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# Some comments and examples on generation of (hyper-)archimedean $\ell$-groups and $f$-rings

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*This paper is dedicated to Mel Henriksen on the occasion of his 80th birthday.*

**Abstract.** — This paper systematizes some theory concerning the generation of $\ell$-groups and reduced $f$-rings from substructures. We are particularly concerned with archimedean and hyperarchimedean groups and rings. We discuss the process of adjoining a weak order unit to an $\ell$-group, or an identity to an $f$-ring and find significant contrasts between these cases. In $\ell$-groups, hyperarchimedeanness and similar properties fail to pass from generating structures to the structures that they generate, as illustrated by a basic example of Conrad and Martinez which we revisit and elaborate. For reduced $f$-rings, on the other hand, these properties do inherit upwards.

**Résumé.** — Dans cet article, nous donnons les bases d'une théorie sur la génération des $\ell$-groupes et des $f$-anneaux réduits à partir de certaines sous-structures. Nous sommes concernés en premier lieu par les groupes et anneaux archimédiens et hyperarchimédiens. Nous discutons le procédé d'adjoindre une unité faible à un $\ell$-groupe ou une identité à un $f$-anneau et nous trouvons des différences significatives entre ces situations. Dans les $\ell$-groupes, certaines propriétés comme celle d'être hyperarchimédien ne sont pas toujours transférées aux structures engendrées, comme le montre l'exemple fondamental de Conrad et Martinez, que nous revisitons et élaborons. Par contre, ces propriétés sont transférées dans le cas des $f$-anneaux réduits.

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This paper proceeds as follows: §1. Notation and definitions. §2. Review of basic procedures of generating an $\ell$-group or $f$-ring from a subgroup or subring by closing under finite meets and joins. §3. Adjoining a weak order unit to an archimedean $\ell$-group. §4. Characterization of sub-$\ell$-groups of hyperarchimedean $\ell$-groups (**HA**-groups) with unit, and various properties of representations. §5. Essential closures of $\ell$-groups with basis. §6. Examples of representations of hyperarchimedean $\ell$-groups without unit. §7. Representing and adjoining an identity to reduced archimedean $f$-rings (**frA**'s), preliminary to §8. Characterizations of **frA**'s that are **HA**: **HA** qua $\ell$-group = **HA** qua $f$-ring; no examples as in §6 are possible.

The paper owes much to Jorge Martinez, who made/asked penetrating remarks/questions about an early version of our paper [10]. This spawned much of §8 here, and caused us to study the Conrad-Martinez papers [6], [7] that are discussed here in §§4, 6, 8.

We are indebted to the referee for identifying an error in our original version of this paper, and for suggestions and comments which led to significant improvements. We are also indebted to Jim Madden and Charles Delzell for suggestions resulting in a streamlining of the proof of Theorem 1.

## 1. Preliminaries

We take standard terms and facts from ordered algebras as familiar; if necessary, see [1]. With the (noted) exception of §2, all the $\ell$-groups and $f$-rings in this paper are archimedean. (Hence the groups are abelian and the rings are commutative.) Rings are not assumed to have a multiplicative identity. A ring is called **reduced** if it contains no non-zero nilpotents.

We will be concerned with the following categories of $\ell$-algebraic objects and their natural homomorphisms:

**Arch**: archimedean $\ell$-groups.

**W**: **Arch**-objects $H$ with distinguished weak order unit $e_H \geqslant 0$ and homomorphisms preserving unit.

**frA**: reduced archimedean $f$-rings.

$\mathbf{\Phi}$: **frA**-objects $A$ with identity element $1_A$ and identity-preserving homomorphisms. Note that $\mathbf{\Phi}$ is a subcategory of **W** (via the functor forgetting multiplication).

Let $[-\infty,+\infty]$ denote the extended reals with the natural order and topology. Let $\mathcal{X}$ be a Tychonoff space. Then, $D(\mathcal{X})$ denotes the collection of all $f \in C(\mathcal{X},[-\infty,+\infty])$ for which $f^{-1}(-\infty,+\infty)$ is dense in $\mathcal{X}$. This is a lattice in the pointwise order. For $f,g,h \in D(\mathcal{X})$, we write $f+g=h$ (respectively, $fg=h$) if $f(x)+g(x)=h(x)$ (respectively, $f(x)g(x)=h(x)$) for all $x \in \mathcal{X}$ such that $f(x)$, $g(x)$ and $h(x)$ are finite. Note that $f+g$ has an unambiguous meaning on the subset $U$ of $\mathcal{X}$ where both $f$ and $g$ are finite, but this function on $U$ may have no continuous $[-\infty,+\infty]$-valued extension to $\mathcal{X}$. If there is a continuous extension $h$, then $h \in D(\mathcal{X})$ and $f+g=h$. Multiplication is treated similarly. A subset $A$ of $D(\mathcal{X})$ is called an $f$-ring ($\ell$-group, $f$-algebra, etc.) in $D(\mathcal{X})$ if it is closed under the lattice operations and the appropriate algebraic operations. Note that such $A$ is archimedean. For any $f : \mathcal{X} \to [-\infty,+\infty]$, the **zero-set** of $f$ is $\mathfrak{z}(f)=\{x \in \mathcal{X} : f(x)=0\}$ and the **cozero-set** of $f$ is $\mathrm{coz}f = \mathcal{X} \setminus \mathfrak{z}(f)$. For $S \subseteq [-\infty,+\infty]^{\mathcal{X}}$, $\mathfrak{z}(S)=\bigcap\{\mathfrak{z}(f) : f \in S\}$ and $\mathrm{coz}S = \mathcal{X} \setminus \mathfrak{z}(S)$.

For any set $\mathcal{X}$, $\mathbf{1}_{\mathcal{X}}$ denotes the $\mathbb{R}$-valued function $\mathbf{1}_{\mathcal{X}}(x)=1$. Let $\mathbf{C}$ be any one of **Arch**, **W**, **frA**, $\mathbf{\Phi}$. The expression "$A \leqslant B$ in $\mathbf{C}$" means that $A$ is a $\mathbf{C}$-subobject of $B$ (i.e., $A \subseteq B$ and the inclusion is a $\mathbf{C}$-morphism). For $\mathbf{C}=\mathbf{W}$ (respectively, $\mathbf{\Phi}$), this includes the datum $e_A=e_B$ (resp., $1_A=1_B$). The expression "$A \leqslant D(\mathcal{X})$ in $\mathbf{C}$" means that $A \subseteq D(\mathcal{X})$ and is closed under the operations in $D(\mathcal{X})$ requisite for membership in $\mathbf{C}$. E.g., "$A \leqslant D(\mathcal{X})$ in $\mathbf{\Phi}$" means that $A$ is a sublattice of $D(\mathcal{X})$, that $f,g \in A \Longrightarrow f+g \in D(\mathcal{X})$ and $f+g \in A$, likewise for $f \cdot g$ and $f-g$, and that $\mathbf{1}_{\mathcal{X}}=1_A$. For each of our categories, $\mathbf{C}$, any $B \in |\mathbf{C}|$ has representations "$B \approx \overline{B} \leqslant D(\mathcal{X})$ in $\mathbf{C}$" meaning $\overline{B} \leqslant D(\mathcal{X})$ in $\mathbf{C}$ and $B \approx \overline{B}$ is a $\mathbf{C}$-isomorphism. See [18] for a catalogue of these, and [14], [15], [16] and [17] for canonical representations for $\mathbf{C}=\mathbf{W},\mathbf{\Phi}$ and $\mathbf{frA}$, respectively.

We comment further on $\mathbf{C}=\mathbf{W}$. Here we have the Yosida representation, which will be used frequently below. For $H \in |\mathbf{W}|$, there is an essentially unique compact space $\mathcal{Y}H$ with $H \approx \widehat{H} \leqslant D(\mathcal{Y}H)$ in $\mathbf{W}$ and distinct points of $\mathcal{Y}H$ are $0-1$ separated by the functions in $\widehat{H}$. Moreover, for any representation $H \approx \overline{H} \leqslant D(\mathcal{X})$ in $\mathbf{W}$, there is a unique continuous $\mathcal{Y}H \xleftarrow{\tau} \mathcal{X}$ with $\tau\mathcal{X}$ dense in $\mathcal{Y}H$ and for which $\overline{h}=\widehat{h}\circ\tau$ for each $h \in H$. See [14] for details. We identify $H$ with $\widehat{H}$ and always use "$H \leqslant D(\mathcal{Y}H)$" to denote the Yosida representation.

Any uses of "$\leqslant$" that are not so carefully labeled will, we hope, be clear from context.

Following [8], $C^*(\mathcal{X})$ denotes $\{f \in C(\mathcal{X}) : f \text{ is bounded}\}$. We carry this notation into our contexts: whenever $S \subseteq D(\mathcal{X})$, $S^* = \{f \in S : f \text{ is bounded}\}$. For $G \in |\mathbf{W}|$, with $G \leqslant D(\mathcal{Y}G)$, the Yosida representation, this meaning coincides with $G^* = \{g \in G : \exists n \in \mathbb{N} \text{ with } |g| \leqslant ne_G\}$.

Definition 1.1. — *$G$ is **hyperarchimedean** (usually abbreviated "is **HA**") if every homomorphic image of $G$ is archimedean.*

The following gleans information mostly from [4]; see also [12].

Proposition 1.2. —

1. *$G$ is **HA** if and only if there is a representation $G \leqslant \mathbb{R}^{\mathcal{X}}$ in **Arch** satisfying the condition $\mathbf{HA}_1$:*

$$\forall 0 < f, g \in G\ \exists n \in \mathbb{N} \text{ such that } f(x) > 0 \Longrightarrow g(x) < nf(x).$$

   (a) *It then follows that every representation in **Arch** of $G$ in a product of reals satisfies $\mathbf{HA}_1$.*

   (b) *If $G$ is **HA**, then any weak order unit of $G$ is a strong unit.*

2. *A representation $G \leqslant \mathbb{R}^{\mathcal{X}}$ in **Arch** is said to satisfy condition $\mathbf{HA}_1^+$ if*

$$0 < g \in G \Longrightarrow \exists 0 < r < s \in \mathbb{R} \text{ so that } (\, g(x) \neq 0 \Longrightarrow g(x) \in (r, s)\,);$$

   *equivalently, $G \leqslant (\mathbb{R}^{\mathcal{X}})^*$ and for every $0 \neq g \in G^+$ we have $\inf g(\operatorname{coz} g) > 0$. It is clear that a group having such representation satisfies $\mathbf{HA}_1$ so is hyperarchimedean.*

3. *Let $H \in |\mathbf{W}|$ with Yosida representation $H \leqslant D(\mathcal{Y}H)$.*

   (a) *If $H$ is **HA** then the Yosida representation satisfies $\mathbf{HA}_1^+$; any $\mathbf{W}$-embedding $H \leqslant \mathbb{R}^{\mathcal{X}}$ satisfies $\mathbf{HA}_1^+$; any sub-$\ell$-group of $H$ inherits the $\mathbf{HA}_1^+$ representation property of $H$. (See also §3 below.)*

   (b) *$H$ is **HA** if and only if $H \leqslant C(\mathcal{Y}H)$ and $\operatorname{coz} h$ is closed for each $h \in H$. (When this is the case, then the collection of cozero sets of members of $H$ coincides with the collection of all clopen sets in $\mathcal{Y}H$.)*

   (c) *If an $\ell$-group $G$ can be embedded in a hyperarchimedean $\mathbf{W}$-object, then $G$ has a representation satisfying $\mathbf{HA}_1^+$. (The converse fails, as is shown by our example in §6.3.)*

4. *(Bigard) $G$ is* **HA** *if and only if there is $G \leqslant \mathbb{R}^{\mathcal{X}}$ in* **Arch** *with $\mathcal{X}$ a Hausdorff space, $G$ separating points in $\mathcal{X}$, and $\{\operatorname{coz} g : g \in G\} \subseteq CO(\mathcal{X})$ (the compact-open sets in $\mathcal{X}$).*

Note that the existence of a representation of $G$ satisfying $\mathbf{HA}_1^+$ does not imply that every $G \leqslant \mathbb{R}^{\mathcal{X}}$ satisfies this condition. Nor does the existence of an embedding $G \leqslant \mathbb{R}^{\mathcal{X}}$ satisfying the condition in (4) mean that every embedding of $G$ in a product $\mathbb{R}^{\mathcal{X}}$, where $\mathcal{X}$ is a Hausdorff space and $G$ separates points in $\mathcal{X}$, satisfies the given condition, or that $G \leqslant C(\mathcal{X})$.

