}\}$. We show

(1) $HSP\mathcal{G} = HSP(HS\mathcal{G} \cap S\mathbb{R})$, and

(2) $W \neq \mathcal{G} = HSP\mathcal{G}$ if and only if $\exists V(\mathcal{G} = HSP\{\frac{1}{v}\mathbb{Z}|v \in V\})$.

Our proofs are, for the most part, simple calculations. There is no real use of methods of universal algebra (e.g., free objects), and only one restricted use of representation theory (Yosida). Note that (1) implies the basic fact that $HSP\mathbb{R} = W$ (which can be proved in several ways). Note that (2) contrasts W with $\mathcal{C}$ = archimedean l-groups, and $\mathcal{C}$ = abelian l-groups, where $HSP\mathbb{Z} = \mathcal{C}$ in each case.

Résumé. — Soit W la classe des algèbres abstraites décrit dans le titre de l'article (avec les homomorphismes qui preservent l'unité). On utilise dans W la notation H, S, et P, familière de l'algèbre universelle. On note par $\mathbb{Z}$ et les nombres entiers et les nombres réels, avec l'unité 1, *qua* W-objets. On note par V un ensemble fini non-vide de nombres entiers. Soit $\mathcal{G} \subseteq W$ en ensemble non-vide, different de $\{\{0\}\}$. Nous montrons que

(1) $HSP\mathcal{G} = HSP(HS\mathcal{G} \cap S)$, and

(1) Department of Mathematics, McMaster University, Hamilton, Ontario 68S4K1 Canada
(2) Department of Mathematics and Computer Science, Wesleyan University, Middletown, CT 06459 USA
ahager@wesleyan.edu

(2) $W \neq \mathcal{G} = HSP\mathcal{G}$ if and only if $\exists V(\mathcal{G} = HSP\{\frac{1}{v}\mathbb{Z} | v \in V\})$.

Pour la plupart, nos démonstrations sont des calculs simples. On n'utilise pas de méthodes de l'algèbre universelle (par exemple, les objets libres); il n'y a qu'un seul endroit où on utilise, d'une façon très restreint la théorie de representations (Yosida). Nous signalons que (1) entraine le fait basique $HSP = W$ (qui peut être démontré de plusieures façons) et aussi que (2) souligne la differénce entre W et $\mathcal{C} = l$-groupes archimédien et $\mathcal{C} = l$-groupes abéliens, où $HSP\mathbb{Z} = \mathcal{C}$ dans chaque cas.

1. Preliminaries

"l-group" means "lattice-ordered group", a set with group operation $+$ and compatible lattice order $\leqslant$. An "l - homomorphism" is a group homomorphism which preserves the binary sup $\vee$ and inf $\wedge$. [D] is a comprehensive reference. Let G be an l-group. G is archimedean if in $G, 0 \leqslant na \leqslant b \ \forall n \in \mathbb{N}$ implies $a = 0$. (This implies commutativity.) A (weak) unit of G is a positive element u for which $a \wedge u = 0$ implies $a = 0$.

W is the class of archimedean l-groups G with a designated unit e_G, and a W-homomorphism $G \overset{\varphi}{\to} H$ is a l-homomorphism with $\varphi(e_G) = e_H$ (analogous to homomorphisms of rings with identity). (Thus, W forms a category.)

Note that $\{0\} \in W$, and $\forall G \in W, G \overset{0}{\to} \{0\}$ is a W-homomorphism; $\{0\}$ is the terminal object of W. Any "ring of continuous functions" $C(X)$, with constant function 1 as unit, is in W, and all ring homomorphisms $C(X) \to C(X)$ are W-homomorphisms; see [GJ], p. 13. (In fact, conversely too; see [HR], p. 420.) W has products: $G = \prod_I G_i$ is the cartesian product with coordinate-wise $+$ and $\leqslant$, and $e_G = (e_{G_i})$. W has subobjects: "G is a subobject of A", written $G \leqslant A$, means G is a sub-l-group of A and $e_G = e_A$ (the inclusion is a W-homomorphism). W has homomorphic images: For any: $\varphi : G \to A$ in W, $\mathrm{Im}(\varphi) \in |W|$ and the corestriction $G \to \mathrm{Im}(\varphi)$ is a surjective W-homomorphism. (The $\ker \varphi$, for $\varphi : G \to A$ in W, are exactly the convex sub-l-groups I of G for which G/I is archimedean with $e_G + I$ a unit in G/I. The requirement that G/I be archimedean is substantive; see [LZ], p. 427.)

