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## Retractions onto the Space of Continuous Divergence-free Vector Fields

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**ABSTRACT.** — We prove that there does not exist a uniformly continuous retraction from the space of continuous vector fields onto the subspace of vector fields whose divergence vanishes in the distributional sense. We then generalise this result using the concept of  $m$ -charges, introduced by De Pauw, Moonens, and Pfeffer: on any subset  $X \subseteq \mathbb{R}^n$  satisfying a mild geometric condition, there is no uniformly continuous representation operator for  $m$ -charges in  $X$ .

**RÉSUMÉ.** — On prouve qu'il n'existe pas de rétraction uniformément continue de l'espace des champs de vecteurs continus à valeurs dans le sous-espace des champs dont la divergence est nulle au sens des distributions. Les  $m$ -charges, telles que définies par De Pauw, Moonens, et Pfeffer permettent de généraliser le résultat précédent. On prouve qu'il n'y a pas d'opérateur de représentation uniformément continu des  $m$ -charges dans une partie  $X \subseteq \mathbb{R}^n$  vérifiant une hypothèse faible.

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### 1. Introduction

According to the theory of elliptic equations, a solution of the Poisson equation  $\operatorname{div} \nabla u = f$  on  $\mathbb{R}^n$  is  $C^{1,1-n/p}$  when  $f \in L^p(\mathbb{R}^n)$  and  $p > n$ . However in the critical case  $p = n$ , such a solution  $u$  need not be  $C^1$ . John Nirenberg gave the following counterexample ( $n \geq 2$ )

$$u(x) := \varphi(x)x_1 |\ln \|x\||^\alpha, \quad \alpha \in \left(0, \frac{n-1}{n}\right),$$

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where  $\varphi$  is a compactly supported smooth function such that  $\varphi(0) \neq 0$ . One checks that  $\Delta u \in L^n(\mathbb{R}^n)$ , yet  $\nabla u$  is not even locally bounded. The papers [8] and [9] deal with characterizing distributions  $F$  such that the divergence equation  $\operatorname{div} v = F$  admits a continuous solution. The paper [9] focuses on solutions vanishing at infinity and we will first work in this setting. Let  $1^*$  be the Sobolev conjugate exponent  $1^* := n/(n-1)$  and  $BV_{1^*}(\mathbb{R}^n)$  be the space of functions  $\varphi \in L^{1^*}(\mathbb{R}^n)$  whose gradient  $\nabla\varphi$  is a finite vector valued measure, *i.e.*

$$\sup \left\{ \int_{\mathbb{R}^n} \varphi \operatorname{div} v : v \in C_c^1(\mathbb{R}^n; \mathbb{R}^n), \|v\|_\infty \leq 1 \right\} < \infty.$$

The following holds [9, Theorem 6.1 and Section 7]:

**THEOREM 1.1.** — *Let  $F$  be a linear functional on  $BV_{1^*}(\mathbb{R}^n)$ . There is a continuous vector field  $v : \mathbb{R}^n \rightarrow \mathbb{R}^n$  vanishing at infinity such that for all  $\varphi \in BV_{1^*}(\mathbb{R}^n)$*

$$F(\varphi) = - \int_{\mathbb{R}^n} v \cdot d(\nabla\varphi)$$

*if and only if  $F(\varphi_i) \rightarrow 0$  whenever  $(\varphi_i)$  is a sequence weakly converging to 0 in  $L^{1^*}(\mathbb{R}^n)$  satisfying  $\sup_i \|\nabla\varphi_i\| < \infty$  — where  $\|\nabla\varphi\|$  denotes the total variation of a function  $\varphi \in BV_{1^*}(\mathbb{R}^n)$ .*

A functional  $F$  satisfying the hypothesis of Theorem 1.1 is called a *charge vanishing at infinity*. A function  $f \in L^n(\mathbb{R}^n)$  is a charge vanishing at infinity, and so is the distributional divergence of a continuous vector field vanishing at infinity (see [9, Propositions 3.4 and 3.5]). Let us rephrase Theorem 1.1: there is continuous surjective linear operator

$$\operatorname{div} : C_0(\mathbb{R}^n, \mathbb{R}^n) \rightarrow CH_0(\mathbb{R}^n),$$

where  $CH_0(\mathbb{R}^n)$  stands for the space of charges vanishing at infinity. Contrary to the Poisson equation, the nonhomogeneous equation  $\operatorname{div} v = F$  lacks of uniqueness properties.

It is necessary to investigate the functional analytical properties of  $\ker \operatorname{div}$  in order to study the equation  $\operatorname{div} v = F$ . For example, it is not possible to select a solution  $v$  depending on  $F$  in a continuous linear manner (equivalently,  $\ker \operatorname{div}$  is not complemented in  $C_0(\mathbb{R}^n, \mathbb{R}^n)$ ). Indeed, it is proved by considering a dual problem. The adjoint mapping of  $\operatorname{div}$  maps a space of  $BV$  functions into the space of vector valued measures  $C_0(\mathbb{R}^n, \mathbb{R}^n)^*$ ; thus we are led to consider the subspace  $\{\nabla\varphi : \varphi \in BV_{1^*}(\mathbb{R}^n)\} \subseteq C_0(\mathbb{R}^n, \mathbb{R}^n)^*$ . In dimension one it is easy to find, in a measure theoretic way, a  $BV$  function whose derivative is a given signed measure with finite total variation.

