Sets in $\mathbb{C}^N$ with vanishing global extremal function and polynomial approximation


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Sets in $\mathbb{C}^N$ with vanishing global extremal function and polynomial approximation

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Abstract. — Let $\Gamma$ be a non-pluripolar set in $\mathbb{C}^N$. Let $f$ be a function holomorphic in a connected open neighborhood $G$ of $\Gamma$. Let $\{P_n\}$ be a sequence of polynomials with $\deg P_n \leq d_n$ ($d_n < d_{n+1}$) such that
$$\limsup_{n \to \infty} |f(z) - P_n(z)|^{1/d_n} < 1, \ z \in \Gamma.$$ We show that if
$$\limsup_{n \to \infty} |P_n(z)|^{1/d_n} \leq 1, \ z \in E,$$ where $E$ is a set in $\mathbb{C}^N$ such that the global extremal function $V_E \equiv 0$ in $\mathbb{C}^N$, then the maximal domain of existence $G_f$ of $f$ is one-sheeted, and
$$\limsup_{n \to \infty} \|f - P_n\|_K^{\frac{1}{d_n}} < 1$$ for every compact set $K \subset G_f$. If, moreover, the sequence $\{d_{n+1}/d_n\}$ is bounded then $G_f = \mathbb{C}^N$.

If $E$ is a closed set in $\mathbb{C}^N$ then $V_E \equiv 0$ if and only if each series of homogeneous polynomials $\sum_{j=0}^{\infty} Q_j$, for which some subsequence $\{s_{n_k}\}$ of partial sums converges point-wise on $E$, possesses Ostrowski gaps relative to a subsequence $\{n_{k_i}\}$ of $\{n_k\}$.

In one-dimensional setting these results are due to J. Müller and A. Yavrian [5].

Résumé. — Soit $\Gamma$ un sous-ensemble non pluripolaire de $\mathbb{C}^N$. Soit $f$ une fonction holomorphe sur un voisinage ouvert connexe $G$ de $\Gamma$. Soit $\{P_n\}$ une suite de polynômes de degré $\deg P_n \leq d_n$ ($d_n < d_{n+1}$) telle que
$$\limsup_{n \to \infty} |f(z) - P_n(z)|^{1/d_n} < 1, \ z \in \Gamma.$$
On démontre que si
\[ \limsup_{n \to \infty} |P_n(z)|^{1/d_n} \leq 1, \ z \in E, \]
où \( E \) est un sous-ensemble de \( \mathbb{C}^N \) tel que la fonction extrémale globale \( V_E \equiv 0 \) sur \( \mathbb{C}^N \), alors le domaine maximal d’existence \( G_f \) de \( f \) est uniforme, et
\[ \limsup_{n \to \infty} \|f - P_n\|_{\frac{1}{d_n}} < 1 \]
pour tout compact \( K \subset G_f \). Si, de plus, la suite \( \{d_{n+1}/d_n\} \) est bornée alors \( G_f = \mathbb{C}^N \).

Si \( E \) est un sous-ensemble fermé de \( \mathbb{C}^N \) alors \( V_E \equiv 0 \) si et seulement si chaque série de polynômes homogènes \( \sum_{j=0}^{\infty} Q_j \), ayant une sous-suite \( \{s_{n_k}\} \) de sommes partielles convergeant ponctuellement sur \( E \), admet des lacunes de type Ostrowski relativement à une sous-suite \( \{n_{k_l}\} \) de \( \{n_k\} \).

En dimension 1, ces résultats sont dus à J. Müller and A. Yavrian [5].

### 1. Introduction

Given an open set \( \Omega \) in \( \mathbb{C}^N \), let \( PSH(\Omega) \) denote the set of all plurisubharmonic (PSH) functions in \( \Omega \). Let \( \mathcal{L} \) be the class of PSH functions in \( \mathbb{C}^N \) with minimal growth, i.e. \( u \in \mathcal{L} \) if and only if \( u \in PSH(\mathbb{C}^N) \) and \( u(z) - \log(1 + ||z||) \leq \beta \) on \( \mathbb{C}^N \), where \( \beta \) is a real constant depending on \( u \).

If \( E \) is a subset of \( \mathbb{C}^N \), the global extremal function \( V_E \) associated with \( E \) is defined as follows.

If \( E \) is bounded, we put
\[
V_E(z) := \sup \{u(z); \ u \in \mathcal{L}, \ u \leq 0 \text{ on } E\}, \ z \in \mathbb{C}^N.
\]
If \( E \) is unbounded, we put (see [7])
\[
V_E(z) := \inf \{V_F(z); \ F \subset E, \ F \text{ is bounded}\}, \ z \in \mathbb{C}^N.
\]

It is known (see e.g. [6, 7]) that \( V_E^* \) (the upper semicontinuous regularization) is a member of \( \mathcal{L} \) iff \( E \) is non-pluripolar (non-plp). \( V_E^* \equiv +\infty \) iff \( E \) is pluripolar (plp).

If \( N = 1 \) and \( E \) is a compact non-polar subset of \( \mathbb{C} \), then \( V_E^* (z, \infty) \equiv g_E(z, \infty) \) for \( z \in D_\infty \), where \( D_\infty \) is the unbounded component of \( \mathbb{C} \setminus E \), and \( g_E \) is the Green function of \( D_\infty \) with the logarithmic pole at infinity.

If \( N \geq 2 \) and \( E \) is non-pluripolar, the function \( V_E^* \) is called pluricomplex Green function (with pole at infinity).
By [5] a closed subset $E$ of $\mathbb{C}$ is non-thin at $\infty$ if and only if $V_E^* \equiv 0$. One can check that for all $E \subset \mathbb{C}^N$, $N \geq 1$, we have $V_E^* \equiv 0$ if and only if $V_E \equiv 0$. Therefore, one can agree with the author of [9] that it is reasonable to say that a set $E \subset \mathbb{C}^N$ is non-thin at infinity (resp., thin at infinity), if $V_E \equiv 0$ (resp., $V_E \not\equiv 0$). In particular, if $V_E^* \equiv \infty$ the set $E$ is thin at infinity.

In chapter 2 of this paper we discuss properties of sets $E$ in $\mathbb{C}^N$ with $V_E \equiv 0$. Similarly, as in [5] and [9], very important role in our applications is played by the necessary and sufficient conditions stated in section 2.18 (which are a slightly modified version of the conditions of Tuyen Trung Truong’s Theorem 2 in [9]).

In chapters 3 and 4 we prove an $N$-dimensional version of the classical Ostrowski Gap Theorems for power series of a complex variable.

In chapters 5 and 6 we show that properties of sets $E \subset \mathbb{C}^N$ with $V_E \equiv 0$ ($N \geq 1$) may be applied to obtain results in $N$-dimensional setting analogous to those obtained earlier by J. Müller and A. Yavrian [5] in the one-dimensional case.

2. Sets in $\mathbb{C}^N$ with $V_E \equiv 0$

Now we shall state several properties of the global extremal function. Most of the properties are known and follow either from the elementary theory of the Lelong class $\mathcal{L}$ and from the definition of the extremal function, or from the Bedford-Taylor theorem on negligible sets in $\mathbb{C}^N$.

In the sequel $F, E, E_n$ (resp., $K, K_n$) are arbitrary (resp., compact) subsets of $\mathbb{C}^N$.

2.1. Monotonicity property of the extremal function. $V_F \leq V_E$, if $E \subset F$.

2.2. $V_E = \lim_{R \to \infty} V_{E_R}$, where $E_R := E \cap B(0, R)$, and $B(0, R) := \{z \in \mathbb{C}^N; \|z\| < R\}$ (resp., $B(0, R) := \{\|z\| \leq R\}$).