Henceforth, W-homomorphisms are called simply "morphisms." All classes $\mathcal{G} \subseteq W$ are supposed isomorphism-closed, non-void, and not $\{\{0\}\}$. Let $\mathcal{G} \subseteq W$. $P\mathcal{G}$ (respectively, S$\mathcal{G}$; H$\mathcal{G}$) consists of all products (respectively, subobjects; morphic images) of objects from $\mathcal{G}$ (all meant in W). $P\{G\}$ is abbreviated to PG, etc... $\mathcal{G}$ is a P (respectively, $S; H$) class if $\mathcal{G} = P\mathcal{G}$ (respectively, $S\mathcal{G}; H\mathcal{G}$). Any $HSP\mathcal{G}$ is an H-, S-, P- class, and is called an HSP-class. (Note that $\{\{0\}\}$ is an HSP-class. We do not mention this further.)

2. Subobjects of $\mathbb{R}$ Generate

Let $\mathcal{G} \subseteq W$; we exclude $\emptyset$ and $\{\{0\}\}$. Let $\mathcal{K}(\mathcal{G})$ =HS$\mathcal{G}$∩SR. For $G \in W$, let $G^* = \{g \in G \mid \exists n(|g| \leqslant ne_G\}$ (the subobject of bounded "functions", in allusion to Yosida Representation (see [HR]); notation follows [GJ]). For $K \in S\mathbb{R}$, we have $K = K^*$. For any morphism $G \xrightarrow{\varphi} L$, we have $G^* \xrightarrow{\varphi^*} L^*$ given by $\varphi^*(g) = \varphi(g)$ (since $\varphi(e_G) = e_L$, and then $\varphi(G^*) \subseteq L^*$) : "( )* is a functor".

Theorem 2.1. — $HSP\mathcal{G} = HSP\mathcal{K}(\mathcal{G})$.

Proof. — The containment $\supseteq$ is clear. For $\subseteq$ we claim, and prove below, that for $G \neq \{0\}$,

(a) $\mathcal{K}(G) = \mathcal{K}(G^*)$,

(b) $G = G^* \Rightarrow G \in SP(HG \cap S\mathbb{R}) \qquad (\subseteq SP\mathcal{K}(G))$,

(c) $G \in HS(G^*)^\omega \qquad (( )^\omega$ the countable power).

And thus $G \in HSP\mathcal{K}(G)$. $\square$

Now, for the general $HSP\mathcal{G} \subseteq HSP\mathcal{K}(\mathcal{G})$: Let $A \in HSP\mathcal{G}$. Then $A \in HSG$ for $G \in P\mathcal{G}$, so $\mathcal{K}(A) \subseteq \mathcal{K}(G)$, and $HSP\mathcal{K}(A) \subseteq HSP\mathcal{K}(G) \subseteq HSP\mathcal{K}(\mathcal{G})$. By the above, $A \in HSP\mathcal{K}(A)$, so $A \in HSP\mathcal{K}(\mathcal{G})$.

We prove the claims (a), (b), (c).

(a) $\mathcal{K}(G) \supseteq \mathcal{K}(G^*)$ since $G^* \in SG$.

$\mathcal{K}(G) \subseteq \mathcal{K}(G^*)$: If $K \in \mathcal{K}(G)$, as $G \geqslant B \overset{\varphi}{\twoheadrightarrow} K$, we have $G^* \geqslant B^* \overset{\varphi^*}{\twoheadrightarrow} K^* = K$. ($\varphi^*$ is onto K since φ is onto: if $\varphi(b) = k$, choose $n \in \mathbb{N}$ with $|k| \leqslant n$, and let $c = (b \wedge ne_G) \vee (-ne_G)$. Then, $c \in B^*$ and $\varphi(c) = k$.)

(b). (This is a version of the classical Yosida Theorem for strong unit [Y]. This is the only place in the paper where the word "ideal" is used (see 5.5, *en passant*). One might also hold the view that this is the only place in the paper where actual Algebra occurs.)

In an abelian l-group, "ideal" means "convex sub-l-group," and ideals are kernels of l-homomorphisms. The ideal P in G is prime if and only if G/P is totally ordered; a maximal ideal is prime. If M is a maximal ideal in G, then G/M has no proper ideals, and thus is archimedean (since "archimedean" means "no infinitesimals" and the infinitesimals form an ideal). Hölder's Theorem: L is archimedean and totally ordered if and only if there is an l-embedding $L \overset{\varphi}{\hookrightarrow} \mathbb{R}$. If $0 \neq r \in \mathbb{R}^+$, then $x \to rx$ defines an l-automorphism of $\mathbb{R}$. Thus:

If $G = G^*$ in W and M is a proper maximal ideal, then $e_G \notin M$ and we have a W-embedding $\alpha_M : G/M \hookrightarrow \mathbb{R}$, $\alpha_M = a \circ \varphi$, where $G/M \overset{\varphi}{\hookrightarrow} \mathbb{R}$ is any Hölder map and a is the automorphism of $\mathbb{R}, x \mapsto (1/\varphi(e_G + M))x$. So $G/M \in HG \cap S\mathbb{R}$.

