ASHISH K. SRIVASTAVA
A Survey of Rings Generated by Units

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Dedicated to Melvin Henriksen on his 80th Birthday

Abstract. — This article presents a brief survey of the work done on rings generated by their units.

Résumé. — Cet article est un bref survey de l'étude des anneaux engendrés par leurs unités.

The study of rings generated additively by their units started in 1953-1954 when K. G. Wolfson [24] and D. Zelinsky [25] proved, independently, that every linear transformation of a vector space V over a division ring D is the sum of two nonsingular linear transformations, except when dim $V = 1$ and $D = \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. This implies that the ring of linear transformations $\text{End}_D(V)$ is generated additively by its unit elements. In fact, each element of $\text{End}_D(V)$ is the sum of two units except for one obvious case when V is a one-dimensional space over $\mathbb{Z}_2$. In 1998 this result was reproved by Goldsmith, Pabst and Scott, who remarked that this result can hardly be new, but they were unable to find any reference to it in the literature [8]. Wolfson and Zelinsky's result generated quite a bit of interest in the study of rings that are generated by their unit elements.

All our rings are associative (not necessarily commutative) with identity element 1. A ring R is called a *von Neumann regular ring* if for each element $x \in R$ there exists an element $y \in R$ such that $xyx = x$. In 1958 Skornyakov ([20], Problem 31, page 167) asked: *Is every element of a von Neumann regular ring (which does not have $\mathbb{Z}_2$ as a factor ring) a sum of units?* This

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question of Skornyakov was answered in the negative by Bergman in 1977 (see [9]). Bergman constructed a von Neumann regular algebra in which not all elements are sums of units.

There are several natural classes of rings that are generated by their unit elements – in particular, rings in which each element is the sum of two units. Let X be a completely regular Hausdorff space. Then every element in the ring of real-valued continuous functions on X is the sum of two units [18]. Every element in a real or complex Banach algebra is the sum of two units [22].

The rings in which each element is the sum of k units were called (S, k)-*rings* by Henriksen. Vámos has called such rings *k-good rings*.

It may be easily observed that if R is a 2-good ring, then the matrix ring $M_n(R)$ is also a 2-good ring. The following result of Henriksen is quite surprising, and it shows that every ring is Morita equivalent to an $(S, 3)$-ring (or a 3-good ring).

THEOREM 0.1. — (*Henriksen, [11]*) *Let R be any ring. Then each element of the matrix ring $M_n(R)$, $n > 1$, can be written as the sum of exactly three units.*

We say that an $n \times n$ matrix A over a ring R *admits a diagonal reduction* if there exist invertible matrices $P, Q \in M_n(R)$ such that PAQ is a diagonal matrix. A ring R is called an *elementary divisor ring* if every square matrix over R admits a diagonal reduction.

THEOREM 0.2. — (*Henriksen, [11]*) *Let R be an elementary divisor ring. Then each element of the matrix ring $M_n(R)$, $n > 1$, can be written as the sum of two units in $M_n(R)$.*

An infinite-dimensional matrix is called *row-and-column-finite* if each of its rows and columns has only finitely many nonzero entries. Recently, Wang and Chen have shown the following.

THEOREM 0.3. — (*Wang and Chen, [23]*) *Let R be a 2-good ring. Then the ring $B(R)$ of all $\omega \times \omega$ row-and-column-finite matrices over R is also a 2-good ring. However, if R is any arbitrary ring, then the ring $B(R)$ is a 3-good ring.*

1. von Neumann Regular Rings Generated by Units

A number of authors have studied von Neumann regular rings in which each element is a sum of units. We give below a short survey of results obtained in this direction.

A ring R is called a *unit-regular ring* if for each element $x \in R$ there exists a unit element $u \in R$ such that $xux = x$. Henriksen had shown that every unit-regular ring is an elementary divisor ring [10].

THEOREM 1.1. — (*Ehrlich, [5]*) *Let R be a unit-regular ring in which 2 is invertible. Then every element of R is the sum of two units.*

The above result follows quickly by observing that if R is a unit-regular ring, then each $x \in R$ can be written as $x = eu$, where e is an idempotent and u is a unit. We may write $e = (1 + e) - 1$. Now, since 2 is a unit in R, $(1 + e)$ is a unit with $(1 + e)^{-1} = 1 - \frac{1}{2}e$. This gives that e is the sum of two units and hence x is the sum of two units.

A ring R is called *right self-injective* if each right R-homomorphism from any right ideal of R to R can be extended to an endomorphism of R. The *index* of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The *index of nilpotence* of a two-sided ideal I in R is the supremum of the indices of all nilpotent elements of I. If this supremum is finite, then I is said to have *bounded index of nilpotence*.