2.3. $V_{E_R}^*(z) = \lim_{R \to \infty} V_{E_R}^*(z) = \sup\{u(z); u \in \mathcal{L}, u \leq 0 \text{ q.a.e. on } E\}$, where “q.a.e. on $E$” means that the corresponding property holds quasi-almost everywhere on $E$, i.e. on $E \setminus A$, where $A$ is a pluripolar set.

Hence, if $E$ is non-pluripolar then the pluricomplex Green function $V_E^*$ is the unique maximal element of the set $\mathcal{W}^*(E) := \{u \in \mathcal{L}, u \leq 0 \text{ q.a.e. on } E\}$ ordered by the condition: if $u_1, u_2 \in \mathcal{W}^*(E)$ then $u_1 \preceq u_2$ if $u_1(z) \leq u_2(z)$ for all $z \in \mathbb{C}^N$. 

– 191 –
2.4. $V_{K_n} \uparrow V_K$, if $K_{n+1} \subset K_n$, $K = \cap K_n$.

2.5. $V^*_E \downarrow V^*_E$, if $E_n \subset E_{n+1}$, $E = \cup E_n$.

2.6. $(\lim V^*_E) = \lim V^*_E = V^*_E \equiv V^*_E \setminus A$.

2.7. If $E, A$ are subsets of $C^N$ and $A$ is pluripolar then $V^*_E \cup A \equiv V^*_E \equiv V^*_E \setminus A$.

2.8. Product property of the extremal function [1]. If $E \subset C^M$, $F \subset C^N$ then

$$V^*_E \times F(z, w) = \max \{V^*_E(z), V^*_F(w)\}, (z, w) \in C^M \times C^N.$$ 

Hence, a product $E \times F$ is non-thin at infinity if and only if the both factors are non-thin at infinity (a different proof of this property was given in [9]).

In the sequel we shall omit "at infinity" while speaking about non-thin (resp., thin) sets at infinity.

2.9. A set $E$ in $C^N$ is non-thin if and only if the set $E \setminus B$ (resp., $E \cup B$) is non-thin for every bounded set $B$.

Without loss of generality we may assume that $B$ is a ball $B(0, R)$. If $E \setminus B$ is non-thin then $E$ is non-thin by the monotonicity property.

Now assume that $E$ is non-thin. Then $E \setminus B$ is non-pluripolar because otherwise we would have $\log^+ \|z\|_R \equiv V^*_B(z) \equiv V^*_{B \cup (E \setminus B)}(z) \equiv V^*_E(z) \equiv 0$. A contradiction. Therefore $V^*_E \setminus B \in \mathcal{L}$. Put $M = \max \|z\|_R V^*_E \setminus B(z)$. Then $u := V^*_E \setminus B - M \in \mathcal{L}$ and $u \leq 0$ q.a.e. on $E$. Hence $u \leq V^*_E \equiv 0$ in $C^N$ which implies that $E \setminus B$ is non-thin.

It is obvious that $E \cup B$ is non-thin if $E$ is non-thin. In order to show the inverse implication, it sufficient to observe that $E \setminus B = (E \cup B) \setminus B$.

2.10. If $E$ is non-pluripolar then the limit

$$\sigma := \lim_{R \uparrow \infty} \max_{\|z\|=R} V^*_E(z) / \log R$$

exists and $\sigma$ either equals 0 (if and only if $E$ is non-thin), or $\sigma = 1$ (if and only if $E$ is thin).

The function $V^*_E$ is a member of the class $\mathcal{L}$. Therefore the limit exists and $0 \leq \sigma \leq 1$. One can check that $\sigma = 0$ if and only if $E$ is non-thin.
We should show that the case $0 < \sigma < 1$ is excluded. Indeed, the function $u := \frac{1}{\sigma} V_E^*$ is a member of $L$, and $u \leq 0$ q.a.e. on $E$. Hence, $\frac{1}{\sigma} V_E^* \leq V_E^*$ on $\mathbb{C}^N$. It follows that $\sigma \geq 1$. Consequently, $\sigma = 1$.

2.11. Robin function, Robin constant and logarithmic capacity. If $E$ is non-pluripolar then there exists a uniquely determined homogeneous PSH function $\tilde{V}_E(\lambda, z)$ of $1 + N$ variables $(\lambda, z) \in \mathbb{C} \times \mathbb{C}^N$ such that $\tilde{V}_E(1, z) = V_E^*(z)$ on $\mathbb{C}^N$. One may check that $\tilde{V}(\lambda, z) = \log |\lambda| + V_E^*(z/\lambda)$ if $\lambda \neq 0$, and $\tilde{V}_E(0, z) = \limsup_{(\lambda, \zeta) \to (0, z)} (\log |\lambda| + V_E(\zeta/\lambda))$.

The homogeneous PSH function $\tilde{V}_E(0, z)$ is called Robin function of $E$, and the set function $\gamma(E) := \max_{\|z\|=1} \tilde{V}_E(0, z)$ - Robin constant of $E$. The set function $c(E) := e^{-\gamma}$ is called logarithmic capacity of $E$. It is clear that the Robin constant and the logarithmic capacity of $E$ depend on the choice of the norm $\| \cdot \|$ in $\mathbb{C}^N$.

2.12. A necessary condition for non-thinness. If $E$ is non-thin then $c(E) = \infty$.

Indeed, if $V_E \equiv 0$ then $\tilde{V}_E(\lambda, z) \equiv \log |\lambda|$. Hence, $\tilde{V}_E(0, z) \equiv -\infty$ which implies that $\gamma(E) = -\infty$, i.e. $c(E) = +\infty$.

It is known that the condition 2.12 is not sufficient for closed subsets of the complex plane (and, consequently, for subsets of $\mathbb{C}^N$ with $N \geq 2$). We shall give a simple example.

2.13. An example of a closed set $E \subset \mathbb{C}$ with $V_E \neq 0$ and $c(E) = \infty$.

Let $\{a_n\}, \{\epsilon_n\}$ be two sequences of real numbers such that:

\[
 a_{n+1} > a_n > 0, \quad \epsilon_n > 0, \quad \sum_{1}^{\infty} \epsilon_n = 1, \quad \lim_{n \to \infty} \sum_{1}^{n} \epsilon_k \log a_k = +\infty,
\]

e.g. $\epsilon_n = 2^{-n}, a_n = e^{2^n}$.

Put

\[ U(z) := \sum_{1}^{\infty} \epsilon_n \log \frac{|z - a_n|}{1 + a_n}, \quad E := \{z; U(z) \leq 0\}. \]

It is clear that $E$ is closed and unbounded. It remains to check that $c(E) = +\infty$ and $V_E(z) = U^+(z)$, where $U^+(z) := \max\{0, U(z)\}$. To this order we put

\[ U_n(z) := (\sum_{1}^{n} \epsilon_k)^{-1} \sum_{1}^{n} \epsilon_k \log \frac{|z - a_k|}{1 + a_k}, \quad E_n := \{z; U_n(z) \leq 0\}. \]

– 193 –
Sets in $\mathbb{C}^N$ with vanishing global extremal function

One can easily check that $E_n$ is compact and regular ($E_n$ is a finite union of non-trivial continua), $E_n \subset E_{n+1}$, $V_{E_n}(z) \equiv U_{\alpha_n}^+(z) \cap U^+(z) \equiv V_{E}(z)$, $\tilde{V}_{E_n}(\lambda, z) = (\sum_{k=1}^{n} \epsilon_k)^{-1} \sum_{k=1}^{n} \epsilon_k \log \frac{|z-\lambda a_k|}{1+|a_k|}$ if $|z/\lambda| \geq R = R(n) = \text{const} > 0$, $\tilde{V}_{E_n}(0, z) \equiv \log \|z\| + \gamma(E_n)$ for all $z \in \mathbb{C}$, and hence $\log c(E_n) = -\gamma(E_n) = (\sum_{k=1}^{n} \epsilon_k)^{-1} \sum_{k=1}^{n} \epsilon_k \log(1+|a_k|)$ for all $n \geq 1$, which gives the required result.