## 2. Join-meet generation

In order to describe the contents of this section, let us begin with some notation. If $S$ is a subset of the lattice $L$, we use $j(S, L)$ (respectively, $m(S, L)$) to denote the collection of all joins (resp., meets) of non-empty finite subsets of $L$. When no confusion is likely, we write merely $jS$ ($mS$). Now, in the present section, we are concerned with the case in which $L$ and $S$ have some additional structure, e.g., $L$ is an $\ell$-group or an $f$-ring and $S$ is a group or a ring. We will prove that in many such cases, $mjS := m(jS)$ is closed under the additional operations in $L$, or inherits other properties.

The main theorem of this section collects several useful and closely related results. Part (A) is well-known: see, e.g., [1], 2.1.4. (B) is Theorem 3.3 in [9]. The proof there was essentially left to the reader. Many authors have cited this result, but no one has ever published a careful proof and experience suggests that there is much misunderstanding surrounding it. We take this occasion to insert a complete proof into the literature. (C) is a new observation. (D) is a combination of Theorem 3.1 in [16] with the Henriksen-Isbell result (B). The proof below is much simpler than the proof in [16]. It provides a powerful tool for exploring the properties of the canonical embedding of a **frA**-object into a $\mathbf{\Phi}$-object (see Section 7). Note that (A) applies to all $\ell$-groups and (B) applies to all $f$-rings (whether reduced or not). Not even commutativity is assumed.

We begin with two technical lemmas.

Lemma 1. — *Let $A$ be an $f$-ring, and let $f, g \in A$. Then:*

$$\begin{aligned} 0 \leqslant g \quad &\Longrightarrow fg \leqslant f^2 g + g, \\ 0 \leqslant f \quad &\Longrightarrow fg \leqslant fg^2 + f. \end{aligned}$$

*Proof.* — It suffices to show that these implications hold in totally ordered rings, which we do. As for the first, if $g \leqslant fg$, then either $f \geqslant 0$ and so $fg \leqslant f^2g \leqslant f^2g + g$, or $g = 0$. In either case, the first implication is satisfied. If, on the other hand, $fg < g$, then $fg < g + f^2g$. This proves the first implication. The second is proved similarly. $\square$

Recall that in any (additively written) $\ell$-group, $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$. It is a fact that $f = f^+ - f^-$.

Lemma 2. — *Let $A$ be an $f$-ring and let $f, g \in A$. Then:*

$$f^+g^+ = (fg \wedge (f^2g + g) \wedge (fg^2 + f)) \vee 0.$$

*Proof.* — It suffices to show that this identity holds in any totally ordered ring. If $f$ and $g$ are both positive, then by the previous lemma, both sides simplify to $fg$. Otherwise, both sides simplify to 0. $\square$

Theorem 2.1. —

*(A)* ([20]) *If $A$ is a sub-semigroup of the $\ell$-group $H$ (not necessarily a sublattice), then $jA$ and $mA$ are each sub-semigroups of $H$. If $A$ is a group, then the sublattice of $H$ generated by $A$, namely $mjA$, is a subgroup of $H$.*

*(B)* ([9]) *If $A$ is a subring of the $f$-ring $H$ (not necessarily a sublattice), then the sublattice of $H$ generated by $A$, namely $mjA$, is a subring of $H$.*

*(C) If $A$ is a subgroup of $D(\mathcal{X})$ (not necessarily a sublattice), then the sublattice of $D(\mathcal{X})$ generated by $A$, namely, $mjA$, is a subgroup of $D(\mathcal{X})$.*

*(D)* (cf. [16]) *If $A$ is a subring of $D(\mathcal{X})$ (not necessarily a sublattice), then the sublattice of $D(\mathcal{X})$ generated by $A$, namely, $mjA$, is a subring of $D(\mathcal{X})$.*

*Proof.* — (A): The first statement is an immediate consequence of the distributivity of the group operation over the lattice operations in the $\ell$-group $H$. That $mjA$ is a lattice is a consequence of the distributivity of the lattice operations in $H$; that it is a group results from the identity $-(a \vee b) = (-a) \wedge (-b)$ and its dual.

For future reference, note that this argument shows that that for any arrays of variables $x = \{x_{ij}\}$ and $y = \{y_{kl}\}$, there is an identity of the form

$$\Big(\bigwedge_i \bigvee_j x_{ij}\Big) + \Big(\bigwedge_k \bigvee_l y_{kl}\Big) = \bigwedge_\alpha \bigvee_\beta a_{\alpha\beta},$$

where each $a_{\alpha\beta}$ is a sum of the form $x_{ij} + y_{kl}$ for some specific indices $i, j, k, l$.

(B): In view of (A), it suffices to show that $mjA$ is closed under multiplication. Suppose $F, G \in mjA$. We must show that $FG \in mjA$. Now, $F = \bigwedge_i \bigvee_j f_{ij}$ and $G = \bigwedge_k \bigvee_l g_{kl}$, with $f_{ij}, g_{kl} \in A$. In $H$,

$$FG = (F^+ - F^-)(G^+ - G^-) = F^+G^+ - F^+G^- - F^-G^+ + F^+G^+.$$

Since $mjA$ is closed under addition and subtraction, it suffices to show that each of the terms is in $mjA$. Now, since the operation $()^+$ and multiplication by positive elements both distribute over suprema and infima,

$$F^+G^+ = F^+ \bigwedge_k \bigvee_l g^+_{kl} = \bigwedge_k \bigvee_l F^+ g^+_{kl},$$

and for each $k$ and $l$,

$$F^+ g^+_{kl} = \left(\bigwedge_i \bigvee_j f^+_{ij}\right) g^+_{kl} = \bigwedge_i \bigvee_j f^+_{ij} g^+_{kl}.$$

Thus, to show $F^+G^+ \in mjA$, it suffices to show that $f^+g^+ \in mjA$ for all $f, g \in A$. This follows from Lemma 2. For the other cases, note that $F^- = (-F)^+$, $G^- = (-G)^+$, and by (A), $-F, -G \in mjA$, so analogous arguments apply.

For future reference, note that we have shown that for any arrays of variables $x = \{x_{ij}\}$ and $y = \{y_{kl}\}$, there are polynomials $p_{\alpha\beta}(x, y)$ with integer coefficients and degree at most 3 such that there is an $f$-ring identity of the form:

$$\left(\bigwedge_i \bigvee_j x_{ij}\right)\left(\bigwedge_k \bigvee_l y_{kl}\right) = \bigwedge_\alpha \bigvee_\beta p_{\alpha\beta}(x, y).$$

We now prove (C) and finally (D). Bear in mind that $mjA$ is a sublattice of $D(\mathcal{X})$. If $F \in mjA$, it is clear that $-F \in mjA$. Let $F, G \in mjA$ and let $\mathcal{U}$ be their common domain of reality. Then the pointwise sum of $F$ and $G$ is unambiguously defined on $\mathcal{U}$. The first note above labeled "for future reference" provides a formula that express the pointwise sum of $F$ and $G$ on $\mathcal{U}$ as the restriction to $\mathcal{U}$ of an element of $mjA \subseteq D(\mathcal{X})$. But, the definition of addition in $D(\mathcal{X})$ is exactly: "form the pointwise sum on the common domain of reality, and then (if possible) extend to $\mathcal{X}$)." Thus $mjA$ is a subgroup of $D(\mathcal{X})$. To prove (D), we need to show that products of elements of $mjA$ are in $mjA$. The proof is precisely analogous to the proof for sums that we have just given. $\square$

## 3. W-generation

This section presents the simple idea of "**W**-generation", and a few easy observations about it, which represent about all we have been able to say. This contrasts with the corresponding idea in **frA**, "**Φ**-generation", which is completely pinned down in Theorem 7.3, *infra.* That **W**-generation says so little and **Φ**-generation says so much seems responsible for the examples in **Arch/W** (see §6) and the lack of such examples in **frA/Φ** (see §§7,8).