If $G = G^*$ in W then, $\forall g \neq 0 \, \exists \, M \, (g \notin M)$ (by Zorn's lemma; see [HR], p. 415). Thus the "reduced product" $< \alpha_M > : G \to \prod\{G/M \mid M \text{ maximal}\}$ is a W-embedding, and $G \in SP(HG \cap S\mathbb{R})$. (See the early parts of [D] for the basic facts quoted above.)

(c). (This proof uses a sequential convergence in W introduced in [BH], injected into an argument from [HIJ], which originally used pointwise convergence on the Yosida space.)

Fix $G \neq \{0\}$ in W. In the following, $a \leqslant \epsilon$ in G means $\epsilon = 1/k$ for $k \in \mathbb{N}$ and $ka \leqslant e_G$; and, for $p \in \mathbb{N}, pe_G \in G$ is just called p.

Definition 2.2. — *([BH], 2.2.4) In G : let $r \geqslant 1$.*
$g_n \xrightarrow{r} g$ means $[\forall \epsilon \, \forall p \in \mathbb{N} \, \exists \, n_0 \, (n \geqslant n_0 \Rightarrow (p - r)^+ \wedge |g_n - g| \leqslant \epsilon)]$ and
$g_n \to g$ means $[\exists r \, (g_n \xrightarrow{r} g)]$.

(Intuition: View G as an l-group of continuous extended real-valued functions on a compact space X, e.g., in its Yosida representation - see [D] or [HR]. Then, $g_n \xrightarrow{r} g$ means, for the functions, g_n converges uniformly to g on each subset of X on which r is bounded - or, on each compact subset of $r^{-1}(\mathbb{R})$.)

The convergence has the following basic features gleaned from [BH].

(i) $g_n \to g$ and $g_n \to g' \Rightarrow g' = g$.

(ii) $g_n \to g$ and $h_n \to h \Rightarrow g_n \otimes h_n \to g \otimes h$ for $\otimes = +, -, \vee, \wedge$.

(iii) For each $g, (g \wedge n) \vee (-n) \to g$ (indeed, $\xrightarrow{r}$ for $r = |g| \vee 1$).

(Here, (i) and (ii) are parts of [BH], 2.2.5, (iii) follows from [BH], 2.1.8. The proofs are not difficult but use "archimedean" several times.)

We now show that $G \in HS(G^*)^\omega$. Define $B \subseteq (G^*)^\omega$ as: $b = (b_n) \in B$ means $\exists g \in G\,(b_n \to g)$. Given $b \in B$, this g is unique by (i); call it $\lambda(b)$. We have a function $B \xrightarrow{\lambda} G$. By (ii), $B \leqslant (G^*)^\omega$ (i.e., B is a W-subobject) and λ is a morphism. By (iii), λ is onto. $\square$

Corollary 2.3. — $W = HSP\mathbb{R}$.

Proof. — By 2.1 any $G \in HSP\mathcal{K}(G)$, while $\mathcal{K}(G) \subseteq HSP\mathbb{R}$. $\square$

There are other quite different proofs of 2.3. See 5.1.

3. Generating $\mathbb{R}$ from $S\mathbb{R}$.

Let K be a subgroup of $\mathbb{R}$. It is elementary that : Topologically, K is either dense or discrete, in the latter case cyclic. If $1 \in K$ (i.e., $K \leqslant \mathbb{R}$), then K is either dense or of the form $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N} - \{0\}$.

In the proof of 3.1 (3) below and later, we use the further easy observations: $\frac{1}{m}\mathbb{Z} \leqslant \frac{1}{n}\mathbb{Z}$ if and only if $m|n$; $S\frac{1}{n}\mathbb{Z} = \{\frac{1}{v}\mathbb{Z} : v \in \text{Div } n\}$. ($m|n$ is "m divides n". Div n is the set of divisors of n.)

Consider finite $V \subseteq \mathbb{N} - \{0\}$, put $\mathcal{K}_V = \{\frac{1}{v}\mathbb{Z} : v \in V\} (\subseteq S\mathbb{R})$ and $B(V) = \Pi\mathcal{K}_V (\in W)$. Any finite $\mathcal{K} \subseteq S\mathbb{R}$ consisting of discrete subgroups is a $\mathcal{K}_V$. Note that $S\mathcal{K}_V = \mathcal{K}_{\text{Div } V}$ (Div $V = \cup\{\text{Div } v | v \in V\}$).

The following does not use 2.1. (Of course though, knowing [2.3: $HSP\mathbb{R} = W$], the point is partly which $\mathcal{K} \subseteq S\mathbb{R}$ do / do not have $HSP\mathcal{K}$ proper; we consider this explicitly in the next section.)

Theorem 3.1. — *Let $K \in S\mathbb{R}$ be dense. Let $\mathcal{K} \subseteq S\mathbb{R}$ consist of discrete subgroups.*

(1) $\mathbb{R} \in HS(K^w)$

(2) If $\mathcal{K}$ is infinite, then $\mathbb{R} \in HS(\prod \mathcal{K})$.