Taking $E \times F$ with $E$ in $\mathbb{C}$ as just above, and with a non-thin subset $F$ of $\mathbb{C}^{N-1}$ ($N \geq 2$), one gets a thin subset of $\mathbb{C}^N$ with $c(E \times F) = \infty$.

2.14. A sufficient condition for non-thinness. Using an inequality due to B. A. Taylor [8] one can show (see [9] for details) that a sufficient condition for $E$ to be non-thin is

$$\limsup_{R \to \infty} \frac{\log c(E_R)}{\log R} > 1 - \frac{1}{C_N},$$

where $C_N$ is a constant depending only on the dimension $N$ with $C_N > 1$ for $N \geq 2$, and $C_1 = 1$.

2.15. Example. Let $\{a_n\}$ be a sequence of distinct points in $\mathbb{C}^N$ with $a_n \neq 0$ ($n \geq 1$). Let $\epsilon_n$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \epsilon_n = 1$. Let $u$ be the function defined by

$$u(z) = \sum_{n=1}^{\infty} \epsilon_n \log \frac{\|z-a_n\|}{1+\|a_n\|}, \quad z \in \mathbb{C}^N.$$  

Then $u$ is a non-constant ($u(0) \geq -\log 2$, $u(a_n) = -\infty$ for every $n \geq 1$) member of the class $L$ such that $E := \{z; u(z) < 0\}$ is an open set containing the unit ball and all points $a_n$. It is clear that $E$ is thin. Moreover, if the sequence $\{a_n\}$ is dense in $\mathbb{C}^N$ then $E$ is a thin unbounded open set dense everywhere.

2.16. Example. Every non-pluripolar real cone $E$ in $\mathbb{C}^N$ (without loss of generality, we assume that $E$ has its vertex at the origin, so that $tz \in E$, if $t \in \mathbb{R}$, $t > 0$, $z \in E$) is non-thin. Indeed, one can check that the sets $E_R := E \cap \{|z| \leq R\}$ are non-pluripolar, and $E_R = RE_1$ for all $R \geq 1$. Observe that $V_E(z) \leq V_{E_R}(z) \equiv V_{E_1}(\frac{1}{R}z)$ for all $z \in \mathbb{C}^N$ and for $R \geq 1$. It follows that $V_E(z) \leq V_{E_1}(0)$ for all $z$ which gives the required result.

2.17. Example. It follows from Wiener Criterion [3] that if $E$ is a countable union of closed (or open) discs $\{z \in \mathbb{C}; |z-a_n| \leq r\}$, where $r = \text{const} > 0$, $a_n \in \mathbb{C}$ and $a_n \to \infty$ as $n \to \infty$, then $E$ is non-thin at infinity.

We shall show that analogous property is no more true in $\mathbb{C}^N$ with $N \geq 2$. Put $E := \bigcup_{1}^{\infty} B_n$ where $B_n := \{(z_1, z_2); |z_1-a_n|^2 + |z_2|^2 \leq 1\}, a_n \in \mathbb{C}$.
and $a_n \to \infty$ as $n \to \infty$. It is sufficient to prove that $V_E(z_1, z_2) = \log^+ |z_2|$ for all $(z_1, z_2)$. It is clear that $\log^+ |z_2| \leq V_E(z_1, z_2)$ on $E$ and hence in the whole space $\mathbb{C}^2$. Now let $u$ be a function of the class $\mathcal{L}$ with $u \leq 0$ on $E$. We want to show that $u(z_1, z_2) \leq \log^+ |z_2|$ in $\mathbb{C}^2$. Without loss of generality we may assume (by taking $\max[u, 0]$) that $u = 0$ on $E$. Fix $z_2^0$ with $|z_2^0| \leq 1$. Then $u(z_1, z_2^0) = 0$ for all $z_1$ in the union of the discs $\{|z_1-a_n| \leq 1\}$. Therefore $u(z_1, z_2) = 0$ for all $(z_1, z_2)$ with $z_1 \in \mathbb{C}$ and $|z_2| \leq 1$. Hence $u(z_1, z_2) \leq \log^+ |z_2|$ in $\mathbb{C}^2$.

2.18. Necessary and sufficient conditions for non-thinness. For a non-pluripolar set $E \subset \mathbb{C}^N$ the following conditions are equivalent.

(1) If $u \in \mathcal{L}$, $u \leq 0$ a.e. on $E$ then $u = \text{const} \leq 0$;

(2) $V_E \equiv 0$;

(3) $V_E^* \equiv 0$;

(4) If $u_k \in \mathcal{L}$ ($k \geq 1$) and $u(z) := \limsup_{k \to \infty} u_k(z) \leq 0$ a.e. on $E$ then $u^* = \text{const} \leq 0$;

(5) If $\{p_k\}$ is a sequence of polynomials of $N$ complex variables and $\{n_k\}$ is a sequence of natural numbers such that $\deg p_k \leq n_k$ and $v := \limsup_{k \to \infty} \frac{1}{n_k} \log |p_k| \leq 0$ a.e. on $E$ then $v^* = \text{const} \leq 0$.

Proof. — The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ are easy to check. In order to show the implication $(5) \Rightarrow (1)$ fix $u \in \mathcal{L}$ with $u \leq 0$ a.e. on $E$. Assuming $(5)$ holds, we need to show that $u = \text{const} \leq 0$.

It is known [6, 7] that there exits a sequence of holomorphic polynomials $\{p_n\}$ such that $\deg p_n \leq n$ and $u = v^*$ where $v := \limsup_{n \to \infty} \frac{1}{n} \log |p_n|$. By theorem on negligible sets [4], we know that $u = v^* \leq 0$ a.e. on $E$. By $(5)$ it follows that $u = v^* = \text{const} \leq 0$. □

2.19. Remark. Consider the following property $(1')$ of $E$

$(1')$ If $u \in \mathcal{L}, u \leq 0$ on $E$ then $u = \text{const} \leq 0$.

It is obvious that if $E$ has the property $(1)$ then $E$ satisfies $(1')$. The inverse implication does not hold for $N \geq 2$ (we do not know if it is true for arbitrary sets on the complex plane). Namely, by Example 1.1. of [2], the set $E := \{(z_1, z_2) \in \mathbb{C}^2; (z_1 \in \mathbb{C}, |z_2| \leq 1) \text{ or } (z_1 = 0, z_2 \in \mathbb{C})\}$ satisfies $(1')$ but it does not satisfy $(1)$, because $V_E^*(z_1, z_2) \equiv \log^+ |z_2|$.

– 195 –
Sets in $\mathbb{C}^N$ with vanishing global extremal function

### 3. Power series with Ostrowski gaps

Let

$$f(z) = \sum_{j=0}^{\infty} Q_j(z), \quad \text{where} \quad Q_j(z) = \sum_{|\alpha|=j} a_\alpha z^\alpha, \quad (3.1)$$

be a power series in $\mathbb{C}^N$, i.e. a series of homogeneous polynomials $Q_j$ of $N$ complex variables of degree $j$.

