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On the Pierce-Birkhoff Conjecture for Smooth Affine Surfaces over Real Closed Fields

Sven Wagner[1]

Abstract. — We will prove that the Pierce-Birkhoff Conjecture holds for non-singular two-dimensional affine real algebraic varieties over real closed fields, i.e., if W is such a variety, then every piecewise polynomial function on W can be written as suprema of infima of polynomial functions on W. More precisely, we will give a proof of the so-called Connectedness Conjecture for the coordinate rings of such varieties, which implies the Pierce-Birkhoff Conjecture.

Résumé. — Nous montrons que la conjecture de Pierce-Birkhoff est vérifiée dans le cas des variétés algébriques affines réelles non-singulières de dimension 2 sur des corps réels clos, c'est-à-dire que si W est une telle variété, alors toute fonction polynômiale par morceaux sur W s'écrit comme un supremum d'infima de fonctions polynômiales sur W. Plus précisément, nous montrons la conjecture dite *de connexité* pour l'anneau des coordonnées d'une telle variété, laquelle implique la conjecture de Pierce-Birkhoff.

1. Introduction

In 1956, G. Birkhoff and R. S. Pierce raised the following question, which is well-known today as the **Pierce-Birkhoff Conjecture**:

Conjecture 1.1 *(Pierce, Birkhoff).* —

Let $t\colon \mathbb{R}^n \to \mathbb{R}$ be a piecewise polynomial function. Then there is a finite family of polynomials $\{h_{ij}\}_{i\in I, j\in J} \subset \mathbb{R}[X_1,\dots,X_n]$ such that

$$t = \sup_{i\in I}\big(\inf_{j\in J} h_{ij}\big).$$

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The statement of this conjecture depends on the following

Definition 1.2. — *A function $t\colon \mathbb{R}^n \to \mathbb{R}$ is said to be piecewise polynomial if there are closed semialgebraic subsets $U_1, \dots, U_m$ of $\mathbb{R}^n$ and polynomials $t_1, \dots, t_m \in \mathbb{R}[X_1, \dots, X_n]$ such that $\mathbb{R}^n = \bigcup_{k=1}^{m} U_k$ and $t = t_k$ on U_k.*

This conjecture was proved in 1984 by Louis Mahé in the case $n = 2$ (see [Mah84]). For $n > 2$, it is still open.

In 1989, J. J. Madden formulated the Pierce-Birkhoff Conjecture for $\mathbb{R}^n$ in terms of the real spectrum of the polynomial ring $\mathbb{R}[X_1, \dots, X_n]$. In doing so, he introduced the concept of a "separating ideal". He used separating ideals to define a property that makes sense for any commutative ring and he showed that this property is satisfied by $\mathbb{R}[X_1, \dots, X_n]$ for $n = 1, 2, 3, \dots$ if and only if the Pierce-Birkhoff Conjecture is true. In 1995, D. Alvis, B. L. Johnston and J. J. Madden applied Zariski's theory of quadratic transformations to study separating ideals in two-dimensional regular local domains, showing that quadratic transforms of separating ideas are well-behaved if the residue field is real closed. In 2007, F. Lucas, J. J. Madden, D. Schaub and M. Spivakovsky introduced the Connectedness Conjecture and showed that it implies the Pierce-Birkhoff Conjecture.

In the present paper, we give a proof of the Connectedness Conjecture for the coordinate ring of any non-singular two-dimensional affine real algebraic variety over a real closed field. Of course, this implies the Pierce-Birkhoff Conjecture for such rings. Our proof rests on the theory developed by Madden and by Alvis, Johnston and Madden. In an attempt to make the present paper self-contained, we include a review the essential results of this theory.

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2. Pierce-Birkhoff Rings

In the next two sections, we will give a short overview of Madden's article [Mad89] and we include a proof of one of the main results.

Let A be a (commutative) ring (with 1). Denote by $\operatorname{Sper} A$ the real spectrum of A. Let $\alpha \in \operatorname{Sper} A$. Then α induces a total ordering on $A(\alpha) := A/\operatorname{supp}(\alpha)$, where $\operatorname{supp}(\alpha) = \alpha \cap -\alpha$. We denote by ρ_α the homomorphism

$A \to A/\mathrm{supp}(\alpha)$ and by $R(\alpha)$ the real closure of the quotient field $k(\alpha)$ of $A(\alpha)$ with respect to the ordering induced by α.

Definition 2.1 *(Schwartz). — An element $s \in \prod_{\alpha\in \mathrm{Sper} A} R(\alpha)$ is called an **abstract semialgebraic function** on Sper A iff the image of s is constructible, i.e., there exists a formula $\Phi(T)$ in the language of ordered rings with coefficients in A containing only one variable, which is also free, such that, for all $\alpha \in$ Sper A, $\Phi_\alpha(s(\alpha))$ holds in $R(\alpha)$ and $s(\alpha)$ is the only solution of the specialization $\Phi_\alpha(T)$ in $R(\alpha)$. We say that s is **continuous** if it satisfies the following compatibility condition regarding specializations: Let $\alpha, \beta \in$ Sper A, β a specialization of α. Then $R(\alpha)$ contains a largest convex subring $C_{\beta\alpha}$ with maximal ideal $M_{\beta\alpha}$ such that $A(\alpha) \subset C_{\beta\alpha}$ and $\rho_\alpha^{-1}(M_{\beta\alpha}) =$ supp(β). Then $C_{\beta\alpha}/M_{\beta\alpha}$ is a real closed field containing $A(\beta)$ and therefore also $R(\beta)$. The condition s has to suffice is*

$$\lambda_{\beta\alpha}(s(\alpha)) = s(\beta) \in R(\beta) \subset C_{\beta\alpha}/M_{\beta\alpha},$$

where $\lambda_{\beta\alpha}\colon C_{\beta\alpha} \to C_{\beta\alpha}/M_{\beta\alpha}$.

We denote by $SA(A)$ the set of all continuous abstract semialgebraic functions on Sper A. This is a subring of $\prod_{\alpha\in \mathrm{Sper} A} R(\alpha)$.

Remarks 2.2. —

(i) Every element $a \in A$ induces an abstract semialgebraic function.

(ii) For $s, t \in SA(A)$, the set $\{\alpha \in \mathrm{Sper}\, A \mid s(\alpha) = t(\alpha)\} \subset \mathrm{Sper}\, A$ is constructible.

Definition 2.3. — *By $\boldsymbol{PW}(A)$, we denote the set of all continuous abstract semialgebraic functions $s \in SA(A)$ with the property that, for all $\alpha \in$ Sper A, there exists an element $a \in A$ such that $s(\alpha) = a(\alpha)$.*

*The elements of $PW(\mathbb{R}[X_1, \dots, X_n])$ are called **abstract piecewise polynomial functions**.*

An abstract piecewise polynomial function has a presentation analogous to that of a piecewise polynomial as in Definition 1.2. To be precise, suppose $t \in \mathrm{PW}(A)$. Let $\alpha \in \mathrm{Sper}\, A$. If $t(\alpha) = a(\alpha)$ for some $a \in A$, then, $t = a$ on a constructible set U_α containing α. By compactness, we have $\mathrm{Sper}\, A = \bigcup_{i=1}^{m} U_i$ for finitely many constructible sets U_i such that $t = a_i$ on U_i for some $a_i \in A$. Since $\{\alpha \in \mathrm{Sper}\, A \mid t(\alpha) = a_i(\alpha)\}$ is closed in the spectral topology, we may assume that each U_i is closed.

Definition 2.4 *(Madden). — Suppose $t \in SA(A)$. We say t is* ***sup-inf-definable over*** *A if there is a finite family $\{h_{ij}\}_{i\in I, j\in J} \subset A$ such that*

$$t = \sup_{i\in I}\big(\inf_{j\in J} h_{ij}\big).$$

A is called a ***Pierce-Birkhoff ring*** *if every $t \in PW(A)$ is sup-inf-definable over A.*

Let R be a real closed field, and let V be an algebraic subset of R^n. We denote by $\mathcal{P}(V)$ its coordinate ring. Then $\mathrm{PW}(\mathcal{P}(V))$ is isomorphic to the ring of piecewise polynomial functions on V. Hence the question whether the Pierce-Birkhoff Conjecture holds for V is equivalent to the question whether $\mathcal{P}(V)$ is a Pierce-Birkhoff ring.

3. Separating Ideals

The present section summarizes the main results of [Mad89], including the definition of separating ideals and their use in describing the abstract Pierce-Birkhoff property. First, using the abstract Pierce-Birkhoff property (Definition 2.4) together with the compactness of the constructible topology of the real spectrum, we show that an abstract piecewise polynomial is globally sup-inf-definable if it is sup-inf-definable on every pair of elements of the real spectrum. This shows that the Pierce-Birkhoff property is local in a very strong sense.