DEFINITION 3.1. — *$S \subseteq H$ is **W-generating** if $H \in |\mathbf{W}|$ and $H$ is generated qua $\ell$-group by $S$ and $e_H$. $G \leqslant H$ (respectively, $\sigma : G \hookrightarrow H$) is said to be **W-generating** if $G \leqslant H$ in **Arch** and $G \subseteq H$ (resp., $\sigma : G \hookrightarrow H$ in **Arch** and $\sigma G \subseteq H$) is **W**-generating. If $G$ is a **W**-generating $\ell$-subgroup of $H$, we write $G \overset{\mathbf{W}}{\leqslant} H$.*

PROPOSITION 3.2. — *(a) $S \subseteq D(\mathcal{X}) \Longrightarrow S' \subseteq D(\mathcal{X})$, where $S'$ is any one of $\mathbb{Z} \cdot S$, $S + \mathbb{Z} \cdot \mathbf{1}_{\mathcal{X}}$, $jS$, $mS$, but it is **not** necessarily true that $S + S \subseteq D(\mathcal{X})$.*

*(b) If $S \subseteq H$ is **W**-generating, then $H = jm(\mathbb{Z} \cdot S + \mathbb{Z} \cdot e_H)$. When $H \leqslant D(\mathcal{Y}H)$, $S$ is **W**-generating in $H$ if and only if $H = jm(\mathbb{Z} \cdot S + \mathbb{Z} \cdot \mathbf{1}_{\mathcal{Y}H})$.*

*(c) If $G \leqslant D(\mathcal{X})$, then there is $H \leqslant D(\mathcal{X})$ in which $G$ is **W**-generating.*

*Proof.* — (a) is immediate. For (b), $\mathbb{Z} \cdot S + \mathbb{Z} \cdot e_H$ is the subgroup of $H$ generated by $S$ and $e_H$; apply Theorem 2.1.A. For (c), apply Theorem 2.1.C: $H = mj(G + \mathbb{Z} \cdot \mathbf{1}_{\mathcal{X}})$. □

PROPOSITION 3.3. — *Suppose $H \in |\mathbf{W}|$. If $S \subseteq H$ is **W**-generating, then $S$ separates points in $\mathcal{Y}H$; it follows that $\mathfrak{z}(S) \equiv \bigcap \{\mathfrak{z}(s) : s \in S\}$ has $|\mathfrak{z}(S)| \leqslant 1$. Thus, either*

1. *$\mathfrak{z}(S) = \varnothing$, or*
2. *$\mathfrak{z}(S) = \{\alpha\}$, and $\mathcal{Y}H = \mathrm{coz}S \cup \{\alpha\}$.*

*Proof.* — Given $x \neq y$ in $\mathcal{Y}H$, then there is $h \in H$ with $h(x) \neq h(y)$. But by Proposition 3.2, $h = \bigvee_{i=1}^{k} \bigwedge_{j=1}^{l} (m_{ij}s_{ij} + n_{ij}\mathbf{1})$, for appropriate $k, l, m_{ij}, n_{ij} \in \mathbb{Z}$ and $s_{ij} \in S$. It is clear that $s_{ij}(x) \neq s_{ij}(y)$ for some $i, j$. □

COROLLARY 3.4. — *If $G \overset{\mathbf{W}}{\leqslant} H$, then $\mathrm{coz}G$ is dense in $\mathcal{Y}H$.*

Proposition 3.5. — *Suppose $G \overset{\mathbf{W}}{\leqslant} H$, and view $G \subseteq D(\mathcal{Y}H)$. If $\mathfrak{z}(G) = \varnothing$, then $G$ contains a weak order unit. The converse fails.*

*Proof.* — The first statement succumbs to a standard compactness argument. For the second: consider $G \equiv \{g \in C([0,1]) : g(1) = 0\} \overset{\mathbf{W}}{\leqslant} H \equiv jm\left(G + \mathbb{Z} \cdot 1_{[0,1]}\right)$: the function $g(x) = 1 - x$ is a weak unit of $G$ while $\mathfrak{z}(G) = \{1\}$. $\square$

## 4. Sub-$\ell$-groups of $HA$'s with unit

In this section, we characterize (in a rather weak sense) those $G \in |\mathbf{Arch}|$ for which there is $G \overset{\mathbf{W}}{\leqslant} H$ where $H$ is **HA** (Corollary 4.7 below). This result "originates" in [7]; see (c) in the remarks at the end of the section.

Definition 4.1. — *A representation $G \leqslant D(\mathcal{X})$ in* **Arch** *is* **BA** *if*

$$\forall g \in G, \exists 0 < \varepsilon = \varepsilon(g, 0) \in \mathbb{R} \text{ such that } g(x) \neq 0 \Longrightarrow |g(x)| \geqslant \varepsilon);$$

*it is* **BA**$\mathbb{Z}$ *if*

$$\forall g \in G,\ \forall n \in \mathbb{Z}, \exists 0 < \varepsilon = \varepsilon(g, n) \in \mathbb{R} \text{ such that } g(x) \neq n \Longrightarrow |g(x) - n| \geqslant \varepsilon.$$

Note that neither of these conditions involves a topology on $\mathcal{X}$. **BA** stands for "bounded away (from zero)"; **BA**$\mathbb{Z}$ stands for "bounded away from the integers", (see [13]). We will, when convenient, also say that subsets, and/or individual members, of $D(\mathcal{X})$ are **BA** or **BA**$\mathbb{Z}$.

Since the conditions **BA**, **BA**$\mathbb{Z}$, and boundedness will appear often, and in varying combinations, we also adopt the convention of saying that a function, or collection of functions, "is **B**" if the function(s) in question is (are) bounded. It should be emphasized that the conditions **B**, **BA**, and **BA**$\mathbb{Z}$ apply to representations of a given group $G$, and not to $G$ itself, although this distinction vanishes in **W**, as noted in Proposition 4.3(3) below.

Proposition 4.2. — *Suppose $G \leqslant H \leqslant D(\mathcal{X})$ in* **Arch**.

1. *If $H \leqslant D(\mathcal{X})$ is* **B** *(respectively,* **BA***, resp.,* **BA**$\mathbb{Z}$*), then so is $G \leqslant D(\mathcal{X})$. These properties "**inherit down**".*
2. *Suppose $H \leqslant D(\mathcal{X})$ in* **W** *and $G \overset{\mathbf{W}}{\leqslant} H$. If $G \leqslant D(\mathcal{X})$ is* **B** *(respectively,* **BA**$\mathbb{Z}$*), then so is $H \leqslant D(\mathcal{X})$. These properties "**inherit up**" under* **W***-generation.*
3. *The statement analogous to (2) for* **BA** *fails: see Example 6.3 below.*

*Proof.* — (1) These are obvious.

(2) Here, $H = jm(G + \mathbb{Z} \cdot \mathbf{1}_{\mathcal{X}})$. So for **B**, the statement is obvious. Now suppose $G \leqslant D(\mathcal{X})$ is $\mathbf{BA}\mathbb{Z}$. It is readily seen that $G + \mathbb{Z} \cdot \mathbf{1}_{\mathcal{X}}$ is $\mathbf{BA}\mathbb{Z}$ and that if $f, g \in D(\mathcal{X})$ are both $\mathbf{BA}\mathbb{Z}$, then both $f \vee g$ and $f \wedge g$ are, also, with $\varepsilon(f \vee g, n) = \varepsilon(f, n) \wedge \varepsilon(g, n) = \varepsilon(f \wedge g, n)$. □

Proposition 4.3. — *Let $G \in |\mathbf{Arch}|$.*

1. (a) *Suppose that $G \leqslant \mathbb{R}^{\mathcal{Y}}$ in* **Arch** *for some set $\mathcal{Y}$. For $x_1, x_2 \in \mathcal{Y}$, set $x_1 \sim_G x_2$ if $g(x_1) = g(x_2)$ for all $g \in G$. This is an equivalence relation: set $\mathcal{X} = Y/\sim_G$. This process results in an* **Arch**-*embedding of $G$ in $\mathbb{R}^{\mathcal{X}}$, where $G$ separates points.*
   (b) *If $G \leqslant \mathbb{R}^{\mathcal{Y}}$ and $G$ separates points in $\mathcal{Y}$, endow $\mathcal{Y}$ with the $G$-weak topology. Then $\mathcal{Y}$ is Tychonoff and $G \leqslant C(\mathcal{Y})$.*
   (c) *If $G \leqslant D(\mathcal{Y})$, and $G$ separates points in $\mathcal{Y}$, where $\mathcal{Y}$ has the $G$-weak topology, then there is a minimal compactification of $\mathcal{Y}$, say $\mathcal{K}$, to which every member of $G$ extends: $G \leqslant D(\mathcal{K})$.*
2. *If $G \leqslant \mathbb{R}^{\mathcal{Y}}$ is* **B** *(respectively,* **BA**, *resp.,* $\mathbf{BA}\mathbb{Z}$*), then there is a compact space $\mathcal{K}$ such that $G \leqslant D(K)$ is* **B** *(resp.,* **BA**, *resp.,* $\mathbf{BA}\mathbb{Z}$*).*
3. *Suppose $H \leqslant D(\mathcal{X})$ in* **W**. *This representation is* **B** *(respectively,* **BA** *resp.,* $\mathbf{BA}\mathbb{Z}$*) if and only if in the Yosida representation $H \leqslant D(\mathcal{Y}H)$ is* **B** *(resp.,* **BA**, *resp.,* $\mathbf{BA}\mathbb{Z}$*).*

*Proof.* — (1) is standard.

(2) Topologize $\mathcal{Y}$ as in (1)(b) above, and let $\mathcal{K} = \beta\mathcal{Y}$, the Čech-Stone compactification of $\mathcal{Y}$. For each $g \in G$, let $\beta g$ denote the extension of $g$ in $D(\beta\mathcal{Y})$ and set $\beta G \equiv \{\beta g : g \in G\}$. Then $\beta G \leqslant D(\beta\mathcal{Y})$ is **B**, **BA** or $\mathbf{BA}\mathbb{Z}$ if and only if $G$ is.

(3) Label the given presentation of $H$ as $\overline{H} \leqslant D(\mathcal{X})$. By the discussion in §1, this is related to the Yosida representation $H \leqslant D(\mathcal{Y}H)$ by continuous dense $\mathcal{Y}H \xleftarrow{\tau} \mathcal{X}$ as $\overline{h} = h \circ \tau$ for each $h \in H$. Then $h$ is **B**, **BA** or $\mathbf{BA}\mathbb{Z}$ if and only if $\overline{h}$ is. □

Proposition 4.4. — *Suppose $G \leqslant D(\mathcal{X})$ in* **Arch**.

1. *This satisfies* **BA** *if and only if* coz$g$ *is closed for each $g \in G$.*
2. $\mathbf{BA}\mathbb{Z}$ *is satisfied if and only if $g^{-1}(n)$ is open, for each $g \in G$ and each $n \in \mathbb{Z}$ (whence $g^{-1}(\mathbb{Z})$ is open).*

The proof is immediate.

Proposition 4.5. — *Suppose $H \in |\mathbf{W}|$ and that, for some space $\mathcal{X}$, we have $H \leqslant D(\mathcal{X})$ in $\mathbf{W}$ (so that $\mathbf{1}_{\mathcal{X}} \in H$). The following are equivalent.*

1. *$H \leqslant D(\mathcal{X})$ is $\mathbf{BA}$.*
2. *$H \leqslant D(\mathcal{X})$ is $\mathbf{BA}\mathbb{Z}$.*
3. *$H^*$ is $\mathbf{HA}$.*
4. *Whenever $H \leqslant D(\mathcal{Y})$ in $\mathbf{W}$ for some $\mathcal{Y}$, this representation is $\mathbf{BA}$.*

Again, the easy verification is omitted.