The set $\mathcal{D}$ given by the formula $\mathcal{D} := \{ a \in \mathbb{C}^N; \text{the sequence (3.1) is convergent in a neighborhood of } a \}$ is called a domain of convergence of (3.1).

It is known that

$$\mathcal{D} = \{ z \in \mathbb{C}^N; \psi^*(z) < 0 \},$$

where

$$\psi(z) := \limsup_{j \to \infty} \sqrt[j]{|Q_j(z)|}. \tag{3.2}$$

If $\psi^*$ is finite then it is PSH and absolutely homogeneous (i.e. $\psi^*(\lambda z) = |\lambda| \psi^*(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^N$). Therefore the domain of convergence $\mathcal{D}$ is either empty, or it is a balanced (i.e. $\lambda z \in \mathcal{D}$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ and $z \in \mathcal{D}$) domain of holomorphy. Every balanced domain of holomorphy is a domain of convergence of a series (3.1).

The number

$$\rho := 1/\limsup_{j \to \infty} \sqrt[j]{\|Q_j\|_{\mathcal{B}}},$$

where $\mathcal{B} := \{ z \in \mathbb{C}^N; \|z\| \leq 1 \}$, is called a radius of convergence of series (3.1) (with respect to a given norm $\|\cdot\|$).

If $N = 1$ then $\mathcal{D} = \rho \mathcal{B}$. If $N \geq 2$ then $\rho \mathcal{B} \subset \mathcal{D}$ but, in general, $\mathcal{D} \neq \rho \mathcal{B}$.

Series (3.1) is normally convergent in $\mathcal{D}$, i.e.

$$\limsup_{j \to \infty} \sqrt[j]{\|Q_j\|_K} < 1, \quad \limsup_{n \to \infty} \sqrt[n]{\|f - s_n\|_K} < 1,$$

for all compact sets $K \subset \mathcal{D}$, where $s_n := Q_0 + \cdots + Q_n$ is the $n$th partial sum of (3.1).

For a strictly increasing sequence $\{n_k\}$ of positive integers we say that a power series (3.1) possesses Ostrowski gaps relative to $\{n_k\}$ if there exists a sequence of real numbers $q_k > 0$ such that $\lim q_k = 0$ and

$$\limsup_{j \to \infty} \sqrt[j]{\|Q_j\|_K} < 1, \quad \limsup_{n \to \infty} \sqrt[n]{\|f - s_n\|_K} < 1,$$
\[
\lim_{j \to \infty, j \in I} \|Q_j\|^{1/j} = 0 \tag{3.2}
\]

where \( \mathbb{B} \) is the unit ball in \( \mathbb{C}^N \), and \( I := \bigcup_k [q_k n_k, n_k] \cap \mathbb{N} \).

We say that a series (3.1) is overconvergent, if a subsequence \( \{s_{n_k}\} \) of its partial sums is uniformly convergent in a neighborhood of some point \( a \in \mathbb{C}^N \setminus \mathcal{D} \).

**Example.** — Consider the function

\[
f(z) = \sum_0^\infty \left( \frac{z(z+1)}{r} \right)^{2^k^2} = \sum_0^\infty r^{-2^k^2} \left( z^{2^k^2} + \cdots + z^{2^k^2+1} \right) = \sum_0^\infty c_j z^j, \text{ where } c_j = 0, \text{ when } 2^{(k-1)^2+1} + 1 \leq j \leq 2^{k^2+1} - 1, \ k \geq 1.
\]

The function \( f \) is given by a power series with Ostrowski gaps relative to the sequence \( n_k = 2^{k^2+1} - 1 \) (with \( q_k := (2^{(k-1)^2+1+1})/(2^{k^2+1}-1) \)). The sequence \( s_{n_k}(z) = \sum_0^{2^{k^2+1}-1} c_j z^j = \sum_0^{2^{(k-1)^2+1}} c_j z^j = \sum_0^{k-1} \left( \frac{z(z+1)}{r} \right)^{2^{j^2}} \) is normally convergent to \( f(z) \) in the lemniscate \( \mathcal{E}_r = \{ z; |z(z+1)| < r \}, \ r > 0 \).

The radius of convergence of our power series is given by the formula \( \rho = \text{dist}(0, \partial \mathcal{E}_r) \). If \( 0 < r \leq \frac{1}{4} \) then \( \mathcal{E}_r \) has two disjoint components. If \( r > \frac{1}{4} \) the lemniscate \( \mathcal{E}_r \) is connected. Our power series is overconvergent at every point of \( \mathcal{E}_r \setminus \{|z| \leq \rho\} \). If \( G \) is a a connected component of \( \mathcal{E}_r \) then the function \( f|_G \) is holomorphic in \( G \) and it has analytic continuation across no boundary point of \( G \).

4. Two Ostrowski Gap Theorems in \( \mathbb{C}^N \)

We say that a compact subset \( K \) of \( \mathbb{C}^N \) is polynomially convex if \( K \) is identical with its polynomially convex hull \( \hat{K} := \{ a \in \mathbb{C}^N; |P(a)| \leq \|P\|_K \text{ for every polynomial } P \text{ of } N \text{ complex variables } \} \).

We say that an open set \( \Omega \) in \( \mathbb{C}^N \) is polynomially convex, if for every compact subset \( K \) of \( \Omega \) the polynomially convex hull \( \hat{K} \) of \( K \) is contained in \( \Omega \).

The aim of this section is to prove the two fundamental Ostrowski gap theorems in \( N \)-dimensional setting, \( N \geq 1 \).

Let \( f \) be a function holomorphic in a neighborhood of the origin of \( \mathbb{C}^N \) whose Taylor series development (3.1) possesses Ostrowski gaps relative to a sequence \( \{n_k\} \).
Sets in $\mathbb{C}^N$ with vanishing global extremal function

Let $\Omega$ be the set of points $a$ in $\mathbb{C}^N$ such that the sequence $\{s_{n_k}\}$ is uniformly convergent in a neighborhood of $a$. By classical theory of envelops of holomorphy, each connected component of $\Omega$ is a polynomially convex domain. Let $G$ be a connected component of $\Omega$ with $0 \in G$.

**Theorem 1.** — $G$ is the maximal domain of existence of $f$. Moreover, $G$ is polynomially convex and

$$\limsup_{k \to \infty} \| f - s_{n_k} \|_{K}^{1/n_k} < 1$$

for every compact subset $K$ of $G$.

**Corollary 4.1.** — The maximal domain of existence $G$ of a function $f$ holomorphic in a neighborhood of the origin of $\mathbb{C}^N$ with Taylor series development possessing Ostrowski gaps relative to a sequence $\{n_k\}$ is a one-sheeted polynomially convex domain of holomorphy.

**Corollary 4.2.** — If a function $f$ holomorphic in a neighborhood of $0 \in \mathbb{C}^N$ has Taylor series development of the form

$$f(z) = \sum_{0}^{\infty} Q_{m_k}(z), \text{ where } m_k < m_{k+1}, \frac{m_{k+1}}{m_k} \to \infty,$$

then the domain of convergence of the series is identical with the maximal domain of existence of $f$.

We need the following lemma (known for $N = 1$, see e.g. [5], Lemma 3).

**Lemma 4.3.** — If a power series (3.1) with positive radius of convergence possesses Ostrowski gaps relative to a sequence $\{n_k\}$ then for every $R > 0$ we have

$$\limsup_{k \to \infty} \| s_{n_k} \|_{BR}^{1/n_k} \leq 1, \quad (4.0)$$

where $B_R := B(0, R)$ is a ball with center $0$ and radius $R$.