Theorem 3.1. — *A is a Pierce-Birkhoff ring if and only if, for all $t \in PW(A)$ and for all $\alpha, \beta \in \operatorname{Sper} A$, there is an element $h \in A$ such that $h(\alpha) \geqslant t(\alpha)$ and $h(\beta) \leqslant t(\beta)$.*

Proof. — Let $t \in \mathrm{PW}(A)$.

Suppose there is a finite family $\{h_{ij}\}_{i\in I, j\in J} \subset A$ such that

$$t = \sup_{i\in I}\big(\inf_{j\in J} h_{ij}\big),$$

and suppose further that there exist $\alpha, \beta \in \operatorname{Sper} A$ such that, for all $h \in A$, $h(\alpha) \geqslant t(\alpha)$ implies $h(\beta) > t(\beta)$. Then, since there is some $i_0 \in I$ such that $\inf_{j\in J} h_{i_0 j}(\alpha) \geqslant t(\alpha)$, we have $\sup_{i\in I}\big(\inf_{j\in J} h_{ij}(\beta)\big) > t(\beta)$, a contradiction.

Suppose now that, for all $\alpha, \beta \in \operatorname{Sper} A$, there is an element $h_{\alpha\beta} \in A$ such that $h_{\alpha\beta}(\alpha) \geqslant t(\alpha)$ and $h_{\alpha\beta}(\beta) \leqslant t(\beta)$. In particular, $h_{\alpha\alpha}(\alpha) = t(\alpha)$.

For each $\alpha, \beta \in \operatorname{Sper} A$, there are closed constructible sets $U(\alpha, \beta)$ and $V(\alpha, \beta)$ such that $\alpha \in U(\alpha, \beta)$, $\beta \in V(\alpha, \beta)$, $h_{\alpha\beta} \geqslant t$ on $U(\alpha, \beta)$ and $h_{\alpha\beta} \leqslant t$ on $V(\alpha, \beta)$. Without loss of generality $U(\alpha, \alpha) = V(\alpha, \alpha)$.

We fix some $\alpha \in \operatorname{Sper} A$. Since $\operatorname{Sper} A$ is the union of the sets $V(\alpha, \beta)$ with $\beta \in \operatorname{Sper} A$, by compactness, there exist $\alpha = \beta_0, \ldots, \beta_r \in \operatorname{Sper} A$ such that $\operatorname{Sper} A = \bigcup_{j=0}^{r} V(\alpha, \beta_j)$. Let $U(\alpha) := \bigcap_{j=0}^{r} U(\alpha, \beta_j)$ and $H_\alpha := \inf_{j \in J} h_{\alpha, \beta_j}$, where $J = \{0, \ldots, r\}$. Then $H_\alpha \leqslant t$ globally and $H_\alpha = t$ on $U(\alpha)$.

$\operatorname{Sper} A$ is the union of the sets $U(\alpha)$ with $\alpha \in \operatorname{Sper} A$, and hence, again by compactness, there exist $\alpha_1, \ldots, \alpha_s \in \operatorname{Sper} A$ such that $\operatorname{Sper} A = \bigcup_{i=0}^{s} U(\alpha_i)$. Then $t = \sup_{i \in I} H_{\alpha_i}$, where $I = \{1, \ldots, s\}$. $\square$

Definition 3.2. — *Let A be a ring. For $\alpha, \beta \in \operatorname{Sper} A$, we denote by $\langle \alpha, \beta \rangle$ the ideal of A generated by all $a \in A$ with the property $a(\alpha) \geqslant 0$ and $a(\beta) \leqslant 0$. We will call $\langle \alpha, \beta \rangle$ the* ***separating ideal*** *of α and β.*

Remarks 3.3. — Let $\alpha, \beta \in \operatorname{Sper} A$.

1. $\operatorname{supp}(\alpha) + \operatorname{supp}(\beta) \subset \langle \alpha, \beta \rangle$.
2. In general, $\langle \alpha, \beta \rangle$ is not a prime ideal.

Lemma 3.4. — *Let $\alpha, \beta \in \operatorname{Sper} A$. Let $a \in A$ such that $a(\alpha) \geqslant 0$. Then we have $a \in \langle \alpha, \beta \rangle$ if and only if there exists an $h \in A$ such that $a(\alpha) \leqslant h(\alpha)$ and $h(\beta) \leqslant 0$.*

Definition 3.5. — *Let $\alpha \in \operatorname{Sper} A$. An ideal I of A is called* ***α-convex*** *if, for all $a, b \in \alpha$, it follows from $a + b \in I$ that $a, b \in I$. The set of α-ideals of A is totally ordered by inclusion. If A is noetherian, there exists a largest proper α-convex ideal in A, called the* ***center*** *of α in A, and denoted by cent(α).*

Proposition 3.6. — *Let $\alpha, \beta \in \operatorname{Sper} A$.*

a) In A, the ideal $\langle \alpha, \beta \rangle$ is convex with respect to α and β.

b) Both α and β induce the same total ordering on $A / \langle \alpha, \beta \rangle$, and $\langle \alpha, \beta \rangle$ is the smallest ideal of A with this property.

c) *If* $\sqrt{\langle\alpha,\beta\rangle}$ *is proper, then it is prime, and* α *and* β *induce the same total ordering on* $A/\sqrt{\langle\alpha,\beta\rangle}$. *Thus, in this case,* $\sqrt{\langle\alpha,\beta\rangle}$ *together with this order is the least common specialization* γ *of* α *and* β *in* $\mathrm{Sper}\,A$.

d) *Let* $t \in PW(A)$. *For* $\delta \in \mathrm{Sper}\,A$, *we denote by* t_δ *any element* $a \in A$ *such that* $t(\delta) = a(\delta)$. *The compatibility condition for* t *gives us* $t_\alpha(\gamma) = t_\gamma(\gamma) = t_\beta(\gamma)$, *hence* $t_\alpha - t_\beta \in \sqrt{\langle\alpha,\beta\rangle}$.

e) *Every ideal of* A *containing* $\langle\alpha,\beta\rangle$ *is* α*-convex if and only if it is* β*-convex.*

f) *Suppose* α *and* β *have no common specialization. Then* $\langle\alpha,\beta\rangle = A$.

g) *Suppose* A *is noetherian and* $\mathrm{cent}(\alpha) \neq \mathrm{cent}(\beta)$. *Then* $\langle\alpha,\beta\rangle = A$, *since otherwise both* $\mathrm{cent}(\alpha)$ *and* $\mathrm{cent}(\beta)$ *would be* α*- and* β*-convex, and therefore equal because of their maximality.*

THEOREM 3.7. — *A is a Pierce-Birkhoff ring if and only if for all* $t \in PW(A)$ *and all* $\alpha, \beta \in \mathrm{Sper}\,A$, *we have* $t_\alpha - t_\beta \in \langle\alpha,\beta\rangle$.

COROLLARY 3.8. — *Every field is Pierce-Birkhoff.*

Remarks 3.9. — Let $\alpha, \beta \in \mathrm{Sper}\,A$.

(i) If $\langle\alpha,\beta\rangle$ is a prime ideal or equal to A, then for each $t \in \mathrm{PW}(A)$, we have $t_\alpha - t_\beta \in \langle\alpha,\beta\rangle$.

(ii) If α and β have no common specialization, then $\langle\alpha,\beta\rangle = A$, so we have to check the condition of the theorem only for α and β having a common specialization.

(iii) If A is a ring with the property that the localization at any real prime ideal is a discrete valuation ring, then A is a Pierce-Birkhoff ring. See, for example, Lemma 5.1 below.

(iv) Any Dedekind ring is a Pierce-Birkhoff ring.

(v) The (real) coordinate ring of any non-singular (real) algebraic curve is a Pierce-Birkhoff ring.

4. Connectedness

The Connectedness Conjecture was introduced by Lucas, Madden, Schaub and Spivakovsky in [LMSS07] who showed that it implies the Pierce-Birkhoff Conjecture. We will review this work. First, in order to state the conjecture, we make the following definition.