Proposition 4.6. — *For $G \in |\mathbf{Arch}|$, the following are equivalent.*

1. *There is a space $\mathcal{X}$ and an embedding $G \leqslant D(\mathcal{X})$ that is $\mathbf{BA}\mathbb{Z}$.*
2. *There is $H \in |\mathbf{W}|$ such that $H^*$ is $\mathbf{HA}$ and $G \leqslant H$.*
3. *There is $J \in |\mathbf{W}|$ with $G \overset{\mathbf{W}}{\leqslant} J$ and such that $J^*$ is $\mathbf{HA}$ (and, consequently, $G \leqslant D(\mathcal{Y}J)$ is $\mathbf{BA}\mathbb{Z}$).*

*Proof.* — If $G \leqslant D(\mathcal{X})$ is $\mathbf{BA}\mathbb{Z}$, set $H = jm\,(G + \mathbb{Z} \cdot \mathbf{1}_{\mathcal{X}})$. Then $G \overset{\mathbf{W}}{\leqslant} H \leqslant D(\mathcal{X})$; apply Proposition 4.2(2): $H \leqslant D(\mathcal{X})$ is $\mathbf{BA}\mathbb{Z}$. By Proposition 4.5, $H^*$ is $\mathbf{HA}$. Thus, (1) implies (2). Now assume (2): we have $G \leqslant H \leqslant D(\mathcal{Y}H)$. Set $J = jm\,(G + \mathbb{Z} \cdot 1_H)$. Then $J^* \leqslant H^*$ is $\mathbf{HA}$, so $J \leqslant D(\mathcal{Y}J)$ is $\mathbf{BA}\mathbb{Z}$, so, $G \leqslant D(\mathcal{Y}J)$ is, also: (2) implies (3), the parenthetical remark in (3) holding, by proposition 4.5. Trivially, (3) implies (1). $\square$

The following Corollary is Proposition 4.6 with boundedness superimposed.

Corollary 4.7. — *For $G \in |\mathbf{Arch}|$, the following are equivalent:*

1. *There is a space $\mathcal{X}$ and an embedding $G \leqslant D(\mathcal{X})$ that is $\mathbf{B}$ and $\mathbf{BA}\mathbb{Z}$.*
2. *There is $H \in |\mathbf{W}|$ that is $\mathbf{HA}$ and $G \leqslant H$.*
3. *There is $H \in \mathbf{W}$ with $G \overset{\mathbf{W}}{\leqslant} H$ and such that $H$ is $\mathbf{HA}$.*

*Remarks 1.* — (a) In contrast with the situation in $\mathbf{W}$, an $\mathbf{Arch}$-object may be $\mathbf{BA}\mathbb{Z}$ in some, but not all, representations, and a representation may be **BA** but not **BA**$\mathbb{Z}$. For example, consider the four representations of the same **Arch** object, $G \cong G_i \leqslant D(\mathbb{N})$ given by

$$G_i = \{f \in C(\mathbb{N}) : \exists n_0 \in \mathbb{N} \text{ so that } n > n_0 \Longrightarrow f(n) = mh_i(n) \text{ for some } m \in \mathbb{Z}\},$$

for $i = 1, 2, 3,4$ where, for each $n \in \mathbb{N}$,

$$\begin{aligned} h_1(n) &= \frac{1}{n}, \\ h_2(n) &= 1, \\ h_3(n) &= n, \text{ and} \\ h_4(n) &= 1 + \frac{1}{n}. \end{aligned}$$

$G$ is **HA** (so each $G_i^*$ is), but $G_1$ is not **BA**, while $G_2$, $G_3$ and $G_4$ are, while $G_4$ is **BA** but not **BA**$\mathbb{Z}$. Although this group has weak order units,

an example that does not is $G \oplus C_K(\mathbb{N})$.

(b) Proposition 4.5 was more-or-less known to the authors of [13]. The equivalence of (1) and (3) in that proposition is due to the second and third authors of [13].

(c) Corollary 4.7 above seems to be exactly the equivalence of (1) and (2) of Theorem 7 in [7]. We have been unable to understand the proof given in [7].

## 5. Essential closure and $\ell$-groups with basis

Here, we collect the tools that will be required to present the examples of the next section.

### 5.1. Essential closure

The results noted here are all from [3]; all objects and maps are in the category **Arch**.

Recall that an embedding $\varphi : G \hookrightarrow H$ is said to be **essential**, or that $H$ is an **essential extension** of $G$, if whenever $I$ is an ideal in $H$ with $\varphi G \cap I = \{0\}$, then $I = \{0\}$. (By **ideal** is meant "convex sub-$\ell$-group".) $G$ is said to be **essentially closed** if it admits no proper essential extension; an essential embedding of $G$ into an essentially closed $H$ is called an **essential closure** of $G$.

1. $G$ is essentially closed if and only if $G \cong D(\mathcal{S})$ for some compact, extremally disconnected space $\mathcal{S}$.

2. For every $G$ there is an essential closure: $\epsilon_G : G \longrightarrow \epsilon G$.

3. If $\varphi : G \longrightarrow H$ is an essential embedding, then there is an embedding $\psi : H \longrightarrow \epsilon G$ such that $\psi\varphi = \epsilon_G$. If $H$ is essentially closed (i.e., if $\varphi : G \longrightarrow H$ is an essential closure), then $\psi$ is an isomorphism.

4. An essential closure of $G$, $\epsilon_G : G \longrightarrow \epsilon G = D(\mathcal{S})$, is obtained by embedding $G$ in $D(\mathcal{S})$, where $\mathcal{S}$ is the Stone space of the Boolean algebra of polars of $G$. If $\varphi : G \hookrightarrow H$ is essential, then $\epsilon H = D(\mathcal{S})$, and there is an automorphism $\alpha : \epsilon G \longrightarrow \epsilon G$ with $\alpha\varphi = \epsilon_G$; the automorphisms of $D(\mathcal{S})$ are precisely the mappings of the form $\alpha_f : h \longmapsto fh$ for some $f \in D(\mathcal{S})^+$ with $f^\perp = \{0\}$ (i.e., coz$f$ is dense in $\mathcal{S}$; or, $f$ is a weak order unit in $D(\mathcal{S})$).

## 5.2. $\ell$-groups with basis

For any set $\mathcal{X}$ and any $\mathcal{S} \subseteq \mathcal{X}$, let $\chi_{\mathcal{S}}$ denote the characteristic function of $\mathcal{S}$; for any $x \in \mathcal{X}$, we let $\chi_x$ denote $\chi_{\{x\}}$.

Definition 5.1. — *Let $G \in |\mathbf{Arch}|$.*

1. *The element $a \in G$ is said to be* ***basic*** *if $a \gneq 0$ and $\langle a \rangle$, the ideal generated by $a$, is totally ordered.*

2. *$\{a_x : x \in \mathcal{X}\}$ is called a* ***basis*** *for $G$ if: $\iota$) it is a maximal set of pairwise disjoint elements of $G$, and $\iota\iota$) each $a_x$ is basic.*

3. *If $G$ has a basis $\{a_x : x \in \mathcal{X}\}$, then an embedding $\tau : G \longrightarrow \mathbb{R}^{\mathcal{X}}$ is called a* ***basic representation*** *of $G$ associated with this basis if for each $x \in \mathcal{X}$ we have $\tau a_x = r_x \chi_x$ for some ($0 \neq$) $r_x \in \mathbb{R}$.*

If $G \in \mathbf{Arch}$ with basis $\{a_x : x \in \mathcal{X}\}$, then $G$ has an associated basic representation, $G \leqslant \mathbb{R}^{\mathcal{X}}$. (Each $\langle a_x \rangle$ is totally ordered and archimedean, so there is a map $\mathbf{0} \neq \tau_x : \mathbf{G} \longrightarrow \mathbb{R}$ whose kernel is $a_x^\perp$. Since $\{a_x : x \in \mathcal{X}\}$ is a maximal pairwise disjoint family in $G$, $\bigcap_{x\in\mathcal{X}} a_x^\perp = \{0\}$. Thus, $\tau : G \ni b \longmapsto \tau b = (\tau_x b) \in \prod\{\mathbb{R}_x : x \in \mathcal{X}\}$ is an embedding.) If we endow $\mathcal{X}$ with the discrete topology, then $G \leqslant C(\mathcal{X}) = \mathbb{R}^{\mathcal{X}}$, and one sees readily that this is an essential embedding, thus an essential closure of $G$.

The foregoing Definition and facts are known, and drawn from the discussion in [6] (see also [1]). The result that follows is a generalization and simplification of the Lemma in [6].

Theorem 5.2. — *Suppose $G \in |\mathbf{Arch}|$ with basis $\{a_x : x \in \mathcal{X}\}$ and that $G \leqslant \mathbb{R}^{\mathcal{X}}$ is an associated basic representation of $G$. If $\sigma : G \hookrightarrow \mathbb{R}^{\mathcal{Y}}$ in* **Arch**, *then there is $\rho : \mathbb{R}^{\mathcal{Y}} \longrightarrow \mathbb{R}^{\mathcal{X}}$ in* **Arch** *such that $\rho \circ \sigma$ is an essential embedding. Consequently, there is a positive weak unit $u$ of $\mathbb{R}^{\mathcal{X}}$ such that*

$$ug = \rho\sigma g \text{ for each } g \in G.$$

*If $h \in \mathbb{R}^{\mathcal{Y}}$ is* **B** *(respectively* **BA**, *resp.,* $\mathbf{BA}\mathbb{Z}$*), then $\rho h$ has the same property, and if $h$ is the constant function $h(y) = r_0 \in \mathbb{R}$ for each $y \in \mathcal{Y}$, then $\rho h\,(x) = r_0$ for each $x \in \mathcal{X}$.*

*Proof.* — For each $x \in \mathcal{X}$, choose $k(x) \in \mathcal{Y}$ with $\sigma a_x(k(x)) \neq 0$, and define

$$\rho : \mathbb{R}^{\mathcal{Y}} \ni h \longmapsto \rho h = h\,|_{k(\mathcal{X})} \in \rho\mathbb{R}^{\mathcal{Y}}.$$

Note, first, that $\rho\sigma$ embeds $G$ in $\mathbb{R}^{k(\mathcal{X})}$. For, if $0 \lneq g \in G$, then $g \wedge a_x \neq 0$ for some $x \in \mathcal{X}$, since $\{a_x : x \in \mathcal{X}\}$ is a maximal set of pairwise disjoint elements of $G$. But $\langle a_x \rangle$ is totally ordered and archimedean, so $n\,(g \wedge a_x) \geqslant a_x$ for some $n \in \mathbb{N}$. Thus, $0 \lneq \rho\sigma a_x \leqslant n(\rho\sigma g \wedge \rho\sigma a_x)$, so $\rho\sigma g \neq \mathbf{0}$.