If series (3.1) possesses Ostrowski gaps relative to $\{n_k\}$, then either $\lim q_k n_k = \infty$, or $\mathbb{N} \setminus I$ is finite and consequently the function $f$ is entire. In the second case (4.0) is obvious. In the first case, we have

$$\epsilon_k := \max\{\|Q_j\|_B^{1/j}; q_k n_k \leq j \leq n_k\} \to 0 \text{ as } k \to \infty.$$
Fix $R > 0$. Since the radius of convergence of the series (3.1) is positive, there exists $M > 1$ such that $RM > 1$, and
\[ \|Q_j\|_{B_R} \leq (MR)^j, \quad j \geq 0, \]
because $|Q_j(z)| \leq \|Q_j\|_B \|z\|^j \leq (M\|z\|)^j$, $j \geq 0$, where $M > 1$ is sufficiently large. Therefore
\[ \|Q_j\|_{B_R} \leq \left( \left( \frac{RM}{M} \right) \right)^j \]
and $k_0$ is so large that $\epsilon_k MR \leq 1$ for $k \geq k_0$, and $\|Q_j\|_B \leq \epsilon_k$ for $k \geq k_0$.

Proof of the Lemma is completed.

Proof of Theorem 1.— In the component $G$ of $\Omega$ the function $f$ is a locally uniform limit of the sequence of polynomials $\{s_{n_k}\}$ of corresponding degrees $\leq n_k$.

The function
\[ u_k := \frac{1}{n_k} \log |f - s_{n_k}| \]
is PSH in $G$. By (4.0), the sequence $\{u_k\}$ is locally uniformly upper bounded in $G$. Therefore, if $u := \limsup_{k \to \infty} u_k$, then $u^* \in \text{PSH}(G)$, $u^* \leq 0$ in $G$ and $u^* < 0$ in a neighborhood of 0. Hence, by the maximum principle for PSH functions, we have $u^* < 0$ in $G$. Hence, by Hartogs Lemma,
\[ \limsup_{k \to \infty} \|f - s_{n_k}\|_{K}^{1/n_k} < 1 \]
for every compact subset $K$ of $G$.

Suppose $G$ is not a maximal domain of existence of $f$. Then, there exist a point $a \in G$, a real number $r > \text{dist}(a, \partial G) =: r_0$, and a function $g$ holomorphic in the ball $B(a, r)$ such that $g = f$ on $B(a, r_0)$. Basing on the inequality (4.0), similarly as just above, we can show that
\[ \limsup_{k \to \infty} \|g - s_{n_k}\|_{K}^{1/n_k} < 1 \]
for every compact subset $K$ of $B(a, r)$. It follows that $s_{n_k} \to g$ locally uniformly in $B(a, r)$ as $k \to \infty$. Therefore the sequence $\{s_{n_k}\}$ converges uniformly in a neighborhood of some boundary point of $G$ which contradicts the definition of $\Omega$. It follows that $G$ is a polynomially convex maximal domain of existence of $f$. The proof of Theorem 1 is completed.

– 199 –
Theorem 2. — For every polynomially convex open set $\Omega \subset \mathbb{C}^N$ with $0 \in \Omega$ there exists a function $f$ holomorphic in $\Omega$ whose Taylor series development around $0$

$$f(z) = \sum_{0}^{\infty} Q_j(z), \quad Q_j(z) := \sum_{|\alpha| = j} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha,$$  (4.1)

possesses Ostrowski gaps relative to a sequence $\{n_k\}$ such that:

(i) Every connected component $D$ of $\Omega$ is the maximal domain of existence of $f|_D$;

(ii) The subsequence $\{s_{n_k}\}$ of partial sums of (4.1) converges locally uniformly to $f$ in $\Omega$; in particular, Taylor series (4.1) is overconvergent at every point $a$ of $\Omega \setminus D$, where $D$ is the domain of convergence of (4.1);

(iii) If $G$ is the component of $\Omega$ with $0 \in G$ then

$$\limsup_{k \to \infty} \|f - s_{n_k}\|_K^{1/n_k} < 1$$

for every compact subset $K$ of $G$.

Proof. — Let $\{\xi^{(\nu)}\}$ ($\xi^{(j)} \neq \xi^{(k)}, j \neq k$) be a countable dense subset of $\Omega$. Put $B_\nu := B(\xi^{(\nu)}, r_\nu)$ with $r_\nu := \text{dist}(\xi^{(\nu)}, \partial \Omega)$. Let $c^{(\nu)}$ be a point of $\partial \Omega \cap \partial B_\nu$, and let $E_\nu = \{a^{(\mu\nu)}\}_{\mu \geq 1}$ be a sequence of points of the ball $B_\nu$ such that $a^{(\mu\nu)} \in (\xi^{(\nu)}, c^{(\nu)}) := \{\xi^{(\nu)} + t(c^{(\nu)} - \xi^{(\nu)}); 0 < t \leq 1\}$ and

$$\|a^{(\mu\nu)} - c^{(\nu)}\| < \frac{1}{\mu\nu}, \quad \mu \geq 1.$$

Let $\{E_\nu^*\}$ denote the sequence

$$E_1; E_1, E_2; E_1, E_2, E_3; E_1, \ldots, E_\nu; \ldots$$  (4.2)

in which every set $E_\nu$ is repeated infinitely many times.

Since $\Omega$ is polynomially convex there exists a sequence of polynomially convex compact sets $\{\Delta_k\}$ such that $\Delta_k$ is contained in the interior of $\Delta_{k+1}$ and $\Omega = \bigcup_{1}^\infty \Delta_k$.

Taking, if necessary, a subsequence of $\{\Delta_k\}$, we may assume that $0 \in \Delta_1$ and

$$E_k^* \cap (\Delta_{k+1} \setminus \Delta_k) \neq \emptyset, \quad k \geq 1.$$

Let $a^{(k)}$ be an arbitrary fixed point of this intersection. Given $k \geq 1$, let $W_k$ be a polynomial such that $d_k := \text{deg} W_k \geq k$, and

$$\|W_k\|_{\Delta_k} < 1 < |W_k(a^{(k)})|.$$  (4.3)
Put $f_0(z) \equiv 0$, $\mu_0 = \nu_0 = 1$, and

\[ f_k(z) = \left( \frac{a_1^{(k)} z_1 + \cdots + a_N^{(k)} z_N}{\|a^{(k)}\|^2} \right)^{\mu_k} (W_k(z))^{\nu_k}, \quad k \geq 1, \quad (4.4) \]

where $\mu_k, \nu_k$ are positive integers. We claim that integers can be chosen in such a way that the following conditions are satisfied for all $k \geq 1$

(a) $\mu_{k-1} + \nu_{k-1} d_{k-1} < \mu_k/k$;

(b) $\|f_k\|_{\Delta_k} \leq 2^{-k}$;

(c) $|f_k(a^{(k)})| \geq k + |\sum_{j=0}^{k-1} f_j(a^{(k)})|$.

Indeed, put $\mu_1 = 1$ and choose $\nu_1 \geq 1$ so large that $\|f_1\|_{\Delta_1} \leq \frac{1}{2}$. Then the conditions are satisfied for $k = 1$. Suppose that $\mu_j, \nu_j$ are already chosen for $j = 0, 1, \cdots, k$ for a fixed $k \geq 1$. Observe that $|f_k(a^{(k)})| = |W_k(a^{(k)})|^{\nu_k}$ tends - by right hand side of (4.3) and (c) - to $\infty$ as $\nu_k \to \infty$ (here $\nu_k$ denotes a positive integer valued variable). It is clear that one can find an integer $\mu_{k+1}$ such that (a) is satisfied with $k$ replaced by $k+1$. Now, applying left hand side (respectively, right hand side) inequality of (4.3) one can find an integer $\nu_{k+1}$ so large that (b) (respectively, (c)) is satisfied for $k$ replace by $k + 1$. By the induction principle, the claim is true.