THEOREM 1.2. — (*Utumi, [21]*) *Every element in a von Neumann regular right self-injective ring having no ideals with index of nilpotence 1, is a sum of units.*

THEOREM 1.3. — (*Raphael, [18]*) *If in a von Neumann regular right self-injective ring R, every idempotent is the sum of two units, then every element can be written as the sum of an even number of units.*

Fisher and Snider [6] showed that a von Neumann regular ring with primitive factor rings artinian is unit-regular. Therefore, in view of the result of Ehrlich, they obtained the following.

THEOREM 1.4. — (*Fisher and Snider, [6]*) *Let R be a von Neumann regular ring with primitive factor rings artinian. If 2 is a unit in R, then each element of R can be expressed as the sum of two units.*

A right R-module M is said to satisfy the *exchange property* if for any right R-module A and any internal direct sum decomposition of A given by

$A = M' \oplus N = \bigoplus_I A_i$ for right R-modules M', N, A_i where $M' \cong M$, there always exist submodules $B_i \subseteq A_i$ for each $i \in I$ such that $A = M' \oplus (\bigoplus_I B_i)$. If the above holds for finite index sets I, then M is said to satisfy the *finite exchange property*. A ring R is called an *exchange ring* if R_R has the (finite) exchange property.

The result of Fisher and Snider has been extended to exchange rings by Huanyin Chen.

THEOREM 1.5. — (*Chen, [4]*) *Let R be an exchange ring with artinian primitive factors. Then every element of R is the sum of two units if and only if R does not have $\mathbb{Z}_2$ as a homomorphic image.*

2. Right Self-injective Rings Generated by Units

As a ring of linear transformations is a right self-injective ring, Wolfson and Zelinsky's result has attracted many researchers toward understanding which right self-injective rings have the property that each element is the sum of two units.

A ring R is called a *Boolean ring* if every element of R is an idempotent. If e is an idempotent in ring R, then the ring eRe is called a *corner ring*. In 2005 Vámos proved the following.

THEOREM 2.1. — (*Vámos, [22]*) *Every element of a right self-injective ring is the sum of two units if the ring has no non-zero corner ring that is Boolean.*

Using type theory of von Neumann regular right self-injective rings and the idea of elementary divisor rings, Khurana and Srivastava extended the result of Wolfson and Zelinsky to the class of right self-injective rings in the following theorem. For details on the type theory of von Neumann regular right self-injective rings, the reader is referred to [7].

THEOREM 2.2. — (*Khurana and Srivastava, [12]*) *Every element of a right self-injective ring is the sum of two units if and only if it has no factor ring isomorphic to $\mathbb{Z}_2$.*

Khurana and Srivastava generalized the above result to endomorphism rings of various classes of modules. Before mentioning those results, let us define some terminology that will be required. For any term not defined here, the reader is referred to [15] and [16].

A right R-module M is called *N-injective* if every right R-homomorphism from a submodule L of N to M can be extended to an R-homomorphism

from N to M. A right R-module M is called an *injective module* if M is N-injective for every right R-module N. If M is M-injective, then M is called a *quasi-injective module*.

Consider the following three properties;
(i) Every submodule of M is essential in a direct summand of M.
(ii) If N_1 and N_2 are direct summands of M with $N_1 \cap N_2 = 0$, then $N_1 \oplus N_2$ is also a direct summand of M.
(iii) Every submodule of M isomorphic to a direct summand of M is itself a direct summand of M.

A right R-module M is called a *quasi-continuous module* if it satisfies (i) and (ii); and M is called a *continuous module* if it satisfies (i) and (iii).

A right R-module M is called a *cotorsion module* if every short exact sequence $0 \to M \to E \to F \to 0$, where F is a flat right R-module, splits.

THEOREM 2.3. — (*Khurana and Srivastava, [12]*) *Let S be any ring, let M be a quasi-continuous right S-module with the finite exchange property and let $R = End_S(M)$. If no factor ring of R is isomorphic to $\mathbb{Z}_2$, then every element of R is the sum of two units.*

THEOREM 2.4. — (*Khurana and Srivastava, [12]*) *In the endomorphism ring of a continuous (in particular, of a quasi-injective or even injective) module, every element is the sum of two units provided no factor of the endomorphism ring is isomorphic to $\mathbb{Z}_2$.*

THEOREM 2.5. — (*Khurana and Srivastava, [12]*) *Every element of the endomorphism ring of a flat cotorsion (in particular, pure injective) module is the sum of two units if no factor of the endomorphism ring is isomorphic to $\mathbb{Z}_2$.*

In the same spirit, Meehan proved the following result for the endomorphism ring of a free module over a principal ideal domain.

THEOREM 2.6. — (*Meehan, [17]*) *Let R be a pricipal ideal domain. Then every element of the endomorphism ring of a free module of infinite rank is the sum of two units.*

Recall that a group G is called a *locally finite group* if every finitely generated subgroup of G is finite.