However in larger dimensions, there are restrictions for a measure to be the gradient of a  $BV$  function; it must be for example absolutely continuous with respect to the integral-geometric measure  $\mathcal{J}^{n-1}$  (see [2] for an extensive study of  $BV$  functions).

The theory of Lipschitz free spaces together with the separability of  $CH_0(\mathbb{R}^n)$  allows us to improve on this result: there is no Lipschitz continuous right inverse to  $\operatorname{div}$  (see [7, Corollary 3.2]). We are therefore interested in the best regularity for such an inverse. It is a general fact that a surjective linear map between Banach spaces has a continuous right inverse. However our main Theorem 3.2 proves that a right inverse to  $\operatorname{div}$  cannot be uniformly continuous. In fact what we prove is slightly stronger:  $\ker \operatorname{div}$  is not a uniform retract of  $C_0(\mathbb{R}^n, \mathbb{R}^n)$ . In Section 4 we prove a related result on the representation of charges with positive codimension, as developed in [8].

In the sequel  $n$  will be a fixed integer larger than 2.  $I$  is the unit segment  $[0, 1]$ . We denote by  $C_0(\mathbb{R}^n, \mathbb{R}^n)$  the space of continuous vector fields vanishing at infinity, *i.e.* continuous maps  $v : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that for any  $\varepsilon > 0$  there exists a compact  $K \subseteq \mathbb{R}^n$  with  $|v| < \varepsilon$  outside  $K$ . We endow  $C_0(\mathbb{R}^n, \mathbb{R}^n)$  with the supremum norm  $\|\cdot\|_\infty$ . A locally integrable map  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to have bounded variation if

$$\|\nabla\varphi\| := \sup \left\{ \int_{\mathbb{R}^n} \varphi \operatorname{div} g : g \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n), \|g\|_\infty \leq 1 \right\} < \infty.$$

If  $X$  is a subset of  $\mathbb{R}^n$  we denote by  $\mathbf{BV}(X)$  the space of maps  $\varphi \in L^1(\mathbb{R}^n)$  of bounded variation and compact support in  $X$ . We endow  $\mathbf{BV}(X)$  with the norm  $\|\varphi\|_{\mathbf{BV}(X)} := \|\varphi\|_{L^1(\mathbb{R}^n)} + \|\nabla\varphi\|$ .

We let  $BV_{1^*}(\mathbb{R}^n)$  be the space of maps  $\varphi \in L^{1^*}(\mathbb{R}^n)$  of bounded variation. We norm it by  $\|\varphi\|_{BV_{1^*}(\mathbb{R}^n)} := \|\nabla\varphi\|$ . Indeed  $\|\cdot\|_{BV_{1^*}(\mathbb{R}^n)}$  is a norm by Gagliardo-Nirenberg-Sobolev inequality in  $BV_{1^*}(\mathbb{R}^n)$  (see [9, Proposition 2.5]).

Note that  $\mathbf{BV}(X)$  is a subset of  $BV_{1^*}(\mathbb{R}^n)$  and  $\|\cdot\|_{BV_{1^*}(\mathbb{R}^n)}$  induces a norm equivalent to  $\|\cdot\|_{\mathbf{BV}(X)}$  in  $\mathbf{BV}(X)$  whenever  $X$  is a Lebesgue measurable set with finite measure and the characteristic function  $\chi_X$  has bounded variation. However  $\|\cdot\|_{\mathbf{BV}(X)}$  has the geometrical meaning of a normal mass and will be useful in Section 4 where we define charges as linear functionals on a space of normal currents.

We let  $CH_0(\mathbb{R}^n)$  be the space of charges vanishing at infinity.  $CH_0(\mathbb{R}^n)$  can be made into a Banach space by norming it with

$$\|F\|_{CH_0(\mathbb{R}^n)} := \sup \{F(\varphi) : \varphi \in BV_1^*(\mathbb{R}^n), \|\nabla\varphi\| \leq 1\}.$$

The operator  $\operatorname{div} : C_0(\mathbb{R}^n, \mathbb{R}^n) \rightarrow CH_0(\mathbb{R}^n)$  is defined by  $(\operatorname{div} v)(\varphi) := -\int_{\mathbb{R}^n} v \cdot d(\nabla\varphi)$ . A subspace  $Y$  of a Banach space  $X$  is said to be complemented (in  $X$ ) whenever there exists a bounded linear retraction onto  $Y$ . Recall that an exact sequence of Banach spaces is a diagram

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0,$$

involving bounded linear maps  $f, g$  such that  $f$  is an embedding into  $Y$ ,  $g$  is onto and  $\operatorname{im} f = \ker g$ . Such a sequence is said to split linearly if there exists a bounded linear right inverse to  $g$ , or equivalently, a bounded linear left inverse to  $f$ .

## 2. Preliminary results

An  $\mathcal{L}^1$  space is a Banach space  $X$  for which there exists  $\lambda > 1$  such that for all finite-dimensional subspaces  $E \subseteq X$ , there exist a finite-dimensional  $F \subseteq X$  containing  $E$  and an isomorphism  $T : F \rightarrow \ell_{\dim F}^1$  (where  $\ell_{\dim F}^1$  denotes the space  $\mathbb{R}^{\dim F}$  with the  $\ell^1$  norm  $\|(x_k)\|_1 := \sum |x_k|$ ) such that  $\|T\| \cdot \|T^{-1}\| < \lambda$ . For any measure space  $(X, \mathfrak{B}, \mu)$ , the space  $L^1(X, \mathfrak{B}, \mu)$  is an  $\mathcal{L}^1$  space (see [3, Theorem F.2]). The following proposition is well-known, but we could not find a reference. We thus give an easy proof.