Definition 4.1. — *Let A be a ring and let $\alpha, \beta \in \mathrm{Sper}\, A$. We say α and β satisfy the* ***connectedness condition*** *if for any $g_1, \dots, g_s \in A \setminus \langle\alpha, \beta\rangle$, there exists a connected set $C \subset \mathrm{Sper}\, A$ such that $\alpha, \beta \in C$ and $C \cap \{\delta \in \mathrm{Sper}\, A \mid g_j(\delta) = 0\} = \emptyset$ for all $j \in \{1, \dots, s\}$.*

Conjecture 4.2 *(Connectedness Conjecture). — Suppose R is a real closed field and $A := R[X_1, \dots, X_n]$. Then every pair $\alpha, \beta \in \mathrm{Sper}\, A$ satisfies the connectedness condition.*

Theorem 4.3 *(Lucas, Madden, Schaub, Spivakovsky). — Let A be a noetherian ring in which the connectedness condition holds for every two points $\alpha, \beta \in \mathrm{Sper}\, A$ with a common center. Then A is a Pierce-Birkhoff ring.*

Proof. — Let $t \in \mathrm{PW}(A)$, and let $(U_j)_{j=1}^m$ be a finite sequence of constructible sets in $\mathrm{Sper}\, A$ such that $\mathrm{Sper}\, A = \bigcup_{j=1}^{m} U_j$ and $t = t_j$ on U_j for some $t_j \in A$. Let $\alpha, \beta \in \mathrm{Sper}\, A$. By Theorem 3.7, we have to show that $t_\alpha - t_\beta \in \langle\alpha, \beta\rangle$, where t_α (resp. t_β) is any element $a \in A$ such that $t(\alpha) = a(\alpha)$ (resp. $t(\beta) = a(\beta)$). We may assume that α and β have a common center, since otherwise $A = \langle\alpha, \beta\rangle$. Now let $T = \{\{j, k\} \subset \{1, \dots, m\} \mid t_j - t_k \notin \langle\alpha, \beta\rangle\}$, and we apply the connectedness condition to the finitely many elements $t_j - t_k$ with $\{j, k\} \in T$ to get a connected set $C \subset \mathrm{Sper}\, A$ such that $\alpha, \beta \in C$ and $C \cap \{\delta \in \mathrm{Sper}\, A \mid (t_j - t_k)(\delta) = 0\} = \emptyset$ for all $\{j, k\} \in T$.

Let K be the set of all indices $k \in \{1, \dots, m\}$ such that there exists a sequence $j_1, \dots, j_s \in \{1, \dots, m\}$ with $\alpha \in U_{j_1}$, $j_s = k$, and, for all $q \in \{1, \dots, s-1\}$, we have $C \cap \{\delta \in \mathrm{Sper}\, A \mid (t_{j_q} - t_{j_{q+1}})(\delta) = 0\} \neq \emptyset$.
Let $F = \bigcup_{k \in K} (U_k \cap C)$. Then $\alpha \in F$ by definition.

We claim that $F = C$. Let $K^c := \{1, \dots, m\} \setminus K$ and $G := \bigcup_{j \in K^{(c)}} (U_j \cap C)$. Clearly, $C = F \cup G$ and both sets F and G are closed in C. Suppose $F \cap G \neq \emptyset$, and let $\delta \in F \cap G$. Then there exists some $k \in K$ and some $j \in K^c$ such that $\delta \in U_k \cap U_j$, and thus $t_k(\delta) = t_j(\delta)$. Therefore, we have

$\delta \in C \cap \{t_k - t_j = 0\}$, and hence $j \in K$, a contradiction. We have shown that F and G are disjoint, and since C is connected and $F \neq \emptyset$, this yields $G = \emptyset$. In particular, we have $\beta \in F$.

Now let $k \in K$ such that $\beta \in U_k$, and hence we can set $t_k =: t_\beta$. Then there exists a sequence $j_1, \dots, j_s \in \{1, \dots, m\}$ such that $\alpha \in U_{j_1}$, $j_s = k$ and, for all $q \in \{1, \dots, s-1\}$, $\{j_q, j_{q+1}\} \notin T$, i.e., $t_{j_q} - t_{j_{q+1}} \in \langle \alpha, \beta \rangle$. Hence, we have obtained $t_\alpha - t_\beta \in \langle \alpha, \beta \rangle$, if we set $t_\alpha := t_{j_1}$. $\square$

If $\langle \alpha, \beta \rangle = \{0\}$, then we have $\alpha = \beta$ and $\operatorname{supp}(\alpha) = \operatorname{supp}(\beta) = \{0\}$. Hence $g(\alpha) \neq 0$ for all $g \in A \setminus \langle \alpha, \beta \rangle$, and therefore the set $C = \{\alpha\}$ fulfills all requirements of the connectedness condition.

In the next sections, we will prove that the connectedness condition holds for every pair of points α, β which are in the real spectrum of a finitely generated two-dimensional regular R-algebra A, where R is a real closed field, and have the same center in A, i.e., $\langle \alpha, \beta \rangle \subsetneq A$. Note that, if $\langle \alpha, \beta \rangle = A$, the connectedness condition holds for α and β if and only if there exists a connected set $C \subset \operatorname{Sper} A$ that contains α and β. This is true, for example, if A is the polynomial ring $R[X_1, \dots, X_n]$.

First, we give some examples for connected sets in the real spectrum of a polynomial ring over a real closed field. Let R be a real closed field, and let $A := R[X_1, \dots, X_n]$ be the polynomial ring in n indeterminates over R.

Definition 4.4. — *A semialgebraic subset S of R^n is called **semialgebraically connected** if there are no two non-empty closed semialgebraic sets S_1 and S_2 in S such that $S_1 \cap S_2 = \emptyset$ and $S = S_1 \cup S_2$.*

Proposition 4.5. — *Let U, V be two semialgebraic sets, and let $\varphi: U \to V$ be a continuous semialgebraic map, i.e., a continuous map whose graph is semialgebraic. Then the image under φ of any semialgebraically connected subset of U is again semialgebraically connected.*

Proposition 4.6. — *Every interval of R is semialgebraically connected.*

From Proposition 4.6, one can immediately derive the following result.

Proposition 4.7. — *Every convex semialgebraic set $S \subset R^n$ is semialgebraically connected.*

Examples 4.8. — Let $\varepsilon \in R$ such that $\varepsilon > 0$, and let $I \subset \{1, \dots, n\}$. Then, by Proposition 4.7, the semialgebraic set

$$C_I^\varepsilon := \{x \in R^n \mid \varepsilon - \sum_{i=1}^n x_i^2 > 0,\ x_i > 0\ (i \in I)\}$$

is semialgebraically connected.

Let $V \subset R^n$ be an algebraic set, and let $\mathcal{P}(V)$ be its coordinate ring. For any semialgebraic set S in V, we denote by $\tilde{S}$ the corresponding constructible set in $\operatorname{Sper}\mathcal{P}(V)$. Let us recall Proposition 7.5.1 of [BCR98], which is an important tool to find connected sets in the real spectrum of $\mathcal{P}(V)$.

Proposition 4.9. — *Let S be a semialgebraic set in V. S is semialgebraically connected if and only if $\tilde{S}$ is connected in the spectral topology of $\mathcal{P}(V)$.*

5. The One-Dimensional Case

Let A be an integral domain. In this section, we treat the case where $\sqrt{\langle\alpha,\beta\rangle}$ is a prime ideal of height one, and the localization of A at $\sqrt{\langle\alpha,\beta\rangle}$ is a regular ring.

Lemma 5.1. — *Let A be a integral domain. Let $\alpha, \beta \in \operatorname{Sper} A$ such that $\sqrt{\langle\alpha,\beta\rangle}$ is a prime ideal of height one and $A_{\sqrt{\langle\alpha,\beta\rangle}}$ is a regular local ring. Let $g_1, \dots, g_s \in A \setminus \langle\alpha,\beta\rangle$. Then $\sqrt{\langle\alpha,\beta\rangle} = \langle\alpha,\beta\rangle$ and there exists a connected set $C \subset \operatorname{Sper} A$ such that $\alpha, \beta \in C$ and $C \cap \{\delta \in \operatorname{Sper} A \mid g_j(\delta) = 0\} = \emptyset$ for all $j \in \{1, \dots, s\}$.*

Proof. — Assume $\langle\alpha,\beta\rangle \subsetneq \sqrt{\langle\alpha,\beta\rangle}$. We consider the one-dimensional regular local ring (i.e., discrete valuation ring) $B := A_{\sqrt{\langle\alpha,\beta\rangle}}$. Both orderings $\alpha, \beta \in \operatorname{Sper} A$ extend uniquely to $\alpha', \beta' \in \operatorname{Sper} B$. In B, the separating ideal $\langle\alpha',\beta'\rangle$ is equal to $\langle\alpha,\beta\rangle B$. By assumption, there exists some $\pi \in \sqrt{\langle\alpha,\beta\rangle} \setminus \langle\alpha,\beta\rangle$ such that $\pi B = \sqrt{\langle\alpha,\beta\rangle} B = \sqrt{\langle\alpha',\beta'\rangle}$.

Every element in $b \in B$ can be written as $\pi^r u$ for some $r \in \mathbb{N}$ and $u \in B^\times = B \setminus \sqrt{\langle\alpha',\beta'\rangle}$. Both π and u do not change sign between α' and β', hence $b = \pi^r u$ does not change sign either. Thus $\sqrt{\langle\alpha',\beta'\rangle} = \langle\alpha',\beta'\rangle = \{0\}$, a contradiction.

Let $\gamma \in \operatorname{Sper} A$ be the least common specialization of α and β. Then $\operatorname{supp}(\gamma) = \sqrt{\langle\alpha,\beta\rangle} = \langle\alpha,\beta\rangle$. Hence, for all $g \in A \setminus \langle\alpha,\beta\rangle$, we have $\gamma \in$

$\operatorname{Sper} A \setminus \{\delta \in \operatorname{Sper} A \mid g(\delta) = 0\}$. For any such g, let $C_{g,\gamma}$ be the connected component of the open set $\operatorname{Sper} A \setminus \{\delta \in \operatorname{Sper} A \mid g(\delta) = 0\}$ that contains γ. Since α and β specialize to γ, they are also contained in $C_{g,\gamma}$.

