On separately subharmonic functions
(Lelong’s problem)

A. Sadullaev

ABSTRACT. — The main result of the present paper is: every separately-subharmonic function \( u(x, y) \), which is harmonic in \( y \), can be represented locally as a sum two functions, \( u = u^* + U \), where \( U \) is subharmonic and \( u^* \) is harmonic in \( y \), subharmonic in \( x \) and harmonic in \( (x, y) \) outside of some nowhere dense set \( S \).

RÉSUMÉ. — Le résultat essentiel de ce papier est le suivant : toute fonction séparément sous-harmonique \( u(x, y) \) qui est harmonique en \( y \) peut être représentée localement comme la somme de deux fonctions \( u = u^* + U \), où \( U \) est sous-harmonique et \( u^* \) est harmonique en \( y \), sous-harmonique en \( x \) et harmonique en \( (x, y) \) en dehors d’une ensemble nulle part dense \( S \).

1. Introduction

We will consider functions \( u(x, y) \) of two groups of variables \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \). If \( u \) is separately harmonic, i.e., harmonic in \( x \) for fixed \( y \) and harmonic in \( y \) for fixed \( x \), then \( u \) will be harmonic in both variables (Lelong [2], see also [1]). Lelong investigated also separately subharmonic functions, and proved a series of special results in this area. Here originates the question about subharmonicity of separately subharmonic functions.
However, Wiegerinck, [3] (see also [4]) has shown that a separately subharmonic function need not to be subharmonic in general. He constructed a separately subharmonic function \( u(x, y) \) in the bidisk \( U^2 = \{ |x| < 1 \} \times \{ |y| < 1 \} \subset R_x^2 \times R_y^2 \approx C \times C \), which is not bounded above near 0.

The problem of subharmonicity of a separately subharmonic function \( u(x, y) \) that is in addition harmonic in \( y \), is still open.

In the present paper we will study the class of these functions. Let us begin by recalling the following well-known results:

1. If a separately subharmonic function is bounded above, then it is subharmonic (Lelong [2], Avanissian [5]);
2. If \( u^+ \in L^1_{loc} \), then \( u \) is subharmonic (Arsove [6]);
3. If \( u^+ \in L^p_{loc}, p > 0 \), then \( u \) is subharmonic (Riihentaus [7]);
4. There are also positive results under weak growth conditions (see [8], [9]).

We note that the conditions in the above results are not separated in \( x \) and \( y \). The following results demand separate conditions:

5. Suppose that \( u(x, y) \) is defined on the product domain \( B = B_1 \times B_2 \subset R_x^m \times R_y^m \). If \( u \) is subharmonic in \( x \) and harmonic in \( y \), then there are nowhere dense closed sets \( S_1 \subset B_1, S_2 \subset B_2 \) such that \( u \) is subharmonic in \( G = (B_1 \times B_2) \setminus (S_1 \times S_2) \) (Cegrell and Sadullaev [10]);
6. If \( u(x, y) \) real analytic, subharmonic in \( x \), and harmonic in \( y \), then \( u \) is subharmonic (Imomkulov [11]);
7. There exists a separately subharmonic function \( u(x, y) \), which is real analytic in \( x \), but which is not subharmonic (Cegrell and Sadullaev [10]);
8. If \( u(x, y) \) is \( C^2 \) and subharmonic in \( x \), harmonic in \( y \), then \( u \) is subharmonic (Kołodziej and Thorbjörnson [12]).

2. Results

Let \( u(x, y) \) be a separately subharmonic function in the product domain \( B = B_1 \times B_2 \), which is harmonic in \( y \). We will assume that \( u \) satisfies
this condition in a slightly larger domain $\tilde{B} = \tilde{B}_1 \times \tilde{B}_2$ such that $\tilde{B} \supset B$. Then $u(x, y)$ is subharmonic in a domain $\left(\tilde{B}_1 \times \tilde{B}_2\right) \setminus \left(S_1 \times S_2\right)$, where $S_1 \subset \tilde{B}_1, S_2 \subset \tilde{B}_2$ are closed, nowhere dense sets. Moreover, for every fixed $y \in \tilde{B}_2$ the Laplacian $\Delta_x u(x, y)$ defines a positive distribution as follows

$$F(\varphi) = \int u(x, y) \Delta_x \varphi(x) \, dx \quad \varphi \in C_0^\infty,$$

thus for every test function $\varphi(x) \in C_0^\infty(B_1)$, $\text{supp}\varphi \subset \subset B_1, \varphi \geq 0$ we have $F(\varphi) \geq 0$. Hence, $\Delta_x u(x, y)$ is a Borel measure, depending on the parameter $y$.

**Theorem 2.1.** — For every test-function $\varphi(x) \in C_0^\infty(B_1)$ $F(\varphi)$ is harmonic in $y$ for $y \in B_2 \setminus S_2$. Moreover, if $\text{supp}\varphi \cap S_1 = \emptyset$ then $F(\varphi)$ is harmonic in $y$ for all $y \in B_2$.