THEOREM 2.7. — (*Khurana and Srivastava, [12]*) *If R is a right self-injective ring and G a locally finite group, then every element of the group ring $R[G]$ is the sum of two units unless R has a factor ring isomorphic to $\mathbb{Z}_2$.*

3. Unit Sum Number

Goldsmith, Pabst and Scott introduced the idea of studying rings whose elements can be written as the sum of a fixed number of units [8]. A ring R is said to have the *n-sum property*, for a positive integer n, if its every element can be written as the sum of exactly n units of R. The *unit sum number* of a ring R, denoted by (R), is the least integer n, if any such integer exists, such that R has the n-sum property.

If R has an element that is not a sum of units, then we set (R) to be ∞, and if every element of R is a sum of units but R does not have n-sum property for any n, then we set $(R) = \omega$.

Examples. —

1. $\text{usn}(\mathbb{Z}_k) = 2$ if k is odd and $\text{usn}(\mathbb{Z}_k) = \omega$ if k is even.

2. If D is a division ring then $\text{usn}(D) = 2$ unless $D = \mathbb{Z}_2$, in which case $\text{usn}(D) = \omega$.

3. $\text{usn}(\text{End}_D(V)) = \omega$ if V is a one-dimensional space over $\mathbb{Z}_2$; otherwise $\text{usn}(\text{End}_D(V)) = 2$.

4. If $R = J_p$, the ring of p-adic integers, then $\text{usn}(R) = 2$ except when $p = 2$, in which case $\text{usn}(R) = \omega$.

The complete characterization of unit sum numbers of right self-injective rings was given by Khurana and Srivastava in [13].

THEOREM 3.1. — (*Khurana and Srivastava, [13]*) *Let R be a nonzero right self-injective ring. Then $\text{usn}(R) = 2$, ω, and ∞. Moreover,*

1. $\text{usn}(R) = 2$ *if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$.*

2. $\text{usn}(R) = \omega$ *if and only if R has a factor ring isomorphic to $\mathbb{Z}_2$, but has no factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case every non-invertible element of R is the sum of either two or three units.*

3. $\text{usn}(R) = \infty$ *if and only if R has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Henriksen's Question (Question E, Page 192, [11]):

Does there exist a von Neumann regular ring generated by its units containing a non-invertible element that is not the sum of two units?

In the following example Khurana and Srivastava answered this question in the affirmative.

Example. — [13] Let S be a nonzero regular right self-injective ring which does not have a factor ring isomorphic to $\mathbb{Z}_2$. For instance, take S to be any field other than $\mathbb{Z}_2$. Let $R = S \times \mathbb{Z}_2$.

Clearly R is a von Neumann regular right self-injective ring. By Theorem 3.1, every element of R is a sum of units. But, the element $(0, 1)$ of R is a non-unit which can't be written as the sum of two units.

Ashrafi and Vámos studied the unit sum numbers of rings of integers of number fields [1]. Let $K = \mathbb{Q}(\xi)$ be a number field (that is, a finite extension of $\mathbb{Q}$) and let $\mathcal{O}_K$ be the ring of integers of K. For details on the ring of integers of a number field, the reader is referred to [14].

THEOREM 3.2. — (*Ashrafi and Vámos, [1]*) *The ring of integers of a quadratic or complex cubic number field is not k-good for any k; that is,* $\mathrm{usn}(\mathcal{O}_K) = \omega$ *or* ∞.

In the quadratic case, Ashrafi and Vámos completely determined $\mathrm{usn}(\mathcal{O}_K)$.

THEOREM 3.3. — (*Ashrafi and Vámos, [1]*) (a) *Let* $d > 0$ *be a square-free integer and let* $K = \mathbb{Q}(\sqrt{d})$ *be the corresponding real quadratic field. Then* $\mathrm{usn}(\mathcal{O}_K) = \omega$ *in precisely the following two cases:*
(i) $d \not\equiv 1 \pmod{4}$ *and* $d = a^2 \pm 1$ *for some* $a \in \mathbb{Z}$;
(ii) $d \equiv 1 \pmod{4}$ *and* $d = a^2 \pm 4$ *for some* $a \in \mathbb{Z}$.
In all other cases $\mathrm{usn}(\mathcal{O}_K) = \infty$.
(b) *Let* $d < 0$ *be a square-free integer and let* $K = \mathbb{Q}(\sqrt{d})$ *be the corresponding imaginary quadratic field. Then* $\mathrm{usn}(\mathcal{O}_K) = \omega$ *if* $d = -1$ *or* $d = -3$; *otherwise* $\mathrm{usn}(\mathcal{O}_K) = \infty$.

Ashrafi and Vámos have also investigated the unit sum number of tensor products of algebras.