Now let $g := g_1 \cdots g_s$. Then $g \notin \langle \alpha, \beta \rangle = \sqrt{\langle \alpha, \beta \rangle}$, so we can take $C := C_{g,\gamma}$. $\square$

6. Valuations, Orderings and Quadratic Transformations

Let R be a real closed field. In order to prove the connectedness condition in the case that $\sqrt{\langle \alpha, \beta \rangle}$ has height two in a finitely generated two-dimensional regular R-algebra, we have to consider so-called quadratic transformations along a valuation of this ring.

Let A be a noetherian ring.

Definition 6.1. — *A **valuation** v of A is a map $A \to \Gamma \cup \{\infty\}$, where Γ is a total ordered abelian group, such that for all $a, b \in A$*

(i) $v(0) = \infty$, $v(1) = 0$,

(ii) $v(ab) = v(a) + v(b)$ *and*

(iii) $v(a + b) \geqslant \min\{v(a), v(b)\}$.

*The prime ideal supp(v) $:= \{a \in A \mid v(a) = \infty\}$ is called the **support** of v in A and, if v is non-negative on A, then cent(v) $:= \{a \in A \mid v(a) > 0\}$ is a prime ideal called the **center** of v in A.*

Let v be a valuation of A that is non-negative on A.

*An ideal I of A is called a v-**ideal** if $I = \{a \in A \mid \exists\, b \in I\ (v(b) \leqslant v(a))\}$. Since A is noetherian, for all v-ideals I, there exists an element $b \in I$ such that $I = \{a \in A \mid v(b) \leqslant v(a)\}$. If I is a v-ideal, then $I^v := \{a \in A \mid v(b) < v(a)\}$ is the largest v-ideal properly contained in I. The set of v-ideals of A is totally ordered by inclusion.*

Note that the support of v is the smallest v-ideal in A and that the center of v is the largest proper v-ideal in A.

*If A is a local ring with maximal ideal $\mathfrak{m}$, we say that a valuation v of A **dominates** A if v is non-negative on A and the center of v in A is equal to $\mathfrak{m}$.*

Remark 6.2. — Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k, and let v be a valuation of A that dominates A. Then v induces a valuation on the quotient field of $A/\mathrm{supp}(v)$. We denote the corresponding valuation ring by $\mathcal{O}_v$, its maximal ideal by $\mathfrak{m}_v$ and its residue field by K_v. Since v dominates A, we have $k \subset K_v$.

Let I be a v-ideal of A where I is different from the support of v. Consider the following composition of k-vector space homomorphisms

$$I \twoheadrightarrow I/\mathrm{supp}(v) \to \mathcal{O}_v/\mathfrak{m}_v = K_v,\ a \mapsto \overline{a} = a + \mathrm{supp}(v) \mapsto \frac{\overline{a}}{\overline{b}} + \mathfrak{m}_v,$$

where b is an element of I having minimal value in I. The kernel of this composition is clearly I^v, hence I/I^v is a sub-k-vector space of K_v.

We will now assign to each ordering $\alpha \in \mathrm{Sper}\, A$ of A a valuation v_α.

Definition 6.3. — *Let $\alpha \in \mathrm{Sper}\, A$. Then α induces a total ordering on the field $k(\alpha) = \mathrm{Quot}(A/\mathrm{supp}(\alpha))$. Now let $\mathcal{O}_\alpha$ be the convex hull of $A(\alpha)$ in $k(\alpha)$. This is a valuation ring of $k(\alpha)$. Let v'_α be a corresponding valuation, then $v_\alpha := v'_\alpha \circ \rho_\alpha$ is a valuation of A. For any v_α-ideal I of A, we write I^α instead of I^{v_α}.*

Remark 6.4. — Let $\alpha, \beta \in \mathrm{Sper}\, A$. An ideal I of A is a v_α-ideal if and only if it is convex with respect to α. Hence $\langle \alpha, \beta \rangle$ is a v_α-ideal.

From now on, let A be an regular local domain with maximal ideal $\mathfrak{m}$ and residue field k. In particular, A is integrally closed. Let ord_A be the order valuation of A (i.e., $\mathrm{ord}_A(a) = \max\{n \in \mathbb{N} \mid a \in \mathfrak{m}^n\}$). In [ZS60] (Appendix 5) Oscar Zariski and Pierre Samuel showed a unique factorization theorem for v-ideals in a two-dimensional regular local ring where v is a dominating valuation.

Definition 6.5. — *An ideal I of A is called* ***simple*** *if it is proper and it cannot be written as a product of proper ideals.*

Remark 6.6. — The ord_A-ideals of A are exactly the powers of the maximal ideal $\mathfrak{m}$, hence $\mathfrak{m}$ is the only simple ord_A-ideal of A.

Theorem 6.7. — *If A has dimension two and v is a valuation of A that dominates A, then a v-ideal of A is simple if and only if it cannot be written as a product of proper v-ideals of A. Moreover, every v-ideal of A different from* (0) *and A has a unique factorization into simple v-ideals.*

Remark 6.8. — Actually, Zariski and Samuel proved this unique factorization theorem for complete ideals ([ZS60], Appendix 5, Theorem 3). In an integrally closed domain A, an ideal I is said to be **complete** if it is integrally closed in $\mathrm{Quot}(A)$, i.e., for all $a \in \mathrm{Quot}(A)$ such that $a^m + b_1 a^{m-1} + \cdots + b_m = 0$, where $b_j \in I^j$, we already have $a \in I$. But if v is a valuation of A that is non-negative on A, then every v-ideal I of A is complete, and if a v-ideal I is a product of simple complete ideals, then every factor is already a v-ideal.

In [AJM95], Alvis, Johnston and Madden give a sufficient condition for the simplicity of the separating ideal of two points in the real spectrum that are centered at the same maximal ideal.

Proposition 6.9. — *Suppose $\alpha, \beta \in \mathrm{Sper}\, A$ are both centered at the maximal ideal $\mathfrak{m}$ of A. Let $k := A/\mathfrak{m}$. If $I/I^\alpha \cong k$ for all v_α-ideals I which properly contain $\langle \alpha, \beta \rangle$, then $\langle \alpha, \beta \rangle$ is simple.*

Proof. — Let $x \in \langle \alpha, \beta \rangle$ such that

(i) $x(\alpha) \geqslant 0$ and $x(\beta) \leqslant 0$,

(ii) x has minimal v_α-value in $\langle \alpha, \beta \rangle$, and

(iii) x has minimal v_β-value in $\langle \alpha, \beta \rangle$.

Note that such an element always exists, since there must be an element x_α that satisfies (i) and (ii) and there must be an element x_β that satisfies (i) and (iii), and if x_α does not satisfy (iii) and x_β does not satisfy (ii), then $x_\alpha + x_\beta$ satisfies all three conditions.

Suppose $\langle \alpha, \beta \rangle$ is not simple. Then, by Theorem 6.7, it can be written as the product of two proper v_α-ideals I and J. Since they (properly) contain $\langle \alpha, \beta \rangle$, they are also v_β-ideals. Now we can write $x = \sum_{k=1}^r a_k b_k$, where, for all $k \in \{1, \ldots, r\}$, $a_k \in I$ and $b_k \in J$. Note that if $a_k \in I^\alpha = I^\beta$ or $b_k \in J^\alpha = J^\beta$, then $a_k b_k \in \langle \alpha, \beta \rangle^\alpha \cap \langle \alpha, \beta \rangle^\beta$. Since x satisfies (ii) and (iii), without loss of generality we can write $x = \sum_{k=1}^s a_k b_k + c$, where $s \in \{1, \ldots r\}$, $a_k \in I \setminus I^\alpha$ and $b_k \in J \setminus J^\alpha$ for all $k \leqslant s$, and $c \in \langle \alpha, \beta \rangle^\alpha \cap \langle \alpha, \beta \rangle^\beta$. Let $a \in I \setminus I^\alpha$ and $b \in J \setminus J^\alpha$. Since $I/I^\alpha \cong k \cong J/J^\alpha$, we have, for $k \leqslant s$, that $a_k = u_k a + a'_k$ and $b_k = v_k b + b'_k$ for some $u_k, v_k \in A^\times$, $a'_k \in I^\alpha$ and $b'_k \in J^\alpha$. Then $x = ab \sum_{k=1}^s u_k v_k + c'$, where $v_\alpha(c') > v_\alpha(\langle \alpha, \beta \rangle) = v_\alpha(x)$ and $v_\beta(c') > v_\beta(\langle \alpha, \beta \rangle) = v_\beta(x)$, and thus we can derive from (i) that $(x - c')(\alpha) \geqslant 0$ and $(x - c')(\beta) \leqslant 0$. Since x satisfies (ii) and (iii), we have $\sum_{k=1}^s u_k v_k \in A^\times$. So a or b must change sign between α and β, but this is impossible, since they are not elements of $\langle \alpha, \beta \rangle$. $\square$

DEFINITION 6.10. — *A* ***quadratic transform*** *of A is a local ring $B = (A[x^{-1}\mathfrak{m}])_{\mathfrak{p}}$, where $ord_A(x) = 1$ and $\mathfrak{p}$ is a prime ideal of $A[x^{-1}\mathfrak{m}] = \{\frac{a}{x^m} \mid ord_A(a) \geqslant m\}$ such that $\mathfrak{p} \cap A = \mathfrak{m}$.*

Under a suitable condition, one can extend orderings of A to a quadratic transform of A ([AJM95]).