(3) If $\mathcal{K}$ is finite, then $\mathcal{K} = \mathcal{K}_V$ for some V, $(HSP\mathcal{K}) \cap S\mathbb{R} = S\mathcal{K}$, and $\mathbb{R} \notin HSP\mathcal{K}$.

Proof. — For (1) and (2), consider $\{K_n\}_{\mathbb{N}} \subseteq S\mathbb{R}$ and $P \equiv \prod_{\mathbb{N}} K_n$. For (1), we take $K_n = K\,\forall n$, and for (2), we take $k_0 < k_1 < ...$ and $K_n = \frac{1}{k_n}\mathbb{Z}$. Then let $A \equiv \{a \in P \,|$ The sequence $(a(n))$ converges in $\mathbb{R}\}$, and define $\lambda : A \to \mathbb{R}$ by $\lambda(a) = \lim a(n)$. Then $A \in W$ and λ is a morphism, as is checked easily. We claim λ is onto in either case : Let $\gamma \in \mathbb{R}$. For (1), $\forall n$ choose $a(n) \in K$ with $|a(n) - \gamma| \leqslant 1/n$, by density. For (2), $\forall n$ choose the least m with $\gamma - 1/k_n \leqslant m/k_n$, then $m/k_n \leqslant \gamma + 1/k_n$; define $a(n) = m/k_n$.

(3) We noted above that $\mathcal{K} = \mathcal{K}_V$ for some V and $S\mathcal{K} = \mathcal{K}_{\text{Div } V}$. Clearly $\mathbb{R} \notin \mathcal{K}_{\text{Div } V}$. We show $HSP\mathcal{K}_V \cap S\mathbb{R} = S\mathcal{K}_V$. The inclusion $\supseteq$ is clear. For $\subseteq$:
$HSP\mathcal{K}_V = HSPB(V)$ of course, so consider $B(V)^\Gamma \geqslant A \xrightarrow{\varphi} \mathbb{R}$.

We are to show $\varphi(A) \subseteq$ some $\frac{1}{v}\mathbb{Z}$. Now $\mathbb{R}^V \geqslant \{f \in \mathbb{R}^V | f(v) \in \frac{1}{v}\mathbb{Z} \forall v\} = B(V)$. Let $X = V \times \Gamma = \dot{\bigcup\limits_v} \pi^{-1}(v)$ (π the first projection, $\dot{\bigcup}$ denoting disjoint union). So

$$\mathbb{R}^X \geqslant \{f \in \mathbb{R}^X | f(v,\gamma) \in \frac{1}{v}\mathbb{Z} \ \forall\, (v,\gamma)\} = B(V)^\Gamma$$

and

$$f \in B(V)^\Gamma \Rightarrow f(\pi^{-1}(v)) \subseteq \frac{1}{v}\mathbb{Z} \ \forall v,$$

and if f is bounded, then $f(\pi^{-1}(v))$ is finite, thus closed. Now

$$(\mathbb{R}^X)^* \geqslant (B(V)^\Gamma)^* \geqslant A^* \xrightarrow{\varphi^*} \mathbb{R},$$

with $\varphi^*(A^*) = \varphi(A)$ (by properties of $(\ )^*$ as in (a) of the proof of 2.1).

Simplify this notation: $B = A^*, \psi = \varphi^*$, so $(\mathbb{R}^X)^* \geqslant B \xrightarrow{\psi} \mathbb{R}$. For $f \in \mathbb{R}^X$, let $Z(f) = \{x | f(x) = 0\}$. Note that $Z(b) = \emptyset$ implies b is bounded away from 0 (since V is finite), so $1 \leqslant n\,|b|$ for some $n \in \mathbb{N}$, thus $\psi(b) \neq 0$. Therefore, $\mathcal{Z}(\psi) = \{Z(b) \,|\, b \in \ker \psi\}$ is a proper filter base on X, so there is an ultrafilter $\mathcal{U} \supseteq \mathcal{Z}(\psi)$, and there is a unique v with $\pi^{-1}(v) \in \mathcal{U}$ (since $X = \dot{\bigcup\limits_v} \pi^{-1}(v)$).

Whenever $\mathcal{F}$ is an ultrafilter on a set Y, for each $f \in (\mathbb{R}^Y)^*$, "f converges along $\mathcal{F}$", i.e., $\cap\{\overline{f(F)} | F \in \mathcal{F}\}$ is a singleton, whose element we denote $e_\mathcal{F}(f)$. This defines a function $e_\mathcal{F} : (\mathbb{R}^Y)^* \to \mathbb{R}$, which is easily seen to be a W-morphism with ker $e_\mathcal{F} \supseteq \{f | \mathcal{Z}(f) \in \mathcal{F}\}$. (These assertions can be verified directly, but their truth is transparent upon recognizing $e_\mathcal{F}(f)$ as $\beta f(\mathcal{F})$, which means: Discrete Y has its Čech-Stone compactification βY, of which $\mathcal{F}$ "is" a point, and each $f \in (\mathbb{R}^Y)^*$ has its extension $\beta f \in C(\beta Y)$. Then $\beta f(\mathcal{F}) = e_\mathcal{F}(f)$. See [GJ], 3.17, 4.12, 6.24.)