**PROPOSITION 2.1.** — *Let  $K$  be a compact Hausdorff topological space.  $C(K)^*$  is an  $\mathcal{L}^1$  space.*

*Proof.* — Recall that  $C(K)^*$  is the space of signed Radon measures on  $K$  normed by the total variation. Fix  $\lambda > 1$ . Let  $E \subseteq C(K)^*$  be a finite dimensional subspace generated by a family  $\mu_1, \dots, \mu_n$ . Each Radon measure  $\mu_i$  ( $1 \leq i \leq n$ ) is absolutely continuous with respect to  $\mu := |\mu_1| + \dots + |\mu_n|$  ( $|\mu_i|$  is the total variation measure). The embedding  $\iota : L^1(K, \mu) \rightarrow C(K)^*$ ,  $f \mapsto f\mu$  is an isometry. We denote by  $f_i \in L^1(K, \mu)$  the Radon-Nikodým derivative of  $\mu_i$  with respect to  $\mu$ . Since  $L^1(K, \mu)$  is an  $\mathcal{L}^1$  space, there exist a finite dimensional subspace  $F \subseteq L^1(K, \mu)$  which contains  $f_i$  ( $1 \leq i \leq n$ ) and an isomorphism  $T : F \rightarrow \ell_{\dim F}^1$ , such that  $\|T\| \cdot \|T^{-1}\| < \lambda$  — where  $\|\cdot\|$  stands for the standard operator norm.  $T\iota^{-1}$  is an isomorphism between  $\iota(F) \supseteq E$  and  $\ell_{\dim F}^1$  such that  $\|T\iota^{-1}\| \cdot \|(T\iota^{-1})^{-1}\| < \lambda$ . Hence  $C(K)^*$  is an  $\mathcal{L}^1$  space.  $\square$

In fact we can prove that  $C(K)^*$  is actually an  $L^1$  space, using Kakutani's theory of abstract Lebesgue spaces or the fact that the parameter  $\lambda$  in the

definition of an  $\mathcal{L}^1$  space is here arbitrary, but Proposition 2.1 is enough for our purposes.

DEFINITION 2.2. — *Let  $X, Y$  be Banach spaces, and  $1 \leq p < \infty$ . A linear operator  $T : X \rightarrow Y$  is called  $p$ -absolutely summing if there exists a constant  $C \geq 0$  such that for any choices of  $(x_k)_{k=1}^n$  in  $X$  we have*

$$\left( \sum_{k=1}^n \|T(x_k)\|^p \right)^{1/p} \leq C \sup \left\{ \left( \sum_{k=1}^n |\langle \xi, x_k \rangle|^p \right)^{1/p} : \xi \in X^*, \|\xi\| \leq 1 \right\}$$

The least such constant  $C$  is denoted  $\pi_p(T)$  and is called the  $p$ -absolutely summing norm of  $T$ .

We now gather useful facts concerning absolutely summing operators and  $\mathcal{L}^1$  spaces. Points 1 to 3 are easy facts, whereas Grothendieck theorem is nontrivial and will be the key element in the proof of Theorem 3.1 below. We refer to [10, III.F] for proofs.

PROPOSITION 2.3. —

1. Let  $U : W \rightarrow X$ ,  $T : X \rightarrow Y$ ,  $V : Y \rightarrow Z$  be bounded operators between Banach spaces. If  $T$  is  $p$ -absolutely summing, then  $VTU$  is  $p$ -absolutely summing and  $\pi_p(VTU) \leq \|V\| \pi_p(T) \|U\|$ .
2. Suppose  $1 \leq r < p < \infty$ . Let  $T$  be an  $r$ -absolutely summing operator between Banach spaces  $X$  and  $Y$ . Then  $T$  is  $p$ -absolutely summing and  $\pi_p(T) \leq \pi_r(T)$ .
3. Let  $H$  be a Hilbert space, and let  $T : H \rightarrow H$  be a bounded operator.  $T$  is a Hilbert-Schmidt operator if and only if  $T$  is 2-absolutely summing (for the definition of a Hilbert-Schmidt operator, see [3, Appendix J]).
4. (Grothendieck theorem) Let  $X$  be an  $\mathcal{L}^1$  space,  $H$  a Hilbert space. Every bounded operator  $T : X \rightarrow H$  is 1-absolutely summing.

### 3. Main theorem

We will need this following theorem, which is of independent interest. Let us recall that  $n \geq 2$ .