LEMMA 6.11. — *Let $\alpha \in \mathrm{Sper}\,A$, and let $B = (A[x^{-1}\mathfrak{m}])_{\mathfrak{p}}$ be a quadratic transform of A. If $supp(\alpha) \neq \mathfrak{m}$, then there is a unique $\alpha' \in \mathrm{Sper}\,B$ such that $\alpha' \cap A = \alpha$.*

DEFINITION 6.12. — *Let v be a non-trivial valuation (i.e., $cent(v) \neq supp(v)$) of A that dominates A. The* ***quadratic transform of** A **along** v* *is defined to be the ring $B = S^{-1}A[x^{-1}\mathfrak{m}]$, where x is an element of $\mathfrak{m}$ of minimal value and $S = \{a \in A[x^{-1}\mathfrak{m}] \mid v(a) = 0\}$. B is again a regular local ring, independent of the choice of x, v is extendable to B and it dominates B.*

We can then iterate this process and derive a sequence of quadratic transformations along v starting from A, denoted by

$$A = A^{(0)} \prec A^{(1)} \prec \cdots.$$

This sequence may be infinite.

DEFINITION 6.13. — *Let $B = (A[x^{-1}\mathfrak{m}])_{\mathfrak{p}}$ be a quadratic transform of A. Let I be an ideal of A with $ord_A(I) = r$. Then $\frac{a}{x^r} \in A[x^{-1}\mathfrak{m}]$ for all $a \in I$. Hence $IA[x^{-1}\mathfrak{m}] = x^r I'$ for some ideal I' of $A[x^{-1}\mathfrak{m}]$. The ideal $T(I) := I'B$ is called the* ***transform of** I **in** B*. Now let J be an ideal of B. Since J is finitely generated, there is a smallest integer $n \in \mathbb{N}$ such that $x^n J = W(J)B$ for some ideal $W(J)$ of A, called the* ***inverse transform of** J*.*

Remarks 6.14. — We consider a quadratic transform B of A.

1. For all ideals I of A and all ideals J of B, we always have $T(W(J)) = J$, but in general only $W(T(I)) \supset I$.

2. The transformation of ideals is not order-preserving.

3. For all ideals I, J of A, we have $T(IJ) = T(I)T(J)$.

Zariski and Samuel showed that in dimension two, any simple $\mathfrak{m}$-primary complete ideal can be transformed into a maximal ideal by a suitable sequence of quadratic transformations (again see [ZS60], Appendix 5). Applying this result to simple $\mathfrak{m}$-primary v-ideals yields the following.

THEOREM 6.15. — *Suppose the dimension of A is two. Let v be a non-trivial valuation of A that dominates A and is different from the order valuation ord_A. Let A' be the quadratic transform of A along v, and let $\mathfrak{m}'$ be its maximal ideal. Let $\mathcal{S}$ be the set of all simple $\mathfrak{m}$-primary v-ideals of A, and let $\mathcal{S}'$ be the set of all simple $\mathfrak{m}'$-primary v-ideals of A'. Then A' has again dimension two, the residue field of A' is an algebraic extension of the residue field of A, every transform of an $\mathfrak{m}$-primary v-ideal is again a v-ideal, and we have that the sets $\mathcal{S} \setminus \{\mathfrak{m}\}$ and $\mathcal{S}'$ are in one-to-one correspondence via the order-preserving maps $\mathcal{I} \mapsto T(\mathcal{I})$ and $\mathcal{J} \mapsto W(\mathcal{J})$.*
Furthermore, for every simple $\mathfrak{m}$-primary v-ideal $\mathcal{I}$, there exists some $s \in \mathbb{N}$ and a sequence $A = A^{(0)} \prec \cdots \prec A^{(s)}$ of quadratic transformations along v such that the iterated transform $T^{(s)}(\mathcal{I})$ of $\mathcal{I}$ equals $\mathfrak{m}^{(s)}$, the maximal ideal in $A^{(s)}$.

Remark 6.16. — Suppose the dimension of A is two. Let v be a non-trivial valuation of A that dominates A. Consider the quadratic transform of A along v. Let I be an $\mathfrak{m}$-primary v-ideal of A, and let $r = \mathrm{ord}_A I$ be the order of I. By Theorem 6.15, the transform $T(I)$ is again a v-ideal, so we may consider the following composition of k-vector space homomorphisms

$$I \leftrightarrow x^{-r}I \hookrightarrow T(I) \twoheadrightarrow T(I)/T(I)^v,\ a \mapsto \frac{a}{x^r} \mapsto \frac{a}{x^r} \mapsto \frac{a}{x^r} + T(I)^v,$$

where x is an element of $\mathfrak{m}$ having minimal value. The kernel of this composition is the ideal I^v, hence $I/I^v \subset T(I)/T(I)^v$.

Although in general, the transformation of ideals is not order-preserving, using the Unique Factorization Theorem 6.7, Theorem 6.15 and the multiplicativeness of the ideal transformation, the following can be observed.

LEMMA 6.17. — *Suppose the dimension of A is two. Let v be a non-trivial valuation of A that dominates A and is different from the order valuation ord_A, and let I be a v-ideal such that I properly contains a simple $\mathfrak{m}$-primary v-ideal $\mathcal{I}$. Then $T^{(s)}(I) = A^{(s)}$, where $T^{(s)}$ denotes the iterated ideal transformation with respect to the sequence of quadratic transformations $A = A^{(0)} \prec \cdots \prec A^{(s)}$ of A along v with the property $T^{(s)}(\mathcal{I}) = \mathfrak{m}^{(s)}$.*

Now we assume that A has dimension two and that the residue field of A is real closed, and we consider quadratic transformations along a valuation corresponding to a point in the real spectrum of A. The next theorem is the main result of [AJM95] and important for the proof of the (two-dimensional) Connectedness Conjecture below.

THEOREM 6.18 *(Alvis, Johnston, Madden). — Let $A = (A, \mathfrak{m}, R)$ be a two-dimensional regular local domain such that R is real closed. Let $\alpha, \beta \in \operatorname{Sper} A$ such that $\operatorname{cent}(\alpha) = \mathfrak{m} = \operatorname{cent}(\beta)$ and $\langle\alpha, \beta\rangle \subsetneq \mathfrak{m} = \sqrt{\langle\alpha, \beta\rangle}$.*

Consider the quadratic transformation along v_α. Then:

$$T(\langle\alpha, \beta\rangle) = \langle\alpha', \beta'\rangle \text{ and } W(\langle\alpha', \beta'\rangle) = \langle\alpha, \beta\rangle,$$

where α' and β' are the unique extensions of α and β with respect to the quadratic transform of A along v_α.
Furthermore, there exists an integer $r \in \mathbb{N}$ and a sequence $A = A^{(0)} \prec \cdots \prec A^{(r)}$ of quadratic transformations along v_α such that the iterated transform $T^{(r)}(\langle\alpha, \beta\rangle)$ of $\langle\alpha, \beta\rangle$ equals $\mathfrak{m}^{(r)}$, the maximal ideal in $A^{(r)}$.

Proof. — Let $\alpha, \beta \in \operatorname{Sper} A$ such that $\operatorname{cent}(\alpha) = \operatorname{cent}(\beta) = \mathfrak{m}$ and $\langle\alpha, \beta\rangle$ is $\mathfrak{m}$-primary, but properly contained in $\mathfrak{m}$.

Let I be a v_α-ideal that properly contains $\langle\alpha, \beta\rangle$. Then I properly contains a simple $\mathfrak{m}$-primary v_α-ideal:
Since $\sqrt{\langle\alpha, \beta\rangle} = \mathfrak{m}$, there are only finitely many v_α-ideals bigger than I. In [ZS60] (Appendix 5), it is shown that v_α is a prime divisor, i.e., its residue field has transcendence degree 1 over R, if and only if there are only finitely many simple $\mathfrak{m}$-primary v_α-ideals. If v_α is not a prime divisor, then one of the infinitely many simple $\mathfrak{m}$-primary v_α-ideals must be properly contained in I. If it is a prime divisor, then, according to [AJM95] (Theorem 4.4), $\langle\alpha, \beta\rangle$ contains a simple $\mathfrak{m}$-primary v_α-ideal, which is therefore properly contained in I.