It is clear that $\rho \circ \sigma$ is a basic representation of $G$ in $\mathbb{R}^{\mathcal{X}}$ (for each $x \in \mathcal{X}$, $\rho\sigma\,(a_x) = \chi_{k(x)} \cdot \sigma a_x\,(k\,(x)))$; hence, it is an essential embedding. The existence of $u$ now follows from (4) in the summary of Conrad's results in the first paragraph of this section. The last sentence merely collects the properties of restriction that we require below. $\square$

Corollary 5.3. — *Suppose $G \in |\mathbf{Arch}|$ with basis $\{a_x : x \in \mathcal{X}\}$ and that $G \leqslant C\,(\mathcal{X})$ is an associated basic representation of $G$. If $\sigma : G \hookrightarrow H$ is* **W***-generating, then there is $\rho : H \longrightarrow C\,(\mathcal{X})$ in* **W** *such that $\rho \circ \sigma$ is an essential embedding: there is a positive weak order unit $u \in C\,(\mathcal{X})$ such that*

$$\rho\sigma\,(g) = ug$$

*for each $g \in G$. If $h \in H$ is* **B** *(respectively* **BA**, *resp.,* $\mathbf{BA}\mathbb{Z}$*) in $D\,(\mathcal{Y}H)$, then $\rho h$ has the same property in $C\,(\mathcal{X})$, and if $h$ is the constant function $h(y) = r_0 \in \mathbb{R}$ for each $y \in \mathcal{Y}$, then $\rho h\,(x) = r_0$ for each $x \in \mathcal{X}$.*

*Proof.* — View $H \subseteq D(\mathcal{Y}H)$; by Corollary 3.4, $\mathrm{coz}\sigma G$ is dense in $\mathcal{Y}H$. But $\bigcup\{\mathrm{coz}\sigma a_x : x \in \mathcal{X}\}$ is dense in $\mathrm{coz}\sigma G$ and each $(\sigma a_x)^{-1}(\mathbb{R})$ is dense in $\mathrm{coz}\sigma a_x$. So, $\sigma G \subseteq \mathbb{R}^{\mathcal{X}_1}$ (where $\mathcal{X}_1 = \bigcup\{(\sigma a_x)^{-1}(\mathbb{R}) : x \in \mathcal{X}\}$), from which it follows that $H = jm(\sigma G + \mathbb{Z} \cdot \mathbf{1}_{\mathcal{X}}) \subseteq \mathbb{R}^{\mathcal{X}_1}$. Apply Theorem 5.2. $\square$

Proposition 5.4. — *Suppose $G \in |\mathbf{Arch}|$ and that $\sigma : G \longrightarrow H$ is* **W***-generating. If $H$ is* **HA**, *then there is $\varphi : H \twoheadrightarrow H'$ in* **W** *such that $\varphi \circ \sigma$ is an essential embedding of $G$ into $H'$. (Of course, $\varphi \circ \sigma$ is* **W***-generating and $H'$ is* **HA***. )*

*Proof.* — We may suppose $G \neq \{0\}$; otherwise, $H' = \{0\}$ works. If $\sigma : G \longrightarrow H$ is not essential, then there is an ideal $\{0\} \neq I$ in $H$ with $I \cap G = \{0\}$. By Zorn's Lemma, there is a maximal such, say $I_0$. Note that $e_H$ is a strong order unit, since $H \in |\mathbf{W}|$ and is $\mathbf{HA}$, and $e_H \notin I_0$, since $G \neq \{0\}$. Set $\varphi : H \longrightarrow H/I_0 \equiv H'$. Then $\varphi \circ \sigma : G \longrightarrow H'$ is $\mathbf{W}$-generating, since $jm\,(\varphi\sigma\,(G) + \mathbb{Z} \cdot \varphi\,(e_H)) = \varphi\,(jm\,(\sigma G + \mathbb{Z} \cdot e_H)) = \varphi\,(H)$. □

## 6. The Examples

We present three examples of $\mathbf{HA}$ $\ell$-groups: (6.1) with $\mathbf{BA}\mathbb{Z}$ representation but no $\mathbf{B}$ representation, hence no embedding into an archimedean $\ell$-group with strong order unit; (6.2) with $\mathbf{B}$ representation but no $\mathbf{BA}$ representation in any $D\,(\mathcal{Y})$; (6.3) with representation that is $\mathbf{B}$ and $\mathbf{BA}$ (i.e., satisfies $\mathbf{HA}_1^+$), but no representation that is $\mathbf{B}$ and $\mathbf{BA}\mathbb{Z}$ in an $\mathbb{R}^{\mathcal{Y}}$. None of these embeds into a $\mathbf{HA}$ $\ell$-group with unit, by Corollary 4.7. The last example also exhibits a situation: $G \overset{\mathbf{W}}{\leqslant} H$, where $G$ is $\mathbf{HA}$ having a representation that is $\mathbf{B}$ and $\mathbf{BA}$, but $H$ has no $\mathbf{BA}$ representation. See Corollary 8.2 for the marked contrast in $\mathbf{frA}$.

All three examples utilize the method of [6] (which method was revisited in [7] to further ends), with the present exposition benefitting from our clarification of $\mathbf{B}$, $\mathbf{BA}$, $\mathbf{BA}\mathbb{Z}$ in §4 and our arrangement of the preliminary steps in §5. Example 6.1 is exactly the example of [6]; example 6.2 answers a natural question; example 6.3 (answers another natural question and) is a counterexample to a parenthetical remark in [6] (II on p. 295) - which slip, one realizes after a certain amount of careful reading, has no further effect on [6].

All three examples have the form $G\,(\mathfrak{A}, \gamma) \leqslant C\,(\mathbb{N})$, which is defined as follows. Let $\mathfrak{A}$ be a family of infinite subsets of $\mathbb{N}$ that is *almost disjoint* (every pair of distinct members has finite intersection). Let $\gamma : \mathfrak{A} \longrightarrow C\,(\mathbb{N})$ be any function, and let $G\,(\mathfrak{A},\gamma)$ denote the subgroup of $C\,(\mathbb{N})$ generated by $\{\chi_A \cdot \gamma\,(A) : A \in \mathfrak{A}\} \cup \{\chi_n : n \in \mathbb{N}\}$. One readily sees that $G\,(\mathfrak{A}, \gamma)$ is an $\ell$-subgroup of $C\,(\mathbb{N})$ and that it is $\mathbf{HA}$. Moreover, $G\,(\mathfrak{A},\gamma)$ has a basis: $\{\chi_n : n \in \mathbb{N}\}$. Note that $G\,(\mathfrak{A}, \gamma) \leqslant C\,(\mathbb{N})$ is an essential closure. Recall that $f \in C\,(\mathbb{N})$ with $f^\perp = \{0\}$ means $\mathrm{coz} f = \mathbb{N}$: $f$ is a weak unit in $C\,(\mathbb{N})$.

There are almost disjoint families of infinite subsets of $\mathbb{N}$ having cardinality $c$, the power of the continuum ([8], 6Q), thus of any smaller cardinal.

### 6.1. The Conrad-Martinez Example

A hyperarchimedean $\ell$-group that is presented as an $\ell$-subgroup of $C(\mathbb{N})$ that is $\mathbf{BA}\mathbb{Z}$, but that has no $\mathbf{B}$ representation.

Let $\mathfrak{A}$ be of cardinality sufficient to allow an injection $\gamma : \mathfrak{A} \longrightarrow C^{\uparrow}(\mathbb{N},\mathbb{N})$ (the strictly increasing functions) having $\gamma(\mathfrak{A})$ cofinal in $C^{\uparrow}(\mathbb{N},\mathbb{N})$ with respect to the order $\overset{*}{<}$, where $f \overset{*}{<} g$ if $f(x) < g(x)$ for almost all $x \in \mathbb{N}$. (See comment below.)

It is clear that $G \equiv G(\mathfrak{A},\gamma)$ is $\mathbf{BA}\mathbb{Z}$; to see that it has the claimed property, suppose otherwise: there is a set $\mathcal{Y}$ and an embedding $\sigma : G \longrightarrow \mathbb{R}^{\mathcal{Y}}$ such that $\sigma g$ is bounded for each $g \in G$. Now apply Theorem 5.2: there are $\rho : \mathbb{R}^{\mathcal{Y}} \longrightarrow \left(\mathbb{R}^{\mathbb{N}}\right)^{*}$ in **Arch** and a positive weak order unit $u \in \mathbb{R}^{\mathbb{N}}$ such that $uG = \rho\sigma G$. Since $uG$ contains only bounded functions, the same will be true of $u_1 G$ for any $u_1 \in \mathbb{R}^{\mathbb{N}}$ with $0 \leqslant u_1 \leqslant u$; since $u$ is a weak order unit in $\mathbb{R}^{\mathbb{N}}$, there is $u_1 \leqslant u$ with $\frac{1}{u_1} \in C^{\uparrow}(\mathbb{N},\mathbb{N})$. Now, $(1/u_1)^2 \in C^{\uparrow}(\mathbb{N},\mathbb{N})$, so there is a $v = \gamma(A_0)$ for some $A_0 \in \mathfrak{A}$ with $v \geqslant (1/u_1)^2$. Then $g_0 \equiv \chi_{A_0} \cdot \gamma(A_0) \in G$, while

$$ug_0 = u \cdot \chi_{A_0} \cdot v \geqslant u \cdot \chi_{A_0} \cdot \frac{1}{u_1^2} \geqslant \chi_{A_0} \cdot \frac{1}{u_1}$$

(since $v \geqslant \frac{1}{u_1^2} \geqslant \frac{1}{u_1} \geqslant \frac{1}{u}$). Since $\frac{1}{u_1} \in C^{\uparrow}(\mathbb{N},\mathbb{N})$, this says that $ug_0$ is unbounded, a contradiction.

Here, $|G(\mathfrak{A},\gamma)| = |\mathfrak{A}|$, and the issue of minimizing this seems interesting. The cardinal $d = \min\left\{|D| : D \text{ is } \overset{*}{<}\text{-cofinal in } C^{\uparrow}(\mathbb{N},\mathbb{N})\right\}$ is due to Katětov, and of course $|\mathfrak{A}| = d$ suffices in the example. It is clear that $\omega < d$, and it is known that each of $[\omega_1 = d < c,\ \omega_1 < d < c,\ \omega_1 < d = c]$ is consistent with ZFC ([5]). So, we ask: is there an example here of size $< d$; even countable?

### 6.2. The second example

A hyperarchimedean $\ell$-group that is presented as an $\ell$-subgroup of $C^{*}(\mathbb{N})$, but that has no $\mathbf{BA}$ representation in any $D(\mathcal{Y})$.

Let $\mathfrak{A}$ be as before and again let $\gamma : \mathfrak{A} \hookrightarrow C^{\uparrow}(\mathbb{N},\mathbb{N})$ have $\gamma(\mathfrak{A})$ cofinal in $C^{\uparrow}(\mathbb{N},\mathbb{N})$ with respect to the order $\overset{*}{<}$. Define $\gamma_1 : \mathfrak{A} \longrightarrow C(\mathbb{N},\mathbb{R})$ by $\gamma_1(A) = \frac{1}{\gamma(A)}$ for each $A \in \mathfrak{A}$.

Clearly, $G \equiv G(\mathfrak{A},\gamma_1) \leqslant C^{*}(\mathbb{N})$. Suppose $\sigma : G \hookrightarrow D(\mathcal{Y})$ for some $\mathcal{Y}$, and that $\sigma G$ is $\mathbf{BA}$ and let $H = jm(\sigma G + \mathbb{Z} \cdot \mathbf{1}_{\mathcal{Y}})$. Now apply Corollary

5.3: there are $\rho : H \to C(\mathbb{N})$ and a weak unit $u$ of $C(\mathbb{N})$ such that $\rho\sigma g = ug$ for each $g \in G$, and $uG$ is **BA** in $C(\mathbb{N})$ since $\sigma G$ is **BA** in $D(\mathcal{Y})$. If $u_1 \in \mathbb{R}^{\mathbb{N}}$ with $u \leqslant u_1$, then $u_1 G$ is also **BA**, clearly. Choose $u \leqslant u_1 \in C^{\uparrow}(\mathbb{N}, \mathbb{N})$ and then choose $v \geqslant u_1^2$ with $v = \gamma(A_1) \in \gamma(\mathfrak{A})$. Then $g_1 \equiv \chi_{A_1} \cdot \gamma_1(A_1) \in G$, while $ug_1 = u \cdot \chi_{A_1} \cdot \frac{1}{v} \leqslant u \cdot \chi_{A_1} \cdot \frac{1}{u_1^2} \leqslant \chi_{A_1} \cdot \frac{1}{u_1}$. Thus, $ug_1$ is not **BA**, a contradiction.