THEOREM 3.1. — *Neither  $\mathbf{BV}(I^n)$  nor  $BV_{1^*}(\mathbb{R}^n)$  is complemented in an  $\mathcal{L}^1$  space. In particular none of these is itself an  $\mathcal{L}^1$  space.*

*Proof.* — The following proof is inspired from [4, second proof of Proposition 2]. Let  $L$  be an  $\mathcal{L}^1$  space containing  $\mathbf{BV}(I^n)$ . Suppose there exists a continuous left inverse  $\pi$  to the inclusion map  $\iota : \mathbf{BV}(I^n) \rightarrow L$ . Let  $f : L^2((0, 1)^n) \rightarrow W^{1,1}((0, 1)^n)$  be the linear map such that for any multi-index  $\alpha \in \mathbb{Z}^n$ ,

$$f : \cos(2\pi\alpha \cdot x) \mapsto \frac{\cos(2\pi\alpha \cdot x)}{\sqrt{1 + |\alpha|^2}}, \quad \sin(2\pi\alpha \cdot x) \mapsto \frac{\sin(2\pi\alpha \cdot x)}{\sqrt{1 + |\alpha|^2}}.$$

$f$  is easily seen to be continuous since it factors through  $W^{1,2}((0, 1)^n)$ :

$$f : L^2((0, 1)^n) \longrightarrow W^{1,2}((0, 1)^n) \xrightarrow{\text{incl.}} W^{1,1}((0, 1)^n)$$

Let  $g : L^{1^*}((0, 1)^n) \rightarrow L^2((0, 1)^n)$  be the bounded (by Sobolev embedding theorem) multiplier operator

$$g : \cos(2\pi\alpha \cdot x) \mapsto \frac{\cos(2\pi\alpha \cdot x)}{(1 + |\alpha|)^{n/2-1}}, \quad \sin(2\pi\alpha \cdot x) \mapsto \frac{\sin(2\pi\alpha \cdot x)}{(1 + |\alpha|)^{n/2-1}},$$

for every  $\alpha \in \mathbb{Z}^n$ . We have the following commutative diagram :

$$\begin{array}{ccccc} & & \Psi & & \\ & \searrow & \text{---} & \swarrow & \\ L^2((0, 1)^n) & & L & & L^2((0, 1)^n) \\ \downarrow f & & \updownarrow \iota \pi & & \uparrow g \\ W^{1,1}((0, 1)^n) & \xrightarrow{\iota_1} & \mathbf{BV}(I^n) & \xrightarrow{\iota_2} & L^{1^*}((0, 1)^n) \end{array}$$

$\iota_1$  is the inclusion map  $W^{1,1}((0, 1)^n) \subseteq \mathbf{BV}(I^n)$ .  $\iota_2$  is the Sobolev inclusion map  $\mathbf{BV}(I^n) \subseteq L^{1^*}((0, 1)^n)$ , and  $\Psi := g\iota_2\iota_1f$ . The boundedness of the inclusion map  $\iota_1$  is guaranteed by the boundedness of the extension by zero operator for  $\mathbf{BV}$  maps (see [5, Section 5.4, Theorem 1]). The linear map  $g\iota_2\pi$  maps an  $\mathcal{L}^1$  space into a Hilbert space; we infer from Grothendieck theorem that it is 1-absolutely summing. So is  $\Psi : L^2((0, 1)^n) \rightarrow L^2((0, 1)^n)$  by virtue of Proposition 2.3 (1). Thus it is 2-absolutely summing according to Proposition 2.3 (2). Then by Proposition 2.3 (3), it is a Hilbert-Schmidt operator. However we have

$$\begin{aligned} \|\Psi\|_{\text{HS}}^2 &= \|\Psi(1)\|^2 + \frac{1}{2^n} \sum_{\alpha \in \mathbb{Z}^n \setminus \{0\}} \left( \|\Psi(\cos(2\pi\alpha \cdot x))\|_{L^2}^2 + \|\Psi(\sin(2\pi\alpha \cdot x))\|_{L^2}^2 \right) \\ &= 1 + \frac{1}{2^{n-1}} \sum_{\alpha \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{(1 + |\alpha|^2)(1 + |\alpha|)^{n-2}} = +\infty, \end{aligned}$$

and this is contradictory.

Now let us remark that  $BV_{1^*}(\mathbb{R}^n)$  contains  $\mathbf{BV}(I^n)$  and the norm induced by  $\|\cdot\|_{BV_{1^*}(\mathbb{R}^n)}$  is equivalent to  $\|\cdot\|_{\mathbf{BV}(I^n)}$  (by Gagliardo-Nirenberg-Sobolev inequality). Let  $\chi_{I^n}$  be the characteristic map of  $I^n$ .  $f \mapsto \chi_{I^n} f$  is a linear retraction from  $BV_{1^*}(\mathbb{R}^n)$  to  $\mathbf{BV}(I^n)$ , and it is bounded (see [5, Section 5.4, Theorem 1] to justify this step). Thus  $\mathbf{BV}(I^n)$  is complemented in  $BV_{1^*}(\mathbb{R}^n)$  and by the above part,  $BV_{1^*}(\mathbb{R}^n)$  cannot be complemented in an  $\mathcal{L}^1$  space.  $\square$

This result is false if  $n = 1$ . In fact, the map  $f \mapsto (f(0), f')$  defines an isomorphism between  $BV[0, 1]$  and  $\mathbb{R} \times C[0, 1]^*$ .  $\mathbb{R} \times C[0, 1]^*$  is a product of  $\mathcal{L}^1$  spaces, and thus is an  $\mathcal{L}^1$  space. We can now prove the main theorem of this paper. If there exists a uniformly continuous right inverse  $s$  to  $\operatorname{div}$ , then  $v \mapsto v - s(\operatorname{div} v)$  is a uniformly continuous retraction of  $C_0(\mathbb{R}^n, \mathbb{R}^n)$  onto its subspace  $\ker \operatorname{div}$ . Such a retraction is forbidden by the following result:

**THEOREM 3.2.** — *There is no uniformly continuous retraction from  $C_0(\mathbb{R}^n, \mathbb{R}^n)$  onto  $\ker \operatorname{div}$ .*

Note that if we replace “uniformly continuous” with “bounded linear”, this theorem is an easy consequence of the preceding result. Indeed, a bounded linear retraction  $r : C_0(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \ker \operatorname{div}$  would split the exact sequence

$$0 \longrightarrow \ker \operatorname{div} \longrightarrow C_0(\mathbb{R}^n, \mathbb{R}^n) \xrightarrow{\operatorname{div}} CH_0(\mathbb{R}^n) \longrightarrow 0.$$

Therefore the dual sequence

$$0 \longrightarrow CH_0(\mathbb{R}^n)^* \longrightarrow C_0(\mathbb{R}^n, \mathbb{R}^n)^* \xrightarrow{\operatorname{div}} (\ker \operatorname{div})^* \longrightarrow 0$$

would also split. The key observation is that  $CH_0(\mathbb{R}^n)^*$  is in fact isomorphic to  $BV_{1^*}(\mathbb{R}^n)$  (see [9, Remark 5.2]).  $BV_{1^*}(\mathbb{R}^n)$  is complemented in the  $\mathcal{L}^1$  space  $C_0(\mathbb{R}^n, \mathbb{R}^n)^*$  (see the following proof), and this is contradictory.

Now, if we suppose there is a uniformly continuous retraction  $r$ , we wish to linearize  $r$  to obtain a contradiction. However, we will not try to differentiate  $r$ ; even differentiation (at one point) of Lipschitz maps between most non reflexive Banach spaces is difficult to obtain, and false in general. Instead we will use the method of invariant means, which applies to general uniformly continuous functions. An invariant mean on  $C_0(\mathbb{R}^n, \mathbb{R}^n)$  is a functional  $M : L^\infty(C_0(\mathbb{R}^n, \mathbb{R}^n)) \rightarrow \mathbb{R}$  which integrates functions with respect to a finitely additive measure (see [3, Appendix C]). If  $X$  is a dual space, one can construct a vector-valued invariant mean  $\hat{M} : L^\infty(C_0(\mathbb{R}^n, \mathbb{R}^n); X) \rightarrow X$  in the same way the Pettis integral is defined from the Lebesgue integral.



However  $\ker \operatorname{div}$  is not a dual space (otherwise, since it is separable, it would be a Radon-Nikodým space by Dunford-Pettis theorem, but the space  $c_0$  of convergent sequences is embeddable in  $\ker \operatorname{div}$ ). This leads to a small complication in the proof below: we will embed  $\ker \operatorname{div}$  into its bidual  $(\ker \operatorname{div})^{**}$ .

*Proof.* — For sake of brevity, we denote  $\ker \operatorname{div}$  by  $Z$ . One argues by contradiction, assuming the existence of a uniformly continuous retraction  $r$ . We refer to [3, Theorem 7.2] to infer the existence of a bounded linear map  $S : C_0(\mathbb{R}^n, \mathbb{R}^n) \rightarrow Z^{**}$  whose restriction to  $Z$  is the evaluation map  $\operatorname{ev}_Z : Z \rightarrow Z^{**}$

$$\operatorname{ev}_Z(z)(z^*) := z^*(z).$$

We let  $\operatorname{ev}_{Z^*}$  be the evaluation map  $Z^* \rightarrow Z^{***}$  and  $\iota : Z \rightarrow C_0(\mathbb{R}^n, \mathbb{R}^n)$  be the inclusion map, and consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & CH_0(\mathbb{R}^n)^* & \xrightarrow{\operatorname{div}^*} & C_0(\mathbb{R}^n, \mathbb{R}^n)^* & \xrightarrow{\iota^*} & Z^* \longrightarrow 0 \\ & & & & & & \downarrow \operatorname{ev}_{Z^*} \\ & & & & & & Z^{***} \\ & & & & S^* \swarrow & & \uparrow \operatorname{ev}_Z \end{array}$$

The map  $S^* \circ \operatorname{ev}_{Z^*}$  is a right inverse to  $\iota^*$ , because one easily checks that  $\operatorname{ev}_{Z^*} \circ \operatorname{ev}_Z = \operatorname{id}_{Z^*}$ . Therefore, the exact sequence splits linearly, and  $CH_0(\mathbb{R}^n)^*$  is complemented in  $C_0(\mathbb{R}^n, \mathbb{R}^n)^*$ . Since  $CH_0(\mathbb{R}^n)^*$  is isomorphic to  $BV_{1^*}(\mathbb{R}^n)$ , it remains to be proven that  $C_0(\mathbb{R}^n, \mathbb{R}^n)$  is an  $\mathcal{L}^1$  space.