THEOREM 3.4. — (*Ashrafi and Vámos, [1]*) *Let* R_1 *and* R_2 *be algebras over a field* F. *If* $\mathrm{usn}(R_1) \leqslant \omega$ *and* $\mathrm{usn}(R_2) \leqslant \omega$; *then* $\mathrm{usn}(R_1 \otimes_F R_2) \leqslant \omega$.

THEOREM 3.5. — (*Ashrafi and Vámos, [1]*) *Let* R *be an algebraic algebra over a field* F *and let* K *be an extension field of* F *different from* $\mathbb{Z}_2$. *Then* $\mathrm{usn}(K \otimes_F R) = 2$.

THEOREM 3.6. — (*Ashrafi and Vámos, [1]*) *Let* K *be a proper pure transcendental field extension of a field* F *and let the integral domain* R *be an* F-*algebra, not algebraic over* F. *Then* $\mathrm{usn}(K \otimes_F R) = \omega$ *or* ∞.

4. Other Related Results

In a somewhat related direction A. W. Chatters, S. M. Ginn, and J. C. Robson considered rings that are generated by their regular elements (see [3], [19]). Recall that an element x in a ring R is called a regular element if x is not a left or right zero-divisor. Note that since every unit element is a regular element, rings generated by their units may obviously be considered as rings generated by their regular elements. In some classes of rings the regular elements and units coincide, for example in von Neumann regular rings.

A ring R is called a *prime ring* if (0) is a prime ideal of R. A ring R is called a *semiprime ring* if the intersection of all prime ideals of R is (0). A ring R is called a *right Goldie ring* if R does not contain an infinite direct sum of right ideals and R has the ascending chain condition on right annihilator ideals.

Chatters and Ginn found the following in case of a prime right Goldie ring.

THEOREM 4.1. — (*Chatters and Ginn, [3]*) *Every element of a prime right Goldie ring is the sum of three or fewer regular elements.*

This was later improved by Robson.

THEOREM 4.2. — (*Robson, [19]*) *Every element of a prime right Goldie ring is the sum of at most two regular elements.*

THEOREM 4.3. — (*Chatters and Ginn [3], Robson, [19]*) *Let R be a semiprime right Goldie ring. Then*

(1) Each element of R is the sum of two regular elements if R does not have $\mathbb{Z}_2$ as a direct summand.

(2) R is generated by its regular elements if and only if R does not have a direct summand isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

THEOREM 4.4. — (*Chatters and Ginn [3]*) *Let R be a left and right Noetherian ring in which 2 is a regular element. Then R is generated by its regular elements.*

It will be interesting to see if the above results on rings generated by their regular elements due to Chatters, Ginn and Robson can be extended to rings generated by their units.

5.   Open Problems

PROBLEM 5.1. — (*Khurana and Srivastava, [13]*) *Let R be a von Neumann regular ring with* $\operatorname{usn}(R) \neq \infty$. *Is every non-invertible element of R the sum of two or three units?*

The answer is in the affirmative if R is, in addition, right self-injective.

PROBLEM 5.2. — (*Henriksen, [11]*) *Let R be a von Neumann regular ring in which 2 is invertible such that* $\operatorname{usn}(R) \neq \infty$. *Is every non-invertible element of R the sum of two units?*

The answer is in the affirmative if R is, in addition, a right self-injective ring.

A ring R is called a *right Hermite ring* if every 1×2 matrix $A = (a \quad b)$ over R with $a \neq 0$ and $b \neq 0$ admits a diagonal reduction, that is, there exists an invertible 2×2 matrix Q over R such that $AQ = (d \quad 0)$ where $d(\neq 0) \in R$.

PROBLEM 5.3. — (*Henriksen, [11]*) *Let R be a right Hermite ring. Is every element in $M_2(R)$ the sum of two units?*

A ring R is called a ring of *type* (m, n) and is denoted by $L(m, n)$ if $R^m \cong R^{m+n}$ as R-modules.

PROBLEM 5.4. — *When is every element of the ring $L(m, n)$ the sum of two units?*

Note that every element of the ring $L(1, 1)$ is a sum of two units if it is an elementary divisor ring. So, it would be useful to study when the ring $L(1, 1)$ is an elementary divisor ring.

CONJECTURE 5.5. — *Let R be a unit-regular ring. Then each element of R is the sum of two units if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$.*

A ring R is called a *clean ring* if for each $r \in R$, $r = e + u$, where e is an idempotent in R and u is a unit in R.

CONJECTURE 5.6. — *Let R be a clean ring. Then each element of R is the sum of two units if and only if R has no factor ring isomorphic to $\mathbb{Z}_2$.*

Note that if R is a clean ring with 2 invertible, then every element of R is the sum of two units [2].

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