We shall prove that the function $f$, given by the formula

\[ f(z) = \sum_{j=0}^{\infty} f_j(z), \quad z \in \Omega, \]

where $f_j$ are defined by (4.4), has the required properties.

It follows from (b) that the series is uniformly convergent on compact subsets of $\Omega$. Hence $f \in \mathcal{O}(\Omega)$. Since for $\nu = 1, 2, \ldots$ the sequence $\{a^{(k)}\}$ contains a subsequence of the sequence $\{a^{(\mu\nu)}\}_{\mu \geq 1}$, we have

\[ \limsup_{t \uparrow 1} |f(\xi^{(\nu)} + t(c^{(\nu)} - \xi^{(\nu)}))| = +\infty. \]

It follows that every connected component $D$ of $\Omega$ is a maximal domain of existence of $f|_D$.

The function $f_k$ is a polynomial given by

\[ f_k(z) = \sum_{j=\mu_k}^{\mu_k+\nu_k d_k} Q_j(z), \]

where $f_0(z) \equiv 0$, $\mu_0 = \nu_0 = 1,$ and $f_k(z) = \left( \frac{a_1^{(k)} z_1 + \cdots + a_N^{(k)} z_N}{\|a^{(k)}\|^2} \right)^{\mu_k} (W_k(z))^{\nu_k}, \quad k \geq 1,$

(b) $\|f_k\|_{\Delta_k} \leq 2^{-k}$;

(c) $|f_k(a^{(k)})| \geq k + |\sum_{j=0}^{k-1} f_j(a^{(k)})|$.
Sets in $\mathbb{C}^n$ with vanishing global extremal function

where $Q_j$ is a homogeneous polynomial of degree $j$. By the condition (a), the Taylor series development of $f$ around 0 is given by

$$f(z) = \sum_{j=0}^{\infty} Q_j(z), \quad \|z\| < \rho,$$  \hspace{1cm} (4.5)  

where $\rho = \text{dist}(0, \partial D)$ and $Q_j = 0$ for $\mu_k - 1 + \nu_k - 1 + 1 \leq j \leq \mu_k - 1$, $k \geq 1$.

Put $n_k := \mu_k - 1$, and $q_k := \frac{\mu_k - 1 + \nu_k - 1 + 1}{\mu_k - 1}$. Then $q_k > 0$ and, by (a), $\lim_{k \to \infty} q_k = 0$. It follows that the series (4.5) has Ostrowski gaps relative to the sequence $n_k := \mu_k - 1$, $k \geq 1$. It is clear that

$$s_{n_k}(z) = \sum_{j=0}^{n_k} Q_j(z) = \sum_{j=0}^{k} f_j(z).$$

Therefore the subsequence $\{s_{n_k}\}$ of partial sums of the Taylor series (4.5) converges locally uniformly to $f$ in $\Omega$. Moreover, by Theorem 1, we conclude that $\{s_{n_k}\}$ satisfies condition (iii), which completes the proof of Theorem 2.

5. Sets $E$ in $\mathbb{C}^n$ with $V_E \equiv 0$ and power series with Ostrowski gaps

The following theorem is an N-dimensional version of Theorem 2 in [4].

**Theorem 3.** — Given a closed subset $E$ of $\mathbb{C}^n$, the following two conditions are equivalent:

(a) $V_E \equiv 0$.

(b) If a subsequence $\{s_{n_k}\}$ of partial sums of a power series (3.1) satisfies the inequality

$$\limsup_{k \to \infty} \|s_{n_k}(z)\|_B(0,R) \leq 1,$$

for every $z \in E$, then series (3.1) possesses Ostrowski gaps relative to a subsequence $\{n_{k_1}\}$ of the sequence $\{n_k\}$.

**Proof of Theorem 3.** — Our proof is an adaptation of the proof in one-dimensional case presented in [5].

First we shall show that (a) \(\Rightarrow\) (b). To this order observe that – by (a) – we have (5) of section 2.18 which implies – by Hartogs Lemma – that

$$\limsup_{k \to \infty} \|s_{n_k}\|_B(0,R) \leq 1,$$

for every $R > 0$.  \hspace{1cm} (5.2)
The implication \((a) \Rightarrow (b)\) follows from

**Lemma 5.1.** — If \(\{s_{n_k}\}\) satisfies (5.2) then the power series (3.1) possesses Ostrowski gaps relative to a subsequence \(\{n_{k_l}\}\) of \(\{n_k\}\).

**Proof of Lemma 5.1.** — By (5.2), for every \(l \geq 1\), we can find \(k_l \in \mathbb{N}\) such that

\[
\|s_{n_{k_l}}\|_{B(0,l)} \leq (1 + \frac{1}{l})^{n_{k_l}}, \quad l \geq 1.
\]

Hence, by Cauchy inequalities, we get

\[
\|Q_j\|^1/j_{B} \leq \frac{1}{l} \left(1 + \frac{1}{l}\right)^{-n_{k_l} j} \leq \frac{e}{l}, \quad \frac{n_{k_l}}{l} \leq j \leq n_{k_l}, \quad l \geq 1,
\]

which (with \(q_l := \frac{1}{l}\)) completes the proof of Lemma 5.1.

\((b) \Rightarrow (a)\). It is enough to prove that \(\text{non}(a) \Rightarrow \text{non}(b)\). Let \(E\) be a thin closed set in \(\mathbb{C}^N\). We shall construct a power series (3.1), for which a subsequence \(\{s_{n_k}\}\) satisfies (5.1), but which does not possess Ostrowski gaps relative to any subsequence of \(\{n_k\}\).

Our construction is based on the following useful known result.

**Lemma 5.2.** — If \(K\) is a compact subset of \(\mathbb{C}^N\) then

\[
V_K(z) = \sup \left\{ \frac{1}{k} \log |P_k(z)|; \|P_k\|_K = 1, \ k \geq 1\right\}, \ z \in \mathbb{C}^N,
\]

where \(P_k\) is a polynomial of \(N\) complex variables of degree at most \(k\).

Without loss of generality we may assume that \(\overline{\mathbb{B}} \subset E\) (because, by property 2.9 we know that \(E\) is thin if and only if \(E \cup \overline{\mathbb{B}}\) is thin).

Choose a point \(a \in \mathbb{C}^N\) such that \(R_0 := \|a\| > 1\) and \(V_E(a) := \eta > 0\). Put \(\epsilon_k := \eta/k\), \(R_k := R_0 + k\), and \(E_k = E \cap \{\|z\| \leq R_k\}\) for \(k \geq 0\). Then \(V_{E_k}(a) \downarrow V_E(a)\).

Let \(p_0, q_0 \geq 1\) be arbitrary integers, and let \(W_{q_0}\) be a polynomial of degree \(\leq q_0\) such that \(\|W_{q_0}\|_{E_0} = 1, |W_{q_0}(a)| > e^{(\eta-\epsilon_0)q_0}, \) where \(0 < \epsilon_0 < 1\).