To do so, remark that the  $n$ -sphere  $\mathbb{S}^n$  is the Alexandroff compactification of  $\mathbb{R}^n$ , so  $C_0(\mathbb{R}^n, \mathbb{R}^n)$  is isomorphic to a (closed) hyperplane of  $C(\mathbb{S}^n, \mathbb{R}^n)$ . This space is isomorphic to its hyperplanes (it is an immediate consequence of [1, Proposition 4.4.1]), thus

$$C_0(\mathbb{R}^n, \mathbb{R}^n)^* \simeq C(\mathbb{S}^n, \mathbb{R}^n)^* \simeq (C(\mathbb{S}^n)^n)^* \simeq (C(\mathbb{S}^n)^*)^n,$$

and  $(C(\mathbb{S}^n)^*)^n$  is an  $\mathcal{L}^1$  space as a finite product of  $\mathcal{L}^1$  spaces.  $\square$

*Remark.* — Let us mention what follows: a corollary of Michael’s selection theorem (see [3, Chapter 1, Section 3]) asserts that each surjective linear map between Banach spaces has a continuous right inverse, or equivalently, each closed subspace of a Banach space is a continuous positively homogeneous retract. But the proof of Michael’s theorem does not provide us with a concrete continuous retraction. One also obtains the existence of a non continuous linear retract  $C_0(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \ker \operatorname{div}$  by elementary linear algebra.

#### 4. Adaptation to charges of positive codimension

We mainly use the notations of [8], which explains in greater details the basic properties of charges in middle dimension. For  $m \geq 0$ , we denote by  $\wedge_m \mathbb{R}^n$  and  $\wedge^m \mathbb{R}^n$  the spaces of  $m$ -vectors and  $m$ -covectors in  $\mathbb{R}^n$ . If  $m > n$  these spaces are null spaces and we let the same be true by definition if  $m < 0$ . The inner product in  $\mathbb{R}^n$  induces canonical inner products in  $\wedge_m \mathbb{R}^n$  and  $\wedge^m \mathbb{R}^n$ , and we still denote by  $|\cdot|$  the associated Euclidean norm. We denote by  $\mathcal{G}^m(\mathbb{R}^n)$  the space of smooth compactly supported  $m$ -forms on  $\mathbb{R}^n$  and we give it the usual locally convex topology of test functions described in [6, 4.1.1]. The space of  $m$ -currents  $\mathcal{G}^m(\mathbb{R}^n)$  is defined to be the dual space  $\mathcal{G}_m(\mathbb{R}^n)^*$  and we endow it with the weak\* topology. The support of  $T \in \mathcal{G}_m(\mathbb{R}^n)$ ,  $\text{supp } T$ , is defined to be the smallest closed subset  $X \subseteq \mathbb{R}^n$  for which  $T(\omega) = 0$  whenever  $\text{supp } \omega \subseteq \mathbb{R}^n \setminus X$ . The extended real number<sup>1</sup>

$$\mathbf{M}(T) := \sup\{T(\omega) : \omega \in \mathcal{G}^m(\mathbb{R}^n), \|\omega\|_\infty \leq 1\},$$

where  $\|\omega\|_\infty := \sup\{|\omega(x)| : x \in \mathbb{R}^n\}$ , is the mass of the  $m$ -current  $T$ . For any subset  $X \subseteq \mathbb{R}^n$ , we introduce the linear space

$$\mathbf{M}_m(X) := \{T \in \mathcal{G}_m(\mathbb{R}^n) : \mathbf{M}(T) < \infty, \text{supp } T \subseteq X\}$$

normed by  $\mathbf{M}$ .  $d : \mathcal{G}^{m-1}(\mathbb{R}^n) \rightarrow \mathcal{G}^m(\mathbb{R}^n)$  denotes the exterior differentiation of smooth forms, and the boundary map  $\partial : \mathcal{G}_m(\mathbb{R}^n) \rightarrow \mathcal{G}_{m-1}(\mathbb{R}^n)$  is defined by  $\partial T(\omega) := T(d\omega)$ . For any  $m$ -current the extended real number  $\mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}(\partial T)$  is its normal mass. We similarly define the space  $\mathbf{N}_m(X)$  of  $m$ -currents of finite normal mass, compactly supported in  $X$ . In top dimension a normal current  $T \in \mathbf{N}_n(X)$  is represented by a function  $f \in \mathbf{BV}(X)$

$$T(\omega dx_1 \wedge \cdots \wedge dx_n) = \int_{\mathbb{R}^n} \omega f.$$

For a normal current  $T \in \mathbf{N}_m(\mathbb{R}^n)$ , we define its flat norm

$$\begin{aligned} \mathbf{F}(T) &:= \inf\{\mathbf{M}(S) + \mathbf{M}(T - \partial S) : S \in \mathbf{N}_{m+1}(\mathbb{R}^n)\} \\ &= \sup\{T(\omega) : \omega \in \mathcal{G}^m(\mathbb{R}^n), \|\omega\|_\infty, \|\partial\omega\|_\infty \leq 1\}. \end{aligned}$$

A  $m$ -charge  $\alpha$  on  $X$  is a linear functional on  $\mathbf{N}_m(X)$  such that for every  $\varepsilon > 0$  and every bounded set  $K \subseteq X$ , there exists  $\theta > 0$  such that

$$\alpha(T) \leq \theta \mathbf{F}(T) + \varepsilon \mathbf{N}(T)$$

for every normal current  $T \in \mathbf{N}_m(K)$ . We denote by  $\mathbf{CH}^m(X)$  the linear space of all  $m$ -charges on  $X$  and give it the Fréchet topology induced by

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<sup>(1)</sup> Note that we do not use Federer's notation here.

the family of seminorms

$$\|\alpha\|_K := \sup\{\alpha(T) : T \in \mathbf{N}(K), \mathbf{N}(T) \leq 1\},$$

indexed by all bounded subsets  $K \subseteq X$ .