Applying this to $(Y, \mathcal{F}) = (X, \mathcal{U})$ with the $\pi^{-1}(v) \in \mathcal{U} \supseteq \mathcal{Z}(\psi)$, we see that $\ker e_{\mathcal{U}} \supseteq \ker \psi$, and that for $b \in B$ (or even $(B(V)^{\Gamma})^*$), $b(\pi^{-1}(v)) = \overline{b(\pi^{-1}(v))} \supseteq \{e_{\mathcal{U}}(b)\}$ so $e_{\mathcal{U}}(b) \in \frac{1}{v}\mathbb{Z}$.

Thus we have the commuting square in W, where α is defined by $\alpha(\psi(b)) = e_{\mathcal{U}}(b)$.

$$\begin{tikzcd} B \arrow[r, "\leqslant"] \arrow[d, "\psi"] & (\mathbb{R}^X)^* \arrow[d, "e_{\mathcal{U}}"] \\ \psi(B) \arrow[r, dashed, "\alpha"] & \mathbb{R} \end{tikzcd}$$

However, α is just the inclusion $\psi(B) \hookrightarrow \mathbb{R}$ (by Hion's Theorem, [D], p. 147), so $\psi(B) = e_{\mathcal{U}}(B) \subseteq \frac{1}{v}\mathbb{Z}$. □

Regarding 3.1(1): J. Madden first pointed out to us that $\mathbb{R} \in HSP\mathbb{Q}$, using a representation of free objects in $HSP\mathbb{Q}$. The referee points out that the corresponding fact in f-rings is noted in [HI], p. 541-542, and that [W], p. III-44 contains "the sequence proof" of a statement like 3.1(1) (we don't know the context; we have not seen [W]).

4. The HSP-classes

We combine sections 2 and 3 as follows. 4.1: 2.3 says $HSP\mathcal{G} = W$ if and only if $\mathbb{R} \in HSP\mathcal{G}$; we take a closer look at that using 3.1. 4.2: 3.1 and 2.3 show that the proper HSP-classes are the $HSP\mathcal{K}_V$; we interpret "$G \in HSP\mathcal{K}_V$."

Proposition 4.1. — *For $\mathcal{G} \subseteq W$, the following are equivalent.*

(1) $\mathbb{R} \in HSP\mathcal{G}$.

(2) $\mathcal{K}(\mathcal{G})$ contains either a dense member or infinitely many discrete members of $S\mathbb{R}$.

(3) There is $\{G_n | n \in \mathbb{N}\} \subseteq \mathcal{G}$ for which $\mathbb{R} \in HS(\Pi G_n)$.

Proof. — (3)$\Rightarrow$(1) is obvious.

(1)$\Rightarrow$(2). If (2) fails, then 3.1 (3) says $\mathbb{R} \notin HSP\mathcal{K}(\mathcal{G})$. But 2.1 says $HSP\mathcal{G} \subseteq HSP\mathcal{K}(\mathcal{G})$.

(2) $\Rightarrow$ (3) Let $K \in HS\mathcal{G} \cap S\mathbb{R}$ be dense. We have $\mathcal{G} \ni G \geqslant A \twoheadrightarrow K$. Now, $\mathbb{R} \in HS(K^\omega)$, by 3.1 (1), as $K^\omega \geqslant B \twoheadrightarrow \mathbb{R}$. Combining these, $G^\omega \geqslant A^\omega \twoheadrightarrow K^\omega \geqslant B \twoheadrightarrow \mathbb{R}$. So $\mathbb{R} \in H(SH)S(G^\omega) \subseteq H(HS)S(G^\omega) = HS(G^\omega)$ (since $SH \leqslant HS$).

Let $\{K_n\}_{\mathbb{N}} \subseteq HS\mathcal{G} \cap S\mathbb{R}$ be distinct, discrete. We have $\forall n\ G_n \geqslant A_n \twoheadrightarrow K_n$. Now, $\mathbb{R} \in HS(\Pi K_n)$, by 3.2 (2), as $\Pi K_n \geqslant B \twoheadrightarrow \mathbb{R}$. Combining these, $\Pi G_n \geqslant \Pi A_n \twoheadrightarrow \Pi K_n \geqslant B \twoheadrightarrow \mathbb{R}$, so $\mathbb{R} \in \cdots HS(\Pi G_n)$ as previously. $\square$