Suppose \(p_j, q_j, W_{q_j} (j = 0, \ldots, k)\) are already chosen in such a way that \(W_{q_j}\) is a polynomial of degree \(\leq q_j\) and

\[
p_{j-1} + q_{j-1} < p_j < q_j/j, \quad (5.3)
\]
Sets in $\mathbb{C}^N$ with vanishing global extremal function

$$\frac{R_{q_j}^j}{(1 + \epsilon_j)^j} \leq \frac{1}{j^2}, \quad (5.4)$$

$$\|W_{q_j}\|_{E_j} = 1, \quad |W_{q_j}(a)| > e^{(\eta - \epsilon_j)q_j}. \quad (5.5)$$

Now, it is easy to find integers $p_{k+1}, q_{k+1}$ and a polynomial $W_{q_{k+1}}$ such that $(5.3), (5.4), (5.5)$ are satisfied for $j = k + 1$.

First choose an arbitrary integer $p_{k+1} > p_k + q_k$, next choose an arbitrary integer $q_{k+1} > (k+1)p_{k+1}$ and a polynomial $W_{q_{k+1}}$ such that $(5.4)$ and $(5.5)$ are satisfied with $j = k + 1$.

Consider the series

$$f(z) = \sum_{k=0}^{\infty} \left( \bar{a}_1 z_1 + \ldots + \bar{a}_N z_N \right)^{p_k} \frac{W_{q_k}(z)}{(1 + \epsilon_k)^{q_k}}. \quad (5.6)$$

From $(5.5)$ it follows that series $(5.6)$ converges uniformly on every $E_k, k \geq 0$. In particular, its sum $f$ is a holomorphic function in the unit ball. The $k$-th component of $(5.6)$ is of the form $\sum_{j=p_k}^{p_k+q_k} Q_j$, where $Q_j$ is a homogeneous polynomial of degree $j$. Hence $f(z) = \sum_{k=0}^{\infty} \left( \sum_{j=p_k}^{p_k+q_k} Q_j(z) \right), z \in \mathbb{B}$. After removing the parentheses we get a power series with positive radius of convergence. Put $n_k = p_k + q_k$. It is clear that for every $k \geq 1$

$$|s_{n_k}(a)| \geq \frac{|W_{q_k}(a)|}{(1 + \epsilon_k)^{q_k}} - |s_{n_{k-1}}(a)| \geq e^{q_k(\eta - \epsilon_k)} \frac{(1 + \epsilon_k)^{q_k}}{(1 + \epsilon_k)^{q_k}} - \sum_{0}^{k-1} e^{q_j V_{E_j}}(a) \geq$$

$$\frac{e^{q_k(\eta - \epsilon_k)}}{(1 + \epsilon_k)^{q_k}} - k M^{q_{k-1}},$$

where $M$ is a positive constant. Taking into account that $\epsilon_k \to 0, (k M^{q_{k-1}})^{1/q_k} \to 1$ and $p_k/q_k \to 0$ as $k \to \infty$, we have

$$\liminf_{k \to \infty} \|s_{n_k}\|_{B(0, R_0)} \geq \liminf_{k \to \infty} |s_{n_k}(a)|^{1/n_k} \geq e^n > 1,$$

which by Lemma 4.3 gives the required result.

**Remark.** — The same idea of proof may be used to show that Theorem 3 remains true if $E \subset \mathbb{C}^N$ is of type $F_\sigma$. The implication $(a) \Rightarrow (b)$ holds for every set $E$ with $V_E \equiv 0$. 

– 204 –
6. Approximation by polynomials
with restricted growth near infinity

Let $E$ be a subset of $\mathbb{C}^N$ with $V_E \equiv 0$. Let $\Gamma$ be a non-pluripolar subset of an open connected set $G$. Let $f$ be a function holomorphic in $G$. The following theorem is an $N$-dimensional counterpart of Theorem 1 in [5].

**Theorem 4.**— If $\{P_n\}$ is a sequence of polynomials of $N$ complex variables with $\deg P_n \leq d_n$ ($d_n < d_{n+1}$, $d_n$ is an integer) such that

$$\limsup_{n \to \infty} |f(z) - P_n(z)|^{1/d_n} < 1, \quad z \in \Gamma,$$

(6.1)

$$\limsup_{n \to \infty} |P_n(z)|^{1/d_n} \leq 1, \quad z \in E,$$

(6.2)

then the maximal domain of existence $G_f$ of $f$ is a polynomially convex open subset of $\mathbb{C}^N$ such that

$$\limsup_{n \to \infty} \|f - P_n\|_K^{1/d_n} < 1$$

(6.3)

for every compact subset $K$ of $G_f$.

If, moreover, the sequence $\{d_{n+1}/d_n\}$ is bounded then $G_f = \mathbb{C}^N$.

Observe that the point-wise geometrical convergence (6.1) of $\{P_n\}$ to $f$ on a non-pluripolar set $\Gamma$ along with the restricted growth (6.2) of $\{P_n(z)\}$ at every point $z$ of a non-thin set $E$ imply the uniform geometrical convergence (6.3) of $\{P_n\}$ to $f$ on every compact subset $K$ of $G_f$.

In Theorem 1 of [5] the authors assume that $\Gamma$ is a nontrivial continuum in $\mathbb{C}$, and $\limsup_{k \to \infty} \|f - P_n\|_\Gamma^{1/d_n} < 1$, which in the case of $N = 1$ is more restrictive than (6.1).

**Proof of Theorem 4.**— 1°. First we shall show that (6.3) is true for every compact subset $K$ of $G$. To this order observe that the function

$$u_n(z) := \frac{1}{d_n} \log |f(z) - P_n(z)|$$

is PSH($G$). The condition (6.2) and property (5) of the necessary and sufficient conditions 2.18 for non-thinness imply that for every compact subset $K$ of $G$ and for every $\epsilon > 0$ there exist a positive constant $M = M(K, \epsilon)$ and a positive integer $n_0 = n_0(K, \epsilon)$ such that $u_n(z) \leq \frac{1}{d_n} \log (M + M(1+\epsilon)^{d_n}) \leq \frac{1}{d_n} \log (2M) + \epsilon, \quad n \geq n_0, \quad z \in K$. Hence $u := \limsup_{n \to \infty} u_n \leq 0$ in $G$, and
$u < 0$ on $\Gamma$ by (6.1). The function $u^*$ is non-positive and plurisubharmonic in $G$, and, by the theorem on negligible sets, we have $u(z) = u^*(z) < 0$ on $\Gamma \setminus A$, where $A$ is pluripolar. By the maximum principle $u^*(z) < 0$ in $G$ which, by the Hartogs Lemma, implies the required inequality (6.3) for compact sets $K \subset G$.

$2^0$. Put $\Omega := \{a \in \mathbb{C}^N; \text{the sequence } \{P_n\} \text{ is uniformly convergent in a neighborhood of } a\}$. It follows from $1^0$ that $G \subset \Omega$. Let $G_f$ denote the connected component of $\Omega$ containing $G$. It is clear that $G_f$ is polynomially convex. We claim that $G_f$ is the maximal domain of existence of $f$. It is clear that $\tilde{f}(z) := \lim_{n \to \infty} P_n(z)$, $z \in G_f$, is holomorphic in $G_f$, and $\tilde{f} = f$ in $G$. We need to show that $G_f$ is the maximal domain of existence of $\tilde{f}$. By $1^0$ we have (6.3) with $G$ replaced by $G_f$ and $f$ by $\tilde{f}$.

Suppose, contrary to our claim, that there exist $a \in G_f, r > \text{dist}(a, \partial G_f) =: r_0$ and a function $g$ holomorphic in the ball $B(a, r)$ such that $g(z) = \tilde{f}(z)$ if $\|z - a\| < r_0$. By $1^0$ we have $\limsup_{n \to \infty} \|g - P_n\|_{K}^{1/d_n} < 1$ for every compact subset $K$ of the ball $B(a, r)$. Therefore the sequence $\{P_n\}$ converges locally uniformly in this ball which contains boundary points of $G_f$. This contradicts the definition of the last set.