### 6.3. The third example

A hyperarchimedean $\ell$-group $G$ that is presented as an $\ell$-subgroup of $C^*(\mathbb{N})$ that is **BA**, but that has no representation that is **B** and **BA**$\mathbb{Z}$ in an $\mathbb{R}^{\mathcal{Y}}$. It follows that $H = jm(G + \mathbb{Z} \cdot \mathbf{1}_{\mathbb{N}})$ fails to satisfy **BA**, so **BA** does not "inherit up" under **W**-generation (Proposition 4.2(3)). (If this representation were **BA**, it would be **B** and **BA**, so it would satisfy $\mathbf{HA}_1^+$. Thus, $H$ would be **HA**. But, $G$ embeds in no **W**-object that is **HA**.)

Here, let $|\mathfrak{A}| = \mathfrak{c}$, and set

$$\Gamma = \{f \in C(\mathbb{N}) : n \in \mathbb{N} \Longrightarrow f(n) > 0, \text{ and } A \in \mathfrak{A} \Longrightarrow f|_A \text{ is } \mathbf{B} \text{ and } \mathbf{BA}\}.$$

Since $|\Gamma| = c$, there is a bijection $\beta : \mathfrak{A} \longrightarrow \Gamma$. Define $\gamma : \mathfrak{A} \longrightarrow C(\mathbb{N})^+$ by

$$\gamma(A) = \beta A + r \cdot \beta A,$$

where $r(n) = \frac{1}{n}$ for each $n \in \mathbb{N}$.

The resulting construct $G = G(\mathfrak{A}, \gamma)$is, in this presentation, both **B** and **BA**, and so satisfies $\mathbf{HA}_1^+$. (Note that it is not **BA**$\mathbb{Z}$: choose $A \in \mathfrak{A}$ with $\beta A(n) = 1$ for each $n \in \mathbb{N}$).

We give two proofs that $G$ has no representation in an $\mathbb{R}^{\mathcal{Y}}$ that is both **B** and **BA**$\mathbb{Z}$, the first using Theorem 5.2, the second using Proposition 5.4 (which avoids bases, providing food for thought).

First proof. Suppose $\sigma : G \hookrightarrow \mathbb{R}^{\mathcal{Y}}$ has $\sigma G$ both **B** and **BA**$\mathbb{Z}$. By Theorem 5.2, there is $\rho : \mathbb{R}^{\mathcal{Y}} \longrightarrow \mathbb{R}^{\mathbb{N}} = C(\mathbb{N})$ with $\rho\sigma$ essential and there is a positive weak unit $u$ in $C(\mathbb{N})$ for which $ug = \rho\sigma g$ for each $g \in G$. Since all $\sigma g$ are **B** and **BA**$\mathbb{Z}$, the same is true for all $\rho\sigma g$, again by Theorem 5.2. It follows that for each $A \in \mathfrak{A}$, $u|_A$ is **B** and **BA**$\mathbb{Z}$. Recall that each $\chi_A \gamma(A)$ is strictly positive on $A$, where it is also **B** and **BA**. It follows that $u \in \Gamma$, as does $\frac{1}{u}$. Hence, $\frac{1}{u} = \beta(A_0)$ for some $A_0 \in \mathfrak{A}$, so $g_0 = \chi_{A_0} \cdot \gamma(A_0) = \chi_{A_0}\left(\frac{1}{u} + \frac{1}{u} \cdot r\right) \in G$. But, $ug_0 = \chi_{A_0}(1 + r)$ is not **BA**$\mathbb{Z}$, a contradiction.

Second proof. By Corollary 4.7, $G$ has a representation that is both **B** and **BA**$\mathbb{Z}$ if and only if there is $\sigma : G \hookrightarrow H$ for some $H$ which is **HA** and which has a unit. Suppose we have such an embedding $\sigma : G \hookrightarrow H$, where $H$ is **HA** and has unit. Without loss of generality, we may assume that $\sigma G$ is a **W**-generating subgroup of $H$. Apply Proposition 5.4 to produce $\varphi : H \twoheadrightarrow H'$, in **W** with $\varphi\sigma$ an essential embedding; by §5.1, there is $\psi : H' \hookrightarrow C(\mathbb{N})$ and a weak unit $u \in C(\mathbb{N})$ such that $ug = \psi\varphi\sigma g$ for each $g \in G$. Now, $H' \leqslant D(\mathcal{Y}H)$ is **B** and $\mathbf{BA}\mathbb{Z}$; so, too, is $\varphi\sigma G \leqslant D(\mathcal{Y}H')$. Hence, $\psi\varphi\sigma G$ is **B** and $\mathbf{BA}\mathbb{Z}$ in $C(\mathbb{N})$. Now proceed exactly as in the first proof.

## 7. Φ-generation

An embedding $\psi : A \hookrightarrow B$ in **frA** is said to be **Φ-generating** if $A \in |\mathbf{frA}|$, $B \in |\mathbf{\Phi}|$, and $B = jm(\psi A + \mathbb{Z} \cdot 1_B)$ (i.e., if $\psi A \overset{\mathbf{W}}{\leqslant} B$ in **Arch**). In this section, after presenting background information, we prove a useful result (Theorem 3) about **Φ-generating** maps.

Recall that for every Tychonoff space $\mathcal{X}$ there is a unique compact space $\beta\mathcal{X}$, called the Čech-Stone compactification of $\mathcal{X}$, in which $X$ is a dense subspace and every $f \in D(\mathcal{X})$ has an extension $f^\beta \in D(\beta\mathcal{X})$. (See, e.g., [8].)

In [16], it was shown that if $A \in |\mathbf{frA}|$ there is a Tychonoff space $\mathcal{X}_A$ and an embedding $A \ni a \longrightarrow \overline{a} \in \overline{A} \leqslant D(\mathcal{X}_A)$ such that:

1. For each $a \in A$, $\overline{a}^\beta(\beta\mathcal{X}_A \setminus \mathcal{X}_A) \subseteq \{0, \pm\infty\}$.

2. For each $x \in \mathcal{X}_A$, there is $a \in A$ with $0 < \overline{a}(x) \in \mathbb{R}$.

3. $\overline{A}$ separates points and closed sets in $\mathcal{X}_A$: given point $x$ and closed set $\mathcal{K}$ with $x \notin \mathcal{K}$, there is $(a_{x,\mathcal{K}} =)\ a \in A$ with $\overline{a}(x) \neq 0$ and $\overline{a}(\mathcal{K}) = \{0\}$.

One readily sees that $\mathcal{X}_A$ is locally compact and that the function(s) $a_{x,\mathcal{K}}$ can always be chosen from $\overline{A}^*$. This representation is unique ([17]): if $\varphi$ is an embedding of $A$ in $D(\mathcal{X})$ for some Tychonoff space $\mathcal{X}$ that satisfies conditions (like) (1), (2), (3) above, then there is a homeomorphism $\tau : \mathcal{X} \longrightarrow X_A$ such that $\overline{a}(\tau x) = \varphi a(x)$ for each $a \in A$ and $x \in \mathcal{X}$. We require the much more general statement about this uniqueness (Proposition 7.1 below), much like that for the Yosida representation enunciated before Definition 1.1.

In an $\ell$-ring $A$, $I \subseteq A$ is called an **$\ell$-ideal** if it is an $\ell$-group ideal and a ring ideal.

Proposition 7.1. — *Suppose $A \in |\mathbf{frA}|$, and that $\sigma : A \hookrightarrow D(\mathcal{Y})$ for completely regular $\mathcal{Y}$. If this embedding satisfies:*

$$y_1,\ y_2 \in \mathcal{Y} \text{ with } y_1 \neq y_2 \Longrightarrow \exists a \in A \text{ with } 0 < \sigma a(y_1) < +\infty \text{ and } \sigma a(y_2) = 0,$$

*then there is a continuous injection $\tau : \mathcal{Y} \longrightarrow X_A$ such that $\tau\mathcal{Y}$ is dense in $\mathcal{X}_A$ and for each $a \in A$, $\sigma a = \overline{a} \circ \tau$.*

*Proof.* — Theorem 3.1 of [17] states that for each point $y \in \beta\mathcal{Y}$,

$$M_y = \left\{a \in A : (\sigma a)^{\beta}(\sigma b)^{\beta}(y) = 0 \text{ for each } b \in A\right\}$$

is either a prime maximal $\ell$-ideal of $A$ or it is all of $A$ (and that every prime maximal $\ell$-ideal has this form). The condition here guarantees that $M_{y_1}$ and $M_{y_2}$ are distinct prime maximal $\ell$-ideals of $A$ whenever $y_1$ and $y_2$ are distinct points of $\mathcal{Y}$. Corollary 3.3 of [17] says that if $y \in \beta\mathcal{Y}$ with $M_y \neq A$, and if $a \in A$, then $\sigma a(y) = \widehat{M_y(a)}$, where if $M_y(a) \geqslant 0$,

$$\widehat{M_y(a)} = \inf\left\{\frac{m}{n} \in \mathbb{Q} : n\left(M_y(a)\right)^2 \leqslant m M_y(a)\right\}$$

and if $M_y(a) < 0$,

$$\widehat{M_y(a)} = -\widehat{M_y(|a|)}.$$

For each $x \in \mathcal{X}_A$, set

$$\overline{M}_x = \left\{a \in A : \overline{a}\overline{b}(x) = 0 \text{ for all } \overline{b} \in \overline{A}\right\}.$$

If $y \in \mathcal{Y}$, there is a point $\tau y \in \mathcal{X}_A$ with

$$\overline{M}_{\tau x} = \sigma(M_y).$$

The two results from [17] cited above guarantee the existence of the $\tau y$ and show that $\overline{a}(\tau y) = \sigma a(y)$ for each $y \in \mathcal{Y}$. The continuity of $\tau$ follows from the fact that the functions in $\overline{A}$ determine the topology of $\mathcal{X}_A$.; since $\sigma$ is an embedding, $\bigcap\{M_y : y \in \mathcal{Y}\} = \{0\}$, so $\tau\mathcal{Y}$ is dense in $\mathcal{X}_A$. □

$\mathcal{X}_A$ is compact if and only if $A$ contains a **superunit** (a positive element $a$ for which $ab \geqslant b$ for each $0 \leqslant b \in A$ (see [9])). Othewise, let $\mathcal{M}_A$ denote the smallest compactification of $\mathcal{X}_A$ over which all of the functions in $\overline{A}$ extend: there is exactly one point $p \in \mathcal{M}_A$ at which $\overline{a}(p) = 0$ for each $a \in A$; and $\overline{A}$, viewed as a sub-$f$-ring of $D(\mathcal{M}_A)$, separates points and closed sets in $\mathcal{M}_A$ in the sense of (3) above - with the exception that if $\mathcal{K}$ is closed and $p \notin \mathcal{K}$, then there is $a \in A$ with $\overline{a}(p) = 0$ and $\overline{a}(\mathcal{K}) \subseteq [1, +\infty]$.