From the fact that I properly contains a simple $\mathfrak{m}$-primary v_α-ideal, one concludes that $I/I^\alpha \cong R$: By Theorem 6.15, there is a sequence of quadratic transformations $A = A^{(0)} \prec \cdots A^{(s)}$ along v_α such that this simple $\mathfrak{m}$-primary v_α-ideal is transformed into the maximal ideal $\mathfrak{m}^{(s)}$ of $A^{(s)}$, and therefore I is transformed into $A^{(s)}$ (Lemma 6.17). Since $A^{(s)}/\mathfrak{m}^{(s)}$ is a real algebraic extension of R, they are equal. Thus, by Remark 6.16, we have $I/I^\alpha \cong R$. (Note that if v_α is not a prime divisor, then $K_{v_\alpha} = R$, and therefore $I/I^\alpha \cong R$ already follows from Remark 6.2.)

By Proposition 6.9, we have that $\langle\alpha, \beta\rangle$ is simple. It is true in general that $T(\langle\alpha, \beta\rangle) \subset \langle\alpha', \beta'\rangle$ ([AJM95], Lemma 3.2), and with the considerations above one easily shows that $W(\langle\alpha', \beta'\rangle) \subset \langle\alpha, \beta\rangle$ ([AJM95], Lemma 4.7). Applying W on the first inclusion and T on the second, one gets, by Theorem

6.15, $\langle\alpha,\beta\rangle \subset W(\langle\alpha',\beta'\rangle)$ and $\langle\alpha',\beta'\rangle \subset T(\langle\alpha,\beta\rangle)$. Altogether, we have the desired equalities.

Again using Theorem 6.15, the last assertion follows from the simplicity of $\langle\alpha,\beta\rangle$. $\square$

7. The Connectedness Conjecture for Smooth Affine Surfaces over Real Closed Fields

In this last section, we shall prove the Connectedness Conjecture for the coordinate ring of a non-singular two-dimensional affine real algebraic variety over a real closed field. That is, we shall show that any two points in the real spectrum of such a coordinate ring which have the same center satisfy the connectedness condition. We can assume that the variety is irreducible. By Lemma 5.1, we then need to consider only points α,β where $\sqrt{\langle\alpha,\beta\rangle}$ has height two. We will use Theorem 6.18 to simplify the problem of constructing a suitable connected set.

Let $A = (A,\mathfrak{m},R)$ be a two-dimensional regular local domain such that $R = A/\mathfrak{m}$ is real closed. Let $\alpha,\beta \in \operatorname{Sper} A$ such that $\operatorname{cent}(\alpha) = \mathfrak{m} = \operatorname{cent}(\beta)$ and $\langle\alpha,\beta\rangle \subsetneq \mathfrak{m} = \sqrt{\langle\alpha,\beta\rangle}$. By Theorem 6.18, there exists a finite sequence $(A,\mathfrak{m},R) =: (A^{(0)},\mathfrak{m}^{(0)},k^{(0)}) \prec \cdots \prec (A^{(r)},\mathfrak{m}^{(r)},k^{(r)})$ of quadratic transformations along the valuation v_α such that $\langle\alpha^{(r)},\beta^{(r)}\rangle = T^{(r)}(\langle\alpha,\beta\rangle) = \mathfrak{m}^{(r)}$, where $\alpha^{(r)}$ and $\beta^{(r)}$ are the unique extensions of α and β to $A^{(r)}$.

At first we take a closer look at these quadratic transformations. The quadratic transformation $(A^{(i)},\mathfrak{m}^{(i)},k^{(i)}) \prec (A^{(i+1)},\mathfrak{m}^{(i+1)},k^{(i+1)})$ is a transformation along $v_{\alpha^{(i)}}$, which is the unique extension of v_α to $A^{(i)}$. We will write v instead of $v_{\alpha^{(i)}}$ and sometimes α and β instead of $\alpha^{(i)}$ and $\beta^{(i)}$.

By Theorem 6.15, we have that, for all $i \in \{0,\dots,r\}$, the regular local ring $A^{(i)}$ has dimension two and residue field $k^{(i)} = R$, since $k^{(i)}$ is a real field. Now let (x_i,y_i) be a regular system of local parameters of $A^{(i)}$, i.e., $m^{(i)} = (x_i,y_i)$. From now on, we will assume that $0 < v(x_i) \leqslant v(y_i)$. Then, a regular system (x_{i+1},y_{i+1}) of parameters of $A^{(i+1)}$ such that $0 < v(x_{i+1}) \leqslant v(y_{i+1})$ can be derived from (x_i,y_i) in the following way (see [ZS60], Appendix 5, proof of Proposition 1):

I. Suppose $v(x_i) < v(y_i)$:

1. If $v(x_i) \leqslant v(\frac{y_i}{x_i})$, then let $x_{i+1} := x_i$ and $y_{i+1} := \frac{y_i}{x_i}$.
2. If $v(x_i) > v(\frac{y_i}{x_i})$, then let $x_{i+1} := \frac{y_i}{x_i}$ and $y_{i+1} := x_i$.

II. Suppose $v(x_i) = v(y_i)$. Pick an element $u \in {A^{(i)}}^\times$ such that $\frac{y_i}{x_i} - u$ has positive value. (This is possible because $v(\frac{y_i}{x_i}) = 0$, hence there exists some element $u \in {A^{(i)}}^\times$ such that $\overline{u}^v = \overline{\frac{y_i}{x_i}}^v \in R$.)

1. If $v(x_i) \leqslant v(\frac{y_i}{x_i} - u)$, let $x_{i+1} := x_i$ and $y_{i+1} := \frac{y_i}{x_i} - u$.
2. If $v(x_i) > v(\frac{y_i}{x_i} - u)$, let $x_{i+1} := \frac{y_i}{x_i} - u$ and $y_{i+1} := x_i$.

Without loss of generality we may assume that $x_i(\alpha) > 0$ and $y_i(\alpha) > 0$. Since $\langle \alpha^{(i)}, \beta^{(i)} \rangle \subsetneq \mathfrak{m}^{(i)}$ for all $i < r$, we also have $x_i(\beta) > 0$ if $i < r$.

Proposition 7.1. — *Suppose A is the localization at the maximal ideal $(z_1, \dots, z_n)$ of a finitely generated two-dimensional regular R-algebra $R[z_1, \dots, z_n]$ without zero divisors, where R is a real closed field. Let $\alpha, \beta \in \operatorname{Sper} A$ both centered at the maximal ideal $\mathfrak{m}$ of A such that $\langle \alpha, \beta \rangle \subsetneq \mathfrak{m} = \sqrt{\langle \alpha, \beta \rangle}$. Let $v := v_\alpha$. Suppose $0 < v(z_1) \leqslant v(z_2), \dots, v(z_n)$. Then there exist elements $u_2, \dots, u_n \in R$ such that the quadratic transform A' of A along v equals the localization of $R[z_1', \dots, z_n']$ at the maximal ideal $(z_1', \dots, z_n')$, where $z_1' := z_1$ and $z_j' = \frac{z_j}{z_1} - u_j$ if $j > 1$. Suppose further that (z_1, z_2) is a regular system of parameters of A. Then (z_1', z_2') is a regular system of parameters of A'.*

Proof. — Let $j > 1$. If $v(z_j) > v(z_1)$, let $u_j := 0$. If $v(z_j) = v(z_1)$, then, since $\mathfrak{m}/\mathfrak{m}^\alpha \cong R$ (as shown in the proof of 6.18), there exists some $u \in R$ such that $v(z_j - uz_1) > v(\mathfrak{m}) = v(z_1)$, and we take $u_j := u$.

Let $B := A[z_1^{-1}\mathfrak{m}] = \{\frac{a}{z_1^m} \mid a \in A,\ \operatorname{ord}_A(a) \geqslant m\}$. Then we have $A' = S^{-1}B$, where $S = \{b \in B \mid v(b) = 0\}$. Let $a \in A$. By assumption $a = \frac{a_1}{a_2}$, where $a_1 \in R[z_1, \dots, z_n]$ and $a_2 \in R[z_1, \dots, z_n] \setminus (z_1, \dots, z_n)$. Let $m \in \mathbb{N}$ such that $m \leqslant \operatorname{ord}_A(a) = \operatorname{ord}_A(a_1)$, hence $a_1' := \frac{a_1}{z_1^m} \in R[z_1, \frac{z_2}{z_1}, \dots, \frac{z_n}{z_1}] = R[z_1', \dots, z_n']$. Further $a_2 \in R[z_1, \dots, z_n] \setminus (z_1, \dots, z_n) \subset R[z_1', \dots, z_n'] \setminus (z_1', \dots, z_n')$. Hence $B \subset R[z_1', \dots, z_n']_{(z_1', \dots, z_n')}$. The valuation v extends uniquely to $R[z_1', \dots, z_n']$, it is non-negative on this ring, and its center is the maximal ideal $(z_1', \dots, z_n')$, therefore it also extends uniquely to $R[z_1', \dots, z_n']_{(z_1', \dots, z_n')}$

Suppose now that $v(\frac{a}{z_1^m}) = 0$, i.e., $v(a_1) = v(a) = mv(z_1)$. Then we have that $v(a_1') = v(a_2) = 0$. Since v is centered on $(z_1', \dots, z_n')$ in $R[z_1', \dots, z_n']$, we have that $a_1' \in R[z_1', \dots, z_n'] \setminus (z_1', \dots, z_n')$. Thus, we have shown that $A' = S^{-1}B = R[z_1', \dots, z_n']_{(z_1', \dots, z_n')}$.