We say that the measure $\Delta_x u(x, y)$ has the harmonic property with respect to $y$ in the domain $G= (B_1 \times B_2) \setminus (S_1 \times S_2)$.

**Proof.** — The result 5) above states that $u(x, y)$ is subharmonic and therefore $u$ is locally bounded above in $G= (B_1 \times B_2) \setminus (S_1 \times S_2)$. Hence the integral

$$F(\varphi)(y) = \int_{B_1} \varphi(x) \Delta_x u(x, y) = \int_{B_1} u(x, y) \Delta_x \varphi(x)$$

is harmonic in $B_2 \setminus S_2$. If $\text{supp}\varphi \cap S_1 = \emptyset$, then this integral is harmonic in all $B_2$. □

**Corollary 2.2** The measure $F_E(y) = \int_E \Delta_x u(x, y)$ is harmonic in $B_2$ for any $E \subset \subset B_1 \setminus S_1$.

**Corollary 2.3.** — The total measure $\|\Delta_x u(x, y)\|_{B_1} = \int_{B_1} \Delta_x u(x, y)$ is finite ($\neq \infty$) for every fixed $y \in B_2$ and is harmonic function in $B_2 \setminus S_2$.

**Theorem 2.4.** — The function $F_{B_1 \setminus S_1}(y) = \int_{B_1 \setminus S_1} \Delta_x u(x, y)$ is bounded and positive harmonic in $B_2$.

**Proof.** — Let us take an increasing sequence of compacts $E_j \subset E_{j+1} \subset B_1 \setminus S_1$ such that $\bigcup_j E_j = B_1 \setminus S_1$. Then the functions $F_{E_j}(y) = \int_{E_j} \Delta_x u(x, y)$
are harmonic in $B_2$ and form an increasing sequence in $j$. By Harnack’s theorem either $F_{E_j}(y) \nearrow +\infty$ or $(F_{E_j})_j$ converges to a harmonic function. The first possibility is ruled out, because Corollary 2.3 provides a bound on the $F_{E_j}(y)$ for every $y \in B_2 \setminus S_2$.

Thus $\lim_{j \to \infty} F_{E_j}(y) = \int_{B_1 \setminus S_1} \Delta_x u(x, y)$ is harmonic in $B_2$, which completes the proof. □

Now we consider the potential

$$U(x, y) = \int_{B_1 \setminus S_1} K(x - w) \Delta_w u(w, y),$$

where $K$ is the Newtonian kernel,

$$K(w) = \begin{cases} 
\frac{1}{2\pi} \ln |w|, & \text{if } n = 2 \\
\frac{1}{(n - 2)\sigma_n |w|^{n-2}}, & \text{if } n > 2.
\end{cases}$$

The measure $\Delta_x u(x, y)$ has the harmonic property in $(B_1 \setminus S_1) \times B_2$. Moreover, for some constant $C$ the total measure $\int_{B_1 \setminus S_1} \Delta_x u(x, y) \leq C$, $y \in B_2$. It follows that the integral $\int_{B_1 \setminus S_1} \varphi(w) \Delta_w u(w, y)$ is harmonic in $y$ for every continuous function $\varphi \in C(\overline{B}_1)$. Let $K_j(w) \in C^\infty(\mathbb{R}^n)$ approximate $K$ from above, $K_j(w) \downarrow K(w)$. Then for every fixed $x \in B_1$ we have

$$\int_{B_1 \setminus S_1} K_j(x - w) \Delta_w u(w, y) \downarrow \int_{B_1 \setminus S_1} K(x - w) \Delta_w u(w, y)$$

for $j \to \infty$, hence $U(x, y)$ is harmonic in $y$ for fixed $x \in B_1$. Moreover, $U$ is subharmonic in $x$ and bounded above in $B_1 \times B_2$. It follows by the theorem of Lelong and Avanissian (1), that $U$ is subharmonic in $B_1 \times B_2$.

Now we take the difference $u^*(x, y) = u(x, y) - U(x, y)$. The function $u^*(x, y)$ is separately subharmonic and is harmonic in $y$. Moreover, $u^*(x, y)$ is harmonic in $x$ outside $S_1$. Thus we have

**Theorem 2.5.** — Every separately subharmonic function, which is harmonic in $y$, can locally be represented as a sum of two functions:

$$u(x, y) = u^*(x, y) + U(x, y),$$

where $U$ is a subharmonic function and $u^*$ is separately subharmonic and harmonic in $y$, such that the associated measure $\Delta_x u^*(x, y)$ is supported on $S_1$ for every fixed $y \in B_2$. 

– 186 –
Problem 2.6. — We finish our discussion by recalling an open problem on the definition of plurisubharmonic functions: in this definition one demands two conditions.

a. The function $u(z)$ is upper semicontinuous;

b. For each complex line $l$ the restriction $u|_l$ is subharmonic.

The above results on separately subharmonic functions seem to indicate, that the condition a. may be implied by b. But this is still open.

Bibliography