We consider the $HSP\mathcal{K}_V$'s (the proper HSP-classes). We noted in and before 3.1 that $\mathcal{K}(\mathcal{K}_V) \equiv HSP\mathcal{K}_V \cap S\mathbb{R} = S\mathcal{K}_V = \mathcal{K}_{\text{Div } V}$, so using 2.1, $HSP\mathcal{K}_V = HSP\mathcal{K}_{\text{Div } V}$. A little further thought shows $HSP\mathcal{K}_U \subseteq HSP\mathcal{K}_V$ if and only if $S\mathcal{K}_U \subseteq S\mathcal{K}_V$ if and only if $U \subseteq \text{Div } V$. In particular, $\frac{1}{6}\mathbb{Z} \notin HSP\{\frac{1}{2}\mathbb{Z}, \frac{1}{3}\mathbb{Z}\}$.

For $G \in W$, let $V(G) = \{n | \frac{1}{n}\mathbb{Z} \in \mathcal{K}(G)\}$. This may be $\emptyset$, or infinite (in which case our reserving "V" for finite sets is abused - continuing the possibility of such abuse) $\mathcal{K}_{V(G)} = \{\frac{1}{n}\mathbb{Z} | n \in V(G)\} \subseteq \mathcal{K}(G)$, with equality if and only if all members of $\mathcal{K}(G)$ are discrete.

Proposition 4.2. — *For finite $V \subseteq \mathbb{N} - \{0\}$, and $G \in W$, these are equivalent.*

(1) $G \in HSP\mathcal{K}_V$.

(2) $\mathcal{K}(G) \subseteq \mathcal{K}_{Div\ V}$.

(3) $\mathcal{K}(G) = \mathcal{K}_{V(G)}$ *and* $V(G) \subseteq Div\ V$.

Proof. — (1)$\Leftrightarrow$ (2). $G \in HSP\mathcal{K}_V$ if and only if $HSPG \subseteq HSP\mathcal{K}_V$ if and only if $\mathcal{K}(G) \subseteq \mathcal{K}(HSP\mathcal{K}_V) = \mathcal{K}_{\text{Div } V}$ using 2.1 and 3.1.

(2) $\Rightarrow$ (3). If $\mathcal{K}(G) \subseteq \mathcal{K}_{\text{Div } V}$, then all members of $\mathcal{K}(G)$ are discrete. So $\mathcal{K}_{V(G)} = \mathcal{K}(G) \subseteq \mathcal{K}_{\text{Div } V}$ and $V(G) \subseteq \text{Div } V$.

(3) $\Rightarrow$ (1). $\mathcal{K}(G) = \mathcal{K}_{V(G)}$ implies $G \in HSP\mathcal{K}_{V(G)}$ (by 2.1), and $V(G) \subseteq \text{Div } V$ implies $HSP\mathcal{K}_{V(G)} \subseteq HSP\mathcal{K}_V$ (above). $\square$

Let us be more explicit about the association $V \mapsto HSP\mathcal{K}_V$.

Let $\mathbb{D}$ be the collection of all finite $V \subseteq \mathbb{N} - \{0\}$ for which $V = \text{Div } V$, together with $\mathbb{N} - \{0\}$, partially ordered by inclusion. Let $\mathbb{C}$ be the collection of all HSP-classes in W, partially ordered by inclusion. Both $\mathbb{D}$ and $\mathbb{C}$ are complete lattices in which inf is intersection, and the sup of any infinite subset is the top. Note $\emptyset \in \mathbb{D}$, and $\mathcal{K}_\emptyset = \{\{0\}\} \in \mathbb{C}$.

COROLLARY 4.3. — $\mathbb{D}$ *and* $\mathbb{C}$ *are lattice-isomorphic by* $V \mapsto HSP\mathcal{K}_V$.

See 5.6 below for a discussion of further ramifications of 2.1, 4.2, and 4.3.

5. Assorted Remarks

5.1. $W = HSP\mathbb{R}$. This fact (2.3) can be obtained in other ways:

[H], 7.1 uses full Yosida Representation to show that the free W-object on the set I is the l-subgroup of $C(\mathbb{R}^I)$ generated by 1 and the projections $\pi_i : \mathbb{R}^I \to \mathbb{R}$. 2.3 follows. J. Madden pointed out (see [H]) a more general algebraic route to this description of the free objects.

Another route to 2.3 is this. (a) If A is countable, then A embeds in some $\mathcal{C}(X)$. From Yosida Representation and the Baire Category Theorem, we can use $X = \bigcap_A a^{-1}\mathbb{R}$. On the other hand, there is a "point-free" argument using as X the frame spectrum of the frame of W-kernels; it is shown that the A-uniformity on the latter is complete [B1], with a countable base, thus spatial [I]. (b) In a SP-class of abstract algebras, if $\mathcal{D} \subseteq \mathcal{E}$ are $HSP-$classes, and each finitely generated member of $\mathcal{E}$ belongs to $\mathcal{D}$, then $\mathcal{D} = \mathcal{E}$. (See [P], p. 134.) 2.3 follows from (a) and (b).