$3^0$. Let us assume that the sequence $\{\frac{d_n+1}{d_n}\}$ is bounded, say $d_{n+1}/d_n \leq \alpha$, $n \geq 1$. By $2^0$, it is sufficient to show that in this case $\Omega = \mathbb{C}^N$. Consider the following sequence of elements of the Lelong class $L$

$$u_n(z) := \frac{1}{d_{n+1}} \log |P_{n+1}(z) - P_n(z)|, \quad z \in \mathbb{C}^N.$$

Put $u(z) := \limsup_{n \to \infty} u_n(z), z \in \mathbb{C}^N$. It follows from (6.1) that for every $z \in \Gamma$ there exist $\epsilon > 0$ and $M > 0$ such that $u_n(z) \leq \frac{1}{d_{n+1}} \log[M e^{-\epsilon d_{n+1}} + M e^{-\epsilon d_n}] \leq \frac{1}{d_{n+1}} \log(2M) - \frac{1}{\alpha} \epsilon, n \geq 1$. Hence, $u(z) < 0$ for every $z \in \Gamma$.

One can easily check that if $z \in E$, then by (6.2) $u(z) \leq 0$. Therefore $u^* \in L$ and $u^*(z) \leq 0$ for all $z \in E \setminus A$, where $A$ is pluripolar. It follows that $u^* \leq V_E^* = 0$ in $\mathbb{C}^N$. Hence $u^* = c = \text{const} \leq 0$. But, by the theorem on negligible sets, $u^*(z) < 0$ on a non-empty subset of $\Gamma$ which implies that $c < 0$. Hence, by Hartogs Lemma, for every compact subset $K$ of $\mathbb{C}^N$ and for $0 < \epsilon < -c$ there exists $n_0 = n_0(K, \epsilon)$ such that $u_n(z) \leq -\epsilon$ for all $z \in K$ and $n \geq n_0$. It follows that the sequence $\{P_n\}$ is uniformly convergent on $K$. By the arbitrary property of $K$ we get $\Omega = \mathbb{C}^N$.

The method of proof of Theorem 4 may be used to show that the following corollaries are true.
Corollary 6.1. — Let $E$ be a subset of $\mathbb{C}^N$ with $V_E \equiv 0$. Let $\Gamma$ be a non-pluriipolar subset of $\mathbb{C}^N$. Let $\{d_n\}$ be a strictly increasing sequence of positive integers such that $d_{n+1}/d_n \leq \alpha$, $n \geq 1$, with $\alpha = \text{const} > 1$.

If $f : \Gamma \to \mathbb{C}$ is a function such that there exists a sequence of polynomials $\{P_n\}$ with $\deg P_n \leq d_n$ such that

$$\limsup_{n \to \infty} |f(z) - P_n(z)|^{1/d_n} < 1, \quad z \in \Gamma,$$

$$\limsup_{n \to \infty} |P_n(z)|^{1/d_n} \leq 1, \quad z \in E,$$

then $f$ extends to an entire function $\tilde{f}$ such that for every compact set $K \subset \mathbb{C}^N$ we have

$$\limsup_{n \to \infty} \|\tilde{f} - P_n\|_{K}^{1/d_n} < 1.$$

Indeed, by (6.4), given $z \in \Gamma$, there are $M > 0$ and $0 < \theta = \theta(z) < 1$ such that $|f(z) - P_n(z)| \leq M \theta^{d_n}, \quad n \geq 1$. Hence $|P_{n+1}(z) - P_n(z)| \leq 2M \theta^{d_n+1}$ which implies

$$\limsup_{n \to \infty} |P_{n+1}(z) - P_n(z)|^{1/d_{n+1}} < 1, \quad z \in \Gamma.$$

By (6.5), given $z \in E$ and $\epsilon > 0$, there is $M > 0$ such that $|P_{n+1}(z) - P_n(z)| \leq |P_{n+1}(z)| + |P_n(z)| \leq Me^{d_n+\epsilon} + e^{d_n\epsilon} \leq 2Me^{\alpha d_n}, \quad n \geq 1$, which implies that

$$\limsup_{n \to \infty} d_n \sqrt{\frac{1}{d_{n+1}} |P_{n+1}(z) - P_n(z)|} \leq 1, \quad z \in E.$$

Put $u(z) := \limsup_{n \to \infty} \frac{1}{d_{n+1}} \log |P_{n+1}(z) - P_n(z)|$, $z \in \mathbb{C}^N$. Then $u^* \in \mathcal{L}$, $u^* \leq 0$ on $E$ and $u^* < 0$ on $\Gamma \setminus A$, where $A$ is pluripolar. Therefore $u^* = \text{const} < 0$. Hence, by Hartogs Lemma, we have $\limsup \|P_{n+1}P_n^{-1/d_{n+1}} < 1$ for every compact subset $K$ of $\mathbb{C}^N$. It follows that $\tilde{f} := P_1 + \sum_1^\infty (P_{n+1} - P_n)$ is an entire function with the required properties.

In the sequel $P_n$ denotes polynomials with $\deg P_n \leq d_n$, where $d_n$ are integers with $1 \leq d_n < d_{n+1} \leq \alpha d_n$, $\alpha = \text{const} > 1$, $\Gamma$ is a non-pluriipolar subset of $\mathbb{C}^N$, and $f$ is a complex valued function defined on $\Gamma$.

Corollary 6.2. — If $f$ is holomorphic in an open connected set $G$ containing $\Gamma$ such that

$$\limsup_{n \to \infty} |f(z) - P_n(z)|^{1/d_n} < 1, \quad z \in \Gamma,$$
Sets in $\mathbb{C}^N$ with vanishing global extremal function

$$\limsup_{n \to \infty} |P_n(z)|^{\frac{1}{d_n}} \leq 1, \quad z \in G, \quad (6.7)$$

then $f$ has a holomorphic extension $\tilde{f}$ to $G$ such that

$$\limsup_{n \to \infty} \|\tilde{f} - P_n\|^{\frac{1}{K}}_K < 1, \quad \limsup_{n \to \infty} \|P_{n+1} - P_n\|^{\frac{1}{K_{n+1}}} < 1, \quad (6.8)$$

for every compact set $K \subset G$. If, moreover, $G$ is non-thin at infinity then there is an entire function $\tilde{f}$ satisfying (6.8) for $G = \mathbb{C}^N$ such that $\tilde{f} = f$ on $\Gamma$.

**Corollary 6.3.** — If

$$\limsup_{n \to \infty} |f(z) - P_n(z)|^{\frac{1}{d_n}} = 0, \quad z \in \Gamma, \quad (6.9)$$

then $f$ extends to a unique entire function

$$\tilde{f}(z) = P_1(z) + \sum_{j=1}^{\infty} (P_{n+1}(z) - P_n(z)), \quad z \in \mathbb{C}^N,$$

and (6.8) is satisfied.

In order to show the last two corollaries, define

$$u(z) := \limsup_{n \to \infty} \frac{1}{d_{n+1}} \log |P_{n+1}(z) - P_n(z)|,$$

observe that $u^* \in L$, and check that $u^*(z) < 0$ on $G$ in the case of Corollary 6.2 (resp., $u^*(z) = -\infty$ on $\mathbb{C}^N$ in the case of Corollary 6.3) which, by Hartogs Lemma, implies Corollary 6.2 (resp., Corollary 6.3).

**Bibliography**