The last assertion follows immediately from the considerations we made above. $\square$

Lemma 7.2. — *Let $\alpha, \beta \in Sper\, A$ such that $cent(\alpha) = \mathfrak{m} = cent(\beta)$ and $\langle\alpha, \beta\rangle \subsetneq \mathfrak{m} = \sqrt{\langle\alpha, \beta\rangle}$. Let $(A, \mathfrak{m}, R) =: (A^{(0)}, \mathfrak{m}^{(0)}, R) \prec \cdots \prec (A^{(r)}, \mathfrak{m}^{(r)}, R)$ be the sequence of quadratic transformations along $v := v_\alpha$ such that $T^{(r)}(\langle\alpha, \beta\rangle) = \mathfrak{m}^{(r)} = \langle\alpha^{(r)}, \beta^{(r)}\rangle$. Then every $g \in A \setminus \langle\alpha, \beta\rangle$ has the form $x_r^e y_r^f w$, where (x_r, y_r) is a regular system of parameters of $A^{(r)}$, $w \in A^{(r)} \setminus \mathfrak{m}^{(r)}$, $e, f \in \mathbb{N}$, and $e = 0$ (resp. $f = 0$) if x_r (resp. y_r) changes sign between α and β.*

Proof. — Let $g \in A \setminus \langle\alpha, \beta\rangle$. Let $I := \{a \in A \mid v_\alpha(a) \geqslant v_\alpha(g)\} = \{a \in A \mid v_\beta(a) \geqslant v_\beta(g)\}$. Since $g \notin \langle\alpha, \beta\rangle$, we have that $T^{(r)}(I) = A^{(r)}$, by Lemma 6.17.
Let $\nu_0 = \mathrm{ord}_A I$ be the order of the ideal I. Since g has minimal value in I, the element $g \cdot x_0^{-\nu_0}$ has minimal value in the transform $T(I)$ of I. By induction, one shows that there exist $\nu_i \in \mathbb{N}$ $(0 \leqslant i < r)$ such that $g \cdot x_0^{-\nu_0} \cdots x_{r-1}^{-\nu_{r-1}}$ has minimal value in $T^{(r)}(I) = A^{(r)}$, hence

$$g = x_0^{\nu_0} \cdots x_{r-1}^{\nu_{r-1}} w'$$

for some for and some unit w' in $A^{(r)}$.

We would like to write g in the form $x_r^e y_r^f w$, where $w \in {A^{(r)}}^\times$, $e, f \in \mathbb{N}$, and $e = 0$ (resp. $f = 0$) if x_r (resp. y_r) changes sign between α and β. In order to see how this can be done, we have to know for all $i < r$ how to represent a product $x_i^s y_i^t$ in terms of x_{i+1} and y_{i+1}. We will look again at the several cases of a quadratic transformation along v.

- I.1 If $x_i = x_{i+1}$ and $y_i = x_{i+1} y_{i+1}$, then $x_i^s y_i^t = x_{i+1}^{s+t} y_{i+1}^t$.
- I.2 If $x_i = y_{i+1}$ and $y_i = x_{i+1} y_{i+1}$, then $x_i^s y_i^t = x_{i+1}^t y_{i+1}^{s+t}$.
- II.1 If $x_i = x_{i+1}$ and $y_i = x_{i+1}(y_{i+1} + u)$ for some $u \in {A^{(i)}}^\times$, then we have $x_i^s y_i^t = x_{i+1}^{s+t}(y_{i+1} + u)^t$. Note that $y_{i+1} + u$ is a unit in $A^{(i+1)}$.
- II.2 If $x_i = y_{i+1}$ and $y_i = y_{i+1}(x_{i+1} + u)$ for some $u \in {A^{(i)}}^\times$, then we have $x_i^s y_i^t = y_{i+1}^{s+t}(x_{i+1} + u)^t$, and $x_{i+1} + u$ is a unit in $A^{(i+1)}$.

Let $0 \leqslant i < r$. If both x_i and y_i are positive at α and β and we are in case I.1 or I.2, then x_{i+1} and y_{i+1} are also positive at α and β. If we are in case II.1 or II.2, it is possible that $\frac{y_i}{x_i} - u$ changes sign between α and β. In case II.2, we then have $x_{i+1} = \frac{y_i}{x_i} - u \in \langle\alpha^{(i+1)}, \beta^{(i+1)}\rangle$, and, since x_{i+1} has minimal value in $\mathfrak{m}^{(i+1)}$ and $\mathfrak{m}^{(i+1)}$ contains $\langle\alpha^{(i+1)}, \beta^{(i+1)}\rangle$, this yields $\mathfrak{m}^{(i+1)} = \langle\alpha^{(i+1)}, \beta^{(i+1)}\rangle$, thus $i + 1 = r$. In case II.1, if $v(x_{i+1}) = v(x_i) = v(\frac{y_i}{x_i} - u) = v(y_{i+1})$ and $y_{i+1} = \frac{y_i}{x_i} - u \in \langle\alpha^{(i+1)}, \beta^{(i+1)}\rangle$, we can use the same arguments as in the last sentence to show that $i + 1 = r$. In the same case, if $v(x_i) < v(\frac{y_i}{x_i} - u)$ and $y_{i+1} = \frac{y_i}{x_i} - u \in \langle \alpha^{(i+1)}, \beta^{(i+1)} \rangle$, we will reach $A^{(r)}$ after a series of I.1 transformations and maybe one last I.2 transformation. The same holds if y_0 changes sign between α and β.

We will now consider the implications of the last considerations for the representation of an element $g \in A \setminus \langle \alpha, \beta \rangle$ in terms of x_r and y_r in the ring $A^{(r)}$.

1. Suppose, for all $i \leqslant r$, both x_i and y_i are positive at α and β. Then, in the representation $g = x_r^e y_r^f w$ with $e, f \in \mathbb{N}$ and $w \in {A^{(r)}}^\times$ of some element $g \in A \setminus \langle \alpha, \beta \rangle$, we do not care whether e or f is zero or not.

2. Now suppose y_j changes sign between α and β for some j and j is minimal with this property. Hence, if $j > 0$, the last quadratic transformation from $A^{(j-1)}$ to $A^{(j)}$ must be of type II.1. In particular, if $j > 0$, $x_j = x_{j-1}$ does not change its sign between α and β, and we have $x_0^{\nu_0} \cdots x_{j-1}^{\nu_{j-1}} = x_j^s (y_j + u)^t$, where $s, t \in \mathbb{N}$, u a unit in $A^{(j-1)}$, and $y_j + u$ is a unit in $A^{(j)}$. As mentioned above, we have $j = r$ or we will reach $A^{(r)}$ after a series of I.1 transformations and maybe one last I.2 transformation. Thus, if $j < r$, we have $x_j = x_{j+1} = \cdots = x_{r-1}$ and each $g \in A \setminus \langle \alpha, \beta \rangle$ has the form $x_r^e w$ or $y_r^f w$ with $w \in {A^{(r)}}^\times$, depending on whether $x_{r-1} = x_r$ (I.1) or $x_{r-1} = y_r$ (I.2). Note that in these representations none of the factors changes sign between α and β. If $j = r$, then $x_{r-1} = x_r$, and therefore each $g \in A \setminus \langle \alpha, \beta \rangle$ has the form $x_r^e w$ with $w \in {A^{(r)}}^\times$, and x_r does not change its sign.

3. Suppose that, for all $i \leqslant r$, y_i does not change sign between α and β, but x_r changes sign between these two points. Then, as seen above, the last transformation must be of type II.2, hence $g = y_r^f w$ with $w \in {A^{(r)}}^\times$.

Note that the last discussion also showed that at least one of the elements x_r and y_r does not change its sign between α and β. $\square$

Finally, we are able to prove our main result.