5.3. $\mathcal{K}(HSP\mathcal{G})$. The question "$\mathcal{K}(HSP\mathcal{G}) = \mathcal{K}(\mathcal{G})$?" is invited by 2.1. This is so for $\mathcal{G} = \mathcal{K}_V$ (by 3.1(3) and the definition of $\mathcal{K}$), and of course it is so if $\mathcal{K}(\mathcal{G}) = S\mathbb{R}$. In all other cases, it is Not so, by 3.1. For example, $G = \mathbb{Z} + \sqrt{2}\mathbb{Z}$ is dense, and $\frac{1}{n}\mathbb{Z} \in \mathcal{K}(G)$ if and only if $n = 1$.

Related to that, the question "$\mathcal{K}_V \subseteq HSP\mathcal{G}$?" is invited as a "dual" to 4.2. It is not hard to see that this is so if and only if $\mathcal{K}(\mathcal{G}) = S\mathbb{R}$ or $\mathcal{K}(\mathcal{G})$ is an $S\mathcal{K}_U$ with $V \subseteq \mathrm{Div}\ U$.

5.4. $HSP\mathbb{Z}$. In W, this is the least HSP-class (besides $\{\{0\}\}$). The members, and the category, are examined in [HM].

Consider the class lAb of lattice-ordered abelian groups, and now H, S, P meant in lAb. In [D], § 52, it is shown that for each set I, a certain subobject $F(I)$ of $\mathbb{Z}^I$ is the free lAb-object on I. Thus $HSP\mathbb{Z} = lAb$, contrasting with W.

Consider the class Arch of archimedean l-groups (no designated unit; perhaps no units at all), and now H, S, P meant in Arch. The above $F(I) \in SP\mathbb{Z}$ in Arch (the lAb S and P preserve Arch, thus applied in Arch are the Arch S and P; not so for H). So $HSP\mathbb{Z} =$ Arch, contrasting with W.

5.5. Subdirectly irreducible. Let $\mathbb{A}$ be, say, an SP-class of abelian groups. In $\mathbb{A}$; a subdirect representation of A is an embedding $A \subseteq \Pi A_i$ with $\pi_i(A) = A_i$ for each i. A is subdirectly irreducible if each subdirect representation $A \subseteq \Pi A_i$ has some $\pi_i|A$ one-to-one, equivalently, A has a smallest $\mathbb{A}$-ideal $\neq (0)$. Let $si\mathbb{A} = \{A \,|\, A$ is subdirectly irreducible$\}$.

If $\mathbb{A}$ is a variety, then (Birkoff; See [P]) : $A \in SP(HA \cap si\mathbb{A}) \forall A$; $\mathbb{A} = HSP(si\mathbb{A})$. For W, $siW = S\mathbb{R}$ (proof below). Thus, by 2.1, $W = HSP(siW)$ (like a variety), but many $A \notin SP(HA \cap S\mathbb{R})$ (unlike a variety). (Those $A \in SP(HA \cap S\mathbb{R})$) may be called real-semi-simple. Some A have $HA \cap S\mathbb{R} = \emptyset$, e.g., Lebesgue measurable functions on $[0,1]$ mod null functions; also the real-valued functions on any countably generated Boolean frame [B2].)

($siW = S\mathbb{R}$: If $K \in S\mathbb{R}$, the only ideals are (0) and K, so $K \in siW$. Conversely, let $G \in siW$. Then, G is totally ordered (because any $a^{\perp\perp}$ is a W-ideal and if G is not totally ordered, there are disjoint $a, b > 0$, thus $a^{\perp\perp} \cap b^{\perp\perp} = (0)$. Thus there is no smallest W-ideal). Now $G \in S\mathbb{R}$, by Hölder's Theorem.)

5.6. With strong unit. For $G \in W$, if $G = G^*$, the unit e_G is called "strong". For $\mathcal{G} \subseteq W$ let $\mathcal{G}^* = \{G \in \mathcal{G} | G = G^*\}$. [HK] studies in some detail the class of l-groups called "uniformly hyperarchimedean", which class turns out to be $\cup\{(HSP\mathcal{K}_V)^* |$ finite $V \subseteq \mathbb{N} - \{0\}\}$. We sketch the connection. (Present notation differs from [HK].)

As a class of algebras, W^* has its H and S inherited from W, but its P comes from the W^*-product $(\Pi G_i)^*$, Π being the W- product.

For brevity, we use generic notation $\mathcal{G}$ for a subclass of W, $\mathcal{S}$ for a subclass of W^*. As usual, V is a finite subset of $\mathbb{N} - \{0\}$. 2.1 and 3.1(and remarks in section 4) easily yield the following.