In [17], Lemma 5.5 and Theorem 5.6 show the following, where $\overline{A}\ |_{\mathcal{S}}$ denotes the **frA**-object consisting of the restrictions of members of $\overline{A}$ to $\mathcal{S}$.

Proposition 7.2. — *If $\psi : A \twoheadrightarrow B$ in* $\mathbf{frA}$*, then there is a closed subspace $\mathcal{S}$ in $\mathcal{X}_A$ such that $\overline{A}\,|_{\mathcal{S}} \ni \overline{a}\,|_{\mathcal{S}} \longmapsto \psi a \in B$ is an isomorphism.*

In [10], it was shown that for each $A \in |\mathbf{frA}|$ there is a canonical embedding $u_A : A \longrightarrow uA \in |\mathbf{\Phi}|$, namely $uA = jm\,(A + \mathbb{Z} \cdot \mathbf{1})$ in $D\,(\mathcal{X}_A)$. There, attention was focused on the class $\mathbf{U}$ of $\mathbf{frA}$-morphisms that are "$u$-extendable": the $\mathbf{frA}$-morphism $A \xrightarrow{\psi} B$ is in $\mathbf{U}\,(A, B)$ if there is $u\psi \in \mathbf{\Phi}\,(uA, uB)$ with $u\psi \circ u_A = u_B \circ \psi$.

Finally, we give the main result of this section.

Theorem 7.3. — *Suppose $\psi : A \hookrightarrow B$ is $\mathbf{\Phi}$-generating.*

1. *Either*
   - (a) *$\psi \in \mathbf{U}$, so that there is an isomorphism $u\psi : uA \longrightarrow B$ in $\mathbf{\Phi}$ with $u\psi \circ u_A = \psi$, or*
   - (b) *$\psi \notin \mathbf{U}$, which occurs if and only if $B = (\psi A)^{\perp\perp} \oplus (\psi A)^{\perp}$, with $(\psi A)^{\perp} \neq \{0\}$. In this case:*
     - (i) *$(\psi A)^{\perp} \cong \mathbb{Z}$, and $A$ has a superunit $a_0$ with $\psi\,(a_0)$ not a superunit in $B$.*
     - (ii) *There is an isomorphism $\gamma : uA \longrightarrow (\psi A)^{\perp\perp}$ in $\mathbf{\Phi}$ with $\gamma \circ u_A = \psi^{\circ}$, where $\psi^{\circ}$ denotes the codomain restriction of $\psi$.*
     - (iii) *$B = (\psi A)^{\perp\perp} \oplus (\psi A)^{\perp}$ is the only proper direct sum decomposition $B = B_1 \oplus B_2$ with $\psi A \subseteq B_1$.*
2. *In either case (a) or case (b) above, there are unique mappings $\alpha : B \twoheadrightarrow uA$ in $\mathbf{\Phi}$ and $\beta : uA \hookrightarrow B$ in $\mathbf{frA}$ satisfying $\alpha \circ \psi = u_A$ and $\beta \circ u_A = \psi$.*

$$
\begin{tikzcd}
A \arrow[r, "u_A"] \arrow[d, "\psi"'] & uA \arrow[dl, dashed, bend right=10, "\beta"'] \\
B \arrow[ur, dashed, bend right=10, "\alpha"'] &
\end{tikzcd}
$$

*Proof.* — (1) (a) This is #7 of Theorem 9 in [10].

(b) This combines Corollary 5 with Proposition 5, both in [10].

(2) In case (a), $\beta = u\psi$ and $\alpha = \beta^{-1}$.

In case (b), $\beta = i_1 \circ \gamma$, where $i_1 : (\psi A)^{\perp\perp} \longrightarrow (\psi A)^{\perp\perp} \oplus (\psi A)^{\perp}$ is the injection $i_1(b) = (b, 0)$; $\alpha = \gamma^{-1} \circ \pi_1$, where $\pi_1 : (\psi A)^{\perp\perp} \oplus (\psi A)^{\perp} \longrightarrow (\psi A)^{\perp\perp}$ is the projection $\pi_1(b_1, b_2) = b_1$. $\square$

## 8. Reduced hyperarchimedean $f$-rings

Let $\mathsf{F}$ denote the forgetful functor from **frA** to **Arch**. For $A \in |\mathbf{frA}|$ we let $\langle a \rangle_{\mathsf{F}(A)}$ (respectively, $\langle a \rangle_A$) denote the ideal in the $\ell$-group $\mathsf{F}(A)$ (resp., the $\ell$-ideal in the $f$-ring $A$) generated by $a$. Note that an ideal in $\mathsf{F}(A)$ is an $\ell$-ideal in $A$ if and only if it is a ring ideal.

By Proposition 7.1 (more specifically, the comment that immediately preceeds it), any of the properties $\mathbf{B}, \mathbf{BA}, \mathbf{BA}\mathbb{Z}$ that is satisfied by some representation of $A$ in a $D(\mathcal{X})$ is satisfied by all such representations.

Lemma 3. — *Suppose $A \leqslant D(\mathcal{X})$ in $|\mathbf{frA}|$.*

1. *The following are equivalent.*
   (a) $A \subseteq C^*(\mathcal{X})$;
   (b) *every ideal in $\mathsf{F}(A)$ is an $\ell$-ideal in $A$;*
   (c) $\langle a^2 \rangle_{\mathsf{F}(A)} \subseteq \langle a \rangle_{\mathsf{F}(A)}$ *for each* $a \in A$.
2. *The following are equivalent.*
   (a) *$A$ is* $\mathbf{BA}$ *in* $D(\mathcal{X})$;
   (b) $\langle a^2 \rangle_{\mathsf{F}(A)} \supseteq \langle a \rangle_{\mathsf{F}(A)}$ *for each* $a \in A$.

*Hence, $A$ is* $\mathbf{B}$ *and* $\mathbf{BA}$ *in $D(\mathcal{X})$ iff $\langle a^2 \rangle_{\mathsf{F}(A)} = \langle a \rangle_{\mathsf{F}(A)}$ for each $a \in A$.*

*Proof.* —

1. $A \subseteq C^*(X) \Longleftrightarrow$
   $\forall b \in A \exists n_b \in \mathbb{N}$ such that $|b| \leqslant \mathbf{n}_b \Longleftrightarrow$
   $\forall a \in A,\ |ab| = |a| \cdot |b| \leqslant n_b\, |a| \Longrightarrow$
   $\langle a \rangle_{\mathsf{F}(A)}$ is an $\ell$-ideal $\forall a \in A$ (so (1) $\Longrightarrow$ (2)) $\Longrightarrow$
   $\forall a \in A,\ a^2 \in \langle a \rangle_{\mathsf{F}(A)}$ (so (2) $\Longrightarrow$ (3)) $\Longrightarrow$

$\forall a \in A \exists n_a \in \mathbb{N}$ with $a^2 \leqslant n_a |a| \Longrightarrow$

$\forall b \in B$, $|ab| = |a| \cdot |b| \leqslant n_a |b|$, since $A$ is reduced, so (3) $\Longrightarrow$ (1).

2. For $a \in A^+$ $\inf \{a(x) : x \in \operatorname{coz} a\} = 0 \Longleftrightarrow$

$a \nleq na^2$ for all $n \in \mathbb{N} \Longleftrightarrow$

$a \notin \langle a^2 \rangle_{\mathsf{F}(A)}$.

□

Theorem 8.1. — *For $A \in |\mathbf{frA}|$, the following are equivalent.*

1. $\mathsf{F}(A)$ *is* $\mathbf{HA}$.
2. *$A$ is $\mathbf{HA}$ as an $\ell$-ring: every $\ell$-ring homomorphic image is archimedean.*
3. *$A \leqslant C^*(\mathcal{X}_A)$ and every $\ell$-ideal in $A$ has the form*

$$\mathfrak{z}^{-1}(\mathcal{S}) = \left\{\overline{a} \in \overline{A} : \overline{a}(\mathcal{S}) = \{0\}\right\}$$

*for some closed subset $\mathcal{S}$ of $\mathcal{X}_A$.*

4. *Every ideal in $\mathsf{F}(A)$ has the form $\mathfrak{z}^{-1}(\mathcal{S})$, for some closed subset $\mathcal{S}$ of $\mathcal{X}_A$.*
5. *$A \leqslant D(\mathcal{X}_A)$ is $\mathbf{BA}$ and every ideal in $\mathsf{F}(A)$ is an $\ell$-ideal in $A$.*
6. *For each $a \in A$, $\langle a \rangle_{\mathsf{F}(A)} = \langle a \rangle_A = \langle a^2 \rangle_{\mathsf{F}(A)}$.*
7. *$A \leqslant C(\mathcal{X}_A)$ and* coz*a is compact for every $a \in A$.*
8. $\mathsf{F}(uA)$ *is* $\mathbf{HA}$.
9. *$A \leqslant D(\mathcal{X}_A)$ satisfies $\mathbf{HA}_1^+$ (i.e., is $\mathbf{B}$ and $\mathbf{BA}$).*
10. *Every $\mathbf{frA}$-embedding of $A$ in a $D(\mathcal{X})$ satisfies $\mathbf{HA}_1^+$ (i.e., is $\mathbf{B}$ and $\mathbf{BA}$).*
11. *$A \leqslant D(\mathcal{X}_A)$ is $\mathbf{B}$ and $\mathbf{BA}\mathbb{Z}$.*
12. *Every $\mathbf{frA}$-embedding of $A$ in a $D(\mathcal{X})$ is $\mathbf{B}$ and $\mathbf{BA}\mathbb{Z}$.*

*Proof.* — That (1) implies (2) is clear.

To see that (2) implies (3), suppose, first, that $a \in A^+ \setminus C^*(\mathcal{X}_A)$. Then there is a point $p \in \beta\mathcal{X}_A$ with $a^\beta(p) = +\infty$, from which it follows that $A/M_p$ is not archimedean. Hence, $A \subseteq C^*(\mathcal{X}_A)$. If $I$ is an $\ell$-ideal in $A$, then $A/I \in |\mathbf{frA}|$, so $A/I \cong A\mid_{\mathcal{S}}$ for some closed subset $\mathcal{S}$ of $\mathcal{X}_A$, by Proposition 7.2. Thus, $I = \mathfrak{z}^{-1}(\mathcal{S})$.

If (3) holds, then every ideal in $\mathsf{F}(A)$ is an $\ell$-ideal in $A$ by Lemma 3, since $A \leqslant C^*(\mathcal{X}_A)$, so (4) holds.