THEOREM 7.3. — *Let W be a non-singular two-dimensional affine real algebraic variety over a real closed field R, and let $\mathcal{P}(W)$ be its coordinate ring. Then every pair of points $\alpha, \beta \in Sper\,\mathcal{P}(W)$ having a common center satisfies the connectedness condition. In particular, $\mathcal{P}(W)$ is Pierce-Birkhoff.*

Proof. — We may assume that W is irreducible. Let $R[x, y, z_1, \ldots, z_m]$ be the coordinate ring $\mathcal{P}(W) = R[X, Y, Z_1, \ldots, Z_m]/\mathcal{I}(W)$ of W, where $R[X, Y, Z_1, \ldots, Z_m]$ is the polynomial ring in $m + 2$ indeterminates over R, and $\mathcal{I}(W)$ the prime ideal of W in $R[X, Y, Z_1, \ldots, Z_m]$. Then $\mathcal{P}(W)$ is a finitely generated two-dimensional regular R-algebra without zero -divisors. Suppose x and y are algebraically independent over R, and that $z_j \notin R$ for all $j \in \{1, \ldots, m\}$.

Let $\alpha, \beta \in \operatorname{Sper} \mathcal{P}(W)$, and assume that they are distinct and have a common center. If the height of $\sqrt{\langle \alpha, \beta \rangle}$ is one, we have shown in Lemma 5.1 that α and β satisfy the connectedness condition. Hence, we can suppose that the height of $\sqrt{\langle \alpha, \beta \rangle}$ is two, i.e., $\sqrt{\langle \alpha, \beta \rangle} = \operatorname{cent}(\alpha) = \operatorname{cent}(\beta)$ is a real maximal ideal $\mathcal{M}$ of $\mathcal{P}(W)$, and the residue field $\mathcal{P}(W)/\mathcal{M}$ equals R. Therefore, we may assume that $\mathcal{M} = (x, y, z_1, \ldots, z_m)$.

Let $g_1, \ldots, g_s \in A \setminus \langle \alpha, \beta \rangle$, and let A be the localization of $\mathcal{P}(W)$ at $\mathcal{M} = \sqrt{\langle \alpha, \beta \rangle}$. Let $\langle \alpha, \beta \rangle_A$ be the separating ideal of (the unique extensions of) α and β in A. Then $\langle \alpha, \beta \rangle_A \cap \mathcal{P}(W) = \langle \alpha, \beta \rangle$. Hence $g_1, \ldots, g_s \notin \langle \alpha, \beta \rangle_A$.

Let $(A, \mathfrak{m}, R) =: (A^{(0)}, \mathfrak{m}^{(0)}, R) \prec \cdots \prec (A^{(r)}, \mathfrak{m}^{(r)}, R)$ be the sequence of quadratic transformations along v_α such that $\mathfrak{m}^{(r)} = T^{(r)}(\langle \alpha^{(0)}, \beta^{(0)} \rangle) = \mathfrak{m}^{(r)} = \langle \alpha^{(r)}, \beta^{(r)} \rangle$. Then 7.1, $A^{(r)} = R[x_r, y_r, z_1^{(r)}, \ldots, z_m^{(r)}]_{(x_r, y_r, z_1^{(r)}, \ldots, z_m^{(r)})}$ for some regular system of parameters (x_r, y_r) of $A^{(r)}$ having the additional property $\mathcal{P}(W) \subset R[x_r, y_r, z_1^{(r)}, \ldots, z_m^{(r)}]$. Recall that x_r and y_r are algebraically independent over R, and we assumed that $x_r(\alpha) > 0$ and $y_r(\alpha) > 0$.

Then, by Lemma 7.2, for all $j \in \{1, \ldots, s\}$, g_j has the form $x_r^{e_j} y_r^{f_j} w_j$, where $w_j \in A^{(r)} \setminus \mathfrak{m}^{(r)}$, and $e_j, f_j \in \mathbb{N}$ are such that $e_j = 0$ (resp. $f_j = 0$) if x_r (resp. y_r) changes sign between α and β. The units w_j can be written as $\frac{w_{j1}}{w_{j2}}$ with $w_{j1}, w_{j2} \in R[x_r, y_r, z_1^{(r)}, \ldots, z_m^{(r)}] \setminus (x_r, y_r, z_1^{(r)}, \ldots, z_m^{(r)})$. Clearly, $0 \notin \mathcal{Z}(\prod_{j=1}^{s} w_{j1} w_{j2})$, the zero set of this element.

Consider the polynomial ring $R[X, Y, Z_1, \ldots, Z_m]$. Since, $z_1^{(r)}, \ldots, z_m^{(r)}$ are algebraic over $R(x_r, y_r)$, there exists a surjective ring homomorphism π from $R[X, Y, Z_1, \ldots, Z_m]$ to $R[x_r, y_r, z_1^{(r)}, \ldots, z_m^{(r)}]$ such that $\mathcal{J} := \pi^{-1}(0)$ is a prime ideal, and $\pi(X) = x_r$, $\pi(Y) = y_r$ and $\pi(Z_j) = z_j^{(r)}$ for all $j \in \{1, \ldots, m\}$. Let $W^{(r)}$ be the irreducible affine real algebraic variety over R corresponding to $R[x_r, y_r, z_1^{(r)}, \ldots, z_m^{(r)}]$. Then 0 is a regular point of $W^{(r)}$ with (x_r, y_r) as a regular system of local parameters, hence there exist $f_1, \ldots, f_m \in \mathcal{I}(W^{(r)}) = \mathcal{J} \subset R[X, Y, Z_1, \ldots, Z_m]$ such that the matrix $\left(\frac{\partial f_j}{\partial Z_k}\right)_{j,k=1,\ldots,m}$ is invertible. By the semialgebraic implicit function theorem, there exist an open semialgebraic neighborhood U of 0 in R^2, an open semialgebraic neighborhood V of 0 in R^m and a continuous semialgebraic map $\varphi\colon U \to V$ such that $\varphi(0) = 0$ and

$$f_1(a,b) = \cdots = f_m(a,b) = 0 \iff \varphi(a) = b$$

for every $(a,b) \in U \times V$. Then the map $\psi\colon U \mapsto \mathcal{Z}(f_1,\ldots,f_m) \cap U \times V$, $a \mapsto (a,\varphi(a))$ is a semialgebraic continuous bijection.

Let $\mathcal{U}$ be an open neighborhood of 0 in R^{m+2} such that $\mathcal{U} \subset U \times V$, $\mathcal{Z}(f_1,\ldots,f_m) \cap \mathcal{U} = W^{(r)} \cap \mathcal{U}$, and $\mathcal{Z}(\prod_{j=1}^{s} w_{j1}w_{j2}) \cap \mathcal{U} = \emptyset$. Let $\varepsilon \in R$ such that $\varepsilon > 0$ and

$$\{t \in R^2 \mid (\varepsilon - x_r^2 - y_r^2)(t) > 0\} \subset \psi^{-1}(\mathcal{U}).$$

Let I be the subset of the two-element set $\{x_r, y_r\}$ that contains all elements which do not change sign between α and β, and let $\mathcal{V} := \{t \in R^2 \mid \varepsilon - \sum_{i=1}^{2} t_i^2 > 0,\ p(t) > 0\ (p \in \{X,Y\},\ \pi(p) \in I)\} \subset \psi^{-1}(\mathcal{U})$. Note that from the proof of Lemma 7.2, it follows that I is not empty. By Proposition 4.7, we have that $\mathcal{V}$ is semialgebraically connected, and therefore, $\psi(\mathcal{V}) \subset W^{(r)} \cap \mathcal{U}$ is also semialgebraically connected (Proposition 4.5). Hence, by Proposition 4.9, $D := \widetilde{\psi(\mathcal{V})}$ is connected in $\operatorname{Sper} \mathcal{P}(W^{(r)})$, and D contains $\alpha^{(r)}$ and $\beta^{(r)}$, since $v_\alpha(x_r), v_\alpha(y_r), v_\beta(x_r)$ and $v_\beta(y_r)$ are all positive, and since we assumed that $x_r(\alpha) > 0$ and $y_r(\alpha) > 0$, and therefore they are also positive at β if they do not change sign.

Let $j \in \{1,\ldots,s\}$. We have $w_{j2}g_j = x_r^{e_j} y_r^{f_j} w_{j1}$, with $e_j = 0$ if $x_r \notin I$ and $f_j = 0$ if $y_r \notin I$, and therefore $g_j(\delta') \neq 0$ for all $\delta' \in D$. Consider the natural continuous map $\operatorname{Sper}(\pi)\colon \operatorname{Sper}(\mathcal{P}(W^{(r)})) \to \operatorname{Sper} \mathcal{P}(W)$, and set $C := \operatorname{Sper}(\pi)(D) = \{\delta' \cap \mathcal{P}(W) \mid \delta' \in D\}$. By Proposition 4.9, D is connected, thus C must also be connected. D contains $\alpha^{(r)}$ and $\beta^{(r)}$, and therefore C contains α and β. Since $g_j(\delta') \neq 0$ for all $j \in \{1,\ldots,s\}$ and all $\delta' \in D$, we have $C \cap \{\delta \in \operatorname{Sper} \mathcal{P}(W) \mid g_j(\delta) = 0\} = \emptyset$. $\square$

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