If  $Y \subseteq \mathbb{R}^p$  and  $f : X \rightarrow Y$  is a Lipschitz map, the pushforward map  $f_{\#} : \mathbf{N}_m(X) \rightarrow \mathbf{N}_m(Y)$  induces a continuous pullback map  $f^{\#} : \mathbf{CH}^m(Y) \rightarrow \mathbf{CH}^m(X)$ . We denote by  $C_u(X, \wedge^m \mathbb{R}^n)$  the linear space of all  $m$ -forms which are uniformly continuous on any bounded subset of  $X$ , and give it the Fréchet topology of uniform convergence on bounded subsets. There is a linear continuous map  $C_u(X, \wedge^m \mathbb{R}^n) \rightarrow \mathbf{CH}^m(X)$  (see [8, Example 5.3]), which allows us to consider each uniformly continuous form on bounded subsets as a charge. There is actually a unique locally convex topology  $\mathfrak{F}_{m,X}$  such that  $\mathbf{CH}^m(X)$  is precisely the set of  $\mathfrak{F}_{m,X}$  continuous linear functionals, but we will not use this fact except indirectly in Proposition 4.2. We denote by  $\mathcal{E}_{m,X} := C_u(X, \wedge^m \mathbb{R}^n) \times C_u(X, \wedge^{m-1} \mathbb{R}^n)$  with the product Fréchet topology. We let  $\Theta_{m,X} : \mathcal{E}_{m,X} \rightarrow \mathbf{CH}^m(X)$  be the map  $(\omega, \eta) \mapsto \omega + d\eta$ . We now state the representation theorem [8, Theorem 6.1], which is a generalization of Theorem 1.1

**THEOREM 4.1 (DE PAUW, MOONENS, PFEFFER).** — *For any  $\alpha \in \mathbf{CH}^m(X)$ , there exists  $(\omega, \eta) \in \mathcal{E}_{m,X}$  such that  $\alpha = \omega + d\eta$ , i.e the map  $\Theta_{m,X}$  is onto.*

In case  $X$  is a bounded set,  $\mathbf{CH}^m(X)$  is a Banach space normed by

$$\|\alpha\|_{\mathbf{CH}^m(X)} := \sup\{\alpha(T) : T \in \mathbf{N}_m(X), \mathbf{N}(T) \leq 1\}.$$

We now state a duality proposition between the dual Banach space  $\mathbf{CH}^m(X)^*$  and the space  $\mathbf{N}_m(X)$  normed by  $\mathbf{N}$ .

**PROPOSITION 4.2.** — *Let  $X \subset \mathbb{R}^n$  be compact. Then the linear map  $\Upsilon : \mathbf{N}_m(X) \rightarrow \mathbf{CH}^m(X)^*$  defined by  $\Upsilon(T)(\alpha) := \alpha(T)$  is an isomorphism of Banach spaces.*

*Proof.* — It is straightforward to check that  $\Upsilon$  is a continuous linear mapping. From the closedness of  $X$  and [8, Theorem 3.16], we deduce that  $\Upsilon$  is a bijection. The proposition follows then from the open mapping theorem.  $\square$

We also need the following elementary lemma (see [6, 4.1.8]), which we state for the sake of clarity.

LEMMA 4.3. — Let  $T \in \mathcal{D}_{m_1}(\mathbb{R}^{n_1})$  and  $S \in \mathcal{D}_{m_2}(\mathbb{R}^{n_2})$ . Let  $p : \mathbb{R}^{n_1+n_2} \rightarrow \mathbb{R}^{n_2}$  be the projection onto the last  $n_2$  coordinates. Suppose that  $\text{supp } T$  is compact. Then

$$p_{\#}(T \times S) = \begin{cases} T(1)S & \text{whenever } m_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The representation theorem 4.1 shows that the map  $\Theta_{m,X}$  is onto. Representing charges deals with picking one right inverse to  $\Theta_{m,X}$ . Michael's selection theorem is also true for Fréchet spaces (the proof of [3, Theorem 1.1] remains valid in the case of Fréchet spaces), therefore the existence of a continuous right inverse to  $\Theta_{m,X}$  is guaranteed. The following theorem proves that one cannot however require such an inverse to be uniformly continuous.

THEOREM 4.4. — Assume  $n \geq 2$  and  $2 \leq m \leq n$ . Let  $X$  a subset of  $\mathbb{R}^n$  which contains a bilipschitz copy of the  $m$ -cube. The linear map

$$\Theta_{m,X} : \mathcal{E}_{m,X} \longrightarrow \mathbf{CH}^m(X)$$

has no uniformly continuous right inverse.

*Proof.* — Suppose that  $\Theta_{m,X}$  has a uniformly continuous left inverse  $S$ . The map

$$(\omega, \eta) \mapsto (\omega, \eta) - S(\omega + d\eta)$$

is a uniformly continuous retraction of  $\mathcal{E}_{m,X}$  onto  $\ker \Theta_{m,X}$ . Proceeding as in Theorem 3.2, one proves that the dual sequence

$$0 \longrightarrow \mathbf{CH}^m(X)^* \xrightarrow{\Theta_{m,X}^*} \mathcal{E}_{m,X}^* \longrightarrow (\ker \Theta_{m,X})^* \longrightarrow 0$$

splits linearly. We now prove Theorem 4.4 in three steps.

*Step 1.* — We prove the theorem in top dimension  $m = n$ , for  $X = I^n$ .