Suppose $a \in A^+ \setminus C^*(\mathcal{X}_A)$. The ideal generated in $\mathsf{F}(A)$ by $a$ is $\langle a\rangle = \{b \in A : |b| \leqslant na \text{ for some } n \in \mathbb{N}\}$, and it is clear that $a^2 \notin \langle a\rangle$. Hence, $\langle a\rangle$ and $\langle a^2\rangle$ are distinct ideals in $\mathsf{F}(A)$ having the same zero set: (4) fails. In a similar fashion, if $a \in A^+$ is not $\mathbf{BA}$, then $a \notin \langle a^2\rangle$, and (4) fails. Thus, if (4) holds, then $A \leqslant C^*(\mathcal{X}_A)$, so $\mathfrak{z}^{-1}(S)$ is a ring ideal in $A$ for every $S \subseteq \mathcal{X}_A$: (5) holds.

By Lemma 3, when (5) holds, $A \leqslant D(\mathcal{X}_A)$ is $\mathbf{B}$ and $\mathbf{BA}$; the equalities of (6) follow by that lemma.

Notice that each of (5) and (6) is equivalent to the statement: "$A \leqslant D(\mathcal{X}_A)$ is $\mathbf{B}$ and $\mathbf{BA}$", by Lemma 3. Suppose (6) holds: then $A \subseteq C^*(\mathcal{X}_A)$. It follows that $a^\beta(\beta\mathcal{X}_A \setminus \mathcal{X}_A)$ is $\{0\}$ or $\varnothing$. But $A$ is $\mathbf{BA}$ in $D(\mathcal{X}_A)$, so $\mathrm{coz}a^\beta = cl_{\beta\mathcal{X}_A}(\mathrm{coz}a)$; it follows that $cl_{\beta\mathcal{X}_A}(\mathrm{coz}a) \subseteq \mathcal{X}_A$, whence (7) holds.

To show (7) implies (8), we show that when (7) holds for $A$, it also holds for $uA$. Since (7) is readily seen to imply that $A$ satisfies $\mathbf{HA}_1^+$, this will show that $\mathsf{F}(uA)$ is $\mathbf{HA}$. Recall the construction of $uA$: the compact space $\mathcal{M}_A$ is $\mathcal{X}_A$ if the latter is compact; otherwise, it is the smallest compactification of $\mathcal{X}_A$ over which all of the functions in $A$ extend. View $A$ as a sub-$f$-ring of $D(\mathcal{M}_A)$, and first form $A_1 = A + \mathbb{Z}\cdot 1_{uA}$, then set $uA = mjA_1$. Note that

$$(a + n1_{uA})(x) = 0 \text{ if and only if } a(x) = -n, \text{ so}$$

if $n = 0$, then $a + n1_{uA} \in A$ and has compact cozero set. Otherwise,

$$\mathrm{coz}(a + n1_{uA}) = (\mathcal{M}_A \setminus \mathrm{coz}a) \cup \mathrm{coz}\left(a^2 + na\right),$$

a compact set.

Next, form $jA_1 = \left\{\bigvee_{i=1}^n f_i : f_i \in A_1, 1 \leqslant i \leqslant n; n \in \mathbb{N}\right\}$. To see that every member of $jA_1$ has compact cozero set, it suffices to show that if $f$ and $g$ each have compact cozero sets, then so does $f \vee g$. If $\mathrm{coz}f$ is compact, then $\mathrm{coz}f^+$ and $\mathrm{coz}f^-$ must be disjoint clopen subsets of $\mathrm{coz}f$ and $\mathrm{coz}fg = \mathrm{coz}f \cap \mathrm{coz}g$, so the equality

$$\mathrm{coz}(f \vee g) = \mathrm{coz}f^+ \cup \mathrm{coz}g^+ \cup \mathrm{coz}fg$$

displays $\mathrm{coz}(f \vee g)$ as the union of three compact sets.

Similarly, $\operatorname{coz}(f \wedge g) = \operatorname{coz} f^{-} \cup \operatorname{coz} g^{-} \cup \operatorname{coz} fg$, which is compact whenever $\operatorname{coz} f$, $\operatorname{coz} g$ are. It follows that every finite meet of functions having compact cozero sets has compact cozero set. But this means that

$$uA = mjA_1$$

is hyperarchimedean.

Finally, if (8) holds, then $\mathsf{F}(uA)$ has a $\mathbf{W}$-embedding in a product of copies of $\mathbb{R}$ that satisfies $\mathbf{HA}_1^+$, from which (9) follows, and (10) now follows by Proposition 7.1.

That (10) implies (1) follows by Proposition 1., so we now know that (1) through (10) are equivalent. It is evident that (12) $\Longrightarrow$ (11) $\Longrightarrow$ (9), so we complete the proof by showing that (8) $\Longrightarrow$ (12). Suppose (8): $\mathsf{F}(uA)$ is $\mathbf{HA}$. Then $\mathsf{F}(uA) \leqslant D(\mathcal{X}_{uA})$ in $\mathbf{W}$ and is $\mathbf{HA}$, so it is $\mathbf{BA}\mathbb{Z}$, by Proposition 4.5: (12) holds. $\square$

Corollary 8.2. — *Suppose $A \in |\mathbf{frA}|$.*

1. *If $A \leqslant B$ is $\mathbf{\Phi}$-generating, then $A$ is $\mathbf{HA}$ if and only if $B$ is $\mathbf{HA}$. (See Remark (e) below.)*
2. *If $A$ is $\mathbf{HA}$ and if $\epsilon_A : A \longrightarrow \epsilon A$ is an essential closure in $\mathbf{frA}$, then $jm(\epsilon_A A + \mathbb{Z} \cdot 1_{\epsilon A})$ is $\mathbf{HA}$.*

*Proof.* — (1) As noted in Theorem 7.3, it was shown in [10] that $B \cong uA$ or $B \cong uA \oplus \mathbb{Z}$.

(2) By Theorem 9 of [10], this construct “is” $uA$, which is $\mathbf{HA}$ by Theorem 8.1. (Note that such essential closures in $\mathbf{frA}$ always exist, by Bernau’s representation theorem.) $\square$

Proposition 8.3. — *Suppose $A$ is $\mathbf{HA}$. Viewing $A \leqslant D(\mathcal{X}_A)$, $A$ is an $\ell$-ideal in $uA$ if and only if it satisfies: $a \in A \Longrightarrow \chi_{\operatorname{coz} a} \in A$.*

*Proof.* — By Theorem 8.1, for each $a \in A$ there is $0 < r \in \mathbb{R}$ such that $a^2(x) \geqslant r$ for each $x \in \operatorname{coz} a$. Thus, there is $n \in \mathbb{N}$ with $na^2(x) \geqslant 1$ on $\operatorname{coz} a$. If $A$ is an $\ell$-ideal in $uA$, then $\chi_{\operatorname{coz} a} = \mathbf{1}_{uA} \wedge na^2 \in A$.

Conversely, suppose $A$ satisfies the stated condition. If $A$ contains a superunit $e$, then $\operatorname{coz} e = \mathcal{M}(uA)$, so $\chi_{\operatorname{coz} e} = 1_{uA} \in A$ and $A = uA$. Otherwise, $A \subsetneq uA$ and Theorem 4 in [10] states that $A$ is an $\ell$-ideal in $uA$ if and only if $uA = \varrho A$, where $\varrho A$ denotes traditional adjunction of a unit in ring theory. (As a group, $\varrho A = A \oplus \mathbb{Z}$, and multiplication is defined by $(a,n)\cdot(b,m) = (ab+ma+nb, nm)$.) Now, $A_1 = \mathbb{Z}\cdot 1_{uA} + A \neq \varrho A$ precisely when $n1_{uA} \in A$ for some $n \in \mathbb{N}$ which cannot occur here since $A$ contains no superunit. Since $uA$ is the sublattice of $D(\mathcal{M}(uA))$ generated by $A_1$, it suffices to show that $A_1$ is closed under the lattice operations. For $m, n \in \mathbb{Z}$ and $a, b \in A$, we have

$$\begin{aligned}
&(n1_{uA}+a)\vee(m1_{uA}+b)\\
&=(n\vee m)\chi_{\mathcal{M}(uA)\smallsetminus \mathrm{coz}(a^2+b^2)} + ((n1_{uA}+a)\vee(m1_{uA}+b))\chi_{\mathrm{coz}(a^2+b^2)}\\
&=(n\vee m)1_{uA} + [(n\chi_{\mathrm{coz}(a^2+b^2)}+a)\vee(m\chi_{\mathrm{coz}(a^2+b^2)}+b)\\
&-(n\vee m)\chi_{\mathrm{coz}(a^2+b^2)}]\\
&\in \mathbb{Z}\cdot 1_{uA} + A = A_1 \quad (\text{since } \chi_{\mathrm{coz}(a^2+b^2)} \in A).
\end{aligned}$$

Similarly for meet. $\square$

*Remarks 2.* — (a) In Theorem 8.1, the conditions "$A \leqslant C^*(\mathcal{X}_A)$" in (3) and "$A \leqslant C(\mathcal{X}_A)$" in (7) are necessary. The sub-$f$-ring $A$ of $C(\mathbb{N})$ consisting of the eventual polynomials (including the constants) satisfies the remaining conditions in both (3) and (7) but is not **HA** (here, $\mathcal{X}_A = \alpha\mathbb{N}$, the one-point compactification of $\mathbb{N}$).

(b) In Theorem 8.1, the equivalence of conditions (1) and (10) was proved by Conrad ([4], Lemma A).

(c) In [7], the result in part 2 of Corollary 8.2 is stated without the condition "$\epsilon_A \in$ **frA**"; it is the first part of their Corollary to Theorem 7. That "$\epsilon_A \in$ **frA**"; is needed is shown by using the following example: the eventual constants in $C(\mathbb{N},\mathbb{N})$ form an **frA**-object, say $B$, that is **HA**; however, $B \ni b \longmapsto fb \in C(\mathbb{N})$, where $f(x) = \frac{1}{x}$ for all $x \in \mathbb{N}$, is an essential **Arch**-embedding of $B$ in $D(\beta\mathbb{N})$ which fails to satisfy $\mathbf{HA}_1$. For an example that does not already contain an identity element, consider $B_1 = B \oplus C_K(\mathbb{N})$.

(d) Proposition 8.3 is the second statement in the Corollary to Theorem 7 of [7]. Actually, by Proposition 7.1, we could employ any **frA**-representation in a $D(\mathcal{Y})$ here.

(e) Question. Suppose $A \leqslant D(\mathcal{X}_A)$ is **BA**. Is $uA \leqslant D(\mathcal{X}_A)$ also **BA**? (Equivalently: when $A \leqslant B$ is $\mathbf{\Phi}$-generating, is it true that $B \leqslant D(\mathcal{X}_B)$ is also **BA**?) This is the same as asking whether **HA** can be replaced by **BA** in Corollary 8.2(1). "Yes" would strengthen Corollary 8.2(1).

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