(a) The HSP-classes of W and W^* are in bijective correspondence via $\mathcal{G} \mapsto \mathcal{G}^*$; $HSP\mathcal{S} \leftarrow \mathcal{S}(HSP$ in $W)$. In W^*, $HSP\mathbb{R} = W^*$. $\mathcal{S}$ is proper HSP in W^* if and only if $\exists V(\mathcal{S} = HSP\mathcal{K}_V$ in $W^*)$.

Connecting with [HK]: An l-group is called hyperarchimedean if each of its l-group quotients is archimedean. In such, any weak unit is strong. HA denotes $\{G \in W \,|\, G$ is hyperarchimedian$\} = \{G \in W^* \,|\, G$ is hyper-archimedean $\}$. $\mathcal{L}_1$ is the class of abelian l-groups with strong unit and unit-preserving homomorphisms; $\mathcal{L}_1$ has its H, S, and P (the product being the l-ideal in ΠG_i generated by (e_i), e_i being the strong unit of G_i and ΠG_i the l-group product). Note $W^* = \mathcal{L}_1 \cap W$.

(b) ([HK],4.4(a)) The following are equivalent. $\mathcal{S}$ is HSP in $\mathcal{L}_1$ and $\mathcal{S} \subseteq HA$; $\mathcal{S}$ is proper HSP in W^* (i.e., (a) above); $\exists V(\mathcal{S} = HSP\mathcal{K}_V$ in $\mathcal{L}_1$).

The most interesting part of the proof of (b) is : $\mathrm{S} \in W^* - HA \Rightarrow \mathbb{R} \in HSPS$ in W^*. The present 3.1(2) is a version of that.

Also note that the present 4.2 is the correspondent in W of ([HK], 4.4(b)).

Finally, we record briefly the connection with MV-algebras. This is discussed further in [HK]. The theory of MV-algebras is exposed in [COM].

There is the Chang-Mundici categorical equivalence between $\mathcal{L}_1$ and MV, the latter being a variety, in consequence of which the HSP classes in $\mathcal{L}_1$ and the subvarieties of MV correspond. The latter have been completely classified (with equations) by Komori. The correspondent of the $\mathcal{S} = HSP\mathcal{K}_V$ in (b) above is called $\mathcal{C}(\{\text{Ł}_v\}_V)$ in [COM], p. 169.

Acknowledgments. — This paper arose from conversations between the authors and J. Madden at the spring 2006 conference in Ordered Algebra in Gainesville, FL. We thank Madden for his contributions (some details in the text). We thank the referee for numerous suggestions which have improved the paper. We thank J. Martinez for organizing the conference, and many others; his contributions are difficult to overstate.

Bibliography

[BH] Ball (R.), Hager (A.). — A new characterization of the continuous functions on a locale, Positivity 10, 165-199 (2006)

[B1] Banaschewski (B.). — On the function ring functor in point-free topology, Appl. Categ. Str. 13, p.305-328 (2005).

[B2] Banaschewski (B.). — On the function rings of point-free topology, Kyungpook Math. J. 48, p.195-206 (2008).

[COM] Cignoli (R.), D'Ottavigno (I.), Mundici (D.). — Algebraic foundations of many-valued reasoning, Kluwer (2000).

[D] Darnel (M.). — Theory of lattice -ordered groups, Dekker (1995).

[GJ] Gillman (L.), Jerison (M.). — Rings of continuous functions, Van Nostrand (1960).

[H] Hager (A.). — Algebraic closures of l-groups of continuous functions, pp. 165 - 193 in Rings of Continuous Functions (C. Aull, Editor), Dekker Notes 95 (1985).

[HK] Hager (A.), Kimber (C.). — Uniformly hyperarchimedean lattice-ordered groups, Order 24, p. 121-131 (2007).

[HM] Hager (A.), Martinez (J.). — Singular archimedean lattice-ordered groups, Alg. Univ. 40, p.119-147 (1998).

[HR] Hager (A.), Robertson (L.). — Representing and ringifying a Riesz space, Symp. Math. 21, p. 411-431 (1977).

[HI] Henriksen (M.), lsbell (J.). — Lattice-ordered rings and function rings, Pac. J. Math. 12, p. 533-565.r (1962).

[HIJ] Henriksen (M.), Isbell (J.), D. Johnson (D.). — Residue class fields of lattice-ordered algebras, Fund. Math. 50, p.107-117 (1965)

[I] Isbell (J.). — Atomless parts of spaces, Math. Scand. 31, p. 5-32 (1972).

[LZ] Luxemburg (W.), Zaanen (A.). — Riesz spaces I, North-Holland (1971).

[P] Pierce (R.). — Introduction to the theory of abstract algebras, Holt, Rinehart and Winston (1968).

[W] Weinberg (E.). — Lectures on ordered groups and rings, Univ. of Illinois (1968).

[Y] Yosida (K.). — On the representation of the vector lattice, Proc. Imp. Acad. (Tokyo) 18, p. 339-343, (1942).