Suppose there exists a continuous right inverse for  $\Theta_{n,I^n}$ , from  $\mathbf{CH}^n(I^n)$  to  $\mathcal{E}_{n,I^n}$ . We dualize it and obtain a continuous left inverse

$$\sigma : \mathbf{M}_n(I^n) \times \mathbf{M}_{n-1}(I^n) \rightarrow \mathbf{BV}(I^n)$$

for the isometry  $\Theta_{n,I^n}^*$ . Note that  $\mathbf{M}_{n-1}(I^n)$  is an  $\mathcal{L}^1$  space. Indeed,

$$\mathbf{M}_{n-1}(I^n) \cong C(I^n, \wedge^{n-1} \mathbb{R}^n)^* \cong C(I^n, \mathbb{R}^n)^* \cong (C(I^n)^*)^n,$$

Proposition 2.1 proves that  $C(I^n)^*$  is an  $\mathcal{L}^1$  space and a finite product of  $\mathcal{L}^1$  spaces is still an  $\mathcal{L}^1$  space. Similarly  $\mathbf{M}_n(I^n)$  is an  $\mathcal{L}^1$  space and so is  $\mathcal{E}_{n,I^n}^* = \mathbf{M}_n(I^n) \times \mathbf{M}_{n-1}(I^n)$ . Thus  $\mathbf{BV}(I^n)$  is a complemented subspace of an  $\mathcal{L}^1$  space and this is contradictory with Theorem 3.1.

*Step 2.* — We prove the theorem under the hypothesis  $2 \leq m < n$  and  $X = I^m \times \{0\} \subseteq \mathbb{R}^n$ .

Suppose  $\Theta_{m, I^m \times \{0\}} : \mathcal{E}_{m, I^m \times \{0\}} \rightarrow \mathbf{CH}^m(I^m \times \{0\})$  has a continuous linear right inverse  $\tau$ , we dualize it and obtain a continuous linear map

$$\tau^* : \mathbf{M}_m(I^m \times \{0\}) \times \mathbf{M}_{m-1}(I^m \times \{0\}) \rightarrow \mathbf{N}_m(I^m \times \{0\})$$

such that  $\tau^*(T, \partial T) = T$  for all  $T \in \mathbf{N}_m(I^m \times \{0\})$ . We let  $\delta_0 \in \mathbf{M}_0(I^{n-m})$  be the Dirac mass at 0, *i.e.*  $\delta_0(\omega) := \omega(0)$  for all  $\omega \in \mathcal{D}^0(\mathbb{R}^{n-m})$ . Let  $p : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be the projection onto the last  $m$  coordinates. We define  $\sigma : \mathbf{M}_m(I^m) \times \mathbf{M}_{m-1}(I^m) \rightarrow \mathbf{N}_m(I^m)$  by setting

$$\sigma(T, S) = p_{\#} \tau^*(\delta_0 \times T, \delta_0 \times S)$$

for all  $T \in \mathbf{M}_m(I^m)$  and  $S \in \mathbf{M}_{m-1}(I^m)$ .  $\sigma$  is linear continuous, with norm bounded by  $\|\tau^*\|$ . Now let  $T \in \mathbf{N}_m(I^m)$ , the following holds

$$\sigma(T, \partial T) = p_{\#} \tau^*(\delta_0 \times T, \delta_0 \times \partial T) = p_{\#} \tau^*(\delta_0 \times T, \partial(\delta_0 \times T)) = p_{\#}(\delta_0 \times T).$$

$\delta_0$  is compactly supported, Lemma 4.3 hence implies  $p_{\#}(\delta_0 \times T) = \delta_0(1)T = T$ . This is contradictory by step 1.

*Step 3.* — We prove the theorem under the general assumption.

Let  $X \subseteq \mathbb{R}^n$  such that  $X$  contains a bilipschitz copy of the  $m$ -cube. Up to a bilipschitz transformation, we suppose that  $X$  contains  $I^m \times \{0\}$ . Suppose there exists a continuous right inverse  $\varphi : \mathbf{CH}^m(X) \rightarrow \mathcal{E}_{m, X}$  for  $\Theta_{m, X}$ . We denote by  $\iota : I^m \times \{0\} \rightarrow X$  the inclusion map and by  $\pi : X \rightarrow I^m \times \{0\}$  the projection onto the compact convex set  $I^m \times \{0\}$ . Note that  $\iota$  and  $\pi$  are Lipschitz continuous. We define  $\psi := \iota_{\#} \varphi \pi_{\#}$ .

$$\begin{array}{ccc} \mathbf{CH}^m(X) & \xrightarrow{\varphi} & \mathcal{E}_{m, X} \\ \pi_{\#} \uparrow & & \downarrow \iota_{\#} \\ \mathbf{CH}^m(I^m \times \{0\}) & \xrightarrow{\psi} & \mathcal{E}_{m, I^m \times \{0\}} \end{array}$$

$\psi$  is a continuous left inverse for  $\Theta_{m, X} : \mathcal{E}_{m, I^m \times \{0\}} \rightarrow \mathbf{CH}^m(I^m \times \{0\})$  since

$$\Theta_{m, X} \psi = \Theta_{m, X} \iota_{\#} \varphi \pi_{\#} = \iota_{\#} \Theta_{m, X} \varphi \pi_{\#} = \iota_{\#} \pi_{\#} = (\pi \iota)_{\#} = \text{id},$$

and this is impossible by step 2.  $\square$

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